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Published in:

Proc. 2009 IEEE Multi-conference on Systems and Control (MSC 2009)

2009

[Link to publication](#)

Citation for published version (APA):

Shiriaev, A., Johansson, R., Robertsson, A., & Freidovich, L. (2009). Criteria for Global Stability of Coupled Systems with Application to Robust Output Feedback Design for Active Surge Control. In *Proc. 2009 IEEE Multi-conference on Systems and Control (MSC 2009)*

Total number of authors:

4

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Criteria for Global Stability of Coupled Systems with Application to Robust Output Feedback Design for Active Surge Control

Anton S. Shiriaev, Rolf Johansson, Anders Robertsson, Leonid B. Freidovich

Abstract—The well-known and commonly accepted finite dimensional model qualitatively describing surge instabilities in centrifugal (and axial) compressors is considered. The problem of global output feedback stabilization for it is solved. The solution relies on two new criteria for global stability proposed for a class of nonlinear systems exploiting quadratic constraints for infinite sector nonlinearities. The constructive steps in developing a family of output feedback controllers based on these stability tests are presented. Performance of the closed-loop systems are illustrated by simulations.

I. INTRODUCTION

This paper is focused on designing globally stabilizing output feedback controllers for the following nonlinear system

$$\frac{d}{dt}\phi = -\psi + \frac{3}{2}\phi + \frac{1}{2}[1 - (1 + \phi)^3] \quad (1)$$

$$\frac{d}{dt}\psi = \frac{1}{\beta^2}(\phi - u) \quad (2)$$

Here ψ and ϕ are state variables, u is a control input, and β is a positive constant.

The model (1)–(2) is known for more than two decades as the Greitzer model and has been used for approximating the coupled behavior of pressure and flow in dynamics of compressor systems [9], [3], [10], [2], [1], [4]. Difficulties in developing feedback controllers are due to the presence of the cubic nonlinearity in the equation (1). The key for our development is the fact that this nonlinearity satisfy certain quadratic constraints [13].

Both variables $\phi(\cdot)$ and $\psi(\cdot)$ have physical meaning being deviations of the averaged flow and pressure from their mean values. Sometimes, both of them can be assumed as outputs of the system. However, on-line measurements of the flow require special instrumentation and are often not feasible. Here we will consider the most difficult case when only ψ -variable is available

$$y = \psi \quad (3)$$

for feedback design. Presence of the nonlinearity in dynamics of (1)–(2) makes the search for an output feedback controller

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The work has been partly supported by the Swedish Research Council (the grants: 2005-4182, 2008-5243) and the Kempe Foundation.

and an associated Lyapunov function for the closed-loop system to be quite a nontrivial mathematical problem. Clearly, since the nonlinearity is not globally Lipschitz and depends on ϕ -variable, designing a globally stabilizing output feedback controller, relying on measurements of $y = \psi$ only, is a challenge. To the best of our knowledge, this problem has remained open until now.

A. The Key Structural Transformation

To solve the posted problems, we consider a very particular class of output feedback controllers. The key assumption is that after an appropriate change of coordinates, the closed-loop systems can be written as

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} w_1(Cx) + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} w_2(x, e) \quad (4)$$

where x and e are components of the new state, $v = Cx$ is the scalar input to the nonlinearity of the first subsystem, A_{11} , A_{12} , A_{22} , B_1 , B_2 , and C are constant matrices of appropriate dimensions, $w_1(\cdot)$ and $w_2(\cdot)$ are static nonlinearities constructed from the cubic nonlinearity present in the original dynamics (1)–(2).

The paper is organized as follows: In Section II we present two criteria to verify global stability of (4) expressed as conditions for stability of the x - and e -subsystems. We show in Section III how both statements can be used for designing output feedback controllers to stabilize (1)–(2). The results of numerical simulation are discussed in Section IV, and concluding remarks are made in Section V. This Section is continued by discussion of important properties of (1)–(2).

B. Quadratic Constraints for the Nonlinearity in the Surge Dynamics (1)–(2)

Let us discuss useful properties of the nonlinearity of the dynamical system (1)–(2)

$$w(v) := 1 - (1 + v)^3 \quad (5)$$

Lemma 1: The static nonlinearity (5) satisfies the incremental quadratic constraint (QC)

$$[w(v_2) - w(v_1)] \cdot (v_1 - v_2) \geq 0, \quad \forall v_1, v_2 \in \mathbb{R}^1 \quad (6)$$

Proof: Substituting (5) into the left-hand side of (6) we obtain

$$\begin{aligned} & [(v_1 + 1)^3 - (v_2 + 1)^3] (v_1 - v_2) = \\ & [(v_2 + 1)^2 + (v_1 + 1)^2 + (v_2 + 1)(v_1 + 1)] (v_1 - v_2)^2 \geq 0 \end{aligned}$$

Consider now the general form of a dynamic output feedback control law

$$u = \mathcal{U}(z, y), \quad \dot{z} = \mathcal{F}(z, y) \quad (7)$$

where $\mathcal{U}(\cdot)$ and $\mathcal{F}(\cdot)$ are smooth.

Lemma 2: Suppose $[0, \tau^{max})$ is the maximal interval of existence of a solution

$$X(t) = [\phi(t); \psi(t); z(t)]$$

of the closed-loop system (1), (2), (7), where $\tau^{max} > 0$ can be finite or not. Let $F(X)$ be a smooth function.

Then, there are following two possible cases.

- 1) There exists a sequence $\{t_k\}_{k=1}^{\infty}$ of time moments with $\lim_{k \rightarrow \infty} t_k = \tau^{max}$ such that the integrals of the quadratic form

$$G_1[v, w(v)] = w(v) \cdot (-v) - \frac{3}{4} v^2 \quad (8)$$

with $w(\cdot)$ defined in (5), along this solution with $v(t) = F(X(t))$ are strictly positive, i.e.

$$\int_{t_{k-1}}^{t_k} G_1[v(t), w(v(t))] dt > 0, \quad k = 1, 2, \dots \quad (9)$$

- 2) Along this solution,

$$\text{either } F(X(t)) \equiv 0 \quad \text{or} \quad F(X(t)) \equiv -\frac{3}{2}$$

Moreover, the integral in (9) of the quadratic form $G_1[v, w]$ identically equals zero for any $t \in [0, \tau^{max})$.

Proof: To check (9), observe that

$$G_1[v, w(v)] = w(v) \cdot (-v) - \frac{3}{4} v^2 = (v + \frac{3}{2})^2 \cdot v^2 \geq 0 \quad (10)$$

Clearly, if integrating this relation along a solution of the closed-loop system over $[t_k, \tau^{max})$ with $v(t) = F(X(t))$ results in zero value for any $t_k \in [0, \tau^{max})$, then $v(t)$ equals either 0 or $-\frac{3}{2}$ on $[t_k, \tau^{max})$ and it is just left to notice that we can shift the time since the system is time-invariant. ■

II. STABILITY CRITERIA FOR DYNAMICAL SYSTEM (4)

Let us postpone the discussion on how to transform the closed-loop system (1), (2), (7) into the form of (4) and search for conditions, under which global stability of (4) follows from properties of the separated x and e -subsystems. Loosely speaking, such conditions could be interpreted as successful state feedback and reduce observer design criteria. However, it is worth noting that neither x -nor e -subsystems are independent, and just assuming asymptotic stability of each of them will not necessary result in asymptotic stability of (4). Stronger properties will be requested and features of the nonlinear functions $w_1(\cdot)$ and $w_2(\cdot)$ will be used. Namely, we will use the following.

Assumption 1: the nonlinearity $w_1(\cdot)$ satisfies the relations (9) and (10) similar to $w(\cdot)$ defined in (5);

Assumption 2: the nonlinearity $w_2(\cdot)$ satisfies an infinite sector quadratic constraint

$$G_2[e, w_2(x, e)] = e^T \Pi_e w_2(x, e) \geq 0, \quad \forall x, e \quad (11)$$

with Π_e being a constant matrix, that is the relation similar to (6) defined for the original nonlinearity $w(\cdot)$.

The first stability condition will rely on quadratic stabilities of both x and e -subsystems.

Theorem 1: Let Assumptions 1 and 2 hold. Suppose that:

- 1) There exist matrices $P_1 = P_1^T$ and $Q_1 = Q_1^T > 0$ such that the following inequality is valid

$$2x^T P_1 (A_{11}x + B_1 \bar{w}_1) + G_1[Cx, \bar{w}_1] < -x^T Q_1 x \quad (12)$$

for all $x \neq 0$ and $\bar{w}_1 \neq 0$. Moreover, there exists a matrix K_1 such that $G_1[Cx, K_1 x] \geq 0, \forall x$, and $(A_{11} + B_1 K_1)$ is Hurwitz.

- 2) There exist matrices $P_2 = P_2^T$ and $Q_2 = Q_2^T > 0$ such that

$$\begin{aligned} e^T (A_{22}^T P_2 + P_2 A_{22}) e &< -e^T Q_2 e, \\ e^T (P_2 B_2 + \Pi_e) \bar{w}_2 &= 0 \end{aligned} \quad (13)$$

for all $e \neq 0$ and $\bar{w}_2 \neq 0$. Moreover, the matrix A_{22} is Hurwitz.

Then, the nonlinear system (4) is quadratically stable, i.e. there are matrices $\mathcal{P} = \mathcal{P}^T > 0$ and $\mathcal{Q} = \mathcal{Q}^T > 0$ such that along any nontrivial solution $[x(t); e(t)]$ of (4) we have

$$\frac{d}{dt} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}^T \mathcal{P} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} < - \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}^T \mathcal{Q} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \quad (14)$$

Proof: It follows from the minimal stability conditions [13] of Theorem 1, that the matrices P_1 and P_2 are positive definite. Let us consider the positive definite quadratic function

$$W(x, e) = \begin{bmatrix} x \\ e \end{bmatrix}^T \mathcal{P} \begin{bmatrix} x \\ e \end{bmatrix}, \quad \mathcal{P} = \begin{bmatrix} P_1 & 0 \\ 0 & \gamma \cdot P_2 \end{bmatrix}, \quad (15)$$

where γ is a positive constant, and compute the time-derivative of $W(\cdot)$ along a solution of (4).

$$\begin{aligned} \frac{d}{dt} W(x(t), e(t)) &= 2x^T P_1 [A_{11}x + A_{12}e + B_1 w_1(Cx)] \\ &\quad + 2\gamma e^T P_2 [A_{22}e + B_2 w_2(x, e)] \end{aligned}$$

Since the inequalities (10) and (11) are valid along any solution of (4), we have

$$\begin{aligned} \frac{d}{dt} W &< 2x^T P_1 [A_{11}x + A_{12}e + B_1 w_1(Cx)] \\ &\quad + 2\gamma e^T P_2 [A_{22}e + B_2 w_2(x, e)] \\ &\quad + G_1[Cx, w_1(Cx)] + 2\gamma G_2[e, w_2(x, e)] \end{aligned}$$

Taking into account the relations (12), (13), one can observe that the right hand side of the last inequality becomes less or equal to

$$-x^T Q_1 x + 2x^T P_1 A_{12}e - \gamma e^T Q_2 e,$$

which after completing the squares becomes

$$-(R_1 x + r_1 e)^2 - e^T (\gamma Q_2 - r_1^T r_1) e \quad (16)$$

Here R_1 is such that $Q_1 = R_1^T R_1$, and $r_1^T = A_{12}^T P_1 R_1^{-1}$. If γ is chosen large enough, then the quadratic form (16), which serves as an upper bound for $\frac{d}{dt} W(x(t), e(t))$, is negative definite. ■

The second stability criteria is again based on quadratic stability of the e -subsystem, but requires weaker properties (only asymptotic stability) of the x -subsystem.

Theorem 2: Let Assumptions 1 and 2 hold. Suppose that:

- 1) There exist a square matrix $P_1 = P_1^T$ and a row matrix q_1 such that the following inequality is valid

$$2x^T P_1 (A_{11}x + B_1 \bar{w}_1) + G_1 [Cx, \bar{w}_1] \leq -|q_1 x|^2 \quad (17)$$

for all $x \neq 0$ and $\bar{w}_1 \neq 0$. Moreover, there exists a matrix K_1 such that $G_1 [Cx, K_1 x] \geq 0$, $\forall x$, the matrix $(A_{11} + B_1 K_1)$ is Hurwitz, and the pair $[q_1, (A_{11} + B_1 K_1)]$ is observable.

- 2) The system (4) has no nontrivial solution $[x(t), e(t)]$ along which $e(t) \equiv 0$ and $Cx(t) \equiv \text{const} \in \{-\frac{3}{2}, 0\}$.
- 3) There exist matrices $P_2 = P_2^T$ and $Q_2 = Q_2^T > 0$ such that (13) is valid for all $e \neq 0$ and $\bar{w}_2 \neq 0$. Moreover, the matrix A_{22} is Hurwitz.

Then, the origin of nonlinear system (4) is locally exponentially stable and globally asymptotically stable. ■

Proof is based on the following four claims.

Claim 1: Under the assumptions of Theorem 2 the x -subsystem with $e(t) \equiv 0$ is globally asymptotically stable. ■

Indeed, the inequality (17) is valid for any vectors x and \bar{w}_1 , hence it is valid when $\bar{w}_1 = K_1 x$ so that

$$2x^T P_1 (A_{11}x + B_1 K_1 x) \leq -|q_1 x|^2$$

This is a matrix Lyapunov inequality, and so the facts that $(A_{11} + B_1 K_1)$ is Hurwitz and $[q_1, (A_{11} + B_1 K_1)]$ is observable imply that $P_1 > 0$. Let us consider the Lyapunov function candidate $V_1(x) = x^T P_1 x$. Due to validity of (17) and (10), its time derivative along a solution $x(t)$ of the x -subsystem with $e(t) \equiv 0$ satisfies the non-strict inequality

$$\begin{aligned} \frac{d}{dt} V_1(x(t)) &= 2x(t)^T P_1 (A_{11}x(t) + B_1 w_1(t)) \\ &\leq 2x(t)^T P_1 (A_{11}x(t) + B_1 w_1(t)) + G_1 [Cx(t), w_1(t)] \\ &\leq -|q_1 x(t)|^2 \leq 0 \end{aligned} \quad (18)$$

It implies existence of the solutions of the x -subsystem with $e(t) \equiv 0$ on an infinite interval of time [5, Theorem 3.3], Lyapunov stability [5, Theorem 4.1], and their boundedness. It is left to verify that $x(t)$ converges to the origin. Let us consider the integral form of (18)

$$\begin{aligned} V_1(x(t_k)) - V_1(x(t_{k-1})) &= \int_{t_{k-1}}^{t_k} 2x^T P_1 (A_{11}x + B_1 w_1) dt \\ &< \int_{t_{k-1}}^{t_k} 2x^T P_1 (A_{11}x + B_1 w_1) + G_1 [Cx(t), w_1(t)] dt \\ &\leq - \int_{t_{k-1}}^{t_k} |q_1 x(t)|^2 dt \leq 0 \end{aligned} \quad (19)$$

where the sequence $\{t_k\}_{k=1}^{\infty}$ is from (9), which by assumption exists for any particular nontrivial solution¹ of (4).

¹Note that using these special intervals of time allows us to make one of the inequalities strict. This is the key point of the proof.

Hence, $V_1(x(T)) \rightarrow c \equiv \text{const}$ as $T \rightarrow \infty$. It follows from (18) that the ω -limit set of any solution is nonempty, compact, and invariant [5, Lemma 4.1]. Taking a solution $x_{\infty}(t)$ of x -subsystem with $e(t) \equiv 0$ from an ω -limit set and applying (18) and (19) to it, we conclude that $c = 0$ and so $x(T) \rightarrow 0$ as $T \rightarrow \infty$. ■

Claim 2: There are no solutions of (4) that escape to infinity in finite time. ■

The time-derivative of the function $W(\cdot)$ defined by (15) with $\gamma = 1$ along any solution of (4) satisfies the inequality

$$\begin{aligned} \frac{d}{dt} W(x(t), e(t)) &\leq -x(t)^T q_1^T q_1 x(t) + 2x(t)^T P_1 A_{12} e(t) \\ &\quad - e(t)^T Q_2 e(t) \leq \varepsilon_1 W(x(t), e(t)) \end{aligned} \quad (20)$$

for some $\varepsilon_1 > 0$. Hence, solutions cannot grow faster than exponentially, see e.g. [5, Lemma 3.4]. ■

Claim 3: Along any (even unbounded) solution $[x(t), e(t)]$ of (4), $e(t)$ exponentially converges to zero. ■

This fact immediately follows from (13).

Claim 4: All solutions of (4) are bounded. ■

The first inequality in (20) can be rewritten as

$$\frac{d}{dt} W(x(t), e(t)) \leq \varepsilon_2 \cdot \sqrt{W(x(t), e(t))} \cdot \beta(t)$$

with $\beta(t) = e(t)$ and some $\varepsilon_2 > 0$. Integrating the last inequality results in the following one

$$\sqrt{W(x(T), e(T))} - \sqrt{W(x(0), e(0))} \leq \varepsilon_3 \cdot \int_0^T \beta(t) dt$$

Exponential convergence of $\beta(t)$ to zero implies that $\beta(\cdot) \in L^1[0, +\infty)$. In turn, integrability of $\beta(t)$ over the interval $[0, +\infty)$ implies the boundedness of $W(\cdot)$ and so of the solution $[x(t), e(t)]$. ■

To finish the proof of Theorem 2, one can observe that any solution $[x(t), e(t)]$ of (4) will have a non-empty ω -limit set, while on this set e -variable should be zero. That is, this set consists of solutions of x -subsystems with $e(t) \equiv 0$, which are asymptotically stable. This implies that all solutions of (4) converge to the origin. Furthermore, it is readily seen that with the conditions of Theorem 2 the origin is locally exponentially stable by linearization². Hence, it is globally asymptotically stable. ■

III. ROBUST OUTPUT FEEDBACK DESIGN FOR (1), (2)

A. Output Feedback Controller Design: Example 1

Consider the family of output feedback controllers

$$\begin{aligned} u &= \lambda_1 \psi + \lambda_2 z + \alpha_u \left(1 - [1 + c_\psi \psi + c_z z]^3 \right) \\ \dot{z} &= \lambda_3 \psi + \lambda_4 z + \alpha_z \left(1 - [1 + c_\psi \psi + c_z z]^3 \right) \end{aligned} \quad (21)$$

²Note that the QC in the form of sector conditions imply that for all the linear functions $w_1(\cdot)$, $w_2(\cdot)$ satisfying the quadratic constraints, i.e. those that appear after linearization, the dynamics of x and e -subsystems are quadratically stable.

where z is a scalar and eight constant parameters λ 's, α 's and c 's are to be determined. The closed loop system (1), (2) with the controller (21) has the form

$$\begin{bmatrix} \dot{\phi} \\ \dot{\psi} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -1 & 0 \\ \frac{1}{\beta^2} & -\frac{\lambda_1}{\beta^2} & -\frac{\lambda_2}{\beta^2} \\ 0 & \lambda_3 & \lambda_4 \end{bmatrix} \begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{\alpha_u}{\beta^2} \\ \alpha_z \end{bmatrix} w_{\psi,z} + \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} w_\phi, \quad (22)$$

$$w_\phi = 1 - (1 + \phi)^3$$

$$w_{\psi,z} = 1 - (1 + c_\psi \psi + c_z z)^3$$

To meet the structure of the system (4), introduce vectors x and e as follows

$$x = [\psi; z], \quad e = \phi - c_\psi \psi - c_z z, \quad (23)$$

and consider, first, e -subsystem of (4), which is now a scalar differential equation

$$\dot{e} = A_{22}e + B_2 w_2(x, e), \quad e \Pi_e w_2(x, e) \geq 0, \quad \forall x, \forall e \quad (24)$$

Definition of $e(\cdot)$ in (23) and equations (22) allow to proceed with computing $\frac{d}{dt}e$ as

$$\begin{aligned} \dot{e} &= \left(\frac{3}{2} - \frac{c_\psi}{\beta^2} \right) \phi + \left(\frac{c_\psi \lambda_1}{\beta^2} - c_z \lambda_3 - 1 \right) \psi + \\ &+ \left(\frac{c_\psi \lambda_2}{\beta^2} - c_z \lambda_4 \right) z + \frac{1}{2} w_\phi + \left(\frac{c_\psi \alpha_u}{\beta^2} - c_z \alpha_z \right) w_{\psi,z} \end{aligned}$$

To match the first term on the right hand side of (24), the coefficients should obey the identities

$$\begin{aligned} A_{22} &= \left(\frac{3}{2} - \frac{c_\psi}{\beta^2} \right), \quad -A_{22} \cdot c_\psi = \left(\frac{c_\psi \lambda_1}{\beta^2} - c_z \lambda_3 - 1 \right) \\ -A_{22} \cdot c_z &= \left(\frac{c_\psi \lambda_2}{\beta^2} - c_z \lambda_4 \right) \end{aligned} \quad (25)$$

To meet the structure of nonlinearity (infinite sector condition) in (24), we obtain one more equation

$$\frac{1}{\beta^2} c_\psi \alpha_u - c_z \alpha_z = -\frac{1}{2} \quad (26)$$

Then, indeed, we can define B_2 and $w_2(\cdot)$ in (24) as $\frac{1}{2}$ and $w_2(\cdot) \triangleq w_\phi(\cdot) - w_{\psi,z}(\cdot)$ respectively, so that the inequality (24) is achieved

$$e \cdot w_2(x, e) = (\phi - c_\psi \psi - c_z z) \times [(1 + c_\psi \psi + c_z z)^3 - (1 + \phi)^3] \leq 0$$

with $\Pi_e = -1$. To match the structure of dynamics of the x -subsystem in (4) with x defined as in (23), the differential equation for ψ in (22) should be rewritten in coordinates ψ , z and e instead of the original ones ψ , z and ϕ . Namely

$$\begin{aligned} \dot{\psi} &= \frac{1}{\beta^2} (\phi - \lambda_1 \psi - \lambda_2 z - \alpha_u w_{\psi,z}) \\ &= \frac{1}{\beta^2} e + \frac{1}{\beta^2} (c_\psi - \lambda_1) \psi + \frac{1}{\beta^2} (c_z - \lambda_2) z - \frac{1}{\beta^2} \alpha_u w_{\psi,z} \end{aligned} \quad (27)$$

Summing up the manipulations made with the closed-loop system (22), we obtain the next claim.

Lemma 3: Suppose coefficients λ 's, α 's and c 's of the controller (21) satisfy the relations (25), (26). Then

- 1) The closed-loop system (22) can be equivalently written as (4), where x and e are defined by (23),

$$A_{22} = \left(\frac{3}{2} - \frac{c_\psi}{\beta^2} \right), \quad B_2 = \frac{1}{2}, \quad w_1(x) = w_{\psi,z}, \quad w_2(x, e) = w_\phi - w_{\psi,z}, \quad \text{and}$$

$$A_{11} = \begin{bmatrix} \frac{c_\psi - \lambda_1}{\beta^2} & \frac{c_z - \lambda_2}{\beta^2} \\ \lambda_3 & \lambda_4 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} \frac{1}{\beta^2} \\ 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -\frac{\alpha_u}{\beta^2} \\ \alpha_z \end{bmatrix};$$

- 2) The nonlinear function $w_2(\cdot)$ is defined by $w_2(x, e) = w_\phi - w_{\psi,z}$ and satisfies the infinite sector condition (24) with $\Pi_e = -1$;
- 3) The nonlinearity $w_1(\cdot)$ defined by $w_1(x) := w_{\psi,z}$ and the linear fictitious output

$$v_1 = -c_\psi \psi - c_z z = C_1 x \quad (28)$$

of the x -dynamics satisfies the QC (9)

$$w_1(x) \cdot v_1 \geq \frac{3}{4} v_1^2, \quad \forall \psi, z$$

with the redefined state and the linear output. ■

Once the coefficients of the controller (21) are found such that following a change of coordinates the closed-loop system (22) can be rewritten as (4), we can apply Theorem 1 and search for a set of parameters corresponding to stabilizing controllers (21). The result based on applying the Frequency Theorem [13] to verify (12) is formulated next.

Proposition 1: Consider the closed-loop system (22). Suppose that the parameters λ 's, α 's, and c 's are such that the relations (25), (26) are satisfied and the following conditions hold:

- 1) The inequality

$$\operatorname{Re} \left\{ T(j\omega) \right\} - \frac{3}{4} |T(j\omega)|^2 < 0 \quad (29)$$

is valid for all $\omega \geq 0$, where

$$\begin{aligned} T(s) &= C_1 \left(sI_2 - A_{11} \right)^{-1} B_1 \\ &= -\frac{1}{2} s + \frac{c_z \alpha_z (c_\psi - \lambda_1) + c_z \alpha_u \lambda_3 - c_\psi \alpha_u \lambda_4 - c_\psi (c_z - \lambda_2) \alpha_z}{\beta^2} \\ &= \frac{-\frac{1}{2} s + \frac{c_z \alpha_z (c_\psi - \lambda_1) + c_z \alpha_u \lambda_3 - c_\psi \alpha_u \lambda_4 - c_\psi (c_z - \lambda_2) \alpha_z}{\beta^2}}{s^2 - s \left\{ \lambda_4 + \frac{c_\psi - \lambda_1}{\beta^2} \right\} + \frac{\lambda_4 (c_\psi - \lambda_1) - \lambda_3 (c_z - \lambda_2)}{\beta^2}} \end{aligned} \quad (30)$$

- 2) The matrix

$$A_{11} + \frac{3}{4} B_1 C_1 = \begin{bmatrix} \frac{c_\psi - \lambda_1}{\beta^2} & \frac{c_z - \lambda_2}{\beta^2} \\ \lambda_3 & \lambda_4 \end{bmatrix} + \frac{3}{4} \begin{bmatrix} -\frac{\alpha_u}{\beta^2} \\ \alpha_z \end{bmatrix} \begin{bmatrix} -c_\psi \\ -c_z \end{bmatrix}^T$$

is Hurwitz;

- 3) The constant $A_{22} = \left(\frac{3}{2} - \frac{1}{\beta^2} c_\psi \right)$ is negative.

Then with any sets of these parameters the closed loop system (22), i.e. the surge subsystem (1), (2) with the defined by these parameters dynamic output feedback controller (21), is quadratically stable. ■

B. Output Feedback Controller Design: Example 2

Consider the following modification of the family of output feedback controllers (21)

$$\begin{aligned} u &= \lambda_1 \psi + \lambda_2 z + \alpha_u (1 - (1 + c_\psi \psi + c_z z)^3) + \varepsilon_u q \\ \dot{z} &= \lambda_3 \psi + \lambda_4 z + \alpha_z (1 - (1 + c_\psi \psi + c_z z)^3) + \varepsilon_z q \\ \dot{q} &= -(c_\psi \psi + c_z z) \end{aligned} \quad (31)$$

where $\lambda_1 - \lambda_4$, α_u , α_z , c_ψ , c_z , ε_u and ε_z are constant parameters to be determined. The closed-loop system (1)–(2) with any of such controllers is then

$$\begin{bmatrix} \dot{\phi} \\ \dot{\psi} \\ \dot{z} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -1 & 0 & 0 \\ \frac{1}{\beta^2} & -\frac{1}{\beta^2}\lambda_1 & -\frac{1}{\beta^2}\lambda_2 & -\frac{1}{\beta^2}\varepsilon_u \\ 0 & \lambda_3 & \lambda_4 & \varepsilon_z \\ 0 & -c_\psi & -c_z & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \psi \\ z \\ q \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{\beta^2}\alpha_u \\ \alpha_z \\ 0 \end{bmatrix} w_{\psi,z} + \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix} w_\phi$$

$$w_\phi = 1 - (1 + \phi)^3 \quad (32)$$

$$w_{\psi,z} = 1 - (1 + c_\psi\psi + c_z z)^3$$

To meet the structure of the system (4), introduce vectors x and e as follows

$$x^T = [\psi, z, q]^T, \quad e = \phi - c_\psi\psi - c_z z. \quad (33)$$

The e -dynamics are then

$$\begin{aligned} \frac{d}{dt}e &= \frac{d}{dt}\phi - c_\psi \frac{d}{dt}\psi - c_z \frac{d}{dt}z \\ &= \underbrace{\left[\frac{3}{2} - \frac{1}{\beta^2}c_\psi \right]}_{A_{22}} e + \underbrace{\frac{1}{2}}_{B_2} \underbrace{[w_\phi - w_{\psi,z}]}_{w_2(x,e)} \end{aligned} \quad (34)$$

where the last equality holds provided that the coefficients λ 's, c 's, α 's, ε 's satisfy (25), (26) and new relation

$$\frac{1}{\beta^2}c_\psi\varepsilon_u = c_z\varepsilon_z \quad (35)$$

Rewriting ψ -dynamics in e instead of ϕ -variable, similar to (27), allows us to meet the structure of the system (4).

Lemma 4: Suppose coefficients – λ 's, c 's, α 's, ε 's – of the controller (31) satisfy the relations (25), (26), (35). Then

- 1) The closed-loop system (32) can be equivalently written as (4), where x and e are defined by (23), $A_{22} = \left(\frac{3}{2} - \frac{c_\psi}{\beta^2}\right)$, $B_2 = \frac{1}{2}$, $w_1(x) = w_{\psi,z}$, $w_2(x, e) = w_\phi - w_{\psi,z}$, and

$$A_{11} = \begin{bmatrix} \frac{1}{\beta^2}[c_\psi - \lambda_1] & \frac{1}{\beta^2}[c_z - \lambda_2] & -\frac{1}{\beta^2}\varepsilon_u \\ \lambda_3 & \lambda_4 & \varepsilon_z \\ -c_\psi & -c_z & 0 \end{bmatrix} \quad (36)$$

$$A_{12} = \begin{bmatrix} \frac{1}{\beta^2} \\ 0 \\ 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -\frac{1}{\beta^2}\alpha_u \\ \alpha_z \\ 0 \end{bmatrix};$$

- 2) The nonlinear function $w_2(\cdot)$ is defined by $w_2(x, e) := w_\phi - w_{\psi,z}$ and satisfies the infinite sector condition (24) with $\Pi_e = -1$;
- 3) The nonlinear function $w_1(\cdot)$ is defined by $w_1(x) := w_{\psi,z}$ and the linear fictitious output

$$v_1 = -c_\psi\psi - c_z z = \underbrace{[-c_\psi, -c_z, 0]}_{C_1} x \quad (37)$$

of the x -dynamics satisfies the QC (9)

$$w_1(x) \cdot v_1 \geq \frac{3}{4}v_1^2, \quad \forall \psi, z, q$$

with the redefined state and the linear output. ■

With coefficients of the controller as in Lemma 4, we can apply Theorem 2 using the Frequency Theorem [13] to verify (17), and describe stabilizing controllers in the family (31).

Proposition 2: Consider the closed-loop system (32). Suppose that the parameters λ 's, α 's, c 's, and ε 's are such that the relations (25), (26), (35) are valid and the next conditions hold:

- 1) The inequality

$$\operatorname{Re} \left\{ T(j\omega) \right\} - \frac{3}{4} |T(j\omega)|^2 \leq 0 \quad (38)$$

is valid for any $\omega \geq 0$, where

$$T(s) = C_1 (sI_2 - A_{11})^{-1} B_1 = \frac{-\frac{1}{2}s^2 + p_1 s}{s^3 + l_2 s^2 + l_1 s + l_0} \quad (39)$$

with $l_0 = \frac{1}{\beta^2}\varepsilon_u$ and

$$p_1 = \frac{1}{2\beta^2} \left[\frac{c_\psi}{c_z} \lambda_2 - \lambda_1 - 2\alpha_u \right]$$

$$l_2 = -\frac{3}{2} + \frac{1}{\beta^2} \left[\lambda_1 - \frac{c_\psi}{c_z} \lambda_2 \right]$$

$$l_1 = \frac{1}{\beta^2} - \frac{3}{2\beta^2} \lambda_1 + \frac{1}{\beta^2} \left[\frac{3c_\psi}{2c_z} - \frac{1}{c_z} \right] \lambda_2$$

- 2) The matrix $A_{11} + \frac{3}{4}B_1C_1$ is Hurwitz, and the pair $([1, 0, 0], A_{11})$ is observable;
- 3) The constant $A_{22} = \left(\frac{3}{2} - \frac{1}{\beta^2}c_\psi\right)$ is negative.

Then, with any sets of these parameters the closed-loop system (32), i.e. the surge subsystem (1), (2) with the defined by these parameters dynamic output feedback controller (31), is globally asymptotically stable. ■

IV. COMPUTER SIMULATIONS

A. Example 1

The set of stabilizing controllers (21) described in Proposition 1 is not empty. For instance, if $\beta = 1$ then the following coefficients

$$\lambda_1 = -17, \lambda_2 = -4, \lambda_3 = -\frac{207}{2}, \lambda_4 = -\frac{47}{2} \quad (40)$$

$$\alpha_u = -1, \alpha_z = -\frac{9}{2}, c_\psi = 5, c_z = 1$$

satisfy all the requirements. In particular, the transfer function (30) is

$$T(s) = \frac{-s/2 - 1/2}{s^2 + 3s/2 + 1/2}.$$

It is positive real, and hence satisfies (29). The eigenvalues of the matrix $A_{11} + \frac{3}{4}B_1C_1$ are $\{-1, -0.875\}$. Fig. 1 depicts the evolution of ϕ and ψ variables for one of a typical solutions of the closed-loop system.

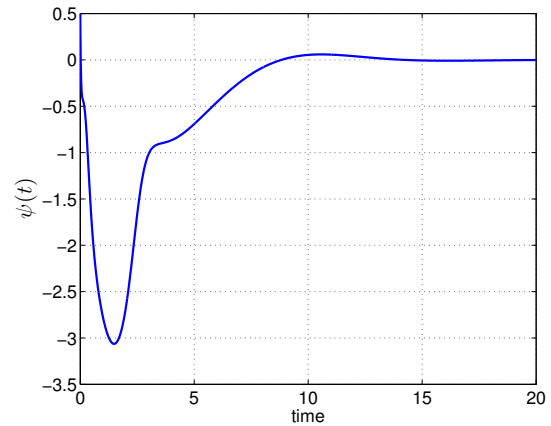
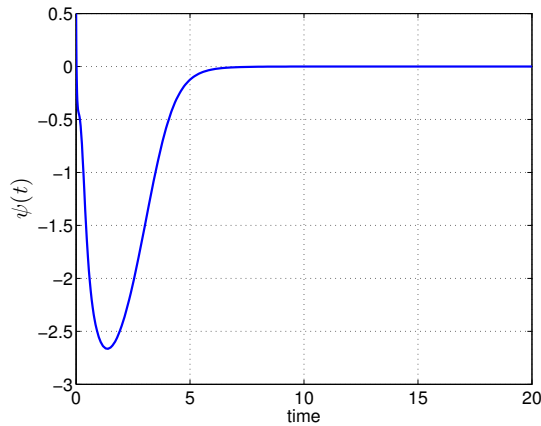
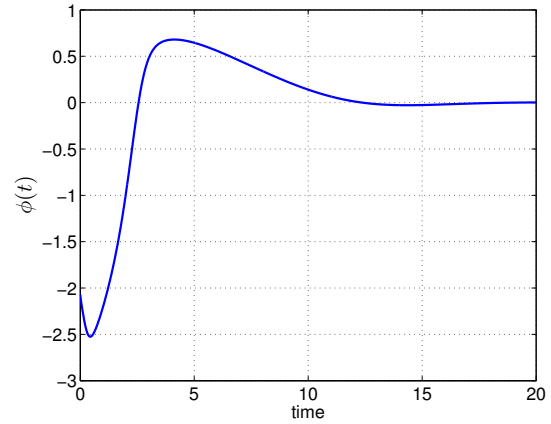
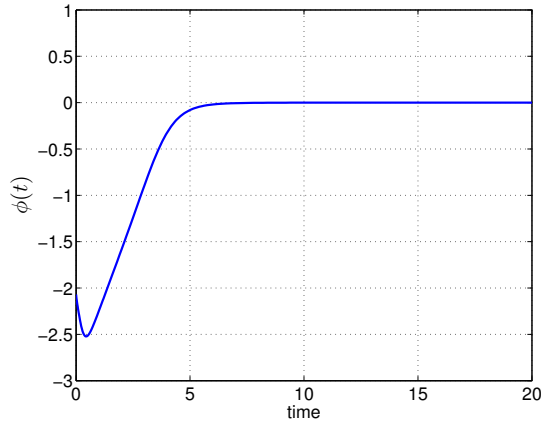


Fig. 1. The solution of the closed-loop system with the dynamical controller (22) when coefficients are chosen as in (40).

Fig. 2. The solution of the closed-loop system with the dynamical controller (32) when coefficients are chosen as in (40), (41).

B. Example 2

The set of stabilizing controllers (21) described in Proposition 2 is not empty. For instance, if $\beta = 1$, then the controllers with coefficients (40) and

$$\varepsilon_u = 4/5, \quad \varepsilon_z = 4 \quad (41)$$

satisfy all the requirements. Fig. 2 depicts the evolution of ϕ and ψ variables for the closed-loop system with the same initial conditions as in Example 1 above.

V. CONCLUDING REMARKS

We suggest here two new families of robust output (drop-in-pressure) feedback controllers that stabilize the well-known finite dimensional model for surge instability of compressor systems. Theoretical results are rigorously proved using quadratic constraints. Controllers from the first family ensure global exponential stabilization. The ones from the second family provide integral action but only ensure local exponential and global asymptotic stability. Performance is verified by simulations.

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