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Evaluation of some integrals relevant to multiple scattering by randomly distributed obstacles

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Abstract

This paper analyzes and solves an integral and its indefinite Fourier transform of importance in multiple scattering problems of randomly distributed scatterers. The integrand contains a radiating spherical wave, and the two-dimensional domain of integration excludes a circular region of varying size. A solution of the integral in terms of radiating spherical waves is demonstrated. The method employs the Erdélyi operators, which leads to a recursion relation. This recursion relation is solved in terms of a finite sum of radiating spherical waves. The solution of the indefinite Fourier transform of the integral contains the indefinite Fourier transforms of the Legendre polynomials, which are solved by a closed formula.

1 Introduction

In recent years, the electromagnetic scattering problem by randomly distributed objects has been successfully formulated and solved. Some important contributions in the field are found in e.g., [3-8, 10, 11, 13, 16-19, 21-25]. These references refer to various aspects of the topic, and more references can be found in these papers. The topic is also treated in several textbooks, see e.g., [12, 14, 20], which can be consulted for a comprehensive treatment of the various multiple scattering theories.

Of critical importance for the solution of a specific scattering problem with hole-corrections (HC) is an integral of the form [9, 18, 20]

$$I_l(z) = \frac{k^2}{2\pi} \iint_{\mathbb{R}^2} H(r-a)h_l^{(1)}(kr)P_l(\cos\theta) \, dx \, dy, \quad z \in \mathbb{R}$$
 (1.1)

where H(x) denotes the Heaviside function, $h_l^{(1)}(kr)$ the spherical Hankel function, and $P_l(x)$ the Legendre polynomial of order l, respectively. We have also adopted the spherical coordinates, $r = \sqrt{x^2 + y^2 + z^2}$ and θ ($\cos \theta = z/r$), and the wave number k. The domain of integration is the plane z = constant, excluding the sphere of radius a > 0 at the center, see Figure 1. For a given value of $|z| \le a$, the radius of the excluded circle is $\sqrt{a^2 - z^2}$. For $|z| \ge a$ the integration is the entire x-y plane. This integral, for a given a > 0, is a non-trivial function of $z \in \mathbb{R}$. To ensure convergence of the integral at infinity, we assume the wave number k has an arbitrarily small imaginary part. The explicit solution of this integral, as a function of z and the index $l = 0, 1, 2, \ldots$, is the aim of this paper, and the goal is to express the solutions in a form that is attractive from a numerical computation point of view.

The solution of the integral $I_l(z)$ is developed in Sections 2 and 3. The indefinite Fourier transform of $I_l(z)$ is also essential for a successful solution of the multiple scattering problem with hole-corrections, and this analysis is found in Sections 4 and 5. The paper is concluded with a short summary in Section 6.

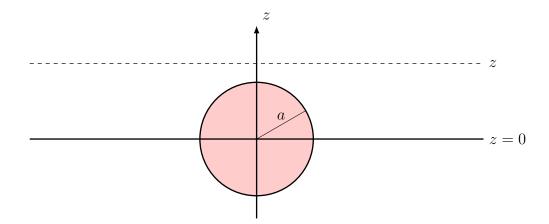


Figure 1: The geometry of the integration domain — the plane z = constant (dotted line), and the exclusion volume — the sphere of radius a located at the origin (in gray).

2 The integral $I_l(z)$

Rewrite the integral $I_l(z)$ in (1.1) in cylindrical coordinates and perform the integration in the azimuthal angle. We get from (1.1)

$$I_l(z) = k^2 \int_{h(z)}^{\infty} h_l^{(1)} \left(k \sqrt{\rho^2 + z^2} \right) P_l \left(z / \sqrt{\rho^2 + z^2} \right) \rho \, d\rho, \quad z \in \mathbb{R}$$
 (2.1)

where

$$h(z) = \begin{cases} \sqrt{a^2 - z^2}, & -a \le z \le a \\ 0, & |z| > a \end{cases}$$

From the parity of the Legendre polynomials, $P_l(-x) = (-1)^l P_l(x)$, we see that also $I_l(-z) = (-1)^l I_l(z)$. Thus, it suffices to evaluate the integral for z > 0. In particular, $I_l(0) = 0$, if l is an odd integer. From (2.1) we also easily compute the integral for l = 0, viz.

$$I_0(z) = \begin{cases} e^{-ikz}, & z \le -a \\ ikah_0^{(1)}(ka) = e^{ika}, & -a \le z \le a \\ e^{ikz}, & z \ge a \end{cases}$$

2.1 Solution outside the interval [-a, a]

In the interval z > a, the integral is evaluated with the use of the transformation of the outgoing scalar spherical wave in terms of planar waves [2, p. 180], *i.e.*, for a

general value of $z \neq 0$

$$\begin{split} h_l^{(1)} \left(k \sqrt{\rho^2 + z^2} \right) P_l \left(\pm |z| / \sqrt{\rho^2 + z^2} \right) \\ &= \frac{\mathrm{i}^{-l}}{2\pi} \iint_{\mathbb{R}^2} P_l \left(\pm k_z / k \right) \mathrm{e}^{\mathrm{i} \mathbf{k}_{\mathrm{t}} \cdot \mathbf{\rho} + \mathrm{i} k_z |z|} \frac{k}{k_z} \frac{\mathrm{d} k_x \, \mathrm{d} k_y}{k^2}, \quad z \gtrless 0 \end{split}$$

where $\boldsymbol{\rho} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}$, $\boldsymbol{k}_{t} = k_{x}\hat{\boldsymbol{x}} + k_{y}\hat{\boldsymbol{y}}$, $k_{t} = |\boldsymbol{k}_{t}|$, and k_{z} is defined by

$$k_z = (k^2 - k_t^2)^{1/2} = \begin{cases} \sqrt{k^2 - k_t^2} & \text{for } k_t < k \\ i\sqrt{k_t^2 - k^2} & \text{for } k_t > k \end{cases}$$

For z > a, we get from (1.1)

$$I_{l}(z) = \frac{k^{2}}{2\pi} \iint_{\mathbb{R}^{2}} \frac{\mathrm{i}^{-l}}{2\pi} \left(\iint_{\mathbb{R}^{2}} P_{l}(k_{z}/k) \, \mathrm{e}^{\mathrm{i}\boldsymbol{k}_{t} \cdot \boldsymbol{\rho} + \mathrm{i}k_{z}|z|} \frac{k}{k_{z}} \frac{\mathrm{d}k_{x} \, \mathrm{d}k_{y}}{k^{2}} \right) \, \mathrm{d}x \, \mathrm{d}y$$
$$= \mathrm{i}^{-l} \iint_{\mathbb{R}^{2}} P_{l}(k_{z}/k) \, \mathrm{e}^{\mathrm{i}k_{z}|z|} \delta(\boldsymbol{k}_{t}) \frac{k}{k_{z}} \, \mathrm{d}k_{x} \, \mathrm{d}k_{y} = \mathrm{i}^{-l} \mathrm{e}^{\mathrm{i}kz}$$

by orthogonality or completeness of the planar waves. This implies that the integral for z > a is

$$I_l(z) = i^{-l}e^{ikz}, \quad z > a$$

and consequently, by parity, or analogous calculations

$$I_l(z) = i^l e^{-ikz}, \quad z < -a$$

We observe that the integral outside the interval [-a, a] is not singular as $a \to 0$. In fact, the module is constant 1.

3 Solution of the integral $I_l(\eta), -a \leq z \leq a$

We have already obtained a solution of the integral in the interval |z| > a, and we now concentrate on finding a solution of the integral in the interval $-a \le z \le a$.

The Erdélyi operators \mathcal{Y}_n^m in Ref. 12 are instrumental in finding a closed formula for the integral $I_l(z)$. From [12, Th. 3.13], we have the following very useful result:

$$D\left(h_l^{(1)}(kr)P_l(\cos\theta)\right) = \frac{l+1}{2l+1}h_{l+1}^{(1)}(kr)P_{l+1}(\cos\theta) - \frac{l}{2l+1}h_{l-1}^{(1)}(kr)P_{l-1}(\cos\theta)$$

where $D = -k^{-1}(\partial/\partial z)$. The D operator and the Erdélyi operators are related by $\mathcal{Y}_1^0 = \sqrt{\frac{3}{4\pi}}D_1^0 = \sqrt{\frac{3}{4\pi}}D$.

 $^{^{1}}$ To ensure convergence of the integral at infinity, assume the wave number k has an arbitrary small, positive imaginary part.

Apply the differential operator D to the integral $I_l(z)$ in (2.1), and use the relation above. We obtain, since h'(z)h(z) = -z, the following recursion relation:²

$$DI_{l}(z) = -kzh_{l}^{(1)}(ka)P_{l}(z/a) + \frac{l+1}{2l+1}I_{l+1}(z) - \frac{l}{2l+1}I_{l-1}(z), \quad -a \le z \le a$$

with initial condition $I_0(z) = ikah_0^{(1)}(ka)$.

In the dimensionless variables $\eta = kz$ and $\xi = ka > 0$, this leads to the recursion relation, $l = 0, 1, 2, \ldots$ (note the mild change in notation)

$$I_{l+1}(\eta) = \frac{2l+1}{l+1} \xi h_l^{(1)}(\xi) \frac{\eta}{\xi} P_l(\eta/\xi) - \frac{2l+1}{l+1} \frac{\mathrm{d}}{\mathrm{d}\eta} I_l(\eta) + \frac{l}{l+1} I_{l-1}(\eta), \quad -\xi \le \eta \le \xi$$

The recursion relation is conveniently put in a more generic form by introducing the variable $x = \eta/\xi \in [-1, 1]$. The dependent variable is now x, and ξ is a parameter. Retaining the same notation for the integral, but with a change of the independent variable, we get

$$I_{l+1}(x) = \frac{2l+1}{l+1} \xi h_l^{(1)}(\xi) x P_l(x) - \frac{2l+1}{\xi(l+1)} I_l'(x) + \frac{l}{l+1} I_{l-1}(x), \quad -1 \le x \le 1$$

The following proposition states the surprisingly simple and elegant solution of this recursion relation.

Proposition 3.1. The recursion relation

$$I_{l+1}(x) = \frac{2l+1}{l+1} \xi h_l^{(1)}(\xi) x P_l(x) - \frac{2l+1}{\xi(l+1)} I_l'(x) + \frac{l}{l+1} I_{l-1}(x), \quad l = 0, 1, 2, \dots$$
(3.1)

with initial condition

$$I_0(z) = i\xi h_0^{(1)}(\xi)$$

has the solution

$$I_{l}(x) = -\xi h_{l+1}^{(1)}(\xi) P_{l}(x)$$

$$+ \sum_{k=0}^{[l/2]} (-1)^{k} \left(\xi h_{l+1-2k}^{(1)}(\xi) + \xi h_{l-1-2k}^{(1)}(\xi) \right) P_{l-2k}(x), \quad l = 0, 1, 2, \dots$$
(3.2)

²Outside the interval $z \in [-a, a]$ the recursion relation reads

$$I_{l+1}(z) = \frac{2l+1}{l+1}DI_l(z) + \frac{l}{l+1}I_{l-1}(z), \quad I_0(z) = e^{ikz} \quad z \ge a$$

which is easily solved by induction over the integer l. The result is

$$I_l(z) = i^{-l} e^{ikz}, \quad z \ge a$$

in agreement with the result above.

Proof. We prove the proposition by induction over the integer l. The recursion relation (3.2) is true for l = 0, due to the properties of the spherical Hankel functions [15, 10.16.1]. We have from (3.2)

$$I_0(x) = \xi h_{-1}^{(1)}(\xi) = \xi \left(\frac{\pi}{2\xi}\right)^{1/2} H_{-1/2}^{(1)}(\xi) = \mathrm{i}\xi \left(\frac{\pi}{2\xi}\right)^{1/2} H_{1/2}^{(1)}(\xi) = \mathrm{i}\xi h_0^{(1)}(\xi)$$

Now assume the solution (3.2) holds for all integers less than or equal to l, and we want to prove that it holds for l + 1. We have from (3.1) and the induction assumption

$$I_{l+1}(x) = \frac{2l+1}{l+1} \xi h_l^{(1)}(\xi) x P_l(x) - \frac{2l+1}{\xi(l+1)} I_l'(x) + \frac{l}{l+1} I_{l-1}(x)$$

$$= \xi h_l^{(1)}(\xi) P_{l+1}(x) + \frac{2l+1}{\xi(l+1)} \xi h_{l+1}^{(1)}(\xi) P_l'(x)$$

$$- \frac{2l+1}{\xi(l+1)} \sum_{k=0}^{[l/2]} (-1)^k \left(\xi h_{l+1-2k}^{(1)}(\xi) + \xi h_{l-1-2k}^{(1)}(\xi) \right) P_{l-2k}'(x)$$

$$+ \frac{l}{l+1} \sum_{k=0}^{[(l-1)/2]} (-1)^k \left(\xi h_{l-2k}^{(1)}(\xi) + \xi h_{l-2-2k}^{(1)}(\xi) \right) P_{l-1-2k}(x)$$

where we used the following recursion relation for the Legendre polynomials:

$$(2l+1)xP_l(x) = (l+1)P_{l+1}(x) + lP_{l-1}(x)$$

We conclude that $I_{l+1}(x)$ is a polynomial in x of the order l+1, and therefore can be expanded in a series of Legendre polynomials. The form is

$$I_{l+1}(x) = \sum_{n=0}^{[(l+1)/2]} a_n P_{l+1-2n}(x)$$

where a_n depends on l and ξ . The coefficients a_n are determined by orthogonality of the Legendre polynomials.

$$a_n = \frac{2l+3-4n}{2} \int_{-1}^{1} I_{l+1}(x) P_{l+1-2n}(x) dx$$

The first coefficient is special.

$$a_0 = \xi h_l^{(1)}(\xi) = -\xi h_{l+2}^{(1)}(\xi) + \left(\xi h_{l+2}^{(1)}(\xi) + \xi h_l^{(1)}(\xi)\right)$$

Proceed in the same way with the remaining coefficients, $n = 1, 2, \dots, [(l+1)/2]$.

$$a_{n} = \frac{2l+1}{l+1} \frac{2l+3-4n}{2} h_{l+1}^{(1)}(\xi) I_{l,l+1-2n}$$

$$- \frac{2l+1}{l+1} \frac{2l+3-4n}{2} \sum_{k=0}^{[l/2]} (-1)^{k} \left(h_{l+1-2k}^{(1)}(\xi) + h_{l-1-2k}^{(1)}(\xi) \right) I_{l-2k,l+1-2n}$$

$$+ \frac{l}{l+1} \sum_{k=0}^{[(l-1)/2]} (-1)^{k} \left(\xi h_{l-2k}^{(1)}(\xi) + \xi h_{l-2-2k}^{(1)}(\xi) \right) \delta_{k,n-1}$$

where we used the notion

$$I_{k,n} = \int_{-1}^{1} P_k'(x) P_n(x) \, dx = \begin{cases} 0, & 0 \le k \le n \\ 1 - (-1)^{k+n}, & 0 \le n < k \end{cases}$$

Use this result, and the following recursion relation for the spherical Hankel functions:

$$(2l+1)h_l^{(1)}(\xi) = \xi h_{l+1}^{(1)}(\xi) + \xi h_{l-1}^{(1)}(\xi)$$
(3.3)

We get

$$a_{n} = \frac{2l+1}{l+1} (2l+3-4n) \left(h_{l+1}^{(1)}(\xi) - \sum_{k=0}^{n-1} (-1)^{k} \left(h_{l+1-2k}^{(1)}(\xi) + h_{l-1-2k}^{(1)}(\xi) \right) \right)$$

$$= h_{l+1}^{(1)}(\xi) + (-1)^{n-1} h_{l+1-2n}^{(1)}(\xi)$$

$$+ \frac{l}{l+1} (-1)^{n-1} \left(\xi h_{l+2-2n}^{(1)}(\xi) + \xi h_{l-2n}^{(1)}(\xi) \right)$$

$$= \frac{2l+1}{l+1} (-1)^{n} (2l+3-4n) h_{l+1-2n}^{(1)}(\xi) - \frac{l}{l+1} (-1)^{n} \left(\xi h_{l+2-2n}^{(1)}(\xi) + \xi h_{l-2n}^{(1)}(\xi) \right)$$

$$= (-1)^{n} \left(\xi h_{l+2-2n}^{(1)}(\xi) + \xi h_{l-2n}^{(1)}(\xi) \right)$$

Collecting the results gives

$$I_{l+1}(x) = \sum_{n=0}^{[(l+1)/2]} a_n P_{l+1-2n}(x)$$

$$= -\xi h_{l+2}^{(1)}(\xi) P_{l+1}(x) + \sum_{n=0}^{[(l+1)/2]} (-1)^n \left(\xi h_{l+2-2n}^{(1)}(\xi) + \xi h_{l-2n}^{(1)}(\xi)\right) P_{l+1-2n}(x)$$

which is the statement (3.2) for l+1, and the proposition is proved.

Alternative expressions of the integral I(z) in the interval $z \in [-a, a]$ can be found. The following corollary shows some.

Corollary 3.1. The integral I(z) in Proposition 3.1 has the following alternative expressions:

$$I_{l}(x) = -\xi h_{l+1}^{(1)}(\xi) P_{l}(x) + \sum_{k=0}^{\lfloor l/2 \rfloor} (-1)^{k} (2l - 4k + 1) h_{l-2k}^{(1)}(\xi) P_{l-2k}(x), \quad l = 0, 1, 2, \dots$$
(3.4)

and

$$I_{l}(x) = i^{1-l} \xi h_{0}^{(1)}(\xi) P_{l-2[l/2]}(x)$$

$$+ \sum_{k=0}^{[l/2]-1} (-1)^{k} \xi h_{l-2k-1}^{(1)}(\xi) \left(P_{l-2k}(x) - P_{l-2k-2}(x) \right), \quad l = 0, 1, 2, \dots \quad (3.5)$$

and

$$I_{l}(x) = i^{1-l} \xi h_{0}^{(1)}(\xi) P_{l-2[l/2]}(x)$$

$$- \sum_{k=0}^{[l/2]-1} (-1)^{k} \xi h_{l-2k-1}^{(1)}(\xi) \frac{2l-4k-1}{(l-2k-1)(l-2k)} P'_{l-2k-1}(x), \quad l = 0, 1, 2, \dots$$
 (3.6)

where the two last sums are zero for l = 0, 1.

Proof. The solution in (3.4) is equivalent to (3.2), which is easily seen since the spherical Hankel functions $h_l^{(1)}(\xi)$ satisfy the recursion relation (3.3). The representation in (3.5) is simply a rearrangement of the sum in (3.2). We obtain from (3.2) $(l=0,1,2,\ldots)$

$$I_{l}(x) = \sum_{k=0}^{\lfloor l/2 \rfloor - 1} (-1)^{k} \xi h_{l-1-2k}^{(1)}(\xi) \left(P_{l-2k}(x) - P_{l-2-2k}(x) \right)$$

$$+ (-1)^{\lfloor l/2 \rfloor} \xi h_{l-1-2\lfloor l/2 \rfloor}^{(1)}(\xi) P_{l-2\lfloor l/2 \rfloor}(x)$$

$$= \sum_{k=0}^{\lfloor l/2 \rfloor - 1} (-1)^{k} \xi h_{l-1-2k}^{(1)}(\xi) \left(P_{l-2k}(x) - P_{l-2-2k}(x) \right)$$

$$+ i^{1-l} \xi h_{0}^{(1)}(\xi) P_{l-2\lfloor l/2 \rfloor}(x)$$

where we used [15, 10.16.1]

$$h_{-1}^{(1)}(\xi) = \mathrm{i} h_0^{(1)}(\xi)$$

Finally, the relation (3.6) from (3.5) with the use of the recursion relation

$$l(l+1) (P_{l+1}(x) - P_{l-1}(x)) = -(2l+1)(1-x^2)P'_l(x)$$

In the original variables z and a, we have

$$I_{l}(z) = i^{1-l}kah_{0}^{(1)}(ka)P_{l-2[l/2]}(z/a) + \sum_{n=0}^{[l/2]-1} (-1)^{n}kah_{l-2n-1}^{(1)}(ka) \left(P_{l-2n}(z/a) - P_{l-2n-2}(z/a)\right), \quad l = 0, 1, 2, \dots$$

or

$$I_{l}(z) = -kah_{l+1}^{(1)}(ka)P_{l}(z/a) + \sum_{k=0}^{[l/2]} (-1)^{k}(2l - 4k + 1)h_{l-2k}^{(1)}(ka)P_{l-2k}(z/a), \quad l = 0, 1, 2, \dots$$

and we see that the integral $I_l(z)$ can be written as a finite sum of spherical waves (except the first term). The most singular term in powers of ka is of the order $(ka)^{1-l}$ (order O(1) if l=0), which is most easily seen from the representation in (3.5).

4 Fourier transform of $I_l(z)$

The indefinite Fourier transform of the function $I_l(z)$ has also importance in the analysis of [9]. More specifically, our goal in this section is to compute

$$\widehat{I}_{l}^{\pm}(z) = k \int_{z_{0}}^{z} I_{l}(t) e^{\pm ikt} dt, \quad z \ge z_{0}, \quad l = 0, 1, 2, \dots$$
 (4.1)

where z_0 is a fixed number such that $z_0 < -a$.

The function $I_l(t)$ has explicit forms in the three intervals $[z_0, -a]$, (-a, a), and $[a, \infty)$. The explicit forms are:

$$I_l(t) = i^l e^{-ikt}, \quad t \le -a$$

and in the interval $t \in (-a, a)$ as a finite sum of spherical waves

$$I_{l}(t) = i^{1-l}kah_{0}^{(1)}(ka)P_{l-2[l/2]}(t/a) + \sum_{n=0}^{[l/2]-1} (-1)^{n}kah_{l-2n-1}^{(1)}(ka) \left(P_{l-2n}(t/a) - P_{l-2n-2}(t/a)\right)$$

In the interval $t \geq a$

$$I_l(t) = i^{-l}e^{ikt}$$

To compute the indefinite Fourier transform we need to calculate the function

$$h_l^{\pm}(z) = k \int_{-a}^{z} P_l(t/a) e^{\pm ikt} dt = ka \int_{-1}^{z/a} P_l(t) e^{\pm ikat} dt, \quad |z| \le a$$
 (4.2)

For z = a the integral is a spherical Bessel function, viz.,

$$h_l^{\pm}(a) = k \int_{-a}^{a} P_l(t/a) e^{\pm ikt} dt = ka \int_{-1}^{1} P_l(t) e^{\pm ikat} dt = 2ka(\pm i)^l j_l(ka)$$

We divide the interval $[z_0, z]$ in three parts. In the interval $z_0 \leq z < -a$, we have

$$\widehat{I}_{l}^{\pm}(z) = i^{l} k \int_{z_{0}}^{z} e^{i(\pm 1 - 1)kt} dt = i^{l} \begin{cases} k(z - z_{0}) \\ \frac{1}{2i} \left(e^{-2ikz_{0}} - e^{-2ikz} \right) \end{cases}$$

and in the interval -a < z < a, we have

$$\widehat{I}_{l}^{\pm}(z) = i^{l} \begin{cases} k(-a - z_{0}) \\ \frac{1}{2i} \left(e^{-2ikz_{0}} - e^{2ika} \right) \end{cases} + i^{1-l}kah_{0}^{(1)}(ka)h_{l-2[l/2]}^{\pm}(z)$$

$$+ \sum_{n=0}^{[l/2]-1} (-1)^{n}kah_{l-2n-1}^{(1)}(ka) \left(h_{l-2n}^{\pm}(z) - h_{l-2n-2}^{\pm}(z) \right)$$

and in the interval a < z, we have

$$\widehat{I}_{l}^{\pm}(z) = i^{l} \begin{cases} k(-a - z_{0}) \\ \frac{1}{2i} \left(e^{-2ikz_{0}} - e^{2ika} \right) \end{cases} + i^{1-l} 2(ka)^{2} h_{0}^{(1)}(ka)(\pm i)^{l-2[l/2]} j_{l-2[l/2]}(ka)$$

$$+ 2(ka)^{2} (\pm i)^{l} \sum_{n=0}^{[l/2]-1} h_{l-2n-1}^{(1)}(ka) \left(j_{l-2n}(ka) + j_{l-2n-2}(ka) \right)$$

$$+ i^{-l} \begin{cases} \frac{1}{2i} \left(e^{2ikz} - e^{2ika} \right) \\ k(z-a) \end{cases}$$

5 Indefinite integral of Legendre polynomials

It remains to find an effective method to compute the functions $h_l^{\pm}(z)$ in (4.2). To this end, define

$$h_l(\eta, \zeta) = \int_{-1}^{\eta} P_l(t) e^{i\zeta t} dt, \quad |\eta| \le 1$$
 (5.1)

We see that $h_l(1,\zeta) = 2i^l j_l(\zeta)$. In terms of the functions $h_l(\eta,\zeta)$, the functions $h_l^{\pm}(z)$ are

$$h_l^{\pm}(z) = kah_l(z/a, \pm ka)$$

Our ambition in this section is to find an efficient method to compute the integrals in (5.1). We express the function $h_l(\eta, \zeta)$ as a recursion relation.

5.1 Solution by recursion

The following recursion relation of Legendre polynomials is useful:

$$P_l(t) = \frac{1}{2l+1} \left(P'_{l+1}(t) - P'_{l-1}(t) \right)$$

Integration by parts then implies $(P_l(-1) = (-1)^l)$

$$h_{l}(\eta,\zeta) = \int_{-1}^{\eta} P_{l}(t) e^{i\zeta t} dt = \frac{1}{2l+1} \int_{-1}^{\eta} \left(P'_{l+1}(t) - P'_{l-1}(t) \right) e^{i\zeta t} dt$$
$$= \frac{1}{2l+1} \left(P_{l+1}(\eta) - P_{l-1}(\eta) \right) e^{i\zeta \eta} - \frac{i\zeta}{2l+1} \left(h_{l+1}(\eta,\zeta) - h_{l-1}(\eta,\zeta) \right)$$

or solving for $h_{l+1}(\eta,\zeta)$

$$h_{l+1}(\eta,\zeta) = \frac{1}{i\zeta} \left(P_{l+1}(\eta) - P_{l-1}(\eta) \right) e^{i\zeta\eta} - \frac{2l+1}{i\zeta} h_l(\eta,\zeta) + h_{l-1}(\eta,\zeta), \quad l = 1, 2, 3, \dots$$

The functions $h_l(\eta,\zeta)$ can therefore be found by iteration with starting values

$$h_0(\eta,\zeta) = \frac{1}{i\zeta} \left(e^{i\zeta\eta} - e^{-i\zeta} \right) = \frac{1}{i\zeta} P_0(\eta) e^{i\zeta\eta} + h_0^{(2)}(\zeta) = \eta h_0^{(1)}(\zeta\eta) + h_0^{(2)}(\zeta)$$

and

$$h_{1}(\eta,\zeta) = \frac{1}{i\zeta} \left(\eta e^{i\zeta\eta} + e^{-i\zeta} \right) + \frac{1}{\zeta^{2}} \left(e^{i\zeta\eta} - e^{-i\zeta} \right)$$

$$= \frac{1}{i\zeta} \left(P_{1}(\eta) - \frac{1}{i\zeta} P_{0}(\eta) \right) e^{i\zeta\eta} + ih_{1}^{(2)}(\zeta) = i\eta^{2} h_{1}^{(1)}(\zeta\eta) + ih_{1}^{(2)}(\zeta)$$

To find the general solution to this recursion scheme, we start by solving the homogeneous difference equation.

Lemma 5.1. The solution to the homogeneous difference equation

$$a_{l+1} + \frac{2l+1}{\mathrm{i}\zeta}a_l - a_{l-1} = 0, \quad l = 1, 2, 3, \dots$$

given the initial values a_0 and a_1 is

$$a_{l} = -\frac{\zeta^{2}}{2i} \left(a_{0} h_{0}^{(2)'}(\zeta) - i a_{1} h_{0}^{(2)}(\zeta) \right) i^{l} h_{l}^{(1)}(\zeta)$$

$$+ \frac{\zeta^{2}}{2i} \left(a_{0} h_{0}^{(1)'}(\zeta) - i a_{1} h_{0}^{(1)}(\zeta) \right) i^{l} h_{l}^{(2)}(\zeta), \quad l = 2, 3, 4, \dots$$

Proof. Two linearly independent solutions to the homogeneous difference equation in the lemma are $i^l h_l^{(1)}(\zeta)$ and $i^l h_l^{(2)}(\zeta)$, which is easily proved by the recursion relation $f_{l+1}(z) - (2l+1)f_l(z)/z + f_{l-1}(z) = 0$, where $f_l(z)$ is any spherical Bessel or Hankel function. The general solution therefore is

$$a_l = c_1 i^l h_l^{(1)}(\zeta) + c_2 i^l h_l^{(2)}(\zeta), \quad l = 2, 3, 4, \dots$$

where c_1 and c_2 are constants determined by the starting values a_0 and a_1 . Explicitly, we get

$$\begin{cases} c_1 h_0^{(1)}(\zeta) + c_2 h_0^{(2)}(\zeta) = a_0 \\ c_1 i h_1^{(1)}(\zeta) + c_2 i h_1^{(2)}(\zeta) = a_1 \end{cases}$$

with solution

$$\begin{cases} c_1 = -\frac{\zeta^2}{2i} \left(a_0 h_0^{(2)'}(\zeta) - i a_1 h_0^{(2)}(\zeta) \right) \\ c_2 = \frac{\zeta^2}{2i} \left(a_0 h_0^{(1)'}(\zeta) - i a_1 h_0^{(1)}(\zeta) \right) \end{cases}$$

where we used the Wronskian of the spherical Hankel functions.

$$h_n^{(2)}(z)h_n^{(1)'}(z) - h_n^{(2)'}(z)h_n^{(1)}(z) = \frac{2i}{z^2}$$

and $h_0^{(1,2)'}(z) = -h_1^{(1,2)}(z)$. This completes the proof of the lemma.

We are now ready for the solution to the inhomogeneous difference equation in $h_l(\eta, \zeta)$ above. We formulate this as a lemma.

Lemma 5.2. Define an iteration scheme by

$$h_{l+1}(\eta,\zeta) = \frac{1}{i\zeta} \left(P_{l+1}(\eta) - P_{l-1}(\eta) \right) e^{i\zeta\eta} - \frac{2l+1}{i\zeta} h_l(\eta,\zeta) + h_{l-1}(\eta,\zeta), \quad l = 1, 2, 3, \dots$$

with starting values

$$h_0(\eta,\zeta) = \eta h_0^{(1)}(\zeta\eta) + h_0^{(2)}(\zeta)$$

and

$$h_1(\eta,\zeta) = i \left(\eta^2 h_1^{(1)}(\zeta\eta) + h_1^{(2)}(\zeta) \right)$$

The solution is

$$h_l(\eta,\zeta) = f_l(\eta,\zeta)e^{i\zeta\eta} + i^l h_l^{(2)}(\zeta), \quad l = 0, 1, 2, 3, \dots$$

where

$$f_{l}(\eta,\zeta) = i^{l} h_{l}^{(1)}(\zeta) \left\{ \sum_{k=1}^{l} \frac{1}{\zeta h_{k-1}^{(1)}(\zeta) h_{k}^{(1)}(\zeta)} \left(-\sum_{n=0}^{k} i^{-n+1} (2n+1) \frac{h_{n}^{(1)}(\zeta)}{\zeta} P_{n}(\eta) + i^{-k+2} h_{k}^{(1)}(\zeta) P_{k-1}(\eta) + i^{-k+1} h_{k+1}^{(1)}(\zeta) P_{k}(\eta) \right) - i \frac{P_{0}(\eta)}{\zeta h_{0}^{(1)}(\zeta)} \right\}, \quad l = 0, 1, 2, \dots$$

Proof. We first subtract the part of the solution that contains the spherical Hankel function of the second kind $h_l^{(2)}(\zeta)$ and the exponential function $e^{i\zeta\eta}$. To this end, let $h_l(\eta,\zeta)=f_l(\eta,\zeta)e^{i\zeta\eta}+i^lh_l^{(2)}(\zeta)$. The recursion relation for $f_l(\eta,\zeta)$ is easily found by the use of the recursion relation $h_{l+1}^{(2)}(z)=(2l+1)h_l^{(2)}(z)/z-h_{l-1}^{(2)}(z)$. We get the new difference equation

$$f_{l+1}(\eta,\zeta) = \frac{1}{\mathrm{i}\zeta} \left(P_{l+1}(\eta) - P_{l-1}(\eta) \right) - \frac{2l+1}{\mathrm{i}\zeta} f_l(\eta,\zeta) + f_{l-1}(\eta,\zeta), \quad l = 1,2,3,\dots$$

with starting values

$$f_0(\eta,\zeta) = \frac{1}{\mathrm{i}\zeta} P_0(\eta)$$

and

$$f_1(\eta,\zeta) = \frac{1}{\mathrm{i}\zeta} \left(P_1(\eta) - \frac{1}{\mathrm{i}\zeta} P_0(\eta) \right)$$

To simplify the notation, we put the difference equation in a standard form [1].

$$a_{n+2} + p_1(n)a_{n+1} + p_0(n)a_n = q(n), \quad n = 1, 2, \dots$$

where

$$\begin{cases} a_n = f_{n-1}(\eta, \zeta) \\ p_1(n) = \frac{2n+1}{i\zeta} \\ p_0(n) = -1 \\ q(n) = \frac{1}{i\zeta} \left(P_{n+1}(\eta) - P_{n-1}(\eta) \right) \end{cases}$$

with initial values

$$\begin{cases} a_1 = \frac{1}{i\zeta} P_0(\eta) \\ a_2 = \frac{1}{i\zeta} \left(P_1(\eta) - \frac{1}{i\zeta} P_0(\eta) \right) \end{cases}$$

A solution to the homogeneous difference equation is (see Lemma 5.1)

$$y_l = i^{l-1} h_{l-1}^{(1)}(\zeta)$$

The final solution then is [1], (n = 3, 4, ...)

$$a_n = \left(\sum_{k=1}^{n-1} \prod_{j=1}^{k-1} \frac{p_0(j)y_j}{y_{j+2}} \left(\sum_{l=1}^{k-1} \frac{q(l)}{y_{l+2}} \left[\prod_{m=1}^{l} \frac{p_0(m)y_m}{y_{m+2}}\right]^{-1} + \frac{a_2}{y_2} - \frac{a_1}{y_1}\right) + \frac{a_1}{y_1}\right) y_n$$

Insert the explicit values, and we obtain

$$f_{l}(\eta,\zeta) = \left\{ \sum_{k=1}^{l} \frac{1}{\zeta h_{k-1}^{(1)}(\zeta) h_{k}^{(1)}(\zeta)} \left(-\sum_{n=1}^{k-1} h_{n}^{(1)}(\zeta) \frac{P_{n+1}(\eta) - P_{n-1}(\eta)}{i^{n}} + i h_{1}^{(1)}(\zeta) P_{0}(\eta) - h_{0}^{(1)}(\zeta) \left(P_{1}(\eta) + i \frac{1}{\zeta} P_{0}(\eta) \right) \right) - i \frac{P_{0}(\eta)}{\zeta h_{0}^{(1)}(\zeta)} \right\} i^{l} h_{l}^{(1)}(\zeta), \quad l = 2, 3, 4, \dots$$

This relation holds also for l = 0, 1, provided the sums with upper limit smaller than the lower limit are interpreted as zero.

We now simplify the sum in this expression.

$$S = -\sum_{n=1}^{k-1} i^{-n} h_n^{(1)}(\zeta) \left(P_{n+1}(\eta) - P_{n-1}(\eta) \right) + i h_1^{(1)}(\zeta) P_0(\eta) - h_0^{(1)}(\zeta) P_1(\eta)$$

$$= i h_1^{(1)}(\zeta) \left(P_2(\eta) - P_0(\eta) \right) + h_2^{(1)}(\zeta) \left(P_3(\eta) - P_1(\eta) \right) - i h_3^{(1)}(\zeta) \left(P_4(\eta) - P_2(\eta) \right)$$

$$+ \dots - i^{-k+2} h_{k-2}^{(1)}(\zeta) \left(P_{k-1}(\eta) - P_{k-3}(\eta) \right) - i^{-k+1} h_{k-1}^{(1)}(\zeta) \left(P_k(\eta) - P_{k-2}(\eta) \right)$$

$$+ i h_1^{(1)}(\zeta) P_0(\eta) - h_0^{(1)}(\zeta) P_1(\eta)$$

$$= -\left(h_0^{(1)}(\zeta) + h_2^{(1)}(\zeta) \right) P_1(\eta) + i \left(h_1^{(1)}(\zeta) + h_3^{(1)}(\zeta) \right) P_2(\eta)$$

$$+ \left(h_2^{(1)}(\zeta) + h_4^{(1)}(\zeta) \right) P_3(\eta) - i \left(h_3^{(1)}(\zeta) + h_5^{(1)}(\zeta) \right) P_4(\eta) + \dots$$

$$- i^{-k+2} \left(h_{k-2}^{(1)}(\zeta) + h_k^{(1)}(\zeta) \right) P_{k-1}(\eta) - i^{-k+1} \left(h_{k-1}^{(1)}(\zeta) + h_{k+1}^{(1)}(\zeta) \right) P_k(\eta)$$

$$+ i^{-k+2} h_k^{(1)}(\zeta) P_{k-1}(\eta) + i^{-k+1} h_{k+1}^{(1)}(\zeta) P_k(\eta)$$

The recursion relation $h_{l+1}^{(1)}(z) + h_{l-1}^{(1)}(z) = (2l+1)h_l^{(1)}(z)/z$ implies

$$S = -3\frac{h_1^{(1)}(\zeta)}{\zeta}P_1(\eta) + 5i\frac{h_2^{(1)}(\zeta)}{\zeta}P_2(\eta) + 7\frac{h_3^{(1)}(\zeta)}{\zeta}P_3(\eta) - 9i\frac{h_4^{(1)}(\zeta)}{\zeta}P_4(\eta) + \dots$$

$$- i^{-k+1}(2k+1)\frac{h_k^{(1)}(\zeta)}{\zeta}P_k(\eta) + i^{-k+2}h_k^{(1)}(\zeta)P_{k-1}(\eta) + i^{-k+1}h_{k+1}^{(1)}(\zeta)P_k(\eta)$$

$$= -\sum_{n=1}^k i^{-n+1}(2n+1)\frac{h_n^{(1)}(\zeta)}{\zeta}P_n(\eta) + i^{-k+2}h_k^{(1)}(\zeta)P_{k-1}(\eta) + i^{-k+1}h_{k+1}^{(1)}(\zeta)P_k(\eta)$$

which gives

$$f_{l}(\eta,\zeta) = \left\{ \sum_{k=1}^{l} \frac{1}{\zeta h_{k-1}^{(1)}(\zeta) h_{k}^{(1)}(\zeta)} \left(-\sum_{n=1}^{k} i^{-n+1} (2n+1) \frac{h_{n}^{(1)}(\zeta)}{\zeta} P_{n}(\eta) + i^{-k+2} h_{k}^{(1)}(\zeta) P_{k-1}(\eta) + i^{-k+1} h_{k+1}^{(1)}(\zeta) P_{k}(\eta) - i \frac{h_{0}^{(1)}(\zeta)}{\zeta} P_{0}(\eta) \right) - i \frac{P_{0}(\eta)}{\zeta h_{0}^{(1)}(\zeta)} \right\} i^{l} h_{l}^{(1)}(\zeta)$$

or

$$f_{l}(\eta,\zeta) = \left\{ \sum_{k=1}^{l} \frac{1}{\zeta h_{k-1}^{(1)}(\zeta) h_{k}^{(1)}(\zeta)} \left(-\sum_{n=0}^{k} i^{-n+1} (2n+1) \frac{h_{n}^{(1)}(\zeta)}{\zeta} P_{n}(\eta) + i^{-k+2} h_{k}^{(1)}(\zeta) P_{k-1}(\eta) + i^{-k+1} h_{k+1}^{(1)}(\zeta) P_{k}(\eta) \right) - i \frac{P_{0}(\eta)}{\zeta h_{0}^{(1)}(\zeta)} \right\} i^{l} h_{l}^{(1)}(\zeta)$$

This completes the lemma.

In conclusion, the functions $h_l^{\pm}(z)$ defined in (4.2) can be expressed in the function $h(\eta, \zeta)$ in (5.1). Specifically, we have

$$h_l^{\pm}(z) = kah_l(z/a, \pm ka)$$

6 Summary and explicit terms

This paper contains an evaluation of a non-trivial integral that occurs in the formulation of scattering by randomly distributed obstacles.

To summarize, the integral $I_l(z)$ in (1.1) has been solved and the solution outside the interval [-a, a] is a simple exponential function in kz, while inside the interval [-a, a], the solution can be found in a finite series of spherical waves. The finite sum of spherical waves depends on the two parameters kz and ka, or, more precisely, the parameter ka and a polynomial of the order l in the parameter z/a. Several equivalent solutions are presented in the paper, one of them is (l = 0, 1, 2, ...)

$$I_{l}(z) = \begin{cases} i^{l} e^{-ikz}, & z \leq -a \\ i^{1-l} kah_{0}^{(1)}(ka) P_{l-2[l/2]}(z/a) \\ + \sum_{n=0}^{[l/2]-1} (-1)^{n} kah_{l-2n-1}^{(1)}(ka) \left(P_{l-2n}(z/a) - P_{l-2n-2}(z/a) \right), & z \in [-a, a] \\ i^{-l} e^{ikz}, & z \geq a \end{cases}$$

The first integrals, l = 0, 1, 2, are of interest for low-frequency expansions. For l = 0 the integral is

$$I_0(z) = \begin{cases} e^{-ikz}, & z \le -a \\ e^{ika}, & z \in [-a, a] \\ e^{ikz}, & z \ge a \end{cases}$$

and for l=1 the result is

$$I_1(z) = \begin{cases} ie^{-ikz}, & z \le -a \\ -ie^{ika} \frac{z}{a}, & z \in [-a, a] \\ -ie^{ikz}, & z \ge a \end{cases}$$

For l=2 the result is

$$I_2(z) = \begin{cases} -e^{-ikz}, & z \le -a\\ e^{ika} \frac{(ka)^2 (3i + ka) - 3(i + ka)(kz)^2}{2(ka)^3}, & z \in [-a, a]\\ -e^{ikz}, & z \ge a \end{cases}$$

and we notice that the integral contains a polynomial in z/a of order l.

Moreover, the indefinite Fourier transform of $I_l(z)$ has also been investigated. More precisely, the integral, see (4.1)

$$\widehat{I}_{l}^{\pm}(z) = k \int_{z_{0}}^{z} I_{l}(t) e^{\pm ikt} dt, \quad z \ge z_{0}, \quad l = 0, 1, 2, \dots$$

is shown to have a solution expressed in spherical waves.

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