Strategies for Computing Switching Feedback Controllers

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Strategies for Computing Switching Feedback Controllers

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***Abstract*— We consider the problem of computing suboptimal feedback switching controllers for discrete dynamical systems. The paper shows how to combine convex optimization techniques with relaxed dynamic programming. We apply the method to several problems that have been considered recently in the literature. A particularly interesting example is given by a DC-DC converter. The proposed algorithm has several interesting properties.**

The main theoretical result of this paper is the introduction of a new approximate policy iteration algorithm which is shown to converge to the optimal cost function.

I. INTRODUCTION

This paper considers the optimal control problem for switched discrete time systems. It is well known that the optimal control problem for non-switched systems, i.e. the computation of the optimal value function, is a very difficult computational tasks in all instances with lack of special structure. Important examples when a solution is readily found includes the linear-quadratic regulator problem with unbounded state and control space. Another example is when the state and control space are finite sets, e.g. Dijkstras algorithm. With this in mind it is not surprising that most of the proposed solution approximations try to mimic these cases. In particular, linearization of non-linear dynamics and gridding are recurring methods in the literature. In recent years, these methods have been extended to switched and hybrid systems, see e.g. [2], [13], [6], [7]. For a recent survey on computational approaches see [17].

A novel approach to overcome some of the difficulties mentioned above was recently proposed in [4], [5], [3], see also [14] for examples from switching systems. The authors to these papers consider problems where the value function can be well approximated by a finite number of linear or quadratic functions. Moreover, bounds on suboptimality are also included in the method. However, the parametrizations used in these papers are restrictive since they can only be applied to problems where the systems are modeled with linear dynamics. Moreover, even in the linear case better parametrizations exists, as we shall see below.This paper presents a computational method and a parametrization so that relaxed value iteration can be applied to large class of problems, in particular non-linear systems. Similar ideas were proposed for continuous-time non-linear regulation problems in [15] and for constrained non-switched discrete-time systems in [16]. It should be noticed that although the underlying idea in the aformetioned references is to find feedback controllers by means of dynamic programming the approaches are quite different. This is due to the fact that the choice of parametrization of the value function has several implications for the resulting algorithms. In this paper, the computations and examples will be performed using relaxed value iteration, applied in a new setting.

This paper also contains theoretical contributions. Aside from value iteration there is also another algorithm, the so called policy-iteration algorithm, [1]. The main result of this paper is the construction of an approximate policy-iteration algorithm. It is shown that the algorithm converges to the optimal value function at linear rate.

The paper is organized as follows. The problem we consider is formulated in section II, where we also define value and relaxed value iteration. The new policy-iteration algorithm is given in section III. Since the choice of parametrization will be polynomials, we give a brief review of some results from the representation of positive polynomials in section IV. We then formulate the approximation algorithm using these results. In the last two sections we show how the proposed algorithm can be applied to several switched control problems.

II. DEFINITIONS AND KNOWN FACTS

Given $f : X \times U \longrightarrow X$ consider the controlled dynamical system

$$x(k+1) = f(x(k), u(k)), \quad x(0) = x_0, \quad k \geq 0 \qquad (1)$$

We denote the set of control sequences $u : \mathbb{N} \longrightarrow U$ by $\mathbb{U}$. For such a sequence u we denote the resulting trajectory by $x_u(k), \quad k \geq 0$. We use the following notation, for each $x \in X$ the subset $U(x) \subset U$ denotes those controls such that $f(x, u) \in X$. We assume that $\forall x \in X, \quad U(x) \neq \emptyset$, thus the system is assumed to be controlled invariant. We also assume that $f(0, 0) = 0$. The total cost associated with a given input sequence is defined by

$$V(x_0, u) = \sum_{k=0}^{\infty} l(x_u(k), u(k)))$$

were the step cost $l : X \times U \longrightarrow \mathbb{R}_+$ is positive definite, i.e. $l(0, 0) = 0$ and $l(x, u) > 0$ if $(x, u) \neq 0$. Our main interest in this paper is to compute approximations to the optimal value function, defined by

$$V^*(x_0) = \inf_{u \in \mathbb{U}(x)} V(x_0, u)$$

The optimal value function can be characterized as the solution to Bellman's equation

$$V^*(x) = \inf_{u \in U(x)} \{V^*(f(x,u)) + l(x,u)\} \tag{2}$$

If we know V^*, the optimal feedback controller is given by

$$\mu^*(x) = \text{argmin}_{u \in U(x)} \{V^*(f(x,u)) + l(x,u)\}$$

A. Exact value iteration

Value iteration derives from Bellman's famous principle of optimality. Consider the cost of controlling system (1) in a finite number of, say N, steps. Doing so in an optimal way would result in a cost

$$V_N(x_0) = \inf_{u \in \mathbb{U}(x)} \sum_{k=0}^{N-1} l(x_u(k), u(k))$$

This is the same as

$$V_N(x) = \inf_{u \in U(x)} \{V_{N-1}(f(x,u)) + l(x,u)\} \tag{3}$$

Together with an initial function V_0, this iterative functional equation defines value iteration. Under suitable conditions the limit $\lim_{N\to\infty} V_N(x)$ exists and coincides with $V^*(x)$. In practice the iteration must, of course, be terminated after a finite number of iterations. Usually one then approximates the optimal controller with $\mu_N(x)$ and then uses this to control the system indefinitely, resulting in cost

$$V_{\mu_N}(x) = \sum_{k=0}^{\infty} l(x_{\mu_N}(k), \mu_N(x_{\mu_N}(k)))$$

We directly see that

$$V_N(x) \le V^*(x) \le V_{\mu_N}(x)$$

Thus, it is in principle possible to check convergence using this simple inequality. The iteration, however, is not easy to perform in practice. In fact, it inherits the difficulties already present in equation (2). It is almost always necessary to make approximations. One particular way of formulating an approximation algorithm is presented next.

B. Relaxed value iteration

The following two statements are slight reformulations from [4], [5]. Let V_N^* be the N'th function obtained using exact value iteration (3). Suppose that $V_N : X \longrightarrow \mathbb{R}$ satisfies the following inequalities

$$\begin{aligned} \inf_{u \in U(x)} \{V_{N-1}(f(x,u)) + \beta l(x,u)\} &\le V_N(x) \\ V_N(x) &\le \inf_{u \in U(x)} \{V_{N-1}(f(x,u)) + \alpha l(x,u)\} \end{aligned} \tag{4}$$

Where $\beta \le 1 \le \alpha \in \mathbb{R}$.

Proposition 1: Suppose that $V_0 = V_0^*$, then

$$\beta V_N^* \le V_N \le \alpha V_N^*, \quad \forall N \in \mathbb{N} \tag{5}$$

We call the iteration (4) relaxed value iteration. It turns out that for some problems it is much easier to find a sequence $\{V_N\}$ that satisfies inequalities (4), compared to the exact iteration. The inequality form also has several other useful properties. The relative bounds obtained can also be used to quantify computation errors made when the exact solution is sought. Convergence can be checked using

Proposition 2: Let $\tilde{X} \subset X$ with $0 \in \tilde{X}$ be any invariant subset. If $V \ge 0$ satisfy

$$\begin{aligned} \inf_{u \in U(x)} \{V(f(x,u)) + \beta l(x,u)\} &\le V(x) \\ V(x) &\le \inf_{u \in U(x)} \{V(f(x,u)) + \alpha l(x,u)\} \end{aligned} \tag{6}$$

Where $\beta \le 1 \le \alpha \in \mathbb{R}$. then

$$\beta V^* \le V \le \alpha V^*, \quad \forall x \in \tilde{X} \tag{7}$$

III. Approximate policy iteration

This section contains the main theoretical result in this paper. Let (μ_0, V_0) be given. The policy iteration algorithm generates a sequence $\{(\mu_j, V_j)\}_{j\ge 1}$ satisfying

$$V_j(x) = V_j(f(x, \mu_j(x))) + l(x, \mu_j(x)) \tag{8}$$

$$\mu_{j+1}(x) = \arg\min_u \{V_j(f(x,u)) + l(x,u)\} \tag{9}$$

For some problems this algorithm is more attractive since it can be shown to converge faster then value iteration. Note that relaxed value iteration converges to something that is close to the optimal cost but in general not equal to. The iteration we propose below also solves a sequence of approximate problems but converges to the exact optimal cost.

A. Main result

Assume that there is a feedback controller μ_0 and a function V_0 such that

$$V_0(x) \ge V_0(f(x, \mu_0(x)) + l(x, \mu_0(x)), \quad \forall x \in X$$

Define

$$T_j(x) = V_{j-1}(x) - V_{j-1}(f(x, \mu_j(x)) - l(x, \mu_j(x))$$

Theorem 1 (Approximate policy iteration): Let $1 \ge \alpha \ge 0$. Suppose that the sequence $\{(\mu_j, V_j)\}_{j\ge 1}$ satisfies

$$V_j(x) \ge V_j(f(x, \mu_j(x))) + l(x, \mu_j(x)) \tag{10}$$

$$V_j(x) \le V_j(f(x, \mu_j(x))) + l(x, \mu_j(x)) + \alpha T_j(x) \tag{11}$$

Then for every $j \ge 1$ it holds

$$\begin{aligned} T_j(x) &\ge 0 \\ V_{j-1} \ge V_j &\ge V_{\mu_j} \end{aligned}$$

Proof: The two inequalities would be inconsistent if $\alpha T_j < 0$, since $\alpha \ge 0$ also $T_j \ge 0$. Now let $x_{\mu_j}(k)$ denote the trajectory as a result of applying μ_j and consider inequality (10)

$$\begin{aligned} &V_j(x_{\mu_j}(0)) - V_j(x_{\mu_j}(t)) \\ &= \sum_{k=0}^{t} (V_j(x_{\mu_j}(k)) - V_j(x_{\mu_j}(k+1))) \\ &\ge \sum_{k=0}^{t} l(x_{\mu_j}(k), \mu_j(k)) \end{aligned}$$

Thus

$$V_j(x_{\mu_j}(0)) \geq V_j(x_{\mu_j}(t)) + \sum_{k=0}^{t} l(x_{\mu_j}(k), \mu_j(k))$$

note that $V_j \geq 0$ and $l(x,u) > 0$ if $(x,u) \neq 0$, hence $x_{\mu_j}(t) \to 0$ as $t \to \infty$ and $V_j \geq V_{\mu_j}$. Put $w = 1-\alpha$, then

$$\begin{aligned}
V_j(x) &\leq V_j(f(x,\mu_j(x))) + l(x,\mu_j(x)) \\
&\quad + \alpha(V_{j-1}(x) - V_{j-1}(f(x,\mu_j(x)) - l(x,\mu_j(x))) \\
&= V_j(f(x,\mu_j(x))) + V_{j-1}(x) - V_{j-1}(f(x,\mu_j(x)) \\
&\quad - w(V_{j-1}(x) - V_{j-1}(f(x,\mu_j(x))) - l(x,\mu_j(x))) \\
&\leq V_j(f(x,\mu_j(x))) + V_{j-1}(x) - V_{j-1}(f(x,\mu_j(x))
\end{aligned}$$

the last inequality implies that

$$\begin{aligned}
&V_{j-1}(x_{\mu_j}(0)) - V_{j-1}(x_{\mu_j}(t)) \\
&= \sum_{k=0}^{t}(V_{j-1}(x_{\mu_j}(k)) - V_{j-1}(x_{\mu_j}(k+1)) \\
&\geq \sum_{k=0}^{t}(V_j(x_{\mu_j}(k)) - V_j(x_{\mu_j}(k+1)) \\
&= V_j(x_{\mu_j}(0)) - V_j(x_{\mu_j}(t))
\end{aligned}$$

by sending $t \to \infty$ we conclude $V_{j-1} \geq V_j$. ■

The result shows that V_j is bounded from below, for by definition $V_{\mu_j} \geq V^*$. Moreover $\{V_j\}_{j \geq 0}$ is monotonically non-increasing. To prove global convergence it is necessary to impose, at least, one more condition on the sequence $\{V_j, \mu_j\}$. In the next result we provide such a condition

Theorem 2: Select $\{\mu_j\}_{j\geq 0}$ according to

$$\mu_{j+1}(x) = \arg\min_u \{V_j(f(x,u)) + l(x,u)\} \tag{12}$$

suppose that $\{V_j\}_{j\geq 1}$ satisfies inequalities (10) and (11). If $V_j = V_{j-1}$ then

$$V_j = V_{j-1} = V^* \text{ and } \mu_j = \mu^*$$

Proof: Consider the proof of $V_{j-1} \geq V_j$. If $V_j = V_{j-1}$ we have

$$\begin{aligned}
0 &= T_j \\
&= V_{j-1}(x) - V_{j-1}(f(x,\mu_j)) - l(x,\mu_j(x)) \\
&= V_{j-1}(x) - \min_u \{V_{j-1}(f(x,u)) + l(x,u)\}
\end{aligned}$$

■

Moreover, if $\{\mu_j\}_{j\geq 0}$ is selected as in the last theorem we can establish a linear convergence rate

Theorem 3 (Speed of convergence): Suppose that there is a parameter $\gamma > 0$ such that $V^*(f(x,u)) \leq \gamma l(x,u)$ for all (x,u) and that $V_0 \leq \delta V^*$ then for every j

$$V_j \leq (1 + (\delta - 1)\left[\tfrac{\gamma+\alpha}{\gamma+1}\right]^j)V^*$$

Proof: Fix $j \geq 1$ and assume that $V_{j-1} \leq \delta_{j-1}V^*$ for all x. First observe that for any numbers a and b

$$\delta_{j-1}a + b + (\gamma b - a)\frac{\delta_{j-1}-1}{\gamma+1} = \frac{\delta_{j-1}\gamma+1}{\gamma+1}(a+b)$$

Set $\alpha = 1 - \hat{\alpha}$ and consider inequality (11)

$$\begin{aligned}
V_j(x) &\leq V_j(f(x,\mu_j(x))) + l(x,\mu_j(x)) \\
&\quad + \alpha(V_{j-1}(x) - V_{j-1}(f(x,\mu_j)) - l(x,\mu_j(x))) \\
&\leq \alpha V_{j-1}(x) + \hat{\alpha}(V_{j-1}(f(x,\mu_j)) + l(x,\mu_j(x))) \\
&= \alpha V_{j-1}(x) + \hat{\alpha}\min_u\{V_{j-1}(f(x,u)) + l(x,u)\} \\
&\leq \alpha\delta_{j-1}V^* \\
&\quad + \hat{\alpha}\min_u\{\delta_{j-1}V^*(f(x,u)) + l(x,u) \\
&\quad + (\gamma l(x,u) - V^*(f(x,u)))\frac{\delta_{j-1}-1}{\gamma+1}\} \\
&= \alpha\delta_{j-1}V^* + (1-\alpha)\frac{\delta_{j-1}\gamma+1}{\gamma+1}V^*
\end{aligned}$$

Hence

$$\delta_j = \frac{\delta_{j-1}(\gamma+\alpha) + 1 - \alpha}{\gamma+1}$$

if we apply this recursion j times, starting at $\delta_0 = \delta$, we get

$$\delta_j = 1 + (\delta-1)\left[\tfrac{\gamma+\alpha}{\gamma+1}\right]^j$$

■

Observe that the case with $\alpha = 0$ corresponds to exact policy-iteration.

IV. Parametrization of the value function

To perform the iteration we must parametrize the value functions in suitable way. Judging the merits of a parametrization several important questions come to mind, e.g. implementation issues and memory requirement. First, however, it must be feasible to computationally verify the inequalities in (4). In this respect multivariate polynomials in combination with recent results in algebraic geometry will be very useful. In particular, positivity on compact sets is easy to formulate and to verify using convex optimization. To state the algorithm in the next section we need to recall some results about the representation of positive polynomials. We provide a very brief review of these ideas, for further information see the references [9], [11].

A. Positive polynomials

$\mathbb{R}[x]$ is the vector space of polynomials in variables $x \in \mathbb{R}^n$. By $\mathbb{R}_d[x]$ we denote the subspace of polynomials of degree at most d. We write $Z_d(x)$ for the column vector consisting of the elements of the canonical basis for $\mathbb{R}_d[x]$. The first simple observation is that if $p \in \mathbb{R}_{2d}[x]$ is a sum of squares $p = \sum_{k=1}^m p_k^2$ for some $p_k \in \mathbb{R}_d[x]$ then $p \geq 0$ for all $x \in \mathbb{R}^n$. We denote the set of all sum of squares of polynomials by $\Sigma[x]$. The following proposition characterizes all such polynomials

Proposition 3: $p \in \Sigma_{2d}[x]$ if and only if

$$p = Z_d(x)^T Q Z_d(x) \tag{13}$$

for some positive semidefinite (psd) matrix Q.

This result is important since it allows us to check in an easy way if a given polynomial is a sum of squares, which was noted in [9]. Given a polynomial p, checking if p is sum of squares can be done using semidefinite programming as follows: Identify coefficients in (13), this gives an affine constraint on Q, taking the intersection with the convex cone of psd matrices results in a convex constraint. Even more useful perhaps, if the coefficients in p are not predetermined but depend on a parametervector via an affine function, i.e. if p_j is the j'th coefficient of p and $t \in \mathbb{R}^w$ is w-dimensional parameter vector we have the relation

$$t \mapsto p_j(t) = c_0 + \sum_{j=1}^{w} c_j t_j \tag{14}$$

where $c_0 ... c_w$ are fixed. In this case we can look for the best $p(t;x)$, as measured using a linear function of the parameter vector, such that $p(t;x) \geq 0$ by solving a semidefinite programming problem.

The other implication is false; if $p \geq 0$ then p is not necessarily a sum of squares. Thus the above procedure gives sufficient conditions for positivity on $\mathbb{R}^n$. This fact shows that checking global positivity of a polynomial using the outlined method can be conservative.

In this paper we focus on positivity on compact sets, this case is less conservative. Consider a set

$$X = \{x : h_k(x) \geq 0, k = 1..m\} \tag{15}$$

with $h_k \in \mathbb{R}[x]$. We associate with X a set of polynomials

$$G_X = \{p : p = \sigma_0 + \sum_{k=1}^{m} \sigma_k h_k, \quad \sigma_k \in \Sigma[x]\} \tag{16}$$

Similar to the global case we clearly have

Lemma 1: If $p \in G_X$ then $p \geq 0$ on X.

The following remarkable partial converse will be useful

Theorem 4 (Putinar[11]): Let X be as in (15). Suppose that there is a real number $r > 0$ such that $r^2 - \sum_{k=1}^{n} x_k^2 \in G_X$, then $p \in \mathbb{R}[x]$ is positive on X only if $p \in G_X$.

There is a gap between lemma 1 and theorem 4. If p is non-negative on X then it is not necessary that $p \in G_X$. For the applications in this paper this gap is not a problem in the following meaning. Suppose that there exists a parameter vector t such that $p(t;x) \geq 0$ on X but $p(t;x) \notin G_X$. Then we know that there is another parameter vector $\hat{t}$ such that $p(\hat{t};x) > 0$ on X and hence $p(\hat{t};x) \in G_X$.

B. Application to relaxed inequalities

In the rest of this paper we assume that f and l are polynomials and that $X = \{x : h_1(x) \geq 0, \ldots, h_p(x) \geq 0\}$ with h_k's polynomials. Let V_{N-1} be given and consider the upper inequality

$$V_N(x) \leq \inf_{u \in U} \{V_{N-1}(f(x,u)) + \alpha l(x,u)\} \tag{17}$$

this inequality holds if and only if

$$V_N(x) \leq \{V_{N-1}(f(x,u)) + \alpha l(x,u)\}, \quad \forall u \in U \tag{18}$$

Here we may consider the right hand side as polynomial in (x,u), and thus we may directly apply the results from the previous section to obtain a finite dimensional constraint on V_N. However, as we are interested in switched problems in this paper we now consider the case with a finite control set U, say $|U| = m$. Application of theorem 4 gives m finite dimensional constraints on V_N

$$-V_N(x) + V_{N-1}(f(x,u_k)) + \alpha l(x,u_k) \tag{19}$$

$$= \sigma_{k0} + \sum_{j=1}^{p} h_j \sigma_{kj} \tag{20}$$

with σ_{kj}'s sum of squares in x.

The lower inequality is more difficult

$$V_N(x) \geq \inf_{u \in U(x)} \{V_{N-1}(f(x,u)) + l(x,u)\} =: g(x)$$

Typically we need to approximate g from above. In the case of a continuous control set a solution to this problem was proposed in [16]. In the case of finite U we may consider a simpler approach. To this end, consider the set of all polynomial partitions of unity

$$W = \{(w_1, .., w_m) : \sum_{k=1}^{m} w_k(x) = 1, \\ 0 \leq w_k(x) \ \forall x \in X, \ w_k \in \mathbb{R}[x]\}$$

We obviously have

Proposition 4: Let $(w_1, .., w_m) \in W$ then $\forall x \in X$

$$g(x) \leq \sum_{k=1}^{m} w_k(x)[V_{N-1}(f(x,u_k)) + l(x,u_k)]$$

We can now replace the lower bound with

$$\sum_{k=1}^{m} w_k(x)[V_{N-1}(f(x,u_k)) + l(x,u_k)] \leq V_N(x), \ \forall x \in X$$

And just as for the upper bound we can write this as

$$V_N(x) - \sum_{k=1}^{m} w_k(x)[V_{N-1}(f(x,u_k)) + l(x,u_k)] \\ = \sigma_0 + \sum_{j=1}^{p} h_j \sigma_j$$

with σ_j's sum of squares in x. This constraint on V_N together with equations (19) defines the constraints that the sequence $\{V_N\}$ in the relaxed value iteration must satisfy.

V. Examples

The following two examples are both application of relaxed value iteration.

A. Example 1

The following example is taken from Lincoln [5], where the synthesis was done using relaxed dynamic programming with a very different parametrization of the value function. We shall see that the resulting control law is much simpler using the method proposed in this paper. The problem involves a DC-DC converter, these circuits are

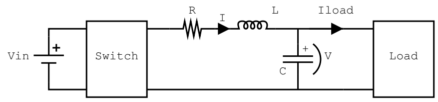

Fig. 1. Circuit in example 1

typical examples of hybrid systems in applications. Consider the continuous-time model

$$\dot{x}_1 = \frac{1}{C}(x_2 - I_{load})$$
$$\dot{x}_2 = \frac{1}{L}(-x_1 - Rx_2 + s(t)V_{in})$$

where x_2 denotes current, x_1 denotes voltage and $s(t) \in \{-1, 1\}$ is the sign of the switch. The primary control objective is to find a feedback switching sequence so that the load voltage is constant despite changes in load current and input load variations. To make it robust, integral action is added to the model

$$\dot{x}_3 = V_{ref} - x_1$$

Switching can only occur at a fixed sampling frequency, so the control problem is to select between to autonomous linear system. After sampling, the system can be written as

$$x_e(k+1) = \Phi_i x_e(k)$$

with $x_e = [x^T \quad 1]^T$
A reasonable step cost is given by

$$l(x) = q_P(x_1 - Vref)^2 + q_I x_3^2 + q_D(x_2 - I_{load})^2$$

with positive weighting constants q_P, q_I and q_D. Giving a total cost

$$V(x) = \sum_{k=0}^{\infty} l(x_k)$$

We solve the problem for states in $\{x : 15 - |x|^2 \geq 0\}$. The constraints take the form

$$-V_N(x) + V_{N-1}(\Phi_1 x_e) + \alpha l(x) = \sigma_{10} + \sigma_{11}(15 - |x|^2)$$
$$-V_N(x) + V_{N-1}(\Phi_2 x_e) + \alpha l(x) = \sigma_{20} + \sigma_{21}(15 - |x|^2)$$

And the lower inequality becomes

$$V_N(x) - \sigma_{32}V_{N-1}(\Phi_1 x_e) - (1 - \sigma_{32})V_{N-1}(\Phi_2 x_e) - l(x)$$
$$= \sigma_{30} + \sigma_{31}(15 - |x|^2)$$

with

$$1 - \sigma_{32} = \sigma_{40} + \sigma_{41}(15 - |x|^2)$$

all σ's being sum of squares in x. After 50 iterations with $\alpha = 4.1$ and $\deg(V_k) = 4$ we have $V_N \approx V_{N-1}$. The controller which is given by

$$s(x) = \text{argmin}_{1,2}\{V_{50}(\Phi_1 x_e), V_{50}(\Phi_2 x_e)\}$$

is almost a switch-plane, see figure 2. The performance of the closed loop is very similar to that in Lincoln [5], but the controller appears much simpler.

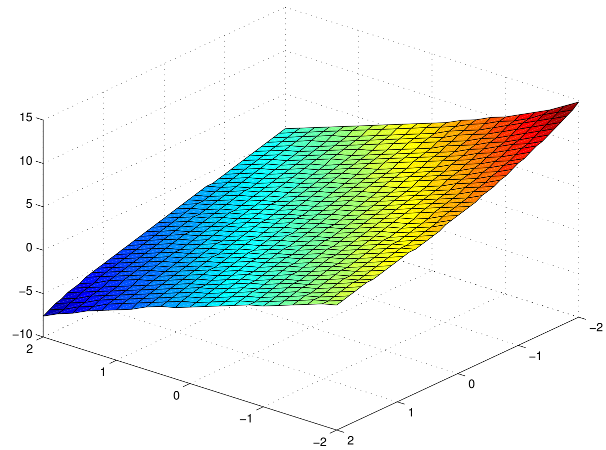

Fig. 2. Each side of the plane corresponds to one switch position

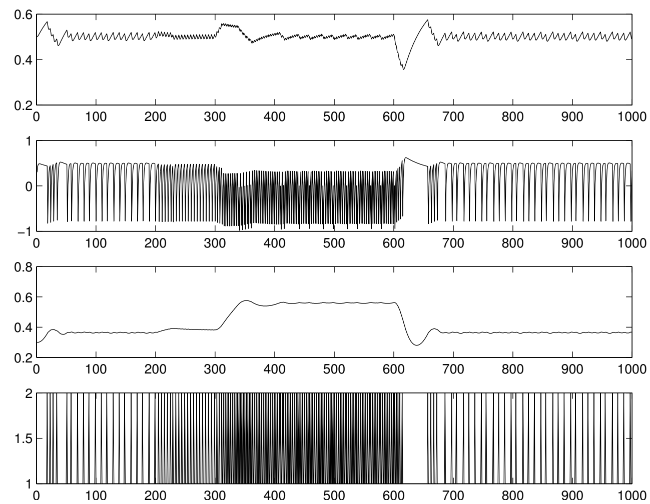

Fig. 3. Top: output voltage. Next to top: current. Next to bottom: integral state. Bottom: Switch position. Reference voltage was $V_{ref} = .5$. At $k = 200$ the load current changes from its nominal value 0.3A to 0.1A, at $k = 300$ it changes to -0.2A and at $k = 600$ it changes back to its nominal value 0.3A

B. Example 2

The following example is from [8]. We consider the following switched discrete-time system

$$x(k+1) = A_q x(k), \quad q \in \{1, 2\} \tag{21}$$

where

$$A_1 = \begin{bmatrix} 1.7 & 4 \\ -0.8 & -1.5 \end{bmatrix}, A_2 = \begin{bmatrix} 0.95 & -1.5 \\ 0.75 & -0.55 \end{bmatrix}$$

We now consider the problem of computing a switching feedback controller for this system. We define the cost as

$$V(x) = \sum_{k=0}^{\infty} x(k)^T x(k)$$

Applying the proposed algorithm the equations are similar to those in the previous example. In this example $X = \{x :$

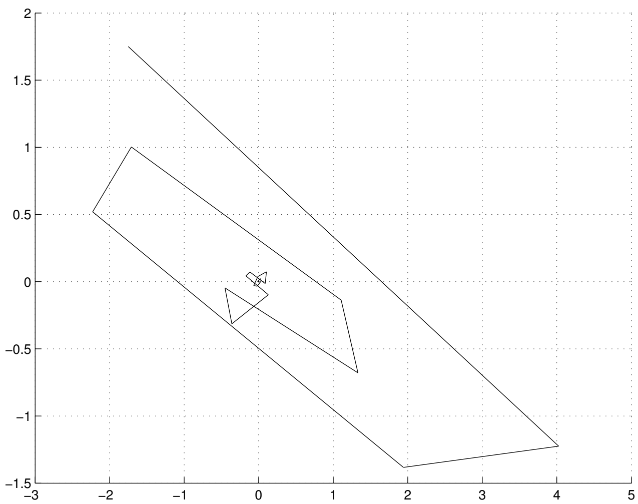

Fig. 4. Closed loop starting from $x = (-1.75, 1.75)$

$10 - |x|^2 \geq 0\}$. After 6 iterations V_N satisfy proposition (2) with $\alpha = 1.7$ and $\deg(V_k) = 4$. The controller is given by

$$q(x) = \text{argmin}_{1,2}\{V_6(A_1x), V_6(A_2x)\}$$

A closed loop trajectory are shown in figure 4.

Remark 1: When doing the required computations on test problems it might be useful to introduce a forgetting factor $0 < \lambda < 1$ in the cost function,

$$\sum_{k=0}^{\infty} \lambda^k l(x(k), u(k))$$

This can be done to ensure a bounded V. In fact, the introduction of λ simplifies the computation of V a lot because the operator $V \mapsto \min_u\{\lambda V(f(x,u)) + l(x,u)\}$ will have a contraction property. This, of course, comes at price: it is not necessary that V is a Lyapunov function.

VI. CONCLUSIONS

A. Conclusions

We proposed an algorithm for feedback synthesis of switching systems, the method gives bounds on optimality. For systems modeled with polynomials the required computations can be done in a tractable way via convex optimization. We have shown, by example, that it can be advantageous to use polynomials as parametrization of the value function for switched control problems. Future work will include case studies of applications to non-linear systems. Application of the policy iteration algorithm on switched control problems will be investigated in the future.

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