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Continuous-Time Model Identification and State Estimation Using Non-Uniformly Sampled Data

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Abstract: This paper presents theory, algorithms and validation results for system identification of continuous-time state-space models from finite non-uniformly sampled input-output sequences. The algorithms developed are methods of model identification and stochastic realization adapted to the continuous-time model context using non-uniformly sampled input-output data. The resulting model can be decomposed into an input-output model and a stochastic innovations model. For state estimation dynamics, we have designed a procedure to provide separate continuous-time temporal update and error-feedback update based on non-uniformly sampled input-output data. Stochastic convergence analysis is provided.

Keywords: System identification, Continuous-time identification, Non-uniform sampling

1. INTRODUCTION

The accurate knowledge of a continuous-time transfer function is a prerequisite to many methods in physical modeling and control system design. System identification, however, is often made by means of time-series analysis applied to discrete-time transfer function models. As yet, there is no undisputed algorithm for parameter translation from discrete-time parameters to a continuous-time description. Problems in this context are associated with translation of the system zeros from the discrete-time model to the continuous-time model whereas the system poles are mapped by means of complex exponentials. As a result, a poor parameter translation tends to affect both the frequency response such as the Bode diagram and the transient response such as the impulse response. One source of error in many existing algorithms is that computation of the system zeros is affected by the assumed and actual inter-sample behavior of the control variables.

There are two circumstances that favor the traditional indirect approach via discrete-time identification: Firstly, data are in general available as discrete measurements. Another problem is the mathematical difficulty to treat continuous-time random processes. In the context of discrete-time measurements, however, it is in many cases sufficient to model disturbances as a noise sequence of finite spectral range. A relevant question is, of course, why there is no analogue to ARMAX models for continuous-time systems. One reason is that polynomials in the differential operator can not be used for identification immediately due to the implementation problems associated with differentiation. The successful ARMAX-models correspond to transfer function polynomials in the z-transform variable z or z^{-1}—*i.e.*, the forward or the backward shift operators, with advantages for modeling and signal processing, respectively, and translation between these two representations is not difficult. A related problem is how to identify accurate continuous-time transfer functions from data and, in particular, how to obtain good estimates of the zeros of a continuous-time transfer function. The difficulties to convert a discrete-time transfer function to continuous-time transfer function are well known and related to the mapping $f(z) = (\log z)/h$—for non-uniform sampling, see (Marvasti 2001; Eng and Gustafsson 2008).

We derive an algorithm that fits continuous-time transfer function models to discrete-time non-uniformly sampled data and we adopt a hybrid modeling approach by means of a discrete-time disturbance model and a continuous-time transfer function.

2. A MODEL TRANSFORMATION

This algorithm introduces an algebraic reformulation of transfer function models. In addition, we introduce discrete-time noise models in order to model disturbances. The idea is to find a causal, stable, realizable linear operator that may replace the differential operator while keeping an exact transfer function. This shall be done in such a way that we obtain a linear model for estimation of the original transfer function parameters a_i, b_i. We will consider cases where we obtain a linear model in all-pass or low-pass filter operators. Actually, there is always a linear one-to-one transformation which relates the continuous-time parameters and the convergence points for each choice of operator (Johansson 1994).

Then follows investigations on the state space properties of the introduced filters and the original model. The convergence rate of the parameter estimates is then considered. Finally, there are two examples with applications to time-invariant and time-varying systems, respectively. Consider a linear $n-$th order transfer operator formulated with a differential operator $p = d/dt$ and unknown coefficients a_i, b_i.

$$G_0(p) = \frac{b_1 p^{n-1} + \cdots + b_n}{p^n + a_1 p^{n-1} + \cdots + a_n} = \frac{B(p)}{A(p)} \tag{1}$$

where it is assumed that A and B are coprime. It is supposed that the usual isomorphism between transfer operators and transfer functions, *i.e.*, the corresponding functions of a complex variable s, is valid. Because of this isomorphism, G_0 will sometimes be regarded as a transfer function and sometimes as a transfer operator. A notational difference will be made with p denoting the differential operator and s denoting the complex frequency variable of the Laplace transform.

On any transfer function describing a physically realizable continuous-time system, it is a necessary requirement that because pure derivatives of the input cannot be implemented. This requirement is fulfilled as $\lim_{s\to\infty} G_0(s)$ is finite, *i.e.*, $G_0(s)$ has no poles at infinity. An algebraic approach to system analysis may be suggested. Let a be point on the positive real axis and define the mapping

$$f(s) = \frac{a}{s+a}, \qquad s \in \mathbb{C}$$

Let $\bar{\mathbb{C}} = \mathbb{C} \cup \infty$ be the complex plane extended with the 'infinity point'. Then f is a bijective mapping from $\bar{\mathbb{C}}$ to $\bar{\mathbb{C}}$ and it maps the 'infinity point' to the origin and $-a$ to the 'infinity point'. The unstable region—*i.e.*, the right half plane (Re $s > 0$)—is mapped onto a region which does not contain the 'infinity point'. Introduction of the operator

$$\lambda = f(p) = \frac{a}{p+a} = \frac{1}{1+p\tau}, \qquad \tau = 1/a \tag{2}$$

This allows us to make the following transformation

$$G_0(p) = \frac{B(p)}{A(p)} = \frac{B^*(\lambda)}{A^*(\lambda)} = G_0^*(\lambda)$$

with

$$A^*(\lambda) = 1 + \alpha_1\lambda + \alpha_2\lambda^2 + \cdots + \alpha_n\lambda^n \tag{3}$$

$$B^*(\lambda) = \beta_1\lambda + \beta_2\lambda^2 + \cdots + \beta_n\lambda^n \tag{4}$$

An input-output model is easily formulated as

$$A^*(\lambda)y(t) = B^*(\lambda)u(t)$$

or on regression form

$$\begin{aligned} y(t) = &-\alpha_1[\lambda y](t) - \cdots - \alpha_n[\lambda^n y](t) \\ &+\beta_1[\lambda u](t) + \cdots + \beta_n[\lambda^n u](t) \end{aligned} \tag{5}$$

This is now a linear model of a dynamical system at all points of time. Notice that $[\lambda u], [\lambda y]$ etc. denote filtered inputs and outputs. The parameters α_i, β_i may now be estimated by any suitable method for estimation of parameters of a linear model. A reformulation of the model (5) to a linear regression form is

$$y(t) = \varphi_\tau^T(t)\theta_\tau, \quad \theta_\tau = (\alpha_1 \; \alpha_2 \; \ldots \; \alpha_n \; \beta_1 \; \ldots \; \beta_n)^T \tag{6}$$

$$\varphi_\tau(t) = (-[\lambda y](t), \ldots - [\lambda^n y](t), \; [\lambda u](t), \ldots [\lambda^n u](t))^T \tag{7}$$

with parameter vector θ_τ and the regressor vector φ_τ. We may now have the following continuous-time input-output relations:

$$\begin{aligned} y(t) &= G_0(p)u(t) = G_0^*(\lambda)u(t), \\ Y(s) &= \mathcal{L}\{\, y(t)\} = G_0^*(\lambda(s))U(s) \end{aligned} \tag{8}$$

$$\begin{aligned} y(t) &= \varphi_\tau^T(t)\theta_\tau \\ Y(s) &= \Phi_\tau^T(s)\theta_\tau \quad \text{where} \quad \Phi_\tau(s) = \mathcal{L}\{\varphi_\tau(t)\}(s) \end{aligned} \tag{9}$$

where $\mathcal{L}$ denotes a Laplace transform. As a consequence of the linearity of the Laplace transform, one can conclude that the same linear relation holds in both the time domain and the frequency domain. Notice that this property holds without any approximation or any selection of data.

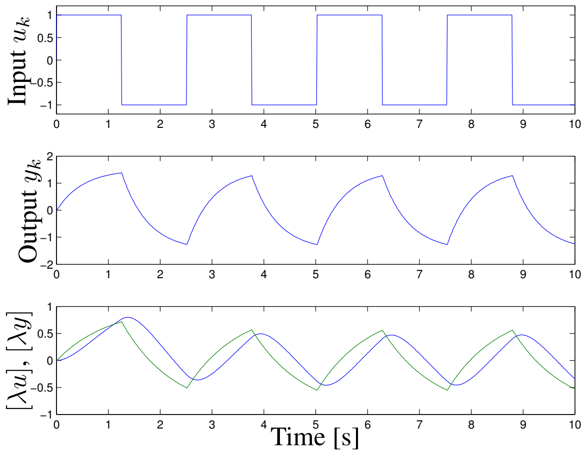

Fig. 1. Non-uniformly sampled data from simulation of continuous-time system of Example 1 with $a_1 = 2, b_1 = 3$ and continuous-time regressors for identification: Input u (*upper*), disturbance-free output y (*middle*), regressors $[\lambda u]$, $[\lambda y]$ (*lower*) for operator λ with $\tau = 1$.

2.1 Example—Estimation of two constant parameters

Consider the system with input u, output y, and the transfer operator G_0

$$y(t) = G_0(p)u(t) = \frac{b_1}{p+a_1}u(t) \tag{10}$$

Use the operator transformation λ of (2) Use the operator transformation λ of (2)

$$\lambda = \frac{1}{1+p\tau} \tag{11}$$

This gives the transformed model

$$G_0^*(\lambda) = \frac{b_1\tau\lambda}{1+(a_1\tau-1)\lambda} = \frac{\beta_1\lambda}{1+\alpha_1\lambda}$$

A linear estimation model of the type (6) is given by

$$y(t) = -\alpha_1[\lambda y](t) + \beta_1[\lambda u](t) = \varphi_\tau^T(t)\theta_\tau(t) \tag{12}$$

with regressor $\varphi_\tau(t)$ and the parameter vector θ_τ and

$$\varphi_\tau(t) = \begin{pmatrix} -[\lambda y](t) \\ [\lambda u](t) \end{pmatrix}, \quad \theta_\tau = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \tag{13}$$

The original parameters are found via the relations

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\tau}(\alpha_1 + 1) \\ \frac{1}{\tau}\beta_1 \end{pmatrix} \tag{14}$$

and their estimates from

$$\begin{pmatrix} \widehat{a}_1 \\ \widehat{b}_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\tau}(\widehat{\alpha}_1 + 1) \\ \frac{1}{\tau}\widehat{\beta}_1 \end{pmatrix} \tag{15}$$

Sampling—*e.g.*, non-uniform sampling—of all variables in Eq.

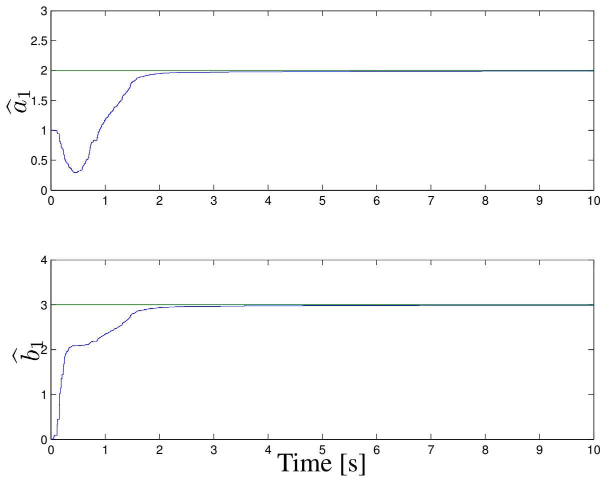

Fig. 2. Continuous-time model identification of Example 1 with $a_1 = 2$, $b_1 = 3$ and recursive least-squares identification using continuous-time regressors for input u and disturbance-free output y. The estimates $\widehat{a}_1, \widehat{b}_1$ converge to the correct values $a_1 = 2$, $b_1 = 3$ for $N = 1000$ samples.

(12) and application of the recursive least-squares estimation algorithm is obviously possible.

The filter constant a (or τ) of the operator λ should be regarded as a design parameter to be chosen appropriately. As the components of the regressor vector φ_τ tend to become small for high frequency input one should match the filter constant with respect to the dynamics of the system investigated.

Remark—Operator Representation Singularities A relevant question is, of course, how general is the choice λ and if it can, for instance, be replaced by some other bijective mapping

$$\mu = \frac{bs + a}{s + a}, \qquad b \in \mathbb{R}, \quad a \in \mathbb{R}^+, \qquad \text{and } s = \frac{\mu a - a}{b - \mu} \tag{16}$$

One can treat this problem by considering the example

$$G_0(s) = \frac{1}{s + a/b + \epsilon} \qquad \text{where} \quad \epsilon \in \mathbb{R} \quad \text{is small}$$

Application of the operator translation μ gives

$$G_0(s) = \frac{1}{s + a/b + \epsilon} = \frac{\mu - b}{-\epsilon b + (a(\frac{1}{b} - 1) + \epsilon)\mu} = G_0^*(\mu)$$

Obviously, the zero-order denominator polynomial coefficient will vanish for $\epsilon = 0$ so that $G_0^*(\mu)$ exhibits a pole at $z = 0$. The corresponding estimation model would be

$$y = \alpha[\mu y] + \beta_1[\mu u] + \beta_0[u] \tag{17}$$

$$= (\frac{1}{\epsilon}\frac{a}{b}(\frac{1}{b} - 1) + \frac{1}{b})[\mu y] - \frac{1}{\epsilon b}[\mu u] + \frac{1}{\epsilon}[u] \tag{18}$$

which exhibits coefficients of very large magnitudes for small ϵ. This would constitute a serious sensitivity problem—at least for $b > 0$ for which $G_0(s)$ is stable. An operator μ with $b < 0$ according to Eq. (16) would give rise to large coefficients of the transformed model only for unstable systems which might be more 'affordable'. By comparison, a model transformation using λ would not exhibit any such singularities. Hence, use of the operator μ should for sensitivity reasons be restricted to cases with $b = 0$ (or $b_{min} < b \leq 0$ for some number b_{min} chosen according to some *a priori* information about the system dynamics). Note that the set of polynomials associated with $b < 0$ is related to the orthogonal Laguerre polynomials.

2.2 *Parameter transformations*

Before we proceed to clearcut signal processing aspects we should make clear the relationship between the parameters α_i, β_i of (2) and the original parameters a_i, b_i of the transfer function (1). Let the vector of original parameters be denoted by

$$\theta = (-a_1 \ -a_2 \ \ldots \ -a_n \ b_1 \ \ldots \ b_n)^T \tag{19}$$

Using the definition of λ (2) and (2) it is straightforward to show that the relationship between (6) and (19) is

$$\theta_\tau = F_\tau \theta + G_\tau \tag{20}$$

where the $2n \times 2n-$matrix F_τ is

$$F_\tau = \begin{pmatrix} M_\tau & 0_{n\times n} \\ 0_{n\times n} & M_\tau \end{pmatrix} \tag{21}$$

and where

$$M_\tau = \begin{pmatrix} m_{11} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ m_{n1} & \cdots & m_{nn} \end{pmatrix}, \quad m_{ij} = (-1)^{i-j}\binom{n-j}{i-j}\tau^j \tag{22}$$

Furthermore, the $2n \times 1-$vector G_τ are given by

$$G_\tau = (g_1 \ \ldots \ g_n \ 0 \ \ldots \ 0)^T; \qquad g_i = \binom{n}{i}(-1)^i \tag{23}$$

The matrix F_τ is invertible when M_τ is invertible, *i.e.* for all $\tau > 0$. The parameter transformation is then one-to-one and

$$\theta = F_\tau^{-1}(\theta_\tau - G_\tau)$$

We may then conclude that the parameters a_i, b_i of the continuous-time transfer function G_0 may be reconstructed from the parameters α_i, β_i of θ_τ by means of basic matrix calculations. As an alternative we may estimate the original parameters a_i, b_i of θ from the linear relation

$$y(t) = \theta_\tau^T \varphi_\tau(t) = (F_\tau\theta + G_\tau)^T\varphi_\tau(t) \tag{24}$$

where F_τ and G_τ are known matrices for each τ. Furthermore, elaborated identification algorithms adapted for numerical purposes sometimes contain some weighting or orthogonal linear combination of the regressor vector components by means of some linear transformation matrix T. Thus, one can modify (24) to

$$y(t) = (T\varphi_\tau(t))^T T^{-T} F_\tau \theta + (T\varphi_\tau(t))^T T^{-T} G_\tau$$

Hence, the parameter vectors θ_τ and θ are related via known and simple linear relationships so that translation between the two parameter vectors can be made without any problem arising. Moreover, identification can be made with respect to either θ or θ_τ.

2.3 *Orthogonalization and Numerics*

In some cases, independent regressor variables are chosen at the onset of identification so as to be orthogonal in order to save computational effort. The case of polynomial regression provides an example of the use of orthogonalized independent variables. Orthogonal expansions are suitable and advantageous for of MA-type (FIR) and AR-type models but cannot be fully exploited in the case of regression vectors containing both input and output data, *e.g*, ARX and ARMAX models. A suitable approach to orthogonalization is to consider an asymptotically stable state space realization (A, B, C) by means of computation of the correlation between the impulse responses of the different state vector components x_i. As these impulse responses can be written $x(t) = e^{At}B$ one finds that

$$P_c = \int_0^\infty x(t)x^T(t)dt = \int_0^\infty e^{At}BB^Te^{A^Tt}dt \tag{25}$$

It is well known that P_c of (25) fulfills the Lyapunov equation

$$AP_c + P_cA^T = -BB^T \tag{26}$$

with a unique positive semidefinite solution P_c. If P_c were diagonal, one could conclude from Eq. (26) that the components of $x(t)$ be orthogonal. To the purpose of orthogonalization, let P_c be factorized according to the Cholesky factorization

$$P_c = R_1^T R_1$$

and choose $T = R_1^{-T}$. Now introduce the state-space transformation

$$z = Tx$$

with the dynamics

$$\dot{z} = A_1 z + B_1, \qquad \text{where } \begin{cases} A_1 = TAT^{-1} \\ B_1 = TB \end{cases} \tag{27}$$

By means of (26) it can be verified that

$$A_1 + A_1^T = -B_1B_1^T, \qquad \text{and } P_z = \int_0^\infty z(t)z^T(t)dt = I$$

which implies that the components of $z = Tx$ are mutually orthogonal over the interval $[0, \infty)$. As P_z is diagonal, it is clear that the components of $z(t)$ be orthogonal.

Thus, the λ-operator is still effective for higher-order systems, the upper order limit of application being determined by the quality of the numerical solvers of Lyapunov equations and numerical Cholesky factorization.

Example—Orthogonalization Let $\lambda(s) = 1/(s+1)$ and consider the third order state-space realization

$$\begin{pmatrix} \lambda(s)U(s) \\ \lambda^2(s)U(s) \\ \lambda^3(s)U(s) \end{pmatrix} = X(s) = \begin{pmatrix} s+1 & 0 & 0 \\ -1 & s+1 & 0 \\ 0 & -1 & s+1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} U(s) \tag{28}$$

The solution of the Lyapunov equation (26) gives

$$P_c = \begin{pmatrix} 0.5000 & 0.2500 & 0.1250 \\ 0.2500 & 0.2500 & 0.1875 \\ 0.1250 & 0.1875 & 0.1875 \end{pmatrix} > 0 \tag{29}$$

so that

$$T = \begin{pmatrix} 1.0000 & 0 & 0 \\ -1.0000 & 2.0000 & 0 \\ 1.0000 & -4.0000 & 4.0000 \end{pmatrix} \tag{30}$$

As a result of orthogonality there is no need to solve a set of linear equation for MA-models with white-noise inputs. This might be advantageous for special purpose hardware implementation or when the linear equations become nearly singular or when the model order increases.

2.4 Non-uniform Sampling

Assume that data acquisition has provided finite sequences of non-uniformly sampled input-output data $\{y(t_k)\}_0^N$, $\{u(t_k)\}_0^N$ at sample times $\{t_k\}_0^N$, where $t_{k+1} > t_k$ for all k.

As the regression model of Eq. (5) is valid for all times, it is also a valid regression model at sample times $\{t_k\}_0^N$

$$\begin{aligned} y(t_k) = &-\alpha_1[\lambda y](t_k) - \cdots - \alpha_n[\lambda^n y](t_k) \\ &+\beta_1[\lambda u](t_k) + \cdots + \beta_n[\lambda^n u](t_k) \end{aligned} \tag{31}$$

Introduce the following brief notation for non-uniformly sampled filtered data

$$[\lambda^j u]_k = [\lambda^j u](t_k),\ 0 \le j \le n,\ 0 \le k \le N \tag{32}$$

$$[\lambda^j y]_k = [\lambda^j y](t_k) \tag{33}$$

so that

$$\begin{aligned} y_k = &-\alpha_1[\lambda y]_k - \cdots - \alpha_n[\lambda^n y]_k \\ &+\beta_1[\lambda u]_k + \cdots + \beta_n[\lambda^n u]_k \end{aligned} \tag{34}$$

Introduce the filter states

$$x_u = \begin{pmatrix} [\lambda^1 u] \\ [\lambda^2 u] \\ \vdots \\ [\lambda^n u] \end{pmatrix}, \qquad x_y = \begin{pmatrix} [\lambda^1 y] \\ [\lambda^2 y] \\ \vdots \\ [\lambda^n y] \end{pmatrix} \tag{35}$$

with dynamics

$$\frac{1}{\tau}\dot{x}_u = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ 1 & -1 & \ddots & \ddots & 0 \\ 0 & 1 & -1 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & 1 & -1 \end{pmatrix} x_u + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u \tag{36}$$

$$\frac{1}{\tau}\dot{x}_y = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ 1 & -1 & \ddots & \ddots & 0 \\ 0 & 1 & -1 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & 1 & -1 \end{pmatrix} x_u + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} y \tag{37}$$

or

$$\frac{1}{\tau}\dot{x}_u = A_\lambda x_u + B_\lambda u, \quad \frac{1}{\tau}\dot{x}_y = A_\lambda x_y + B_\lambda y \tag{38}$$

Adopting a zero-order-hold (ZOH) approximation, the non-uniformly sampled discretized model will be

$$x_u(t_{k+1}) = A_k x_u(t_k) + B_k u(t_k) \tag{39}$$

$$x_y(t_{k+1}) = A_k x_y(t_k) + B_k y(t_k) \tag{40}$$

where

$$\begin{aligned} A_k &= e^{A_\lambda(t_{k+1}-t_k)/\tau} \\ B_k &= \int_0^{t_{k+1}-t_k} e^{A_\lambda s/\tau}\frac{1}{\tau}B_\lambda ds = \int_0^{(t_{k+1}-t_k)/\tau} e^{A_\lambda s}B_\lambda ds \end{aligned} \tag{41}$$

Summarizing the regressor model of Eq. (34) including the regressor filtering, we have

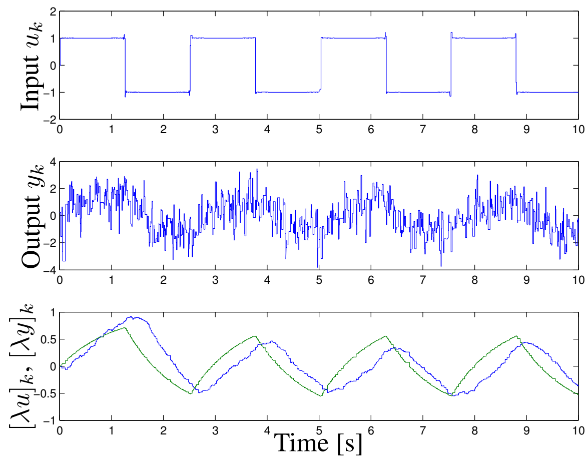

Fig. 3. Non-uniformly sampled data used for continuous-time model identification: Input $\{u_k\}$ (*upper*), output $\{y_k\}$ with stochastic disturbance (*middle*), regressors $\{[\lambda u]_k\}$, $\{[\lambda y]_k\}$ (*lower*).

$$\phi(t_k) = \begin{pmatrix} x_y(t_k) \\ x_u(t_k) \end{pmatrix} \tag{42}$$

$$\phi(t_{k+1}) = \begin{pmatrix} A_k & 0 \\ 0 & A_k \end{pmatrix} \phi(t_k) + \begin{pmatrix} -B_k & 0 \\ 0 & B_k \end{pmatrix} \begin{pmatrix} y(t_k) \\ u(t_k) \end{pmatrix} \tag{43}$$

$$\theta = (\alpha_1 \ \cdots \alpha_n \ \beta_1 \ \cdots \beta_n)^T \tag{44}$$

$$y(t_k) = \phi(t_k)\theta + w(t_k) \tag{45}$$

where $\{w(t_k)\}$ represents an uncorrelated non-uniformly sampled noise sequence.

2.5 Example (cont'd)—Identification of a first-order system

A simulated example of Ex. 2.1 is shown in Figs. 1-4 for parameters $a_1 = 2$, $b_1 = 3$ and with operator time constant $\tau = 1$. A histogram of the sampling intervals is shown in Fig. 5. Whereas a least-squares estimate based on the $N = 1000$ deterministic data of Figs. 1-2 reproduced the exact parameters $a_1 = 2$, $b_1 = 3$, the estimates $\widehat{a}_1 = 1.988$, $b_1 = 3.168$ were obtained for a signal-to-noise ratio equal to one of inputs (input u and noise w) in Figs. 3-4.

2.6 Innovations Model and Prediction

Adopting a standard continuous-time innovations model to complement the system model of Eq. (1), we have

$$\dot{x} = Ax + Bu + Kw, \qquad G_0(s) = C(sI - A)^{-1}B \tag{46}$$

$$y = Cx + w \tag{47}$$

with the innovations model inverse (or Kalman filter)

$$\dot{\widehat{x}} = (A - KC)\widehat{x} + Bu + Ky \tag{48}$$

$$\widehat{w} = y - C\widehat{x} \tag{49}$$

Updates for non-uniformly sampled input-output data can be made as the non-uniformly sampled discrete-time system

$$\widehat{x}(t_{k+1}) = F_k \widehat{x}(t_k) + (G_k \ \ H_k) \begin{pmatrix} u(t_k) \\ y(t_k) \end{pmatrix} \tag{50}$$

$$\widehat{w}(t_k) = y(t_k) - C\widehat{x}(t_k) \tag{51}$$

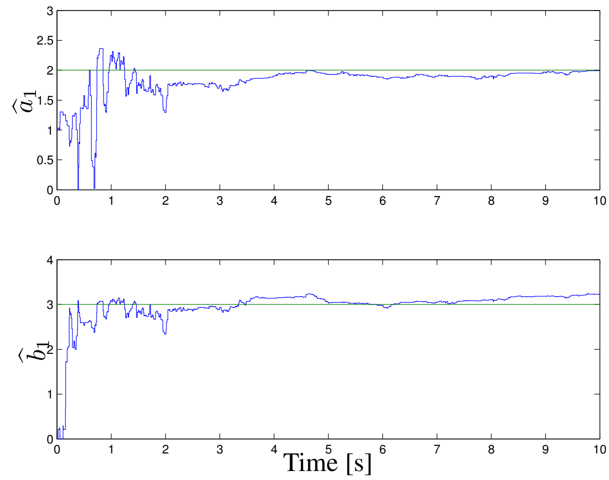

Fig. 4. Continuous-time model identification of Example 1 with $a_1 = 2$, $b_1 = 3$ and recursive least-squares identification using non-uniformly sampled input u and disturbance-contaminated output y. The estimates $\widehat{a}_1$, $\widehat{b}_1$ converge towards the correct values $a_1 = 2$, $b_1 = 3$ ($N = 1000$).

for

$$F_k = e^{(A-KC)(t_{k+1}-t_k)} \tag{52}$$

$$G_k = \int_0^{t_{k+1}-t_k} e^{(A-KC)s} B ds \tag{53}$$

$$H_k = \int_0^{t_{k+1}-t_k} e^{(A-KC)s} K ds \tag{54}$$

which permits state estimation and standard residual analysis for purposes of validation (Johansson 1993; Ch. 9). In prediction-correction format, separate time update and error update can be made as follows

$$\widehat{x}(t_{k|k}) = \widehat{x}(t_{k|k-1}) + \kappa_k(y(t_k) - C\widehat{x}(t_{k|k-1})) \tag{55}$$

$$\widehat{x}(t_{k+1|k}) = \Phi_k \widehat{x}(t_{k|k}) + \Gamma_k u(t_k) \tag{56}$$

$$\widehat{w}(t_{k|k}) = y(t_k) - C\widehat{x}(t_{k|k}) \tag{57}$$

for $\{\Phi_k, \Gamma_k, \kappa_k\}_{k=0}^N$ obtained from non-uniform discretization of Eqs. (46-47).

3. DISCUSSION AND CONCLUSIONS

We have formulated an identification method for continuous-time transfer function models and equivalent to ARMAX models for discrete-time systems. The main differences between this method and traditional models of ARMAX-model identification consists of a different estimation model and a new parametrization of the continuous-time transfer function whereas the parameter estimation method—*i.e.*, least-squares estimation or maximum-likelihood estimation, is the same as that used in ARMAX-model identification. The hybrid approach involves a discrete-time model of the stochastic disturbances with little specification of the continuous-time noise, the properties of which are not known or measured in detail. With reference to the central limit theorem, this approach appears to be an appropriate assumption in physical modeling contexts where small continuous-time disturbances add up to a normally distributed disturbance on the sampled signals. The resulting model thus has the same modeling power of the stochastic environment as that of an ARMAX model.

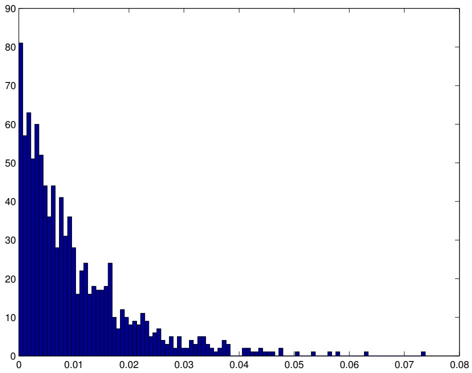

Fig. 5. Histogram of sampling intervals

The methodology presented requires implementation of operators $\lambda^1, \ldots, \lambda^n$ which serve as filters in the estimation model by operations on input-output data. Maximum-likelihood identification based on the parametric model derived results in consistent and asymptotically efficient estimates under the assumption of normally distributed noise (Johansson 1994).

The transformation by means of λ allows an *exact* reparametrization of a continuous-time transfer function. High-frequency dynamics and low-frequency dynamics thus appear without distortion in the mapping from input to output. The low-pass filters implemented for the estimation model have a filtering effect in producing regressor variables for identification. Also orthogonal regressor variables can be used in this context. Both the operator translation and filtering approaches such as the Poisson moment functional (PMF) or the Laguerre polynomials give rise to similar estimation models for the deterministic case (Saha and Rao 1983), (Unbehauen and Rao 1987), (Haber and Unbehauen 1990), (Young 1969, 1981), (Garnier 2003). Implementation of the operator λ may be done as continuous-time filters, discrete-time filters or by means of numerical integration methods. Other discretization policies may be considered. As elaborated by Middleton and Goodwin (1990), it may be valuable to replace z by the $\delta-$operator

$$\delta = \frac{z-1}{h} \tag{58}$$

in order to improve on numerical accuracy in discrete-time implementation. Whereas ZOH only was studied here, intersample behavior is significant for approximation properties.

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