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Observer Synthesis for Switched Discrete-Time Linear Systems using Relaxed Dynamic Programming

Peter Alriksson[1] and Anders Rantzer[1]

***Abstract*— In this paper, state estimation for Switched Discrete-Time Linear Systems is performed using relaxed dynamic programming. Taking the Bayesian point of view, the estimation problem is transformed into an infinite dimensional optimization problem. The optimization problem is then solved using relaxed dynamic programming. The estimate of both the mode and the continuous state can then be computed from the value-function. From an unknown initial state the estimation error goes to zero as more measurements are collected.**

I. Introduction

The subject of this paper is state estimation for a class of discrete-time hybrid systems referred to as Switching Discrete-Time Linear Systems or SLS for short:

$$\begin{aligned} x_{k+1} &= A(\theta_k)x_k + B(\theta_k)u_k + \text{noise} \\ y_k &= C(\theta_k)x_k + \text{noise} \end{aligned} \tag{1}$$

In a SLS the discrete mode θ_k of the system switches in an unknown way, which makes them somewhat different from other types of hybrid systems such as piecewise-linear systems or Markov Jump Linear Systems. The goal is to compute an estimate of both the continuous state x_k and the discrete mode θ_k given measurements y_k.

The area of Markov Jump Linear Systems has received considerable attention in the literature the last three decades, starting with the paper by Ackerson and Fu [1]. However, the filter derived in [1] requires infinite memory and thus much attention has been devoted to deriving finite memory approximations. One popular algorithm is the Interacting Multiple Model (IMM) proposed by [2]. Other papers in this area include [3] and [4].

The estimation problem for a SLS without disturbances has been studied under various assumptions on the discrete modes. In [5] the modes are assumed to be known and a classical Luenberger observer is designed for the SLS using an LMI formulation. The assumption of known modes is relaxed in [6] and [7], where observers are derived using linear algebra methods. In [8] and [9] the mode is estimated by comparing the residuals from a bank of Luenberger observers. The continuous state is then estimated using a Luenberger observer for the resulting time-varying linear system, where the uncertainty in the mode estimate is ignored.

The problem has also been approached from a receding horizon point of view. In [10] the moving horizon estimation problem is solved using mixed integer quadratic program solvers. Recently, Alessandri ([11]) used linear algebra methods to form a set of possible mode sequences, each for which a quadratic program is solved. The method was simplified further in [12].

When designing observers for SLS the concept of observability plays a central role. However the notion of observability for a SLS is far more complex than for linear systems. The concept has been treated in numerous papers including [13] and [14].

The method proposed in this paper is not based on residuals so no minimum number of time steps between switches need to be assumed, as is the case in for example [8]. Also there is no need to specify a time horizon as in the receding horizon approaches. The complexity of the algorithm can be influenced by the choice of a slack parameter $\overline{\alpha}$.

The paper is organized as follows: Section II presents the model under consideration and introduces some notation. In the main Section III, the estimation problem is first transformed to an optimization problem which is then solved using relaxed dynamic programing. Section IV deals with stability of the proposed observer. Finally Section V demonstrates the algorithm for a simple example.

II. Problem Statement

The system under consideration is

$$\begin{aligned} x_{k+1} &= A(\theta_k)x_k + B(\theta_k)u_k + w_k \\ y_k &= C(\theta_k)x_k + v_k \end{aligned} \tag{2}$$

where $x_k \in \mathbf{R}^n$ denotes the continuous state, $u_k \in \mathbf{R}^m$ a known input, $w_k \in \mathbf{R}^n$ the process disturbance. The mode of the system at time k is denoted θ_k and takes values from a finite set $\mathcal{I} = \{1, \ldots, M\}$. The system is observed through a continuous measurement $y_k \in \mathbf{R}^p$ which is corrupted by a measurement disturbance $v_k \in \mathbf{R}^p$.

The process and measurement disturbances are assumed to be white zero mean Gaussian stochastic processes with covariance $Q \in \mathbf{R}^{n\times n}$ and $R \in \mathbf{R}^{p\times p}$ respectively. It is also assumed that w_k is independent of v_k.

The mode variable θ_k is modeled as an exogenous variable governed by some unknown external process.

Now introduce the following notation:

$$X_{[t_0,T]} := \{x_{t_0}, \ldots, x_T\} = \{x_k\}_{k=t_0}^{T}$$

denotes the sequence of continuous states at time t_0 up to time T. Further let $X_T = X_{[0,T]}$ and define Θ_T and Y_T in the same way.

III. Observer Synthesis Using Relaxed Dynamic Programming

In a general estimation problem the conditional probability

$$P[X_T, \Theta_T | Y_T] \tag{3}$$

of the unknown variables, in this case the state and mode, given all measurements up to now plays a central roll. If this quantity is known all different types of estimates can be computed. Here we focus on the maximum a posteriori Bayesian estimate

$$\{\hat{X}_T, \hat{\Theta}_T\} = \arg\max_{X_T, \Theta_T} P[X_T, \Theta_T | Y_T] \tag{4}$$

If the disturbances acting on the system are assumed to be Gaussian and independent the estimation problem can be cast as the following optimization problem (see [15])

$$\begin{array}{ll} \min_{X_T, \Theta_T} & J_T \\ \text{subject to} & (6) \end{array} \tag{5}$$

where

$$\begin{aligned} J_T &= ||x_0 + \check{x}_0||^2_{P_0^{-1}} + \sum_{k=0}^{T} ||v_k||^2_{R^{-1}} + \sum_{k=0}^{T-1} ||w_k||^2_{Q^{-1}} \\ v_k &= y_k - C(\theta_k) x_k \\ w_k &= x_{k+1} - A(\theta_k) x_k - B(\theta_k) u_k \end{aligned} \tag{6}$$

and $\|x\|_Q^2 = x^T Q x$. However this optimization problem grows with time making it practically impossible to solve. Using forward dynamic programming the problem can be transformed to (see [16])

$$V^*_{j+1}(x_{j+1}) = \min_{x_j, \theta_j} \left\{ V^*_j(x_j) + L(v_j, w_j) \right\} \tag{7}$$

As the value function,V_j^*, summarizes all previous information up to now it can be interpreted as a measure of the unlikeliness of a particular estimate. In the case of a linear system without switches the conditional probability is a Gaussian distribution and the corresponding value function just a quadratic function. This allows for the value iteration to be solved analytically which yields the Kalman filter. Next, we will show that the value function for the SLS estimation problem can be expressed as a minimum of quadratic functions

$$V^*_j(x_j) = \min_{\pi \in \Pi_j} \begin{bmatrix} x_j \\ 1 \end{bmatrix}^T \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{12}^T & \pi_{22} \end{bmatrix} \begin{bmatrix} x_j \\ 1 \end{bmatrix} \tag{8}$$

However the size of the finite set of matrices Π_j grows exponentially with time which makes an exact value iteration practically impossible. As proposed in [17] the value iteration can be relaxed if the Bellman equality is replaced by two inequalities instead. This allows for the optimization problem to be *solved* efficiently. The resulting value function fulfills the following inequalities compared to the optimal value function.

$$\underline{\alpha} V^*_j(x_j) \leq V_j(x_j) \leq \overline{\alpha} V^*_j(x_j) \tag{9}$$

A smoothed estimate can be computed by solving a small optimization problem involving the computed value function. Next the different steps described above will be discussed in detail.

A. Bayesian Estimation

Following the developments in for example [16] and [15], Bayes rule is used to rewrite (3) as

$$P[X_T, \Theta_T | Y_T] = \frac{P[Y_T | X_T, \Theta_T] \; P[X_T, \Theta_T]}{P[Y_T]} \tag{10}$$

Expressions for the different parts of (10) will be derived next. Using (2) and the fact that v_k are mutually independent gives

$$P[Y_T | X_T, \Theta_T] = \prod_{k=0}^{T} p_v(v_k) \tag{11}$$

Here $p_v(\cdot)$ denotes the probability density function of the measurement disturbance v_k.

Because of the Gaussian measurement noise assumption the conditional probability (11) reduces to

$$P[Y_T | X_T, \Theta_T] = k_1 \exp\left\{ -\frac{1}{2} \sum_{k=0}^{T} ||v_k||^2_{R^{-1}} \right\} \tag{12}$$

where

$$k_1 = \left(\frac{1}{(2\pi)^{n/2} \sqrt{\det R}} \right)^{T+1}$$

is a data and state independent normalization constant.

Developing an expression for $P[X_T, \Theta_T]$ requires some additional probability theory, which will be repeated here for clarity. Recall the following relation

$$P[A \cap B \cap C] = P[A] \; P[B|A] \; P[C|A \cap B]$$

and identify the different parts of $P[X_T, \Theta_T]$ as

$$A = \{X_{T-1}, \Theta_{T-1}\} \quad B = x_T \quad C = \theta_T \tag{13}$$

The joint probability $P[X_T, \Theta_T]$ can thus be rewritten as

$$\begin{aligned} P[X_T, \Theta_T] = P[X_{T-1}, \Theta_{T-1}] \; P[x_T | X_{T-1}, \Theta_{T-1}] \\ \times P[\theta_T | X_T, \Theta_{T-1}] \end{aligned} \tag{14}$$

Using the Markov property of a state-space equation and that θ_T is assumed to change arbitrary, it is possible to remove the dependence on old state information

$$\begin{aligned} P[X_T, \Theta_T] = P[X_{T-1}, \Theta_{T-1}] \\ \times P[x_T | x_{T-1}, \theta_{T-1}] P[\theta_T] \end{aligned} \tag{15}$$

Repeating this procedure one obtains

$$\begin{aligned} P[X_T, \Theta_T] = P[x_0, \theta_0] \\ \times \prod_{k=0}^{T-1} P[x_{k+1} | x_k, \theta_k] \; P[\theta_{k+1}] \end{aligned} \tag{16}$$

Now assuming that that all modes are equally probable, that x_0 and θ_0 are independent and that w_k are mutually independent makes it is possible to rewrite (16) as

$$P[X_T,\Theta_T] = \frac{1}{M}p_{x_0}(x_0)\prod_{k=0}^{T-1}\frac{1}{M}p_w(w_k) \qquad (17)$$

where p_{x_0} and p_w are probability density functions and M is the number of possible modes.

Under the assumption that w_k and x_0 are normally distributed (17) can be rewritten as

$$P[X_T,\Theta_T] = k_2\exp\bigg\{-\frac{1}{2}||x_0-\check{x}_0||^2_{P_0^{-1}} - \frac{1}{2}\sum_{k=0}^{T-1}||w_k||^2_{Q^{-1}}\bigg\} \qquad (18)$$

where

$$k_2 = \frac{1}{M^{T+1}\sqrt{\det P_0}(2\pi)^{n/2}}\left(\frac{1}{\sqrt{\det Q}(2\pi)^{n/2}}\right)^T$$

Above P_0 denotes the covariance matrix of the initial distribution of x_0 and $\check{x}_0$ its mean.

B. The optimization problem

Now all the pieces are there to formulate the optimization problem. Given the conditional probability of the state given the measurements a maximum a posteriori Bayesian estimate can be formulated as

$$\{\hat{X}_T,\hat{\Theta}_T\} = \arg\max_{X_T,\Theta_T} P[X_T,\Theta_T|Y_T]$$

which is equivalent to the following minimization problem

$$\{\hat{X}_T,\hat{\Theta}_T\} = \arg\min_{X_T,\Theta_T} -\log P[X_T,\Theta_T|Y_T] \qquad (19)$$

After noting that $P[Y_T]$ in the expression (10) for $P[X_T,\Theta_T|Y_T]$ does not depend on X_T or Θ_T we can rewrite (19) as

$$\{\hat{X}_T,\hat{\Theta}_T\} = \arg\min_{X_T,\Theta_T} -\log P[Y_T|X_T,\Theta_T]\ P[X_T,\Theta_T] \qquad (20)$$

Using the previously derived expressions for $P[Y_T|X_T,\Theta_T]$ and $P[X_T,\Theta_T]$, that is (12) and (18), the following optimization problem can be stated:

$$\begin{array}{cc} \min_{X_T,\Theta_T} & J_T \\ \text{subject to} & (22) \end{array} \qquad (21)$$

where

$$J_T = ||x_0+\check{x}_0||^2_{P_0^{-1}} + \sum_{k=0}^{T}||v_k||^2_{R^{-1}} + \sum_{k=0}^{T-1}||w_k||^2_{Q^{-1}} \qquad (22)$$

The maximum a posteriori Bayesian estimation problem has now been transformed into a minimization problem.

C. Forward dynamic programming formulation

The optimization problem (21) grows with time which makes it practically unsolvable. Instead it is desirable to proceed sequentially and compute an estimate of the present state given measurements up to (and including) now. This problem is referred to as the filtering problem. The optimization problem (21) can be solved using forward dynamic programing.

Using the same arguments as in [16] it is convenient to first consider the prediction problem, that is to estimate x_{T+1} given measurements up to and including y_T. So instead of minimizing J_T we minimize

$$J_T + ||w_T||^2_{Q^{-1}} \qquad (23)$$

To proceed sequentially a value function is introduced

$$V^*_{j+1}(x_{j+1}) = \min_{X_j,\Theta_j}\left\{\sum_{k=0}^{j}L(v_k,w_k)+\Gamma(x_0)\right\} \qquad (24)$$

where

$$\begin{aligned} L(v_k,w_k) &= ||v_k||^2_{R^{-1}} + ||w_k||^2_{Q^{-1}} \\ \Gamma(x_0) &= ||x_0-\check{x}_0||^2_{P_0^{-1}} \end{aligned} \qquad (25)$$

Using forward dynamic programing it is possible to write (24) as

$$V^*_{j+1}(x_{j+1}) = \min_{x_j,\theta_j}\{V^*_j(x_j)+L(v_j,w_j)\} \qquad (26)$$

with

$$V^*_0(x_0) = \Gamma(x_0)$$

To find a good parameterization of the value-function first assume that the value function is parameterized as follows:

$$V^*_j(x_j) = \min_{\pi\in\Pi^*_j}\begin{bmatrix}x_j\\1\end{bmatrix}^T\begin{bmatrix}\pi_{11} & \pi_{12}\\ \pi_{12}^T & \pi_{22}\end{bmatrix}\begin{bmatrix}x_j\\1\end{bmatrix} \qquad (27)$$

Note that the value-function depends on the sequence Θ_{j-1} implicitly in the matrices $\pi\in\Pi_j$, that is, each matrix in Π_j is associated with a particular sequence Θ_{j-1}. With this parameterization, (26) can be written as

$$V^*_{j+1}(x_{j+1}) = \min_{\theta_j,\pi\in\Pi^*_j}\min_{x_j}\begin{bmatrix}x_{j+1}\\1\\x_j\end{bmatrix}^T\begin{bmatrix}U_{11} & U_{12} & U_{13}\\ U_{12}^T & U_{22} & U_{23}\\ U_{13}^T & U_{23}^T & U_{33}\end{bmatrix}\begin{bmatrix}x_{j+1}\\1\\x_j\end{bmatrix} \qquad (28)$$

If U_{33} is positive definite the solution is unique and the minimization over x_j gives a new V_{j+1} on the form (27). That is

$$V^*_{j+1}(x_{j+1}) = \min_{\pi\in\Pi^*_{j+1}}\begin{bmatrix}x_{j+1}\\1\end{bmatrix}^T\begin{bmatrix}\pi_{11} & \pi_{12}\\ \pi_{12}^T & \pi_{22}\end{bmatrix}\begin{bmatrix}x_{j+1}\\1\end{bmatrix} \qquad (29)$$

The expressions for the matrices $\pi\in\Pi^*_{j+1}$ and U are given in Appendix B. Thus, the parameterization (27) yields a new value-function on the same form, so theoretically it is possible to continue the value iteration. However the size of the set Π^*_{j+1} could possibly be M times the size of Π^*_j, so the size will grow exponentially with j.

D. Relaxing the Value Iteration

As proposed in [17] the value iteration (26) can be relaxed if the Bellman equality is replaced by two inequalities instead. First define upper and lower bounds on V_{j+1} as

$$\begin{aligned}\overline{V}_{j+1}(x_{j+1}) &= \min_{x_j,\theta_j} V_j(x_j) + \overline{\alpha} L(v_j, w_j) \\ \underline{V}_{j+1}(x_{j+1}) &= \min_{x_j,\theta_j} V_j(x_j) + \underline{\alpha} L(v_j, w_j)\end{aligned} \tag{30}$$

Now it is possible to replace (26) with the two inequalities

$$\underline{V}_{j+1}(x_{j+1}) \leq V_{j+1}(x_{j+1}) \leq \overline{V}_{j+1}(x_{j+1}) \tag{31}$$

Here the scalars $\overline{\alpha} > 1$ and $\underline{\alpha} < 1$ are a slack parameters that can be chosen to determine the distance to optimality. By the introduction of inequalities instead of equalities it is in principle possible to fit a simpler cost-to-go function between the upper and lower bounds.

If, in each step, (31) holds with the upper ($\overline{V}_{j+1}$) and lower ($\underline{V}_{j+1}$) bounds computed as in (30) the obtained solution will satisfy

$$\underline{\alpha} V_j^*(x_j) \leq V_j(x_j) \leq \overline{\alpha} V_j^*(x_j) \tag{32}$$

which gives guarantees on how far from the optimal solution the approximate solution is.

If the approximate value function is parameterized in the same way as V_j^*, the approximate value function V_{j+1} can be constructed by selecting matrices from the lower bound $\underline{V}_{j+1}$ until (31) is satisfied, see Figure 1 for an illustration.

Note that adding a matrix to the set Π_{j+1} decreases the overall function value because of the parametrization used. As in the exact value iteration, each matrix in the parameterization of V_{j+1} is associated with a particular sequence $\hat{\Theta}_j$. For a detailed discussion on how to check that (31) is fulfilled see for example Procedure 1 in [17].

The one step head prediction of the state x_{j+1} is given by the value for which value function $V_{j+1}(x_{j+1})$ has its minimum. As each matrix $\pi \in \Pi_{j+1}$ is associated with a particular sequence Θ_j an estimate of the most probable mode sequence is given by the sequence associated with the minimizing value of $V_{j+1}(x_{j+1})$.

E. Filtering and Smoothing

To solve the filtering problem information about the last measurement needs to be incorporated in the minimization problem. An estimate of x_{j+1} and the sequence Θ_{j+1} given all data up to y_{j+1} is thus given by the values of which the following function has its minimum

$$S_{j+1} = V_{j+1}(x_{j+1}) + ||v_{j+1}||^2_{R^{-1}} \tag{33}$$

Note that this is somewhat different from how the value-function is normally used. Here each matrix in the parameterization of S_{j+1} provides the history of θ and not just the last value θ_{j+1}.

To achieve an estimate of x_{T-N} given measurements up to y_{T-1}, that is smoothing, the following procedure can be used. If the value-function from time $T-N$ together with the measurement sequence $Y_{[T-1,T-N]}$ is saved, a smoothed estimate can be computed as the solution of the following optimization problem

$$S_T^N = \min_{\substack{X_{[T-N,T]} \\ \Theta_{[T-N,T-1]}}} \Big\{ V_{T-N}(x_{T-N}) + \sum_{k=T-N}^{T-1} L(v_k, w_k) \Big\} \tag{34}$$

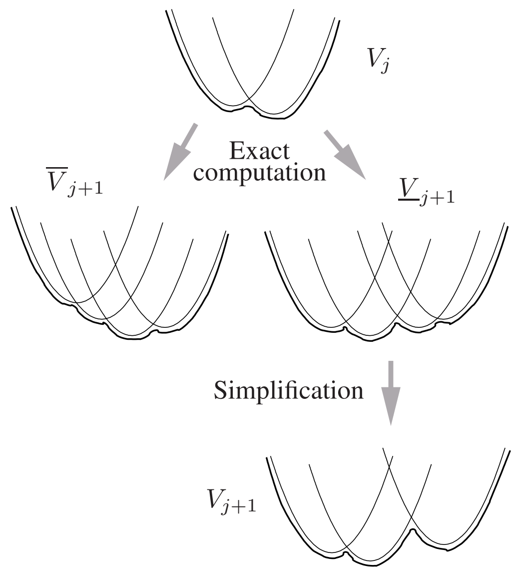

Fig. 1. 1-D illustration of how the value function is approximated using quadratic functions.

IV. Observer Stability

Switched systems have two very different types of states, the continuous state x_k and the discrete mode θ_k, thus what is meant by stability needs to be specified. In this section, the properties of the continuous state estimate will be studied. Particularly the following type of stability will be addressed:

Definition 1: An estimator is an asymptotically stable observer for the noise-free system

$$\begin{aligned}x_{k+1} &= A(\theta_k)x_k + B(\theta_k)u_k \\ y_k &= C(\theta_k)x_k\end{aligned} \tag{35}$$

if there exists an integer N such that for every initial condition x_0 and θ_0 the estimate $\hat{x}_{T|T+N} \rightarrow x_T$ as $T \rightarrow \infty$.

Next, a few concepts concerning observability need to be established. The field of observability for switched systems is very young and thus there are quite a few notions of observability. Here the definition of State Observability from [13] will be used.

Definition 2: (State Observability (SO)) The system (2) is SO if there exists an integer N_0 (the smallest being the index) such that $\forall x \in \mathbf{R}^n$ and all Θ_{N_0},

$$x \neq x' \Rightarrow Y(\Theta_{N_0}, x) \neq Y(\Theta'_{N_0}, x') \quad \forall \Theta'_{N_0} \tag{36}$$

where $Y(\Theta_{N_0}, x)$ denotes the noise free output of (2) for the mode sequence Θ_{N_0} and initial state x when $w_k = 0$. That is, a system is SO if any N_0 consecutive measurements $Y(\Theta_{N_0}, x)$ yield x uniquely without knowledge of Θ_{N_0}. According to [13] adding a known input does not change the SO of a system.

In the following proposition a technical condition on the function $L(v_k, w_k)$ is needed, so we first give the following definition.

Definition 3: A function $\eta : \mathbf{R}_+ \to \mathbf{R}_+$ is a $\mathcal{K}$-function if it is continuous, strictly monotone increasing, $\eta(x) > 0$ for $x \neq 0$, $\eta(0) = 0$ and $\lim_{x\to\infty} \eta(x) = \infty$.

We are now ready to formulate a proposition claiming that, that if the system is initiated in an unknown initial condition and is *not* subject to any disturbances the estimation error approaches zero as more measurements are collected.

Proposition 1: Suppose that the system is State Observable with index N_0, there exists a class $\mathcal{K}$-function $\eta(\cdot)$ such that $\eta(\|v, w\|) \leq L(v, w)$, $v_k = w_k = 0$, $\underline{\alpha} = 1$ and that $N \geq N_0$. Then the proposed method yields an asymptotically stable observer for (35).

Proof: The idea is to bound S_T^N from above and prove that the sequence S_T^N is non-decreasing. First introduce the notation $L_k = L(v_k, w_k)$. Using the assumption that $\underline{\alpha} = 1$ the difference $S_T^N - S_{T-N}^N$ can be bounded by

$$\begin{aligned}
&S_T^N - S_{T-N}^N \\
&\geq \min_{\substack{X_{[T-2N,T]}\\ \Theta_{[T-2N,T-1]}}} \Big\{ V_{T-N}(x_{T-N}) + \sum_{k=T-N}^{T-1} L_k \\
&\quad - V_{T-2N}(x_{T-2N}) - \sum_{k=T-2N}^{T-N-1} L_k \Big\} \\
&\geq \min_{X_{[T-2N,T]}} \Big\{ V_{T-2N}(x_{T-2N}) + \sum_{k=T-2N}^{T-N-1} \underline{\alpha} L_k \\
&\quad + \sum_{k=T-N}^{T-1} L_k - V_{T-2N}(x_{T-2N}) - \sum_{k=T-2N}^{T-N-1} L_k \Big\} \\
&= \sum_{k=T-N}^{T-1} L(\hat{v}_{k|T-1}, \hat{w}_{k|T-1})
\end{aligned} \tag{37}$$

To bound the sequence S_T^N from above, an upper bound on the sequence $V_{T,N}^*$ is first derived as

$$V_T^*(x_T) \leq \sum_{k=0}^{T-1} L(v_k, w_k) + \Gamma(x_0) = \Gamma(x_0) \tag{38}$$

The first inequality above holds for any v_k and w_k due to optimality, and in particular it holds for the actual noise sequences, which are assumed to be zero. Using the argument above rewrite (34) using (38) as

$$\begin{aligned}
S_T^N &\leq V_{T-N}(\hat{x}_{T-N|T-1}) + \sum_{k=T-N}^{T-1} L(v_k, w_k) \\
&\leq V_{T-N}(\hat{x}_{T-N|T-1}) \leq \overline{\alpha} V_{T-N}^*(\hat{x}_{T-N|T-1}) \\
&= \overline{\alpha}\Gamma(x_0).
\end{aligned} \tag{39}$$

Thus S_T^N is bounded from above and non-decreasing so

$$\sum_{k=T-N}^{T-1} L(\hat{v}_{k|T-1}, \hat{w}_{k|T-1}) \to 0 \tag{40}$$

as $T \to \infty$. If $N \geq N_0$ Lemma 1 in Appendix A with $\delta_w = \delta_v = 0$ states that

$$\left\| x_{T-N} - \hat{x}_{T-N|T-1} \right\| \to 0 \tag{41}$$

as claimed. □

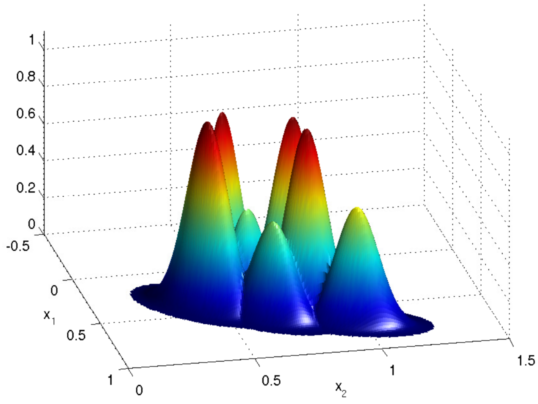

Fig. 2. Plot of the function $\tilde{I}_5 \sim \exp(-V_5(x_5))$ which represents the conditional probability of x_5. The non-Gaussian and multi modal shape is due to the discrete modes in the system.

V. Numerical Example

In this section the proposed algorithm was applied to a oscillatory second order system with two modes. The system used is a continuous time system

$$\begin{aligned}
\dot{x} &= \begin{bmatrix} -2\zeta\omega & 1 \\ -\omega^2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \omega^2 \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x
\end{aligned} \tag{42}$$

with two modes; $\omega = 1$ and $\omega = 2$. The relative damping ζ is equal to 0.4 in both modes. The system was sampled with sampling interval $h = 0.5$ under the assumption of constant input signal between samples. This resulted in a system is on the form (35). Note that the system has unit stationary gain in both modes.

The system was simulated for 90 samples with measurement noise covariance $R = 0.01$ and process noise covariance $Q = 0.001$. The discrete mode was randomly generated as uniformly distributed number rounded to either 1 or 2. The input was filtered white noise with unit variance.

In the algorithm the slack parameters where chosen as $\underline{\alpha} = 1$ and $\overline{\alpha} = 1.5$. The parameters Q and R where chosen equal to the real noise covariance matrices. The smoothing parameter N was chosen equal to 2.

Recall that the value function V_k is proportional to the negative logarithm of the function $I_k(x_k) = \max_{X_{k-1},\Theta_{k-1}} P[X_k, \Theta_k | Y_k]$. Now introduce a function $\tilde{I}_k = c \exp(-V_k(x_k))$, where c is a normalization constant chosen such that $\max_{x_k} \tilde{I}_k(x_k) = 1$. Thus, the function $\tilde{I}_k$ is a scaled approximation of the true conditional probability function $I_k(x_k)$. Figure 2 shows the function $\tilde{I}_k$ for $k = 5$. Note the non-Gaussian and multi modal shape of the function.

One way to evaluate the performance of the algorithm is to compare the norm of the continuous estimation error with a time-varying Kalman filter, where the mode is assumed to be known. This of course gives an upper bound on achievable performance.

In Figure 3 the output y_k, the norm of the continuous estimation error $\|x_k - \hat{x}_{k-2|k}\|$ with unknown modes, the norm of the continuous estimation error for the time-varying Kalman filter, the estimate of the mode $\hat{\theta}_{k-2|k}$ together with the actual mode θ_k are plotted.

Note that when the mode is estimated correctly, the error is close to the lower bound achieved by the time-varying Kalman filter.

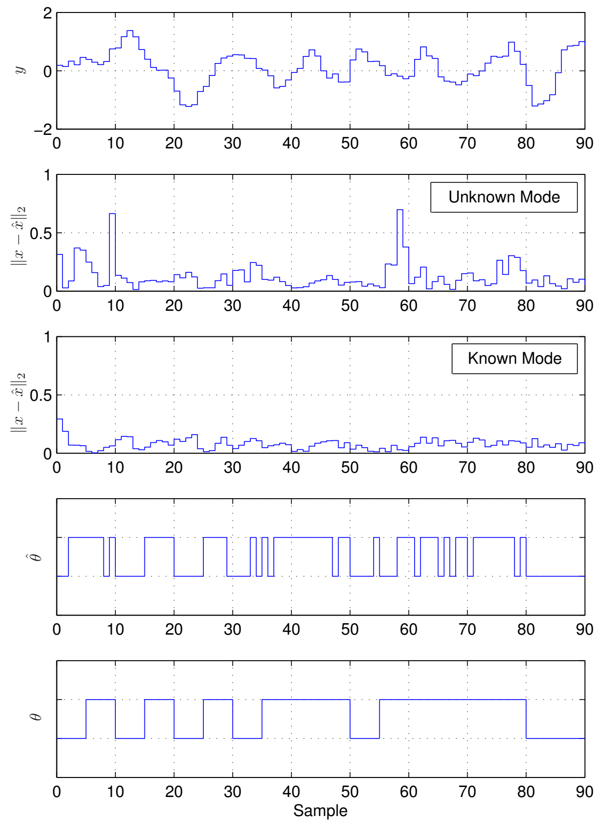

Fig. 3. Plots of the output y_k, the norm of the continuous estimation error $\|x_k - \hat{x}_{k-2|k}\|$ with unknown modes, norm of the continuous estimation error for a standard time-varying Kalman filter, the estimate of the mode $\hat{\theta}_{k-2|k}$ and the actual mode θ_k. Note that the performance is comparable to the case of known modes when the mode is estimated correctly.

VI. Conclusions and Future Work

In this paper we synthesize an observer by solving the optimization problem associated with maximum posteriori Bayesian estimation for switched discrete-time linear systems. The optimization problem is solved using relaxed dynamic programming. The resulting value function stays within a pre-specified factor from the optimal one.

The main contribution is to revisit the reformulation of maximum posteriori Bayesian estimation as dynamic programming and apply relaxed dynamic programming to the resulting optimization problem.

The dynamic programming formulation presented here is somewhat different from how dynamic programming is normally used. Here the value function, even theoretically, can *not* be computed off-line because the set of matrices Π_j depends on measurements. Thus the iteration described in Section III-D must be performed on-line. This can be view in relation to moving horizon estimation where a finite dimensional optimization is solved on-line in each step. Here we have moved the computational effort from the optimization problem to the problem of finding a good representation of past data, often referred to as arrival cost in moving horizon estimation literature.

We also prove that for a system where the only unknown parameter is the initial condition the continuous estimation error goes to zero as more measurements are collected. The method is demonstrated using a second order example, where the system is subject to both measurement and process noise.

Several extensions to the proposed algorithm are possible. For example, the model of the discrete mode could easily be changed to restrict the set of possible sequences using for example a state machine. More difficult modifications includes modeling the discrete variable with a Markov chain or allowing the noise covariances to be dependent on the discrete variable.

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Appendix

A. Lemma 1

Lemma 1: Suppose that the system is State Observable with index N_0, there exists a class $\mathcal{K}$-function $\eta(\cdot)$ such that $\eta(\|v, w\|) \leq L(v, w)$, $\|w_k\| \leq \delta_w$, $\|v_k\| \leq \delta_v$ and

$N > N_0$ then there exists a class $\mathcal{K}$-function κ such that

$$\|x_{T-N} - \hat{x}_{T-N|T-1}\| \leq \kappa\left(\left\|\sum_{k=T-N}^{T-1} L(\hat{v}_{k|T-1}, \hat{w}_{k|T-1}), \delta_w, \delta_v\right\|\right) \quad (43)$$

Proof: The proof is inspired by the work of Rao in [18]. However some assumptions used in the proof of Lemma 2.5 in [18] are not fulfilled for the system (2), thus some modifications had to be made. In particular, the time index of the estimation error is restricted to $T-N$ instead of $\{T-N, T-N+1, \ldots, T\}$. Also the notion of observability is different which changes some technical details in the proof. The complete proof will be given here for clarity and it will be pointed out where proofs the differ.

First recall that x_k denotes the actual state of the system (2) and y_k the measured output of it. Next some notation is introduced; $x(k, x_0, t_0, \Theta_{[t_0,k]}, U_{[t_0,k]})$ denotes the solution of (35) at time k when the system is initialized with x_0 at time t_0 and subject to the input sequence $U_{[t_0,k]}$ and mode sequence $\Theta_{[t_0,k]}$. Further introduce $y(k, x_0, t_0, \Theta_{[k,t_0]}, U_{[t_o,k]}) = C(\theta_k)x(k, x_0, t_0, \Theta_{[t_0,k]}, U_{[t_0,k]})$ in the same way. Using the definitions above introduce

$$\begin{aligned}
\overline{x}_k &= x(k, \hat{x}_{T-N|T-1}, T-N, \hat{\Theta}_{[T-N,k]}, U_{[T-N,k]}) \\
\overline{y}_k &= y(k, \hat{x}_{T-N|T-1}, T-N, \hat{\Theta}_{[T-N,k]}, U_{[T-N,k]}) \\
\tilde{x}_k &= x(k, x_{T-N}, T-N, \Theta_{[T-N,k]}, U_{[T-N,k]}) \\
\tilde{y}_k &= y(k, x_{T-N}, T-N, \Theta_{[T-N,k]}, U_{[T-N,k]}) \\
\hat{v}_{k|T-1} &= C(\theta_k)x_k + v_k - C(\hat{\theta}_k)\hat{x}_{k|T-1} \\
\hat{w}_{k|T-1} &= \hat{x}_{k+1|T-1} - A(\hat{\theta}_{k|T-1})\hat{x}_{k|T-1} - B(\hat{\theta}_{k|T-1})u_k
\end{aligned} \quad (44)$$

Using the triangle inequality a bound for $\|x_i - \hat{x}_{i|T-1}\|$ is obtained as

$$\|x_i - \hat{x}_{i|T-1}\| \leq \|x_i - \tilde{x}_i\| + \|\hat{x}_{i|T-1} - \overline{x}_i\| + \|\overline{x}_i - \tilde{x}_i\| \quad (45)$$

Next upper bounds for the right hand terms of (45) will be derived. Starting with $\|\hat{x}_{i|T-1} - \overline{x}_i\|$, first note that using the definition of $\hat{w}_{k|T-1}$, the estimate $\hat{x}_{i|T-1}$ can be expressed as

$$\begin{aligned}
\hat{x}_{i|T-1} = &\left(\prod_{m=T-N}^{i-1} A(\hat{\theta}_{m|T-1})\right)\hat{x}_{T-N|T-1} + \\
&+ \sum_{m=T-N}^{i-1}\left(\prod_{n=m+1}^{i-1} A(\hat{\theta}_{n|T-1})\right)\hat{w}_{m|T-1} \\
&+ \sum_{m=T-N}^{i-1}\left(\prod_{n=m+1}^{i-1} A(\hat{\theta}_{n|T-1})\right)B(\hat{\theta}_{m|T-1})u_m
\end{aligned} \quad (46)$$

and that

$$\begin{aligned}
\overline{x}_i = &\left(\prod_{m=T-N}^{i-1} A(\hat{\theta}_{m|T-1})\right)\hat{x}_{T-N|T-1} \\
&+ \sum_{m=T-N}^{i-1}\left(\prod_{n=m+1}^{i-1} A(\hat{\theta}_{n|T-1})\right)B(\hat{\theta}_{m|T-1})u_m
\end{aligned} \quad (47)$$

which gives

$$\|\hat{x}_{i|T-1} - \overline{x}_i\| \leq \sum_{m=T-N}^{i-1} \tilde{A}^{i-m-1}\|\hat{w}_{m|T-1}\| \quad (48)$$

where $\tilde{A} = \max_{\theta\in\mathcal{I}} \|A(\theta)\|$. Using similar arguments a bound for $\|x_i - \tilde{x}_i\|$ is given by

$$\|x_i - \tilde{x}_i\| \leq \sum_{m=T-N}^{i-1} \tilde{A}^{i-m-1}\|w_m\| \quad (49)$$

Both (48) and (49) holds for $i = T-N\ldots T$. Because the state update equation is not Lipschitz continuous, the techniques used to bound $\|\hat{x}_{i|T-1} - \overline{x}_i\|$ and $\|x_i - \tilde{x}_i\|$ differs from the proof in [18].

Observing that L is bounded below by a $\mathcal{K}$-function

$$\eta(\|w, v\|) \leq L(v, w) \quad (50)$$

Assumption 2 gives the following bounds

$$\begin{aligned}
\|\hat{w}_{k|T-1}\| &\leq \eta^{-1}\left(\sum_{k=T-N}^{T-1} L(\hat{w}_{k|T-1}, \hat{v}_{k|T-1})\right) \\
\|\hat{v}_{k|T-1}\| &\leq \eta^{-1}\left(\sum_{k=T-N}^{T-1} L(\hat{w}_{k|T-1}, \hat{v}_{k|T-1})\right)
\end{aligned} \quad (51)$$

From the definition (44) of $\hat{v}_{k|T-1}$ the following equality is obtained

$$\sum_{k=T-N}^{T-1} \|\hat{v}_{k|T-1}\| = \sum_{k=T-N}^{T-1} \|C(\theta_k)x_k + v_k - C(\hat{\theta}_{k|T-1})\hat{x}_{k|T-1}\| \quad (52)$$

Applying the inverse triangle inequality on (52) results in the following inequality

$$\sum_{k=T-N}^{T-1} \|\hat{v}_{k|T-1}\| + \|v_k\| + \|C(\hat{\theta}_{k|T-1})\hat{x}_{k|T-1} - \overline{y}_k\| + \|C(\theta_k)x_k - \tilde{y}_k\| \geq \sum_{k=T-N}^{T-1} \|\overline{y}_k - \tilde{y}_k\| \quad (53)$$

Because the notion of observability differs from the one used in [18] the bound for $\|\overline{x}_i - \tilde{x}_i\|$ is restricted to $i = T-N$. Now introduce the vector notation

$$\overline{Y} = \begin{bmatrix} \overline{y}_{T-N} \\ \vdots \\ \overline{y}_{T-1} \end{bmatrix} \quad \tilde{Y} = \begin{bmatrix} \tilde{y}_{T-N} \\ \vdots \\ \tilde{y}_{T-1} \end{bmatrix} \quad (54)$$

From the assumption that the system is SO with index $N_0 < N$ it is possible to bound $\|\overline{x}_{T-N} - \tilde{x}_{T-N}\|$ with

$$\sum_{k=T-N}^{T-1} \|\overline{y}_k - \tilde{y}_k\| \geq \left\|\overline{Y} - \tilde{Y}\right\| \geq \epsilon \|\overline{x}_{T-N} - \tilde{x}_{T-N}\| \quad (55)$$

where ϵ is defined as

$$\epsilon = \min_{\Theta,\hat{\Theta}} \min_{\|\overline{x}_{T-N} - \tilde{x}_{T-N}\|=1} \frac{\|\overline{Y} - \tilde{Y}\|}{\|\overline{x}_{T-N} - \tilde{x}_{T-N}\|} \quad (56)$$

Using the definition of $\overline{y}_k$ and $\tilde{y}_k$ the following bounds are obtained

$$\begin{aligned} \left\|C(\hat{\theta}_{k|T-1})\hat{x}_{k|T-1} - \overline{y}_k\right\| &\leq \tilde{C}\|\hat{x}_{k|T-1} - \overline{x}_k\| \\ \|C(\theta_k)x_k - \tilde{y}_k\| &\leq \tilde{C}\,\|x_k - \tilde{x}_k\| \end{aligned} \quad (57)$$

where $\tilde{C} = \max_{\theta \in \mathcal{I}} \|C(\theta)\|$. Combining (45), (55), (53), (57) and (51) the smoothed estimation error can be bounded by a $\mathcal{K}$-function κ

$$\left\|x_{T-N} - \hat{x}_{T-N|T-1}\right\| \leq \kappa\left(\left\|\sum_{k=T-N}^{T-1} L(\hat{w}_{k|T-1}, \hat{v}_{k|T-1}), \delta_w, \delta_v\right\|\right) \quad (58)$$

For the details of the last step see [18]. □

B. Expressions for U and π

For each element $\pi \in \Pi_j^*$ the matrix U in (28) is computed as

$$\begin{aligned} U_{11} &= Q^{-1} \\ U_{12} &= -Q^{-1}B(\theta_j)u_j \\ U_{13} &= -Q^{-1}A(\theta_j) \\ U_{22} &= u_j^T B(\theta_j)^T Q^{-1} B(\theta_j) u_j + y_j^T R^{-1} y_j + \pi_{22} \\ U_{23} &= u_j^T B(\theta_j)^T Q^{-1} A(\theta_j) - y_j^T R^{-1} C(\theta_j) + \pi_{12}^T \\ U_{33} &= \pi_{11} + C(\theta_j)^T R^{-1} C(\theta_j) + A(\theta_j)^T Q^{-1} A(\theta_j) \end{aligned} \quad (59)$$

For time $j \geq 1$ the matrices $\pi \in \Pi_j^*$ in (29) are given by

$$\pi = \begin{bmatrix} U_{11} & U_{12} \\ U_{12}^T & U_{22} \end{bmatrix} - \begin{bmatrix} U_{13} \\ U_{23} \end{bmatrix} U_{33}^{-1} \begin{bmatrix} U_{13} \\ U_{23} \end{bmatrix}^T$$

and for $j = 0$ by

$$\begin{aligned} \pi_{11} &= P_0^{-1} + C(\theta_0)R^{-1}C(\theta_0) \\ \pi_{12} &= P_0^{-1}x_0 - C(\theta_0)^T R^{-1} y_0 \\ \pi_{22} &= x_0^T P_0^{-1} x_0 + y_0^T R^{-1} y_0 \end{aligned} \quad (60)$$

When computing the upper bound (30), the matrices Q^{-1} and R^{-1} are replaced with $\overline{\alpha}Q^{-1}$ and $\overline{\alpha}R^{-1}$ respectively. The lower bound is computed in the same way but with $\overline{\alpha}$ replaced by $\underline{\alpha}$.

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