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Abstract

An ensemble of LDPC convolutional codes with parity-check matrices composed of permutation matrices is introduced. The convergence of the iterative belief propagation based decoder for terminated convolutional codes in the ensemble when operating on the erasure channel is analyzed. The structured irregularity in the Tanner graph of the codes leads to significantly better thresholds when compared to the corresponding LDPC block codes.

1 Introduction

Low-density parity-check (LDPC) block codes, invented by Gallager [1], have been shown to achieve excellent performance on a wide class of channels. The convolutional counterpart of LDPC block codes, LDPC convolutional codes, have been described in [2][3][4]. Both LDPC block and convolutional codes are defined by sparse parity-check matrices and can be decoded iteratively using message passing algorithms (e.g., belief propagation) with complexity per bit per iteration independent of the block length or constraint length. This makes iterative decoding of LDPC codes with large block length or constraint length feasible.

If all messages exchanged during the iterations are independent, it is possible to analyze the performance of the decoder. For a simple hard-decision algorithm, an upper bound on the bit error probability, as function of the number of iterations, was derived in [1]. This can be used to find a lower bound on the maximum channel parameter (convergence threshold) for which the error probability goes to zero as the number of iterations goes to infinity. An analysis of iterative decoding for the binary erasure channel was given in [5][6]. For this channel, the convergence threshold for the belief propagation algorithm can be described analytically. The density evolution technique, proposed in [7][8], generalizes the ideas in [1] and [6] to a wider class of channels and message passing

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algorithms. In general, however, numerical methods are required to determine tight bounds on the convergence thresholds.

In [9], the existence of a sequence of (J, K) regular[1] LDPC convolutional codes for which an arbitrary number of independent iterations is possible was demonstrated. Based on this result, it follows that the threshold of (J, K) regular LDPC block codes is a lower bound on the threshold of (J, K) regular LDPC convolutional codes for any message passing algorithm and channel. Moreover, simulation results on the additive white Gaussian noise channel (see [3][4]) indicate the possibility that LDPC convolutional codes may have better thresholds than corresponding LDPC block codes.

In this paper we consider a class of regular LDPC convolutional codes with parity-check matrices composed of blocks of $M \times M$ permutation matrices. (A permutation matrix P is a matrix consisting of zeros and ones such that each row and column has exactly a single one.) Iterative belief propagation decoding of terminated convolutional codes on the erasure channel is analyzed, where as will be seen, the termination leads to a structured irregularity in the Tanner graph. We find that this structured irregularity leads to significantly better thresholds compared to the corresponding randomly constructed regular and irregular LDPC block codes.

2 Convolutional Code Ensemble

A rate $R = b/c$ binary convolutional code can be defined as the set of sequences $\boldsymbol{v} = (\ldots, \boldsymbol{v}_{-1}, \boldsymbol{v}_0, \boldsymbol{v}_1, \ldots)$, $\boldsymbol{v}_t \in \mathbb{F}_2^c$, satisfying the equality $\boldsymbol{v}\boldsymbol{H}^{\mathrm{T}} = \boldsymbol{0}$, where the infinite syndrome former matrix $\boldsymbol{H}^{\mathrm{T}}$ is given by

$$
\boldsymbol{H}^{\mathrm{T}} = \begin{pmatrix}
\ddots & & & \ddots & & \\
\boldsymbol{H}_0^{\mathrm{T}}(0) & \ldots & \boldsymbol{H}_{m_{\mathrm{s}}}^{\mathrm{T}}(m_{\mathrm{s}}) & & & \\
& \ddots & & & \ddots & \\
& & \boldsymbol{H}_0^{\mathrm{T}}(t) & \ldots & \boldsymbol{H}_{m_{\mathrm{s}}}^{\mathrm{T}}(t + m_{\mathrm{s}}) \\
& & \ddots & & & \ddots
\end{pmatrix} ,
\tag{1}
$$

and each $\boldsymbol{H}_i^{\mathrm{T}}(t + i)$ is a $c \times (c - b)$ binary matrix. If $\boldsymbol{H}^{\mathrm{T}}$ defines a rate $R = b/c$ convolutional code, the matrix $\boldsymbol{H}_0^{\mathrm{T}}(t)$ must have full rank for all time instants t. In this case, by suitable row permutations we can ensure that the last $(c - b)$ rows are linearly independent. Then the first b symbols at each time instant are information symbols and the last $(c - b)$ symbols the corresponding parity symbols. The largest i such that $\boldsymbol{H}_i^{\mathrm{T}}(t + i)$ is a nonzero matrix for some t is called the syndrome former memory m_s.

LDPC convolutional codes have sparse syndrome former matrices. A (J, K) regular LDPC convolutional code is defined by a syndrome former that contains exactly J ones in each row and K ones in each column.

We now define the ensemble of LDPC convolutional codes of interest. Though the ensemble can be defined more generally, in this paper we focus on the case $K = 2J$, $J > 2$. We construct LDPC convolutional codes defined by syndrome formers $\boldsymbol{H}^{\mathrm{T}}$ with syndrome former memory $m_{\mathrm{s}} = J - 1$. For $i = 0, 1, \ldots, J - 1$, the sub-matrices $\boldsymbol{H}_i^{\mathrm{T}}(t + i)$ of the syndrome former are the matrices $\left(\boldsymbol{P}_i^{(0)}(t + i), \boldsymbol{P}_i^{(1)}(t + i)\right)^{\mathrm{T}}$, where $\boldsymbol{P}_i^{(h)}(t + i)$, $h = 0, 1$,

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[1](J, K) regular LDPC codes are defined by parity-check matrices having J ones in each column of the matrix and K ones in each row of the matrix.
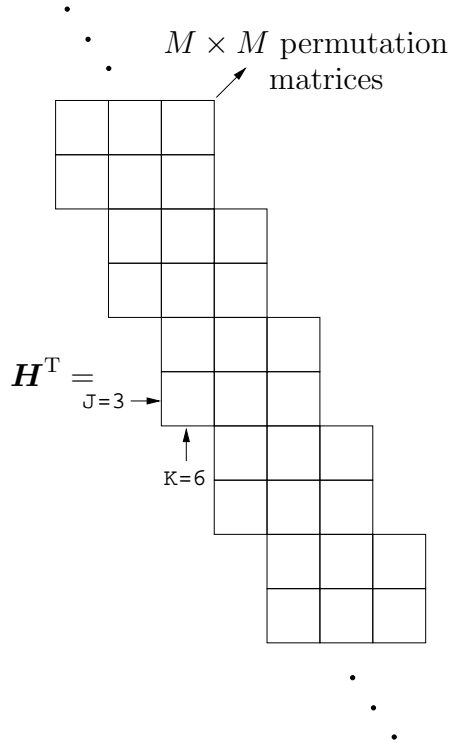
$\boldsymbol{H}^{\mathrm{T}} =$

$\mathrm{J=3} \rightarrow$

$\mathrm{K=6}$

$M \times M$ permutation matrices

PSfrag replacements

Figure 1: Syndrome former matrix for a code in $C(M, 2M, 3)$.

is an $M \times M$ permutation matrix. All other entries of the syndrome former are zero matrices. Equivalently, each $\boldsymbol{H}_i^{\mathrm{T}}(t+i)$, $i = 0, 1, \ldots, J-1$, is a $c \times (c-b)$ binary matrix, where $c = 2M$ and $b = M$. By construction it follows that each row of the syndrome former $\boldsymbol{H}^{\mathrm{T}}$ has J ones and each column K ones. Let $\mathcal{C}(M, 2M, J)$ denote this ensemble of $(J, 2J)$ regular LDPC convolutional codes. Note that the ensemble of codes $\mathcal{C}(M, 2M, J)$ is time-varying. Figure 1 shows the syndrome former matrix of a $(3, 6)$ regular LDPC convolutional code in $\mathcal{C}(M, 2M, 3)$.

Since $\boldsymbol{H}_0^{\mathrm{T}}(t)$ consists of two non-overlapping permutation matrices, it has full rank. Hence $\boldsymbol{H}^{\mathrm{T}}$ defines a rate $R = \frac{M}{2M}$ code. Further, the constraint imposed by the syndrome former, i.e.,

$$\boldsymbol{v}_t \boldsymbol{H}_0^{\mathrm{T}}(t) + \boldsymbol{v}_{t-1} \boldsymbol{H}_1^{\mathrm{T}}(t) + \cdots + \boldsymbol{v}_{t-m_s} \boldsymbol{H}_{m_s}^{\mathrm{T}}(t) = \boldsymbol{0}, \quad \boldsymbol{v}_t \in \mathbb{F}_2^{2M}, t \in \mathbb{Z} \quad (2)$$

can be used to perform a systematic encoding of the code [2]. The constraint length of codes in $\mathcal{C}(M, 2M, J)$ is defined as $\nu = (m_s + 1) \cdot c = J \cdot 2M = KM$. Thus, the constraint length of codes in the ensemble $\mathcal{C}(M, 2M, 3)$ is $6M$.

The Tanner graph for a code in $\mathcal{C}(M, 2M, J)$ can be obtained from its syndrome former matrix. The graph consists of symbol and check nodes, each symbol node corresponding to a particular row and each check node corresponding to a column of the syndrome former matrix $\boldsymbol{H}^{\mathrm{T}}$. There is an edge between a symbol node and a check node if the corresponding symbol takes part in the respective parity-check equation. For the Tanner graph of a convolutional code we can associate a notion of time. At each time instant t the sub-matrices $\boldsymbol{H}_i(t)$ of the syndrome former $\boldsymbol{H}^{\mathrm{T}}$ (see (1)) lead to $c-b$ check nodes in the Tanner graph. Similarly, for each time instant t, we get c symbol nodes in the Tanner graph. Observe that $\boldsymbol{H}_i(t)$ is non-zero only from $i = 0, 1, \ldots, m_s$, hence nodes in the Tanner graph can be connected at most m_s time units away. The Tanner graph of a code in the ensemble $C(M, 2M, 3)$ is comprised of $c = 2M$ symbol nodes and $c-b = M$ check
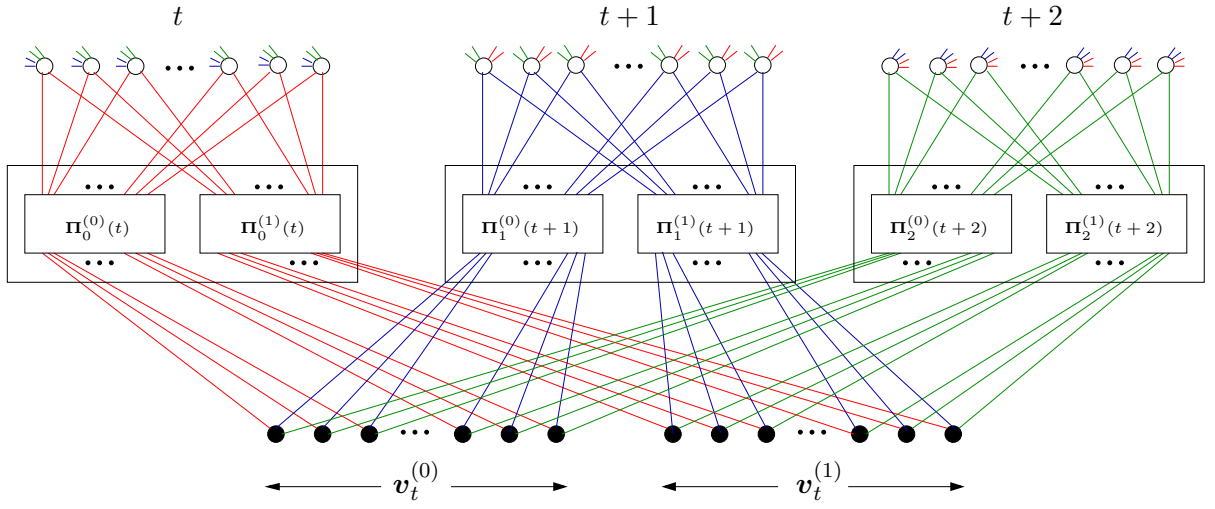
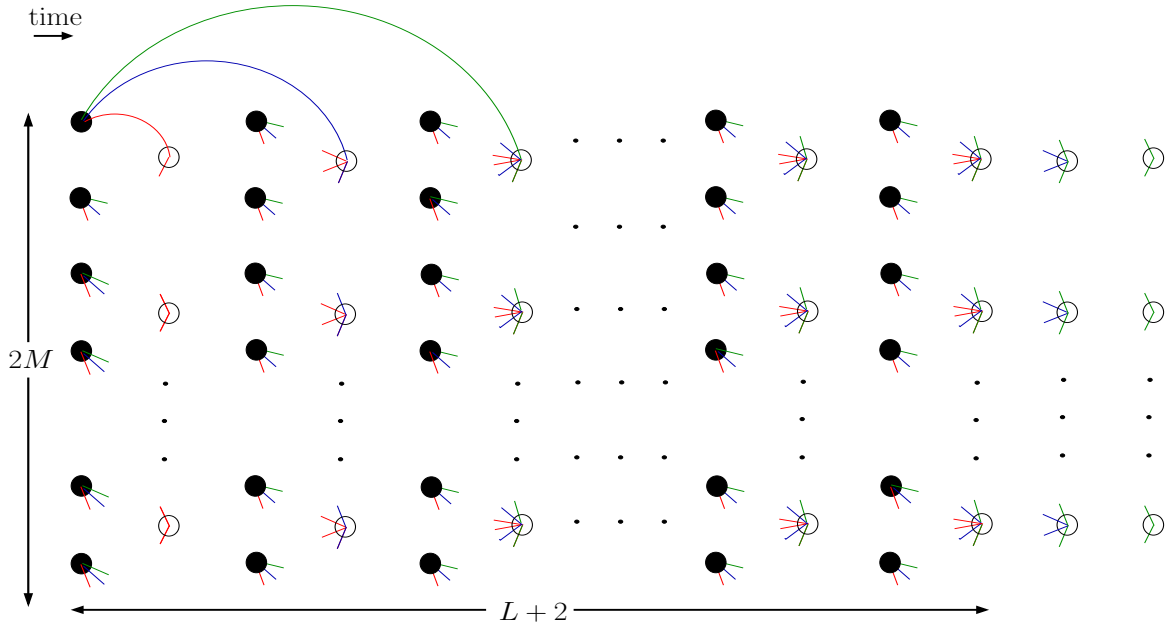Figure 2: Tanner graph connections of the $2M$ symbol nodes at time t.

Figure 3: Tanner graph of a terminated convolutional code obtained from $\mathcal{C}(M, 2M, 3)$.

nodes for each time instant. Further, each node can be connected at most $m_s = 2$ time units away. Figure 2 illustrates how the information symbols $\boldsymbol{v}_t^{(0)}$ and parity symbols $\boldsymbol{v}_t^{(1)}$ at time t are connected through different permutation matrices to parity-check equations at time t, $t+1$, and $t+2$.

For practical applications, a convolutional encoder starts from a known state (usually the all-zero state) and after the data to be transmitted has been encoded, the encoder is terminated back to the all-zero state. It can be shown that for the ensemble $\mathcal{C}(M, 2M, J)$ we need a tail for no more than m_s time instants, i.e., $m_s M$ information bits to return the encoder back to the all-zero state [10].

Suppose that we wish to transmit LM information bits using a code from $\mathcal{C}(M, 2M, J)$. It follows that the terminated code has rate $R = L/2(L+m_s) = 0.5/(1+\frac{J-1}{L})$. Note that for $L \gg J-1$, the rate loss is negligible. In Figure 3 we show the Tanner graph of a terminated code obtained from a convolutional code in the ensemble $C(M, 2M, 3)$. Observe that symbols are zero both before encoding begins, i.e., $t = 1$, and after termination, i.e.,

$t = L + 2$. Hence in obtaining the Tanner graph of the terminated convolutional code edges connecting check nodes to any of the symbol nodes that are known to be zero can be omitted. For example, we can disconnect the check nodes at time $t = 1$ from symbol nodes at time $t < 1$, since these are known to be zero. It follows that, while all symbol nodes in Figure 3 have degree three, the check nodes can have degree either two, four, or six. Note that even though the convolutional code is regular, knowing bits perfectly before encoding and after termination leads to a slight irregularity in the Tanner graph of the terminated convolutional code.

3   Decoding Analysis on the Erasure Channel

Figure 4 shows the binary erasure channel. The probability of an erasure is p and with probability $1 - p$ we receive the transmitted symbol correctly. In each iteration a
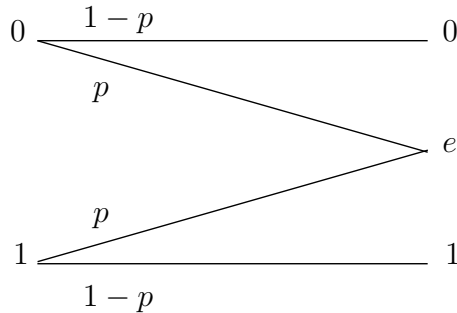

Figure 4: Binary Erasure Channel

message passing decoder exchanges messages between the symbol nodes and the check nodes. On the erasure channel, a symbol node can be recovered correctly if its channel value or any of the messages from the check nodes to which it connects is not an erasure. Thus convergence of belief propagation decoding on the erasure channel can be analyzed by tracking the probability of erasure of the messages. This is straightforward to do as long as the messages exchanged during the iterations are independent. The next theorem guarantees that the number of independent iterations possible on the Tanner graph of the block code, produced by terminating convolutional codes from the ensemble $\mathcal{C}(M, 2M, J)$, can be made arbitrarily large.

Theorem 1 *Let*

$$\ell_0 = \left\lfloor \frac{\log 2JM + \log \varepsilon - \log a}{2 \log[(J-1)(2J-1)]} \right\rfloor \tag{3}$$

where

$$a = \frac{2(2J-1)J^3}{[(J-1)(2J-1)-1]^2} \tag{4}$$

is a constant independent of M and $0 < \varepsilon < 1/2$. Then there exists an M_0 such that, for all $M > M_0$, the probability that the number of independent iterations ℓ, for any symbol in the block code with $L \geq J$ information blocks obtained by terminating a randomly chosen code from $\mathcal{C}(M, 2M, J)$, satisfies $\ell < \ell_0$ is less than ε.

The proof of this theorem is based on an analogous theorem for LDPC block codes given in [11]. Theorem 2 implies that the fraction of symbols for which the number of
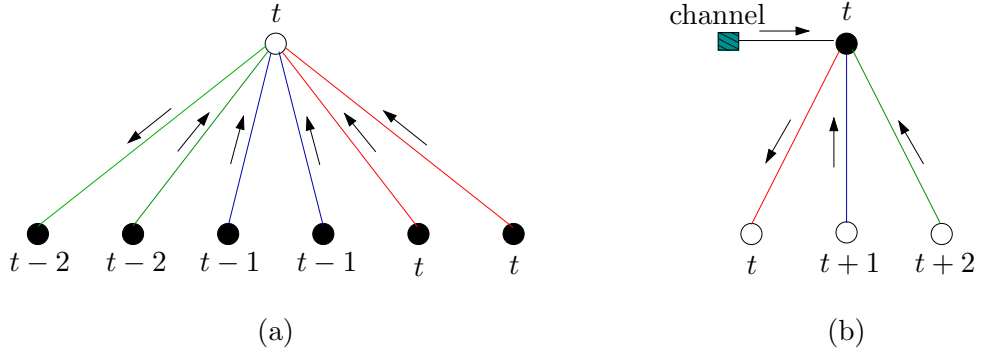
Figure 5: Illustration of the messages (a) to a symbol node and (b) to a check node for the case $J = 3$.

independent iterations is less than ℓ_0 is at most ε. We then fix these symbols, for example by setting them to bit value '0', and do not transmit them. Fixing these symbols leads to a decrease in rate by a factor $(1 - \varepsilon)$, but all remaining symbols are then guaranteed to have at least ℓ_0 independent iterations. By making M sufficiently large ε can be made arbitrarily small, and the rate loss is therefore negligible. The asymptotic growth in ℓ_0 indicated by (3) is exactly the same as obtained for LDPC block codes [1], except that the block length is replaced by the constraint length $\nu = 2JM$.

Consider now the ℓth iteration of the decoding procedure, where $1 \leq \ell < \ell_0$. The message sent from a check node at time t to a connected symbol node is an erasure if at least one of the symbols represented by the other neighboring nodes has been erased. As shown in Figure 5 (a), these symbol nodes belong to different time instants. It follows that the probability $q_{t,t-k}^{(\ell)}$ that the message from a check node at time t to a symbol node at time $t - k$ is an erasure is equal to

$$q_{t,t-k}^{(\ell)} = 1 - \left[\left(1 - p_{t-k,t}^{(\ell-1)} \right) \prod_{k' \neq k} \left(1 - p_{t-k',t}^{(\ell-1)} \right)^2 \right] \ , \quad k, k' \in \{0, m_{\mathrm{s}}\} \ . \tag{5}$$

Here $p_{t-k,t}^{(\ell-1)}$ denotes the probability that the message sent in the previous iteration $\ell - 1$ from a symbol node at time $t - k$ to a check node at time t corresponds to an erasure. For $\ell = 0$ these values are initialized as $p_{t',t}^{(0)} = 0$ for all t' and t. For terminated convolutional codes we also have $p_{t',t}^{(\ell)} = 0$ if $t' < 1$ or $t' > L + m_{\mathrm{s}}$. This condition takes into account the lower check node degrees at the beginning and the end of the Tanner graph. The message from a symbol node to a check node is an erasure if all incoming messages from the neighboring check nodes and that from the channel are erasures. The flow of these messages is shown in Figure 5 (b). Thus we have

$$p_{t,t+k}^{(\ell)} = p \prod_{k' \neq k} q_{t+k',t}^{(\ell)} \ , \quad k, k' \in \{0, m_{\mathrm{s}}\} \ . \tag{6}$$

For regular LDPC block codes, the distribution of the messages exchanged in iteration ℓ are the same for all nodes regardless of their position within the graph. Likewise, for the random irregular code ensembles considered in [5], the message distributions are averaged over all codes and only a single mixture density has to be considered for all check nodes and all symbol nodes, respectively. In our case, while nodes at the same time instant behave identically, the messages from nodes at different times behave differently and must
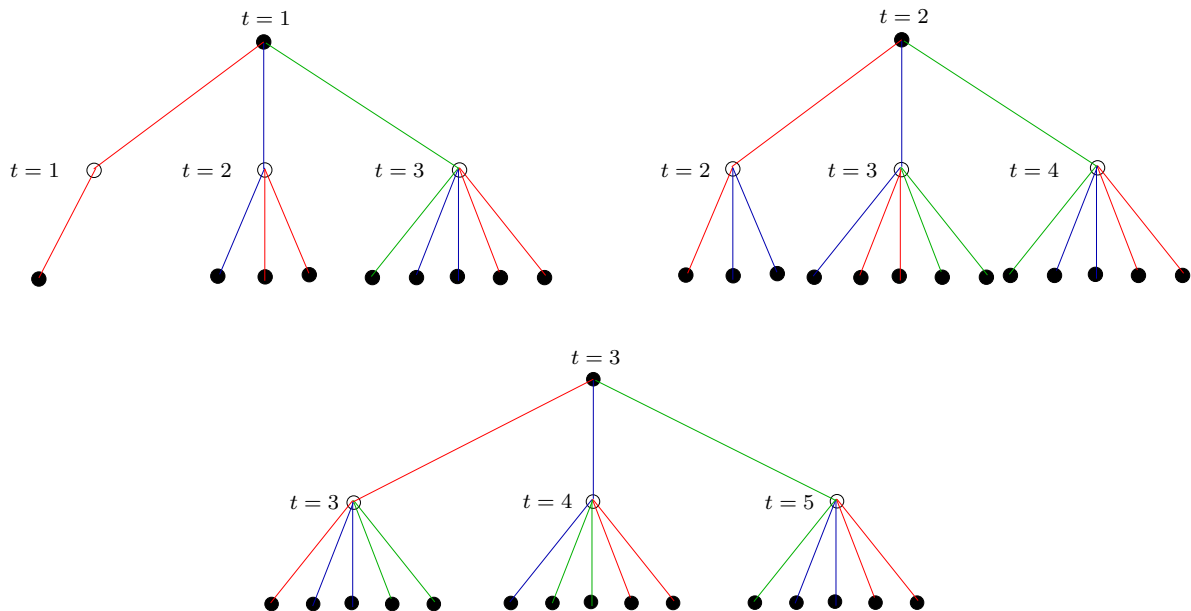
Figure 6: The first level of computation trees for $t = 1, 2, 3$ with $J = 3$.

all be tracked separately. Figure 6 shows the first level of the decoding computation trees for the first three symbol levels in the case $J = 3$. Although only the first and last m_s levels of check nodes have lower degrees, their effect evolves through the complete Tanner graph. To take this structure into account, in each iteration $\ell = 1, \ldots, \ell_0 - 1$, first (5) and then (6) is applied for all $t = 1, \ldots, L + m_\mathrm{s}$. Finally, for $\ell = \ell_0$ the product in (6) is taken over all k' without exclusion of k.

4 Results

To obtain convergence thresholds for terminated codes from an ensemble $\mathcal{C}(M, 2M, J)$ the recursive equations (5) and (6) can be evaluated numerically for all time instants. For proving that the erasure probabilities of all symbols converge to zero, as described in [11], it is sufficient to check if they reach a certain breakout value. The convergence threshold for an ensemble of codes can be found by testing this condition for different channel values p.

Note that in addition to the node degrees J and K the value L is another parameter that influences the result. In Table 1 we present the thresholds obtained for different L for the $(3, 6)$ case. The first column shows $L + m_\mathrm{s} = L + 2$ (the number of information bits per block is LM), the second column shows the rate of the terminated convolutional code, and the third column gives the threshold p^*_conv.[2] For $L + m_\mathrm{s} = 10$, the threshold is quite high, in fact larger than the capacity of rate $R = 1/2$ codes. However, in this case there is a significant rate loss and the terminated code has rate only $R = 0.35$. For larger L the threshold remains constant at $p^*_\mathrm{conv} = 0.488$. Tracking messages becomes increasingly difficult as we increase L but the behavior in Table 1 suggests that the rate of the terminated code can be made arbitrarily close to 0.5 without affecting the threshold.

The fourth column of Table 1 shows the thresholds for random irregular LDPC block codes having the same degree distributions as the terminated convolutional codes (see [5]

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[2]Strictly speaking what we obtain is only a lower bound on the threshold.

$L + m_s$	R_{conv}	p^*_{conv}	$p^*_{\text{irr-blk}}$	Iter.
10	0.400	0.504	0.501	161
25	0.460	0.488	0.456	3423
50	0.480	0.488	0.442	15912
100	0.490	0.488	0.436	40899
150	0.493	0.488	0.433	65884

Table 1: Thresholds for the ensemble $\mathcal{C}(M, 2M, 3)$ with different L and for the corresponding irregular block codes.

or [8] for a definition of degree distributions). Note that the degree distribution of the irregular code is determined by L and that they have the same rate as the terminated convolutional codes. The thresholds of the terminated convolutional codes are better than those of the irregular LDPC block codes. With increasing L the degree distribution of the terminated convolutional code tends to that of a $(3, 6)$ regular LDPC block code. Therefore it is not surprising that the thresholds of the corresponding irregular LDPC block codes tend to the threshold of $(3, 6)$ regular LDPC block codes. However, the thresholds of the terminated LDPC convolutional codes remain unchanged. Hence, the improvement in threshold can be attributed to the structure imposed on the Tanner graph by the convolutional nature of the code.

The fifth column in Table 1 shows for different L the number of iterations with density evolution until the erasure probability of all messages reaches the breakout value. This reflects that for larger L the messages from the stronger nodes at the ends need more time to affect the symbols in the middle. Since for large L the required number of iterations seems to grow linearly with L, according to Theorem 1, L cannot be increased more than logarithmically with M to guarantee a certain target bit erasure probability with density evolution. While the threshold itself is independent of the length L (if large enough), we cannot prove that for a fixed M the effect from the ends of the Tanner graph carries through for arbitrary lengths. On the other hand, the logarithmic relationship between ℓ_0 and M is due to the independence assumption required for the particular method of analysis and may be too restrictive in practice. For LDPC block codes, e.g., excellent performance can be observed with iteration numbers that exceed those given in Theorem 1 by far.

In Table 2 thresholds are presented for different J. In each case L is chosen so that there is a rate loss of 0.2%. The first column in Table 2 shows the values J and K of the underlying convolutional code, the second column the rate of the terminated code, the third column the threshold obtained, and the fourth column the threshold for randomly chosen regular LDPC block codes with the same J and K as the convolutional code. The thresholds of the terminated convolutional codes are much better than for the corresponding block codes. This is reasonable since the constraint nodes at either end of the terminated convolutional codes have lower degrees than in the block code. Our observation has been that if the probability of erasure of symbol nodes at either ends tends to zero then after a sufficient number of iterations the probability of erasure of symbol nodes at all time instants tends to zero.

Interestingly, the terminated convolutional codes with higher J have thresholds better than those with lower J. This behavior is different from that of randomly constructed (J, K) regular LDPC block codes, where for a fixed rate increasing J usually worsens the

(J, K)	R_{conv}	p^*_{conv}	p^*_{blk}
(3,6)	0.499	0.488	0.429
(4,8)	0.499	0.497	0.383
(5,10)	0.499	0.499	0.341

Table 2: Thresholds for the ensembles $\mathcal{C}(M, 2M, J)$ with different J.

threshold. This is the case since for maintaining the same rate it is necessary to increase K accordingly. Note that symbol nodes with higher degrees are stronger than those with lower degrees, however lower degree check nodes are stronger than higher degree check nodes. For randomly constructed LDPC block codes larger check node degrees counteract the gain resulting from higher symbol node degrees, adversely affecting performance. However, in our case the codes with higher J still have strong check nodes of low degrees at either ends. Thus, the symbols at the ends are better protected for codes with larger symbol degrees and, hence, result in better thresholds.

Oswald and Shokrollahi have constructed LDPC block code ensembles from sequences of right regular degree distributions[3] for which the convergence thresholds of the bit erasure probability approach capacity as the maximum symbol node degree tends to infinity [12]. According to Table 2, with increasing J the convergence thresholds of the terminated LDPC convolutional codes also seem to tend to the capacity limit of rate $R = 1/2$ codes. However, in this case the degree distributions are not right regular but left regular, i.e., all symbol nodes have the same degree. Further, following the analysis in [11], for the terminated convolutional codes it can be shown that at the calculated thresholds not only the bit but also the block erasure probability with iterative belief propagation decoding goes to zero as M goes to infinity.

5   Conclusions

In this paper we show that terminated LDPC convolutional codes from the ensemble $\mathcal{C}(M, 2M, J)$ have check nodes of different degree. The terminated codes obtained from $(J, 2J)$ regular LDPC convolutional codes have better thresholds than the corresponding $(J, 2J)$ regular LDPC block codes on the erasure channel. It was observed that the thresholds of the terminated LDPC convolutional codes are even better than those of random irregular block LDPC codes with the same degree distribution. Thus, the improvement in performance is not only due to the irregular graph structure but also due to the special convolutional structure of the codes.

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[3]with fixed check node degrees

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