On the cavity problem for the general linear medium in Electromagnetic Theory

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Abstract

In this paper we study the propagation problem of a time harmonic electromagnetic field inside a cavity filled with a generic bianisotropic medium. We define the concepts of eigenfrequencies and modes of the cavity and we propose a method to prove their existence and countability. We extend, in this respect, the theory for the isotropic, homogeneous, lossless cavity.

1 Introduction

Every electromagnetic phenomenon is described by four time dependent vector fields $\boldsymbol{E}$, $\boldsymbol{H}$, $\boldsymbol{D}$, $\boldsymbol{B}$, adjacent in a set $\Omega \subset \mathbb{R}^3$. For our purposes, Ω represents a cavity *i.e.*, it is a bounded open set having, in addition, a Lipschitz (at least) boundary $\Gamma := \partial\Omega$. This assumption has as a consequence that the exterior normal $\hat{\boldsymbol{n}}$ is defined almost everywhere (with respect to the Hausdorff measure) on Γ. We assume that Γ is metallic; this leads to the perfect electric conductor boundary condition

$$\hat{\boldsymbol{n}} \times \boldsymbol{E} = \boldsymbol{0} \ \text{, in } \Omega \tag{1.1}$$

We use a compact notation to represent the fields, the six–vector notation [10]

$$\mathsf{e} := \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} \quad , \quad \mathsf{d} := \begin{pmatrix} \boldsymbol{D} \\ \boldsymbol{B} \end{pmatrix}$$

Namely, each of e, d is an $\mathbb{R}^6$-vector dependent on $t \in \mathbb{R}$ (time) and $\boldsymbol{r} \in \Omega$ (position). The dynamics inside Ω are described by the Maxwell system

$$\frac{\partial \mathsf{d}}{\partial t} = \nabla \times \mathsf{J}\mathsf{e} \tag{1.2}$$

We have assumed the absence of charges and currents and we denote

$$\mathsf{J} := \begin{bmatrix} 0 & I_3 \\ -I_3 & 0 \end{bmatrix}$$

I_n being the $n \times n$ identity matrix. The field d is incompressible; this is the Gauss law

$$\nabla \cdot \mathsf{d} = \boldsymbol{0} \tag{1.3}$$

Observe that here the divergence operator acts as a scalar in each component $\boldsymbol{D}$, $\boldsymbol{B}$. We need also the constitutive relation

$$\mathsf{d} = F\mathsf{e} \tag{1.4}$$

The operator F is, roughly speaking, the mathematical description of the medium that occupies Ω. As it is proposed in [9], a physical sound set of properties of F is the following:

- F is linear.

- F is causal: if $\mathsf{e}(t) = \mathbf{0}$ for $t \leqslant \tau$, then $(F\mathsf{e})(t) = \mathbf{0}$ for $t \leqslant \tau$.
- F is time invariant: for all $\tau \geqslant 0$, $[F(\mathsf{e}(\cdot - \tau))](t) = (F\mathsf{e})(t - \tau)$.
- F is local: $[F(\mathsf{e}(\cdot))](\boldsymbol{r}) = f(\boldsymbol{r})$.

Note that causality and time invariance refer to time variable t, whereas locality refers to spatial variable $\boldsymbol{r}$. It is quite well known and is proven in detail in [8], see also [7], that such a constitutive law is given in a convolution form

$$\mathsf{d}(\boldsymbol{r},t) = \mathsf{A}(\boldsymbol{r})\mathsf{e}(\boldsymbol{r},t) + \int_0^t \mathsf{K}(\boldsymbol{r},t-s)\mathsf{e}(\boldsymbol{r},s)\,ds \ \ ,\ t \geqslant 0\ ,\ \boldsymbol{r} \in \Omega \tag{1.5}$$

The 6×6 matrix $\mathsf{A}(\boldsymbol{r})$ is called the optical response of the medium and models the instantaneous effects, whereas $\mathsf{K}(t, \boldsymbol{r})$ is called the susceptibility kernel and models the memory effects.

It is also possible to build a theory without assuming causality even if this axiom is the most physically indicated. E.g. this is the case when the phenomenon is periodic with period T. Then we have to consider the constitutive relation as follows

$$\mathsf{d}(\boldsymbol{r},t) = \mathsf{A}(\boldsymbol{r})\mathsf{e}(\boldsymbol{r},t) + \int_0^T \mathsf{K}(\boldsymbol{r},t-s)\mathsf{e}(\boldsymbol{r},s)\,ds \ \ ,\ 0 \leqslant t \leqslant T\ ,\ \boldsymbol{r} \in \Omega \tag{1.6}$$

Note that (1.6) uses the whole history and this is the significant difference with (1.5). Substituting (1.5) to (1.2) and posing initial conditions, we obtain an evolution problem for an integro–differential equation of neutral type. The study of this equation and the well–posedness of the relevant problem is a large part of the author's PhD thesis [7]. In this reference (1.6) is not considered.

The purpose of this paper is to look at special solutions of this equation. More precisely, we search for time–harmonic solutions of (1.2), a well known concept which is further discussed in Section 2. The idea is to transform the evolution problem into a spectral one, where the angular frequency will serve as the eigenvalue.

2 Time harmonic fields

For a thorough discussion of time harmonic fields in the E/M Theory we refer to [4, Ch.1]. Here we collect (and occasionally prove) a variety of facts that are needed in the development of our theory. We start with an arbitrary $\alpha \in \mathbb{R}$ and a $\varphi \in [0, 2\pi)$. The question is whether there exists a $z \in \mathbb{C}$ such that

$$\alpha = \mathrm{Re}\,(z\mathrm{e}^{-\mathrm{i}\varphi}) \tag{2.1}$$

It is direct that, if $z = z_1 + \mathrm{i}z_2$,

$$\alpha = z_1 \cos\varphi + z_2 \sin\varphi = |\,z\,| \cos(\varphi - \arg z) \tag{2.2}$$

The above equation (2.2) shows that, given $\varrho \geqslant |\,\alpha\,|$, there are always exists z, with $|\,z\,| = \varrho$, that do the job.

On the other direction, there is a result of uniqueness.

Lemma 2.1. *Let z_1, $z_2 \in \mathbb{C}$ be such that*

$$\operatorname{Re} z_1 e^{-i\varphi} = \operatorname{Re} z_2 e^{-i\varphi}$$

for every $\varphi \in [0, 2\pi)$. Then $z_1 = z_2$.

Proof. For $\varphi = 0$, we find $\operatorname{Re} z_1 = \operatorname{Re} z_2$, whereas for $\varphi = \pi/2$, $\operatorname{Re}(-iz_1) = \operatorname{Re}(-iz_2)$, that is $\operatorname{Im} z_1 = \operatorname{Im} z_2$. $\square$

Fix now a number $\omega \in \mathbb{R}$ (frequency). Define a real function of the argument $t \in \mathbb{R}$ (time) by the formula

$$f(t) := \operatorname{Re}[f(\omega)e^{-i\omega t}] \tag{2.3}$$

Observe that for different choices of ω (2.3) defines different functions; all of them are periodic with period $2\pi/|\omega|$. This is the reason why we allow the "phasor" $f(\omega)$ to depend on ω.

Lemma 2.2. *Let $f(t) := \operatorname{Re}[f(\omega)e^{-i\omega t}]$ be a time harmonic field. Then its derivative is also a time harmonic field. Actually*

$$f'(t) = \operatorname{Re}[-i\omega f(\omega)e^{-i\omega t}]$$

Lemma 2.3. *Let $f(t) := \operatorname{Re}[f(\omega)e^{-i\omega t}]$, $g(t) := \operatorname{Re}[g(\omega)e^{-i\omega t}]$ be two time harmonic fields of frequency ω. Then the convolution*

$$h(t) := \int_0^{2\pi/\omega} f(t-s)g(s)\,ds$$

is also a time harmonic field of frequency ω. Actually

$$h(t) = \operatorname{Re}\left[\frac{\pi f(\omega)g(\omega)}{\omega}e^{-i\omega t}\right]$$

Proof. Write

$$f(t) = |f(\omega)|\cos[\omega t - \arg f(\omega)] \ , \ g(t) = |g(\omega)|\cos[\omega t - \arg g(\omega)]$$

By using the trigonometric identity

$$\cos a \cos\beta = \frac{1}{2}[\cos(a+\beta) + \cos(a-\beta)]$$

we find that

$$h(t) := \frac{|f(\omega)g(\omega)|}{2}\left[\int_0^{2\pi/\omega} \cos[\omega t - \arg f(\omega) - \arg g(\omega)]\,ds + \right.$$
$$\left. + \int_0^{2\pi/\omega} \cos[\omega t - \arg f(\omega) + \arg g(\omega) - 2\omega s]\,ds\right]$$

Due to the fact that $\arg(wz) = \arg w + \arg z$, the upper integral is equal to

$$\frac{2\pi}{\omega} cos\,[\omega t - \arg[f(\omega)g(\omega)]]$$

whereas the lower integral vanishes. $\square$

An obvious remark is that a time harmonic–field cannot be used for a causal model since it is not vanished before any time instant τ. That is, if we consider time harmonic E/M fields

$$\mathsf{e}(\boldsymbol{r},t) := \mathrm{Re}\left[\mathsf{e}(\boldsymbol{r},\omega)\mathrm{e}^{-\mathrm{i}\omega t}\right] \quad , \quad \mathsf{d}(\boldsymbol{r},t) := \mathrm{Re}\left[\mathsf{d}(\boldsymbol{r},\omega)\mathrm{e}^{-\mathrm{i}\omega t}\right] \tag{2.4}$$

then we take (1.6) as the constitutive relation with $T := 2\pi/|\,\omega\,|$. We assume the same type of harmonicity for the susceptibility kernel as well

$$\mathsf{K}(\boldsymbol{r},t) := \mathrm{Re}\left[\mathsf{K}(\boldsymbol{r},\omega)\mathrm{e}^{-\mathrm{i}\omega t}\right]$$

Then lemma 2.3 shows that (1.6) is written

$$\mathsf{d}(\boldsymbol{r},\omega) = \mathsf{M}(\boldsymbol{r},\omega)\mathsf{e}(\boldsymbol{r},\omega) \tag{2.5}$$

where

$$\mathsf{M}(\boldsymbol{r},\omega) := \mathsf{A}(\boldsymbol{r}) + \frac{\pi}{\omega}\mathsf{K}(\boldsymbol{r},\omega) \tag{2.6}$$

and the Maxwell system (1.2) becomes

$$Q\mathsf{e} = \omega\mathsf{M}(\omega)\mathsf{e} \tag{2.7}$$

with

$$Q := \mathrm{i}\nabla \times \mathsf{J}$$

to denote the Maxwell operator. The idea is to look at (2.7) as an eigenvalue problem; ω would serve as the eigenvalue and e as the eigenvector. This is not a standard eigenvalue problem but rather one for an operator pencil. The Gauss law (1.3) now becomes

$$\nabla \cdot \mathsf{M}(\omega)\mathsf{e} = \mathbf{0} \tag{2.8}$$

Remark 2.1. *Equation* (2.6) *says that if* $\mathsf{K}(\boldsymbol{r},\omega) = o(\omega)$ *when* $\omega \to \infty$*, then the optical response represents the medium for very high frequencies. This is in accordance with the model given in [3, Section 2.2].*

Remark 2.2. *In order for our model to support static electromagnetism* ($\omega = 0$)*, we have to assume that the limit*

$$\lim_{\omega\to 0} \frac{\mathsf{K}(\boldsymbol{r},\omega)}{\omega} \tag{2.9}$$

exists. Then we can define $\mathsf{M}(\boldsymbol{r},0)$ *as the limit value.*

3 Auxiliary results

In this section we collect various results from Functional Analysis which form the mathematical background and are used freely in the sequel. Our main references are [1] and [5]. Let X be a normed linear space and A a linear operator in X having $D(A)$ as its domain of definition and $R(A)$ as its range. The kernel (null space) of A

is written as $\ker A$. The spectrum and the resolvent set of A are denoted by $\sigma(A)$ and $\rho(A)$ respectively. The approximate point spectrum $\sigma_{ap}(A)$ consists of all $\lambda \in \sigma(A)$ such that there exists a sequence $(x_n) \subset D(A)$, $\|x_n\| = 1$ and $(\lambda I - A)x_n \to 0$. For $\lambda \in \rho(A)$, we define the resolvent operator $R(\lambda; A) := (\lambda I - A)^{-1} \in \mathcal{B}(X)$. I is of course the identity operator and $\mathcal{B}(X)$ the algebra of bounded operators in X.

A is called bounded below if there is a positive constant c such that

$$\|Ax\| \geqslant c\,\|x\| \quad , \quad x \in D(A) \tag{3.1}$$

We tacitly denote a positive constant by c. Equation (3.1) is equivalent with $0 \notin \sigma_{ap}(A)$.

Lemma 3.1. *Let X be a Banach space and A be closed and bounded below. Then $R(A)$ is closed.*

Proof. Let $(y_n) \subset R(A)$ such that $y_n \to y$ for some $y \in X$. Choose $(x_n) \subset D(A)$ such that $y_n = Ax_n$. (y_n) is a Cauchy sequence and (3.1) implies that

$$\|y_n - y_m\| \geqslant c\,\|x_n - x_m\|$$

thereby (x_n) is a Cauchy sequence also. Thus $x_n \to x$ for some $x \in X$. But since A is closed we have $x \in D(A)$ and $y = Ax \in R(A)$. $\square$

Lemma 3.2. *Let X be a Banach space and A be closed. The following are equivalent:*
a) A is bounded below.
b) The inverse operator $A^{-1} : R(A) \to D(A)$ is bounded.

Proof. For the implication (a)→(b) observe first that $R(A)$ is a Banach space and the same is true for $D(A)$ if is equipped with the graph norm. The operator $A : D(A) \to R(A)$ is $1-1$, onto and bounded and the consequence comes from the Open Mapping Theorem. The converse implication is trivial. $\square$

Let now X be a complex Hilbert space, $\langle \cdot, \cdot \rangle$ stands for its inner product. It is well known that A is called symmetric if, for every x, $y \in D(A)$,

$$\langle Ax, y \rangle = \langle x, Ay \rangle$$

If A is densely defined, A is called self-adjoint when $A = A^*$. A self-adjoint operator is always symmetric; the converse is generally not true (unlike the bounded case).

Proposition 3.1. *If A is symmetric then $\langle Ax, x \rangle \in \mathbb{R}$, $x \in D(A)$. Conversely, if $\langle Ax, x \rangle \in \mathbb{R}$ for every $x \in D(A)$ and A is densely defined, then A is symmetric.*

Proposition 3.2. *Let A be densely defined and symmetric. If $\rho(A) \neq \emptyset$ then A is self-adjoint. Conversely, if A is self-adjoint, then $\sigma(A) \subset \mathbb{R}$ and consequently $\{\lambda \in \mathbb{C} : \operatorname{Im}\lambda \neq 0\} \subset \rho(A)$.*

We now consider a subspace H. We say that H is A-invariant if $Ax \in H$ whenever $x \in D(A) \cap H$. If H is A-invariant, we can define the part A_H of A in H with domain of definition $D(A_H) := D(A) \cap H$ and formula $A_H x = Ax$. Actually it is the restriction of A in H.

Proposition 3.3. *Let A be a self adjoint operator and H a closed subspace such that either $H^{\perp} \subset \ker A$ or $\overline{R(A)} \subset H$. Then each of these inclusions implies the other, H is A-invariant and the part A_H is again self adjoint.*

Proof. The decompositions

$$X = H \oplus H^{\perp} = \overline{R(A)} \oplus \ker A$$

show that the aforementioned inclusions are equivalent. We denote by P, $P^{\perp}$ the orthogonal projections onto H, $H^{\perp}$ respectively. For $x \in D(A)$, write $x = Px + P^{\perp}x$ and since $P^{\perp}x \in D(A)$ we deduce that

$$P[D(A)] = D(A) \cap H = D(A_H)$$

A_H is densely defined: let $x \in H$. There exists $(x_n) \subset D(A)$ such that $x_n \to x$ (A is densely defined). Thereby $Px_n \to x$. A_H is trivially symmetric *i.e.*, $A \subset A^*$. Let $y^* \in D(A_H^*)$. Then the functional $\phi(x) := \langle Ax, y^* \rangle$, $x \in D(A_H)$, is bounded. Moreover, for $x \in D(A)$, one has

$$\langle Ax, y^* \rangle = \left\langle A(Px + P^{\perp}x), y^* \right\rangle = \langle APx, y^* \rangle = \phi(Px)$$

Thus the functional $D(A) \ni x \mapsto \langle Ax, y^* \rangle \in \mathbb{C}$ is bounded and $y^* \in D(A^*) = D(A)$. Consequently $y^* \in D(A_H)$. □

Let $\rho(A) \neq \emptyset$ and $R(\lambda; A)$ be compact for some $\lambda \in \rho(A)$. The resolvent identity shows then that every resolvent of A is compact. In this case we say that A has compact resolvent. The following is a useful (and easy to prove) characterization.

Proposition 3.4. *The following are equivalent:*
a) A has compact resolvent.
b) $\rho(A) \neq \emptyset$ and $D(A)$, equipped with the graph norm

$$\|x\|_{D(A)} := \sqrt{\|x\|^2 + \|Ax\|^2}$$

is compactly injected in X.

The following is a well known result.

Proposition 3.5. *Let A be unbounded, self-adjoint with compact resolvent. Then $\sigma(A)$ consists of eigenvalues which form an unbounded real sequence with no accumulation point. Each corresponding eigenspace is of finite dimension.*

Let now $B \in \mathcal{B}(X)$. Define the sesquilinear form in X,

$$\langle x, y \rangle_B := \langle Bx, y \rangle$$

It is direct that $\langle \cdot, \cdot \rangle_B$ is symmetric if and only if B is self-adjoint. Furthermore, $\langle \cdot, \cdot \rangle_B$ is positive definite, and in turn an inner product, if and only if B is strictly positive *i.e.*,

$$\langle Bx, x \rangle > 0 \ \text{, for every } x \neq 0.$$

For the corresponding norm $\|\cdot\|_B$ the following inequality holds

$$\|x\|_B \leqslant \sqrt{\|B\|}\,\|x\| \tag{3.2}$$

Equipped with $\langle\,\cdot,\cdot\,\rangle_B$, X is not necessarily a Hilbert space.

Lemma 3.3. *The following are equivalent.*
a) The inner product $\langle\,\cdot,\cdot\,\rangle_B$ turns X into a Hilbert space.
b) B is coercive i.e., there exists $a > 0$ such that

$$\langle\, Bx, x\,\rangle \geqslant a\,\|x\|^2 \tag{3.3}$$

c) B is positive and bounded below.
In this case, the norms $\|\cdot\|$, $\|\cdot\|_B$ are equivalent.

Proof. (a)→(b). Thanks to (3.2) and by the Open Mapping Theorem we have that $c\,\|x\| \leqslant \|x\|_B$, so $a = c^2$.

(b)→(c). Obvious by Cauchy–Schwarz inequality.

(c)→(b). Let $\|Bx\| \geqslant c\,\|x\|$. Then, for $x \neq 0$,

$$\frac{\langle\, Bx, x\,\rangle}{\|x\|^2} \geqslant \frac{1}{c^2}\frac{\langle\, Bx, x\,\rangle}{\|Bx\|^2} \tag{3.4}$$

According to [1, Lemma 2.7.4], one has

$$\|Bx\|^2 \leqslant \|B\|\,\langle\, Bx, x\,\rangle$$

and with this inequality (3.4) becomes

$$\frac{\langle\, Bx, x\,\rangle}{\|x\|^2} \geqslant \frac{1}{c^2\,\|B\|}$$

(b)→(a). (3.2) and (3.3) imply that the norms $\|\cdot\|$, $\|\cdot\|_B$ are equivalent and thus $\|\cdot\|_B$ is complete. □

Remark 3.1. *Since a self adjoint operator has only approximate point spectrum lemma 3.3 admits one more equivalent proposition*
d) B is positive and $0 \in \rho(B)$.

4 Multiplication operators

Another important notion is the matrix multiplication operator in $L^2(\Omega;\mathbb{C}^N)$. A detailed account for abstract matrix multiplication operators is given in [6]. Here we deal mainly with the invertibility of such operators. We denote by $M_N(\mathbb{C})$ the vector space of complex $N \times N$ matrices. The vectors of $\mathbb{C}^N$ are written as $\boldsymbol{x}$, $\boldsymbol{y}$,..., the (complex) inner product as $\boldsymbol{x} \cdot \boldsymbol{y}$ and we use the symbol of the absolute value

$|\cdot|$ for the vector and the corresponding matrix norm. The functions of $L^2(\Omega;\mathbb{C}^N)$ are denoted by $\boldsymbol{U}$, $\boldsymbol{V}$,... and the inner product is

$$\langle \boldsymbol{U}, \boldsymbol{V} \rangle_0 := \int_\Omega \boldsymbol{U}(\boldsymbol{r}) \cdot \boldsymbol{V}(\boldsymbol{r})\, d\boldsymbol{r}$$

Consider a measurable function

$$m \;:\; \Omega \to M_N(\mathbb{C})$$

Note that, since $M_N(\mathbb{C})$ coincides with $\mathbb{C}^{N^2}$, measurabilty of m coincides with measurability of each entry of m. This function gives rise to an operator T_m in $L^2(\Omega;\mathbb{C}^N)$ with domain of definition

$$D(T_m) := \big\{\boldsymbol{U} \in L^2(\Omega;\mathbb{C}^N) \;:\; m(\cdot)\boldsymbol{U}(\cdot) \in L^2(\Omega;\mathbb{C}^N)\big\}$$

and formula

$$(T_m\boldsymbol{U})(\boldsymbol{r}) := m(\boldsymbol{r})\boldsymbol{U}(\boldsymbol{r})$$

Definition 4.1. *T_m is called a matrix multiplication operator corresponding to m.*

Note that m characterizes T_m, in the sense that $T_{m_1} = T_{m_2}$ implies that $m_1 = m_2$ a.e., and is often customary to identify the matrix multiplication operator with the corresponding function. Define now the space

$$L^\infty(\Omega; M_N(\mathbb{C})) := \big\{m \in L^0(\Omega; M_N(\mathbb{C})) \;:\; |\,m(\cdot)\,| \in L^\infty(\Omega)\big\}$$

(L^0 stands for classes of equivalence of measurable functions). The above space becomes a Banach space with the natural norm

$$\|m\|_\infty := \|\,|\,m(\cdot)\,|\,\|_{L^\infty(\Omega)}$$

The following is proved just as the well known case $N = 1$ by substituting the absolute value by the norm.

Proposition 4.1. *A matrix multiplication operator is always closed and densely defined. Especially, T_m is bounded if and only if $m \in L^\infty(\Omega; M_N(\mathbb{C}))$ and in this case $\|T_m\| = \|m\|_\infty$.*

Incidentally, the correspondence

$$L^\infty(\Omega; M_N(\mathbb{C})) \ni m \mapsto T_m \in \mathcal{B}(L^2(\Omega;\mathbb{C}^N))$$

defines a C^*-algebra isometry. A direct consequence of this fact is the following.

Proposition 4.2. *T_m is invertible if and only if $m(\boldsymbol{r})^{-1}$ exists a.e. $\boldsymbol{r} \in \Omega$. In this case $T_m^{-1} = T_{m^{-1}}$. Here $m^{-1}(\boldsymbol{r}) := m(\boldsymbol{r})^{-1}$.*

A necessary and sufficient condition for the existence of function m^{-1} is

$$|\, m(\boldsymbol{r})\boldsymbol{x}\,| \geqslant c(\boldsymbol{r})\,|\,\boldsymbol{x}\,| \quad \text{a.e. } \boldsymbol{r} \in \Omega \tag{4.1}$$

We have then that T_m^{-1} is bounded if and only if m^{-1} is essentially bounded. Since we have

$$|\, m(\boldsymbol{r})\,| \leqslant \frac{1}{c(\boldsymbol{r})}$$

one sees that $c(\boldsymbol{r})$ must be bounded away from zero. Equivalently,

Proposition 4.3. *T_m^{-1} is bounded if and only if m is uniformly bounded below i.e.,*

$$|\, m(\boldsymbol{r})\boldsymbol{x}\,| \geqslant c\,|\,\boldsymbol{x}\,| \quad \textit{a.e. } \boldsymbol{r} \in \Omega \tag{4.2}$$

In the light of lemma 2.3, we have the following useful result.

Proposition 4.4. *Let $m(r)$ be positive i.e., $m(\boldsymbol{r})\boldsymbol{x}\cdot\boldsymbol{x} > 0$ for $\boldsymbol{x} \neq 0$ a.e. $\boldsymbol{r} \in \Omega$ and m be uniformly bounded below. Then m is uniformly coercive i.e., $m(\boldsymbol{r})\boldsymbol{x}\cdot\boldsymbol{x} \geqslant a\,|\,\boldsymbol{x}\,|^2$ a.e. $\boldsymbol{r} \in \Omega$ and*

$$\langle\, \boldsymbol{U}, \boldsymbol{V}\,\rangle_{0m} := \langle\, m\boldsymbol{U}, \boldsymbol{V}\,\rangle$$

defines an equivalent inner product in $L^2(\Omega; \mathbb{C}^N)$.

Remark 4.1. *Note that if a matrix from $M_N(\mathbb{C})$ is positive then it is invertibe and thus bounded below.*

5 Back to the problem

We now return to our main problem (2.7). Our purpose is to establish a discrete spectrum with no accumulation point; this makes the propagation problem well posed from an "engineering" point of view. It also shows that the propagation is possible only at certain isolated values of the frequency; they define the *eigenfrequencies* of the cavity. The corresponding eigenvectors are the *modes* of the cavity.

Let us start with the state space of the problem. As it is usual for the time-harmonic electromagnetics, we work in the complex product space

$$\mathscr{X} := L^2(\Omega; \mathbb{C}^3) \times L^2(\Omega; \mathbb{C}^3)$$

This choice has to do with energy considerations. The inner product in $\mathscr{H}$ is defined naturally as

$$\langle\, \mathsf{e}_1, \mathsf{e}_2\,\rangle := \langle\, \boldsymbol{E}_1, \boldsymbol{E}_2\,\rangle_0 + \langle\, \boldsymbol{H}_1, \boldsymbol{H}_2\,\rangle_0$$

Note that the latter space is an ismorphic realization of $L^2(\Omega; \mathbb{C}^6)$.

The Maxwell operator Q is realized in its weak sense in $\mathscr{X}$ as the formal matrix

$$Q := \mathrm{i}\begin{bmatrix} 0 & \mathrm{curl} \\ -\mathrm{curl} & 0 \end{bmatrix}$$

and with the maximal domain of definition

$$D(Q) := H_0(\operatorname{curl} 0; \Omega) \times H(\operatorname{curl} 0; \Omega)$$

For the notation and the general theory of the relevant Sobolev spaces we refer to [2, Ch. IX, Part A]. The following theorem is from the same reference [Sec. IX.3].

Proposition 5.1. *Q is an unbounded self-adjoint operator. The orthogonal complement of the closed subspace*

$$\mathscr{H} := H(div\, 0; \Omega) \times H_0(div\, 0; \Omega)$$

is contained in $\ker Q$. *More precisely,*

$$\mathscr{H} = (\ker Q)^{\perp} \oplus (\mathbb{H}_2(\Omega) \times \mathbb{H}_1(\Omega))$$

The cohomology space $\mathbb{H}_1(\Omega)$, resp. $\mathbb{H}_2(\Omega)$, is of finite dimension and characterizes magnetostatics, resp. electrostatics, in Ω.

The hypothesis we are going to pose now has to do again with the E/M energy and express the fact that the medium is *lossless.*

Assumption [M]. For each $\omega \in \mathbb{R}$ the matrix $\mathsf{M}(\omega)$ has entries $L^\infty(\Omega)$ functions. In addition, $\mathsf{M}(\boldsymbol{r}, \omega)$ is positive a.e. $\boldsymbol{r} \in \Omega$ and $\mathsf{M}(\omega)$ is uniformly bounded below. Equivalently, $\mathsf{M}(\cdot, \omega)$ is uniformly coercive:

$$\mathsf{M}(\boldsymbol{r}, \omega)\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix} \cdot \begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix} \geqslant a(\omega)\left|\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix}\right|^2 \quad \text{, a.e. } \boldsymbol{r} \in \Omega,\ \boldsymbol{x},\ \boldsymbol{y} \in \mathbb{R}^3 \tag{5.1}$$

We represent M by using the standard notation in E/M theory

$$\mathsf{M}(\boldsymbol{r}, \omega) := \begin{bmatrix} \varepsilon(\boldsymbol{r}, \omega) & \xi(\boldsymbol{r}, \omega) \\ \zeta(\boldsymbol{r}, \omega) & \mu(\boldsymbol{r}, \omega) \end{bmatrix}$$

by noting that, in general, ε, ξ, ζ, μ are 3×3 matrices. The positivity of M already implies that M is Hermitian (a.e. $\boldsymbol{r} \in \Omega$) *i.e.*, $\varepsilon = \varepsilon^*$, $\mu = \mu^*$, $\zeta = \xi^*$ and this fact reduces the number of material parameters from 36 to 21 (one only needs to know the upper triangular part of M). Equation (5.1) is written

$$\varepsilon \boldsymbol{x} \cdot \boldsymbol{x} + 2 \operatorname{Re} \xi^* \boldsymbol{x} \cdot \boldsymbol{y} + \mu \boldsymbol{y} \cdot \boldsymbol{y} \geqslant a(\omega)(|\,\boldsymbol{x}\,|^2 + |\,\boldsymbol{y}\,|^2) \tag{5.2}$$

Proposition 5.2. *$\varepsilon(\cdot, \omega)$ and $\mu(\cdot, \omega)$ are uniformly coercive.*

Proof. Put $\boldsymbol{y} = 0$ and $\boldsymbol{x} = 0$ respectively in (5.2). □

Proposition 5.3. *Both $\mathsf{M}(\omega)$ and $\mathsf{M}(\omega)^{-1}$ define bounded block multiplication operators in $\mathscr{X}$.*

After this, the spectral problem (2.7) can be written

$$Q(\omega)\mathsf{e} = \omega\mathsf{e} \tag{5.3}$$

where we have put formally $Q(\omega) := \mathsf{M}(\omega)^{-1}Q$ and we consider as its domain of definition $D(Q(\omega)) = D(Q)$. The relation

$$\langle\, \mathsf{e}_1, \mathsf{e}_2 \,\rangle_\omega := \langle\, \mathsf{M}(\omega)\mathsf{e}_1, \mathsf{e}_2 \,\rangle \tag{5.4}$$

defines an equivalent inner product in $\mathscr{X}$. Note that all the notions which involve the inner product are, from now on, taken with respect to the inner product (5.4).

Proposition 5.4. *$Q(\omega)$ is an unbounded self-adjoint operator,*

Proof. For $\mathsf{e} \in D(Q(\omega)) = D(Q)$ we have

$$\langle\, Q(\omega)\mathsf{e}, \mathsf{e} \,\rangle_\omega = \langle\, Q\mathsf{e}, \mathsf{e} \,\rangle \in \mathbb{R}$$

and, since $Q(\omega)$ is densely defined, it is symmetric. Let $\mathsf{f}^* \in D(Q(\omega)^*)$. Then the functional

$$\phi(\mathsf{e}) = \langle\, Q(\omega)\mathsf{e}, \mathsf{f}^* \,\rangle_\omega = \langle\, Q\mathsf{e}, \mathsf{f}^* \,\rangle \quad , x \in D(A)$$

is bounded. Thus $\mathsf{f}^* \in D(Q^*) = D(Q) = D(Q(\omega))$. $\square$

Define now the space

$$\mathscr{H}(\omega) := \mathsf{M}(\omega)^{-1}[\mathscr{H}]$$

This is a closed subspace in $\mathscr{H}$. Note that a vector $\mathsf{e} \in \mathscr{H}(\omega)$ satisfies the Gauss law (2.8) and this show that $\mathscr{H}(\omega)$ is the "correct" space to pose the problem. From the decompositions

$$\mathscr{X} = \overline{R(Q(\omega))} \oplus \ker Q(\omega) = \mathscr{H}(\omega) \oplus \mathscr{H}(\omega)^\perp$$

The obvious inclusion $R(Q) \subset \mathscr{H}$ gives $R(Q(\omega)) \subset \mathscr{H}(\omega)$ and thus $\mathscr{H}(\omega)^\perp \subset \ker Q(\omega)$.

Proposition 5.5. *The part $T(\omega) := Q(\omega)_{\mathscr{H}(\omega)}$ of $Q(\omega)$ in $\mathscr{H}(\omega)$ is self-adjoint.*

This way, the spectral problem (5.3) is restated in $\mathscr{H}(\omega)$

$$T(\omega)\mathsf{e} = \omega\mathsf{e} \tag{5.5}$$

We now make two hypotheses which are of technical nature.

Assumption [T]. $D(T(\omega)) = D(Q) \cap \mathscr{H}(\omega)$ with the graph norm is a closed subspace of the Sobolev space $H^1(\Omega; \mathbb{C}^3) \times H^1(\Omega; \mathbb{C}^3)$.

Assumption [Om]. The injection $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact.

These two assumptions imply in fact that the injection $D(T(\omega)) \hookrightarrow \mathscr{H}(\omega)$ is compact and we obtain our basic result.

Proposition 5.6. *The operator $T(\omega)$ has compact resolvent and its spectrum consists of an unbounded real sequence $(\lambda_n(\omega))$ with no accumulation point. Each corresponding eigenspace is of finite dimension.*

That is, solving the spectral problem (5.5) means to solve the equation

$$\lambda_n(\omega) = \omega \tag{5.6}$$

Equation (5.6) is referred as the *dispersion relation* and it connects the medium with the eigenfrequencies. One expects (5.6) to have countably many solutions for every $n = 1, 2, ...$; this is a way to order the modes of the cavity.

6 Conclusion

This paper studies the time harmonic electromagnetic propagation problem in a cavity filled with a bianisotropic (general linear) medium. We formulate the problem as a non-standard eugenvalue problem where the eigenvalue serves as the eigenfrequency and the corresponding eigenvector as the mode of the cavity. By using techniques of Hilbert space theory, we describe a method of solving the problem and prove the desired discrete spectrum.

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References

[1] M. S. Birman and M. Z. Solomjak, *Spectral Theory of Self-Adjoint Operators in Hilbert Spaces*, D. Reidel Publishing Company, Doordecht, 1987

[2] R. Dautray and J.–L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology vol. 3: Spectral Theory and Applications*, Springer, Berlin, 1990.

[3] M. Gustafsson, *Wave Splitting in Direct and Inverse Scattering problems*, PhD Thesis, Department of Electrical and Information Technology, Lund University, Sweden, 2000

[4] R.F. Harrington, *Time–Harmonic Electromagnetic Fields (Reissued Edition)*, IEEE Press, New Jersey, 2001.

[5] G. Helmberg, *Introduction to Spectral Theory in Hilbert Space*, North Holland, Amsterdam, 1969.

[6] A. Holderieth, *Matrix multiplication operators generating one parameter semigroups*, Semigroup Forum **142**, 155–166, 1991.

[7] A.D. Ioannidis, *Mathematical Problems of Propagation and Scattering of Electromagnetic Waves in Complex Media* (in Greek), PhD Thesis, Department of Mathematics, University of Athens, Greece, 2006.

[8] A.D. Ioannidis, A. Karlsson and G. Kristensson, *The general medium in electromagnetic theory*, Preprint.

[9] A. Karlsson and G. Kristensson, *Constitutive relations, dissipation and reciprocity for the Maxwell equations in the time domain*, J. Electro. Waves Appl. **6**, 537–551, 1992.

[10] I.V. Lindell, A.H. Sihvola and K. Suchy, *Six–vector formalism in electromagnetics of bi–anisotropic media*, J. Electro. Waves Appl. **9**, 887–903, 1995.