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2001

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Abstract

In this paper, we approach a multi-objective $\mathbf{H}^p$ problem with several $\mathbf{H}^p$ constraints from the Banach duality point of view. The problem is reduced to an abstract norm minimization problem, and the dual problem is derived using the classical Banach result. It completes the primal-dual pair which can be used to solve the problem numerically by finite-dimensional approximations. While the approximation of the primal problem gives an upper bound of the optimal value, the approximation of the dual problem provides a lower bound, and the gap between them can be done arbitrary small. In view of growing computational power of modern computers, it gives a good alternative to the standard mixed-objective approach.

Keywords: linear system, multiple objectives, convex duality.

1. Introduction

Many control problems are multi-objective by nature. For example, a good performance (such as tracking of low-frequency signals) can be achieved by imposing specifications on the sensitivity function S at low frequencies whereas a suitable robustness property can be described by specifications on the complimentary sensitivity T at high frequencies (uncertainty increases with increasing frequencies). In the intermediate frequency region peaking of both S and T should be minimized to prevent overly large sensitivity to disturbances and the measurement noise.

In this paper, we consider the following control problem that involves multiple $\mathbf{H}^p$ objectives

$$\inf\{\|T_{z_0w_0}\|_{p_0} \;\mid\; \|T_{z_jw_j}\|_{p_j} \le \gamma_j,\; 1 \le j \le J\}, \tag{1}$$

where $T_{z_jw_j}$ is the closed-loop transfer function from the input w_j to the output z_j and $\|\cdot\|_{p_j}$ is the $\mathbf{L}^{p_j}$ norm.

The standard way to solve the multi-objective problem is to reduce it to a mixed one-objective problem with auxiliary weights

$$\inf \left\| \begin{pmatrix} W_0 T_{z_0w_0} \\ \vdots \\ W_J T_{z_Jw_J} \end{pmatrix} \right\|_p$$

and then to apply one of the standard design procedures like LQG or H^∞ optimization. This approach usually suffers from the following drawbacks

- The reduction often introduces conservatism to the initial problem in view of certain algebraic constraints, for instance, $S + T = 1$.
- As a matter of fact, we replace our problem with another one which we know how to solve. We hope to obtain a solution to the former via that to the latter with some appropriate choice of weights. There is always a problem how to choose the weights properly to capture most desirable initial objectives. This makes very often the design procedure iterative, when after getting a solution we check the initial requirements, adjust the weights manually and redesign.
- A controller given by an optimization procedure has usually the same dimension as the augmented plant (the plant + the weights) which might be too high, and there is no easy way to include constraints on the controller dimension to the optimization problem.
- The standard optimization procedures deal with real-rational finite-dimensional plants only, and it is impossible to use them directly for even simple (like time-delay) infinite-dimensional plants.

Such kind of drawbacks forces to look for a solution to the multi-objective problem other than the mixed optimization. In view of growing computational power of modern computers, a straightforward solution via finite-dimensional approximations of the problem (1) appears to be a good alternative. The problem (1) becomes convex in terms of Youla parameter Q

$$\inf\{\|T_{10} - T_{20}QT_{30}\|_{p_0} \;\big|\; \|T_{1j} - T_{2j}QT_{3j}\|_{p_j} \leq \gamma_j, \forall j\}$$

but infinite dimensional. Then by restricting Q to lie, for example, in the space of n-dimensional trigonometric polynomials and discretizing the unit circle sufficiently fine with respect to n, we come up with a finite-dimensional convex approximation of the initial problem. This idea is not new [1, 4, 6, 7] but not immediately amenable because for any n, we can get only an upper bound on the optimum value in (1). A good numerical algorithm must have a stopping criteria which says, for example, how far we are from the real optimum.

An efficient way to get this information is to look at a dual problem. The duality relation plays a role analogous to the inner product in Hilbert space. By suitable interpretation, the dual space provides the setting for Lagrange multipliers, fundamental for a study of constraint optimization problems.

If the initial problem is a minimization, the dual one is a maximization. If both problems have the same optimal value, i.e. sup = inf, we say that there is no duality gap. This means that any finite-dimensional approximation of the dual problem will give a lower bound on the optimum, and solving both primal and dual approximations in parallel yields a nonincreasing sequence of upper bounds and a nondecreasing sequence of lower bounds that converge to the optimal value.

In this paper we derive a (convex infinite-dimensional) dual problem to the multi-objective one (1) in one particular case where all exogenous signals w_j are the same scalar input w. The dual problem completes the primal-dual pair and can be used to solve the problem (1) via successive finite-dimensional approximations. The method developed here is similar to that presented in [7] where the authors use the Banach duality to solve the following nonstandard $\mathbf{H}^\infty$ problem

$$\inf_{Q \in \mathbf{H}^\infty} \||T_1 - T_2 Q| + |T_3 Q|\|_\infty$$

which appears in the robust performance problem. We will show that the multi-objective problem (1) can be considered as an abstract minimum norm optimization and hence has a dual problem with no duality gap. In case of the single scalar input w, we obtain the dual problem explicitly.

A different approach to find a sequence of lower bounds for the multi-objective $\mathbf{H}^2/\mathbf{H}^\infty$ problem was proposed in [8]. The author uses the standard $\mathbf{H}^\infty$ algebra representation by the linear operator space on $\mathbf{H}^2$ along with a projection technique and LMI. The algorithm suggested in [8] uses the Fourier coefficients of the functions T_{ij}, so they are assumed to be easy to calculate. However, it may be a nontrivial problem for nonrational functions. Our method uses only the values of T_{ij} on the unit circle (or the imaginary axis).

Another method to find an approximate solution to the multi-objective $\mathbf{H}^2/\mathbf{H}^\infty$ problem is presented in [2]. The idea is to replace all $\mathbf{H}^\infty$ constraints with $\mathbf{H}^2$ ones and to exploit the relation

$$\|h\|_\infty = \sup\{\|hw\|_2 \mid \|w\|_2 \leq 1\} \tag{2}$$

to construct a sequence $\{w_j\}$ that approximates this supremum. The algorithm gives an optimal solution *provided* that the approximate solutions converge in $\mathbf{H}^\infty$ sense. These solutions are, in general, of a high order since high order weights w_j are needed for a good approximation of (2).

The paper is organized as follows. In Section 2 we give a necessary mathematical background about the Banach duality and collect notations used. The primal multi-objective problem is stated in Section 3 and converted to an abstract norm minimization problem in a Banach space. In Section 4 we derive the dual problem and present the main result. All proofs are moved to the Appendix.

2. Preliminaries and notations

All below in this section is mainly extracted from [5].

Let us consider a normed linear space X. A dual space to X, denoted by X^*, is the space of all linear bounded functionals on X equipped with the standard norm: if $x^* \in X^*$ then

$$\|x^*\| = \sup_{\|x\| \leq 1} |x^*(x)|.$$

The space X^* with this definition of norm becomes a Banach space.

Below we shall use more symmetric notation for the value of the linear functional x^* at the point x

$$x^*(x) = \langle x, x^* \rangle.$$

Definition: A vector $x^* \in X^*$ is said to be *aligned* with a vector $x \in X$ if $\langle x, x^* \rangle = \|x^*\|\|x\|$.

Definition: The vectors $x \in X$ and $x^* \in X^*$ are said to be *orthogonal* if $\langle x, x^* \rangle = 0$.

Definition: Let S be a subset of a normed linear space X. The *annihilator* of S, denoted $S^\perp$, consists of all elements $x^* \in X^*$ orthogonal to every vector in S

$$S^\perp = \{x^* \in X^* \mid \langle x, x^* \rangle = 0,\ \forall x \in S\}.$$

THEOREM 1
Let x be an element in a normed linear space X and let M be a subspace in X. Then

$$\inf_{m\in M}\|x-m\| = \max_{\substack{\|x^*\|\le 1\\ x^*\in M^\perp}} \operatorname{Re}\langle x, x^*\rangle$$

where the maximum on the right is achieved for some $x_0^*\in M^\perp$. If the infimum on the left is achieved for some $m_0\in M$ then x_0^* is aligned with $x-m_0$. □

If the space X is not reflexive, i.e. $X\neq X^{**}$, we can relax the notion of annihilator in the dual space X^* as follows.

Definition: Given a subspace U of the dual space X^*, we define the *pre-annihilator* of $U\subset X$ as a subspace ${}^\perp U$ such that

$$({}^\perp U)^\perp = U.$$

Then we have the following stronger result on duality in space X^*.

THEOREM 2
Let X be a normed linear space. Let M^* be a subspace in X^* and x^* be an element of X^*. If there exists a pre-annihilator of M^* then

$$\min_{m^*\in M^*}\|x^*-m^*\| = \sup_{\substack{\|x\|\le 1\\ x\in {}^\perp M^*}} \operatorname{Re}\langle x, x^*\rangle$$

where the minimum on the left is achieved for some $m_0^*\in M^*$. If the supremum on the right is achieved for some $x_0\in {}^\perp M^*$ then $x^*-m_0^*$ is aligned with x_0. □

We shall use the following notations throughout the paper. The unit circle in the complex plane $\mathbb{C}$ will be denoted as

$$\mathbb{T}=\{z\in\mathbb{C}\mid |z|=1\}.$$

By $\mathbb{A}_{n\times m}$ we denote the class of all $n\times m$ holomorphic functions in the unit disc. The notation $\mathbf{L}^p_{n\times m}$ stands for the standard Lebesgue space $\mathbf{L}^p$ of $n\times m$ matrix functions on $\mathbb{T}$ with Lebesgue measure dm. Denote the standard norm in $\mathbf{L}^p_{n\times m}$ by $\|\cdot\|_p$. The subspace $\mathbf{L}^p_{n\times m}\cap\mathbb{A}_{n\times m}$ is the Hardy space $\mathbf{H}^p_{n\times m}$. An analytical continuation of the transposition with complex conjugation is denoted by $*$

$$f(z)^* = \overline{f(\bar z^{-1})}^{\,T}.$$

3. Problem setup. Primal problem.

To be specific we deal with $\mathbf{L}^p$ and $\mathbf{H}^p$ spaces on the unit circle $\mathbb{T}$ which corresponds to linear discrete-time systems. All results remain true for continuous-time systems also by modulo of the standard bilinear transformation.

Let J be a natural number. Consider a linear time-invariant plant

$$z=\begin{pmatrix} z_0\\ \vdots\\ z_J\end{pmatrix} = P\begin{pmatrix} w\\ u\end{pmatrix}$$

where u is the control and w is the exogenous signal. We shall assume that w is a scalar. The multi-objective problem can be formulated as:

Given integer numbers $0\le j\le J$ and real numbers $1<p_j\le+\infty$, $\gamma_j>0$, find a stabilizing controller K that minimizes the $\mathbf{L}^{p_0}$ norm of the closed loop transfer function T_{z_0w} subject to $\mathbf{L}^{p_j}$ norm constraints on the other transfer functions T_{z_jw}, i.e.

$$\inf_K\{\|T_{z_0w}\|_{p_0}\ \big|\ \|T_{z_jw}\|_{p_j}\le\gamma_j,\ 1\le j\le J\}.$$

With the standard Youla parameterization of all stabilizing controllers $K=K(Q)$, the problem becomes convex since the closed-loop transfer functions depend affinely on the parameter $Q\in\mathbb{A}_{m\times m}$

$$T_{zw}=T_1-T_2QT_3.$$

A simple trick allows us to get rid of T_3 when w is a scalar.

LEMMA 1
If w is a scalar then the Youla parameterization of the closed-loop transfer function takes the form

$$T_{zw}=T_1-T_2Q,\quad Q\in\mathbb{A}_{m\times 1}.$$

□

Thus we consider the following multi-objective optimization problem.

Given $T_{1j}\in\mathbf{H}^\infty_{n_j\times 1}$, $T_{2j}\in\mathbf{H}^\infty_{n_j\times m}$ the multi-objective problem is to find

$$\mu_0=\inf_{Q\in\mathbb{A}_{m\times 1}}\|T_{10}-T_{20}Q\|_{p_0}$$

subject to

$$\|T_{1j}-T_{2j}Q\|_{p_j}\le\gamma_j,\quad j=1,\dots,J.$$

Since the cost function and all specifications are convex we can use Lagrange multiplier technique to find an equivalent problem

$$\begin{aligned}\mu_0 \;=\; \max_{\tau_j\ge 0}\Bigg(\inf_{Q\in\mathbb{A}_{m\times 1}}\Bigg[&\|T_{10}-T_{20}Q\|_{p_0}+\\ &+\sum_{j=1}^J\tau_j\|T_{1j}-T_{2j}Q\|_{p_j}\Bigg]-\sum_{j=1}^J\tau_j\gamma_j\Bigg).\end{aligned}$$

Assuming that all τ_j are absorbed by corresponding T_{ij} we state the primal problem as follows:

Given $T_{1j} \in \mathbf{H}^{\infty}_{n_j\times 1}$ and $T_{2j} \in \mathbf{H}^{\infty}_{n_j\times m}$ the problem is to find

$$\inf_{Q\in \mathbb{A}_{m\times 1}} \sum_{j=0}^{J} \|T_{1j} - T_{2j}Q\|_{p_j}.$$

Introduce a notation for the Banach space

$$\mathbf{F} = \mathbf{L}^{p_0}_{n_0\times 1} \times \mathbf{L}^{p_1}_{n_1\times 1} \times \ldots \times \mathbf{L}^{p_J}_{n_J\times 1}$$

equipped with the norm

$$\|f\|_{\mathbf{F}} = \sum_{j=0}^{J} \|f_j\|_{p_j}$$

and denote $N = \sum_{j=0}^{J} n_j$ and

$$T_1 = \begin{pmatrix} T_{10} \\ \vdots \\ T_{1J} \end{pmatrix} \in \mathbf{H}^{\infty}_{N\times 1}, \quad T_2 = \begin{pmatrix} T_{20} \\ \vdots \\ T_{2J} \end{pmatrix} \in \mathbf{H}^{\infty}_{N\times m}.$$

To simplifiy the exposition we make the following assumption.

Assumption 1: There exists $\varepsilon > 0$ such that

$$T_2(z)^* T_2(z) > \varepsilon I, \quad \forall z \in \mathbb{T}.$$

Remark: Assumption 1 can be extended to allow a finite number of zeros on the unit circle. However we consider the simplest case here for clarity.

Denote

$$p = \min_{0\le j\le J} p_j.$$

Lemma 2

Let Assumption 1 holds. If $\|T_1 - T_2 Q\|_{\mathbf{F}}$ is finite then $Q \in \mathbf{H}^{p}_{m\times 1}$. □

Now we can formulate a final version of the primal problem.

Primal problem: Given $T_1 \in \mathbf{H}^{\infty}_{N\times 1}$, $T_2 \in \mathbf{H}^{\infty}_{N\times m}$ the problem is to find

$$\inf_{Q\in \mathbf{H}^{p}_{m\times 1}} \|T_1 - T_2 Q\|_{\mathbf{F}}. \tag{3}$$

This is the standard minimum norm problem: to find a distance in the space $\mathbf{F}$ from the given element T_1 to the subspace $\mathbf{X} = T_2\mathbf{H}^{p}_{m\times 1} \cap \mathbf{F}$ and Theorem 2 immediately implies that if there exist a pre-dual normed space $\mathbf{G}$ for $\mathbf{F}$ (i.e. $\mathbf{G}^* = \mathbf{F}$) and a pre-annihilator ${}^{\perp}\mathbf{X}$ then

1. there exists at least one $Q_{opt} \in \mathbf{H}^{p}_{m\times 1}$ such that

$$\inf_{Q\in \mathbf{H}^{p}_{m\times 1}} \|T_1 - T_2 Q\|_{\mathbf{F}} = \|T_1 - T_2 Q_{opt}\|_{\mathbf{F}}.$$

2. $$\min_{Q\in \mathbf{H}^{p}_{m\times 1}} \|T_1 - T_2 Q\|_{\mathbf{F}} = \sup_{\substack{x\in {}^{\perp}\mathbf{X} \\ \|x\|_{\mathbf{G}}\le 1}} \operatorname{Re}\langle x, T_1\rangle.$$

In the next section we show that the pre-dual does exist and give an explicit description to it as well as to the set ${}^{\perp}\mathbf{X}$.

4. The main result. Dual problem.

It is relatively easy to obtain a pre-dual to $\mathbf{F}$. Denote by p' the adjoint index to p, i.e. $1/p + 1/p' = 1$.

Lemma 3

A pre-dual space to $\mathbf{F}$ is a linear space

$$\mathbf{G} = \mathbf{L}^{p_0'}_{n_0\times 1} \times \mathbf{L}^{p_1'}_{n_1\times 1} \times \ldots \times \mathbf{L}^{p_J'}_{n_J\times 1}$$

equipped with the norm

$$\|g\|_{\mathbf{G}} = \max_{0\le j\le J} \|g_j\|_{p_j'}.$$

□

To obtain a pre-annihilator, we first derive an equivalent description of the subspace $\mathbf{X}$.

Lemma 4

Let Assumption 1 holds. Then there exists an inner function $\Theta \in \mathbf{H}^{\infty}_{N\times m}$ such that $T_2\mathbf{H}^{p}_{m\times 1} = \Theta\mathbf{H}^{p}_{m\times 1}$. □

Lemma 5

Under Assumption 1 the pre-annihilator of $\mathbf{X}$ is

$${}^{\perp}\mathbf{X} = (I - \Theta\Theta^*)\mathbf{L}^{p'}_{N\times 1} \;\oplus\; \Theta\, \overline{z\mathbf{H}^{p'}_{m\times 1}}.$$

□

Corollary: The pre-annihilator can be represented in terms of T_2 as follows

$${}^{\perp}\mathbf{X} = (I - T_2(T_2^*T_2)^{-1}T_2^*)\mathbf{L}^{p'}_{N\times 1} \;\oplus\; T_2(T_2^*T_2)^{-1}\, \overline{z\mathbf{H}^{p'}_{m\times 1}}.$$

Now we are in a position to present the dual problem to (3).

Dual problem: Given $T_1 \in \mathbf{H}^{\infty}_{N\times 1}$, $T_2 \in \mathbf{H}^{\infty}_{N\times m}$ the problem is to find

$$\sup \operatorname{Re} \int_{\mathbb{T}} x(z)^* T_1(z)\, dm(z)$$

subject to

$$x = (I - T_2(T_2^*T_2)^{-1}T_2^*)g + T_2(T_2^*T_2)^{-1}h,$$

$$\|x\|_{\mathbf{G}} \le 1, \quad g \in \mathbf{L}^{p'}_{N\times 1}, \quad h \in \overline{z\mathbf{H}^{p'}_{m\times 1}}.$$

Using the definition of norm in $\mathbf{G}$ we can rewrite the dual problem in more explicit form: given $T_1 \in \mathbf{H}^\infty_{N\times 1}$, $T_2 \in \mathbf{H}^\infty_{N\times m}$ the problem is to find

$$\sup \operatorname{Re} \int_{\mathbb{T}} \left(g^* T_1 + (h^* - g^* T_2)(T_2^* T_2)^{-1} T_2^* T_1\right) dm$$

subject to

$$\left\| g_j - T_{2j}(T_2^* T_2)^{-1}(T_2^* g + h) \right\|_{p_j'} \le 1, \quad 0 \le j \le J,$$

$$g \in \mathbf{L}^{p'}_{N\times 1}, \quad h \in \overline{z\mathbf{H}^{p'}_{m\times 1}}.$$

This is also a convex optimization problem and, therefore, any finite-dimensional approximation of it will provide a lower bound on the optimal value. Running both approximations in parallel will give a sequence of upper and lower bounds and a distance between them can serve as a decision for a stopping criteria.

5. An Example

Consider the following optimization problem

$$\inf_{q\in\mathbb{A}} \{ \|aq + b\|_\infty \ \mid \ \|q\|_\infty \le 1 \}$$

where a and b are given functions in $\mathbf{H}^\infty$. In this case, all p_j and hence p are $+\infty$. Suppose that the norm in $\mathbf{L}^\infty$ is defined as

$$\|f\|_\infty = \sup_{z\in\mathbb{T}} |f(z)|_\infty = \sup_{z\in\mathbb{T}} \max\{|\operatorname{Re} f(z)|, |\operatorname{Im} f(z)|\}.$$

Then the problem becomes a linear optimization

$$\mu_0 = \inf \left\{ \mu \ \middle| \ \begin{array}{l} \pm\operatorname{Re}(aq+b) \le \mu, \ \pm\operatorname{Re} q \le 1, \\ \pm\operatorname{Im}(aq+b) \le \mu, \ \pm\operatorname{Im} q \le 1 \end{array} \right\}.$$

Using the Lagrange multiplier method this can be written as

$$\mu_0 = \sup_{\tau\ge 0} \left(\inf_{q\in\mathbb{A}} (\|aq+b\|_\infty + \tau\|q\|_\infty) - \tau \right).$$

The functions T_1 and T_2 can be defined as

$$T_1 = \begin{pmatrix} b \\ 0 \end{pmatrix}, \quad T_2 = -\begin{pmatrix} a \\ \tau \end{pmatrix}.$$

Due to Lemma 3 a pre-dual space to the space $\mathbf{F} = \mathbf{L}^\infty \times \mathbf{L}^\infty$ is $\mathbf{G} = \mathbf{L}^1 \times \mathbf{L}^1$ where the norm in $\mathbf{L}^1$ is defined as

$$\|g\|_1 = \int_{\mathbb{T}} |g|_1 \, dm = \int_{\mathbb{T}} (|\operatorname{Re} g| + |\operatorname{Im} g)|) \, dm.$$

Let us obtain an explicite expression for the pre-annihilator from Lemma 5. Here $p' = 1$, so we have

$$\left(I - \frac{T_2 T_2^*}{|T_2|^2}\right) \mathbf{L}^{p'}_{N\times 1} = \frac{1}{|a|^2+\tau^2} \begin{pmatrix} \tau^2 & -\tau a \\ -\tau\bar{a} & |a|^2 \end{pmatrix} \mathbf{L}^1 \times \mathbf{L}^1 =$$
$$= \begin{pmatrix} \tau \\ -\bar{a} \end{pmatrix} (\tau \quad -a) \frac{1}{|a|^2+\tau^2} \mathbf{L}^1 \times \mathbf{L}^1 = \begin{pmatrix} \tau \\ -\bar{a} \end{pmatrix} \mathbf{L}^1.$$

This gives the following form of the pre-annihilator

$$^{\perp}X = \begin{pmatrix} \tau \\ -\bar{a} \end{pmatrix} g + \frac{1}{|a|^2+\tau^2} \begin{pmatrix} a \\ \tau \end{pmatrix} h$$

where $g \in \mathbf{L}^1$ and $h \in \overline{z\mathbf{H}^1}$. After a complex conjugation of all functions, the dual problem takes the form

$$\gamma_0(\tau) = \sup \operatorname{Re} \int_{\mathbb{T}} b x_1 \, dm$$

subject to $g \in \mathbf{L}^1$, $h \in z\mathbf{H}^1$,

$$x_1 = \tau g + \frac{\bar{a}}{|a|^2+\tau^2} h, \quad x_2 = -ag + \frac{\tau}{|a|^2+\tau^2} h,$$
$$\int_T (|\operatorname{Re} x_i| + |\operatorname{Im} x_i|) \, dm \le 1, \quad i = 1, 2.$$

It also has the linear optimization structure

$$\gamma_0(\tau) = \sup \operatorname{Re} \int_{\mathbb{T}} \left(\tau b g + \frac{\bar{a} b}{|a|^2+\tau^2} h \right) dm$$

subject to $g \in \mathbf{L}^1$, $h \in z\mathbf{H}^1$,

$$\pm\operatorname{Re}\left(\tau g + \frac{\bar{a}h}{|a|^2+\tau^2}\right) \le r_1, \ \pm\operatorname{Im}\left(\tau g + \frac{\bar{a}h}{|a|^2+\tau^2}\right) \le r_2,$$
$$\pm\operatorname{Re}\left(ag - \frac{\tau h}{|a|^2+\tau^2}\right) \le s_1, \ \pm\operatorname{Im}\left(ag - \frac{\tau h}{|a|^2+\tau^2}\right) \le s_2,$$
$$\int_{\mathbb{T}} (r_1 + r_2) \, dm \le 1, \quad \int_{\mathbb{T}} (s_1 + s_2) \, dm \le 1.$$

Finally the relation

$$\mu_0 = \sup_{\tau\ge 0} (\gamma_0(\tau) - \tau)$$

gives the possibility to obtain a lower bound on μ_0 using the dual problem.

6. Conclusion

In this paper we have derived the dual convex problem to the multi-objective $\mathbf{H}^p$ control problem in case where all objectives are $\mathbf{H}^p$ norm bounds (with possibly different p's) on the closed-loop transfer functions from the same scalar exogenous signal. It completes the primal-dual pair and gives rise to a number of finite-dimensional algorithms that find the optimal value with a guaranteed accuracy. This approach is straightforward compared to a mixed one-objective problem which may suffer from several attendant drawbacks such as conservatism, manual weight tuning etc.

Appendix

Proof of Lemma 1

To begin with, let us perform a factorization of the function $T_3 \in \mathbf{H}^\infty_{m\times 1}$ as

$$T_3 = T_3^i T_3^o$$

where the scalar $T_3^i \in \mathbf{H}^\infty$ absorbs all common zeros of the entries of T_3 in the closed unit disc and $T_3^o \in \mathbf{H}^\infty_{m\times 1}$ satisfies the inequality

$$\inf_{|z|\le 1} |T_3^o(z)| > 0.$$

By the Carleson corona theorem [3] there exists a function $g \in \mathbf{H}^\infty_{1\times m}$ such that $gT_3^o = 1$, that is T_3^o is left invertible and $\mathbb{A}_{m\times m}T_3^o = \mathbb{A}_{m\times 1}$. Then the Youla parameterization becomes

$$\begin{aligned} T_1 - T_2\mathbb{A}_{m\times m}T_3 &= T_1 - T_2\mathbb{A}_{m\times m}T_3^iT_3^o = \\ &= T_1 - T_3^iT_2\mathbb{A}_{m\times 1}, \end{aligned}$$

and we get the claim of the Lemma where $T_3^iT_2$ is replaced by a new T_2.

Proof of Lemma 2

The proof is given by simple bounds as

$$\begin{aligned} \|Q\|_p &= \|(T_2^*T_2)^{-1}T_2^*T_2Q\|_p \le C\,\|T_2Q\|_p \le \\ &\le C\sum_{j=0}^J \|T_{2j}Q\|_p \le C\sum_{j=0}^J \|T_{2j}Q\|_{p_j} = \\ &= C\,\|T_2Q\|_{\mathbf{F}} \le C\,(\|T_1 - T_2Q\|_{\mathbf{F}} + \|T_1\|_{\mathbf{F}}). \end{aligned}$$

Proof of Lemma 4

Denote the outer factor of $T_2^*T_2$ by W

$$T_2^*T_2 = W^*W, \quad \det W(z) \ne 0,\ \forall |z| \le 1$$

and $\Theta = T_2W^{-1}$. Then $\Theta^*\Theta = I$. Then we have $W^{-1} \in \mathbf{H}^\infty_{m\times m}$ and $W\mathbf{H}^p_{m\times 1} = \mathbf{H}^p_{m\times 1}$. Finally

$$T_2\mathbf{H}^p_{m\times 1} = T_2W^{-1}W\mathbf{H}^p_{m\times 1} = \Theta\mathbf{H}^p_{m\times 1}.$$

Proof of Lemma 3

We show first that any linear bounded functional on $\mathbf{G}$ takes the form

$$\langle f, g\rangle = \int_T f^*g\,dm = \sum_{j=0}^J \int_{\mathbb{T}} f_j^*g_j\,dm. \tag{4}$$

where $f \in \mathbf{F}$. Note that the subspace of $\mathbf{G}$, spanned by only one component g_j with all others being zero, is isometrically isomorphic to $\mathbf{L}^{p'_j}_{n_j\times 1}$. Then by the relation $\left(\mathbf{L}^{p'_j}_{n_j\times 1}\right)^* = \mathbf{L}^{p_j}_{n_j\times 1}$ and linearity we immediately get the representation (4) together with uniqueness of the function $f \in \mathbf{F}$. The inequality

$$\begin{aligned} \left|\int_{\mathbb{T}} f^*g\,dm\right| &\le \sum_{j=0}^J \int_T |f_j^*g_j|\,dm \le \sum_{j=0}^J \|f_j\|_{p_j}\|g_j\|_{p'_j} \le \\ &\le \sum_{j=0}^J \|f_j\|_{p_j} \max_{0\le j\le J} \|g_j\|_{p'_j} = \|f\|_{\mathbf{F}}\|g\|_{\mathbf{G}} \end{aligned}$$

proves that with our choice of norms f gives, in fact, a linear bounded functional. It gives also that

$$\|f\| \le \|f\|_{\mathbf{F}}. \tag{5}$$

We now show that the identity $\|f\| = \|f\|_{\mathbf{F}}$ takes place. Note first that if $\|f\|_{\mathbf{F}} = 0$ then all is proven. So we shall assume that

$$\|f\|_{\mathbf{F}} > 0. \tag{6}$$

By the norm definition we have

$$\left|\int_T f^*g\,dm\right| \le \|f\|\,\|g\|_{\mathbf{G}}. \tag{7}$$

In particular, this is true for g with j-th component equal to

$$g_j(z) = \begin{cases} c_j^{-1}f_j(z)|f_j(z)|^{p_j-2}, & \text{if } f_j(z) \ne 0, \\ 0 & \text{if } f_j(z) = 0 \end{cases}$$

where

$$c_j = \|f_j\|_{p_j}^{p_j-1}.$$

It belongs to $\mathbf{G}$ and $\|g\|_{\mathbf{G}} = 1$. Indeed by the assumption (6) we have at least one $c_j \ne 0$ and then for all those j's

$$\begin{aligned} \|g_j\|_{p'_j} &= \frac{1}{c_j}\left(\int_{\mathbb{T}} |f_j|^{(p_j-1)p'_j}\,dm\right)^{\frac{1}{p'_j}} = \\ &= \frac{1}{c_j}\left(\int_{\mathbb{T}} |f_j|^{p_j}\,dm\right)^{\frac{1}{p'_j}} = \frac{1}{c_j}\|f_j\|_{p_j}^{\frac{p_j}{p'_j}} = 1. \end{aligned}$$

For this g the inequality (7) becomes

$$\sum_{j:\ c_j\ne 0} \frac{1}{c_j}\|f_j\|_{p_j}^{p_j} \le \|f\| \quad \Leftrightarrow \quad \|f\|_{\mathbf{F}} \le \|f\|$$

which together with the opposite one (5) proves the equality.

Proof of Lemma 5

Let $f \in \mathbf{F}$. Then the following statements are equivalent

$$\begin{aligned} f \in ({}^\perp X)^\perp \quad &\Leftrightarrow \quad \int_{\mathbb{T}} f^*((I - \Theta\Theta^*)g + \Theta h)\,dm = 0 \\ &\qquad \forall g \in \mathbf{L}^{p'}_{N\times 1} \quad \forall h \in \overline{z\mathbf{H}^{p'}_{m\times 1}}, \quad \Leftrightarrow \\ &\Leftrightarrow \left[\begin{array}{ll} \int_{\mathbb{T}} f^*(I - \Theta\Theta^*)g\,dm = 0 & \forall g \in \mathbf{L}^{p'}_{N\times 1}, \\ \int_{\mathbb{T}} f^*\Theta h\,dm = 0 & \forall h \in \overline{z\mathbf{H}^{p'}_{m\times 1}} \end{array}\right. \quad \Leftrightarrow \\ &\Leftrightarrow \left[\begin{array}{l} f^*(I - \Theta\Theta^*) = 0 \text{ a.e. on } \mathbb{T}, \\ f^*\Theta \in \overline{\mathbf{H}^p_{1\times m}} \end{array}\right. \quad \Leftrightarrow \ f \in \Theta\mathbf{H}^p_{m\times 1}. \end{aligned}$$

References

[1] Boyd, S. P. and C. H. Barratt, *Linear Controller Design — Limits of Performance*, Prentice-Hall, Englewood Cliffs, NJ, 1991.

[2] Corrêa G. O., "Approximate Solutions to $\mathbf{H}_2$ Optimization Problem with $\mathbf{H}_2$ and $\mathbf{H}_\infty$ Constraints", *Int. J. Control*, vol. 72, No. 14, pp. 1260–1270.

[3] Garnett, J. B., *Bounded Analytical Functions*, Academic Press, New York, 1981.

[4] Ghulchak A. and A. Rantzer, "Robust Controller Design via Linear Programming", *Proc. 38th IEEE Conf. Dec. Contr.*, Phoenix, USA, 1999.

[5] Luenberger, D. G., *Optimization by Vector Space Methods*, John Wiley, New York, 1969.

[6] Megretski A. and A. Rantzer, "Robust Control Synthesis by Convex Optimization: Projective Parametrization and Duality", *Proc. IFAC Congress*, Sydney, Australia, 1993.

[7] Owen, L. G. and G. Zames, "Duality Theory of Robust Disturbance Attenuation", *Automatica* 1993 Vol. 29, No. 3, pp. 695–705.

[8] Scherer, C. W., "Lower Bound in Multi-Objective $\mathbf{H}_2/\mathbf{H}_\infty$ Problems", *Proc. IEEE Conf. Dec. Contr.*, USA, Phoenix, 1999.