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Evaluation of some integrals relevant to multiple scattering by randomly distributed obstacles

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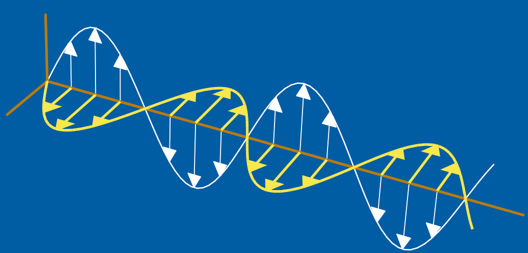

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Abstract

This paper analyzes and solves an integral and its indefinite Fourier transform of importance in multiple scattering problems of randomly distributed scatterers. The integrand contains a radiating spherical wave, and the two-dimensional domain of integration excludes a circular region of varying size. A solution of the integral in terms of radiating spherical waves is demonstrated. The method employs the Erdélyi operators, which leads to a recursion relation. This recursion relation is solved in terms of a finite sum of radiating spherical waves. The solution of the indefinite Fourier transform of the integral contains the indefinite Fourier transforms of the Legendre polynomials, which are solved by a closed formula.

1 Introduction

In recent years, the electromagnetic scattering problem by randomly distributed objects has been successfully formulated and solved. Some important contributions in the field are found in *e.g.*, [3–8, 10, 11, 13, 16–19, 21–25]. These references refer to various aspects of the topic, and more references can be found in these papers. The topic is also treated in several textbooks, see *e.g.*, [12, 14, 20], which can be consulted for a comprehensive treatment of the various multiple scattering theories.

Of critical importance for the solution of a specific scattering problem with hole-corrections (HC) is an integral of the form [9, 18, 20]

$$I_l(z) = \frac{k^2}{2\pi} \iint\limits_{\mathbb{R}^2} H(r-a) h_l^{(1)}(kr) P_l(\cos\theta)\, \mathrm{d}x\, \mathrm{d}y, \quad z \in \mathbb{R} \tag{1.1}$$

where $H(x)$ denotes the Heaviside function, $h_l^{(1)}(kr)$ the spherical Hankel function, and $P_l(x)$ the Legendre polynomial of order l, respectively. We have also adopted the spherical coordinates, $r = \sqrt{x^2+y^2+z^2}$ and θ ($\cos\theta = z/r$), and the wave number k. The domain of integration is the plane $z =$ constant, excluding the sphere of radius $a > 0$ at the center, see Figure 1. For a given value of $|z| \leq a$, the radius of the excluded circle is $\sqrt{a^2 - z^2}$. For $|z| \geq a$ the integration is the entire x-y plane. This integral, for a given $a > 0$, is a non-trivial function of $z \in \mathbb{R}$. To ensure convergence of the integral at infinity, we assume the wave number k has an arbitrarily small imaginary part. The explicit solution of this integral, as a function of z and the index $l = 0, 1, 2, \ldots$, is the aim of this paper, and the goal is to express the solutions in a form that is attractive from a numerical computation point of view.

The solution of the integral $I_l(z)$ is developed in Sections 2 and 3. The indefinite Fourier transform of $I_l(z)$ is also essential for a successful solution of the multiple scattering problem with hole-corrections, and this analysis is found in Sections 4 and 5. The paper is concluded with a short summary in Section 6.

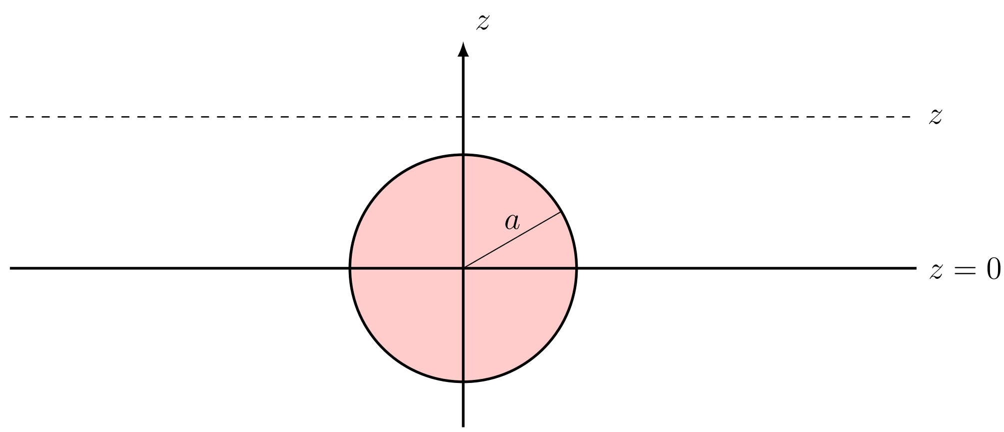

Figure 1: The geometry of the integration domain — the plane z = constant (dotted line), and the exclusion volume — the sphere of radius a located at the origin (in gray).

2 The integral $I_l(z)$

Rewrite the integral $I_l(z)$ in (1.1) in cylindrical coordinates and perform the integration in the azimuthal angle. We get from (1.1)

$$I_l(z) = k^2 \int_{h(z)}^{\infty} h_l^{(1)}\left(k\sqrt{\rho^2+z^2}\right) P_l\left(z/\sqrt{\rho^2+z^2}\right) \rho \, \mathrm{d}\rho, \quad z \in \mathbb{R} \tag{2.1}$$

where

$$h(z) = \begin{cases} \sqrt{a^2-z^2}, & -a \le z \le a \\ 0, & |z| > a \end{cases}$$

From the parity of the Legendre polynomials, $P_l(-x) = (-1)^l P_l(x)$, we see that also $I_l(-z) = (-1)^l I_l(z)$. Thus, it suffices to evaluate the integral for $z > 0$. In particular, $I_l(0) = 0$, if l is an odd integer. From (2.1) we also easily compute the integral for $l = 0$, *viz.*,

$$I_0(z) = \begin{cases} \mathrm{e}^{-\mathrm{i}kz}, & z \le -a \\ \mathrm{i}ka h_0^{(1)}(ka) = \mathrm{e}^{\mathrm{i}ka}, & -a \le z \le a \\ \mathrm{e}^{\mathrm{i}kz}, & z \ge a \end{cases}$$

2.1 Solution outside the interval $[-a, a]$

In the interval $z > a$, the integral is evaluated with the use of the transformation of the outgoing scalar spherical wave in terms of planar waves [2, p. 180], *i.e.*, for a general value of $z \neq 0$

$$h_l^{(1)}\left(k\sqrt{\rho^2+z^2}\right)P_l\left(\pm|z|/\sqrt{\rho^2+z^2}\right)$$
$$=\frac{\mathrm{i}^{-l}}{2\pi}\iint\limits_{\mathbb{R}^2}P_l\left(\pm k_z/k\right)\mathrm{e}^{\mathrm{i}\boldsymbol{k}_\mathrm{t}\cdot\boldsymbol{\rho}+\mathrm{i}k_z|z|}\frac{k}{k_z}\frac{\mathrm{d}k_x\,\mathrm{d}k_y}{k^2},\quad z\gtrless 0$$

where $\boldsymbol{\rho}=x\hat{\boldsymbol{x}}+y\hat{\boldsymbol{y}}$, $\boldsymbol{k}_\mathrm{t}=k_x\hat{\boldsymbol{x}}+k_y\hat{\boldsymbol{y}}$, $k_\mathrm{t}=|\boldsymbol{k}_\mathrm{t}|$, and k_z is defined by

$$k_z=\left(k^2-k_\mathrm{t}^2\right)^{1/2}=\begin{cases}\sqrt{k^2-k_\mathrm{t}^2}\text{ for }k_\mathrm{t}<k\\ \mathrm{i}\sqrt{k_\mathrm{t}^2-k^2}\text{ for }k_\mathrm{t}>k\end{cases}$$

For $z>a$, we get from (1.1)

$$I_l(z)=\frac{k^2}{2\pi}\iint\limits_{\mathbb{R}^2}\frac{\mathrm{i}^{-l}}{2\pi}\left(\iint\limits_{\mathbb{R}^2}P_l\left(k_z/k\right)\mathrm{e}^{\mathrm{i}\boldsymbol{k}_\mathrm{t}\cdot\boldsymbol{\rho}+\mathrm{i}k_z|z|}\frac{k}{k_z}\frac{\mathrm{d}k_x\,\mathrm{d}k_y}{k^2}\right)\mathrm{d}x\,\mathrm{d}y$$
$$=\mathrm{i}^{-l}\iint\limits_{\mathbb{R}^2}P_l\left(k_z/k\right)\mathrm{e}^{\mathrm{i}k_z|z|}\delta(\boldsymbol{k}_\mathrm{t})\frac{k}{k_z}\,\mathrm{d}k_x\,\mathrm{d}k_y=\mathrm{i}^{-l}\mathrm{e}^{\mathrm{i}kz}$$

by orthogonality or completeness of the planar waves.[1] This implies that the integral for $z>a$ is

$$I_l(z)=\mathrm{i}^{-l}\mathrm{e}^{\mathrm{i}kz},\quad z>a$$

and consequently, by parity, or analogous calculations

$$I_l(z)=\mathrm{i}^{l}\mathrm{e}^{-\mathrm{i}kz},\quad z<-a$$

We observe that the integral outside the interval $[-a,a]$ is not singular as $a\to 0$. In fact, the module is constant 1.

3 Solution of the integral $I_l(\eta)$, $-a\leq z\leq a$

We have already obtained a solution of the integral in the interval $|z|>a$, and we now concentrate on finding a solution of the integral in the interval $-a\leq z\leq a$.

The Erdélyi operators $\mathcal{Y}_n^m$ in Ref. 12 are instrumental in finding a closed formula for the integral $I_l(z)$. From [12, Th. 3.13], we have the following very useful result:

$$D\left(h_l^{(1)}(kr)P_l(\cos\theta)\right)=\frac{l+1}{2l+1}h_{l+1}^{(1)}(kr)P_{l+1}(\cos\theta)-\frac{l}{2l+1}h_{l-1}^{(1)}(kr)P_{l-1}(\cos\theta)$$

where $D=-k^{-1}(\partial/\partial z)$. The D operator and the Erdélyi operators are related by $\mathcal{Y}_1^0=\sqrt{\frac{3}{4\pi}}D_1^0=\sqrt{\frac{3}{4\pi}}D$.

[1]To ensure convergence of the integral at infinity, assume the wave number k has an arbitrary small, positive imaginary part.

Apply the differential operator D to the integral $I_l(z)$ in (2.1), and use the relation above. We obtain, since $h'(z)h(z) = -z$, the following recursion relation:[2]

$$DI_l(z) = -kzh_l^{(1)}(ka)P_l(z/a) + \frac{l+1}{2l+1}I_{l+1}(z) - \frac{l}{2l+1}I_{l-1}(z), \quad -a \leq z \leq a$$

with initial condition $I_0(z) = \mathrm{i}kah_0^{(1)}(ka)$.

In the dimensionless variables $\eta = kz$ and $\xi = ka > 0$, this leads to the recursion relation, $l = 0, 1, 2, \ldots$ (note the mild change in notation)

$$I_{l+1}(\eta) = \frac{2l+1}{l+1}\xi h_l^{(1)}(\xi)\frac{\eta}{\xi}P_l(\eta/\xi) - \frac{2l+1}{l+1}\frac{\mathrm{d}}{\mathrm{d}\eta}I_l(\eta) + \frac{l}{l+1}I_{l-1}(\eta), \quad -\xi \leq \eta \leq \xi$$

The recursion relation is conveniently put in a more generic form by introducing the variable $x = \eta/\xi \in [-1, 1]$. The dependent variable is now x, and ξ is a parameter. Retaining the same notation for the integral, but with a change of the independent variable, we get

$$I_{l+1}(x) = \frac{2l+1}{l+1}\xi h_l^{(1)}(\xi)xP_l(x) - \frac{2l+1}{\xi(l+1)}I_l'(x) + \frac{l}{l+1}I_{l-1}(x), \quad -1 \leq x \leq 1$$

The following proposition states the surprisingly simple and elegant solution of this recursion relation.

Proposition 3.1. *The recursion relation*

$$I_{l+1}(x) = \frac{2l+1}{l+1}\xi h_l^{(1)}(\xi)xP_l(x) - \frac{2l+1}{\xi(l+1)}I_l'(x) + \frac{l}{l+1}I_{l-1}(x), \quad l = 0, 1, 2, \ldots \tag{3.1}$$

with initial condition

$$I_0(z) = \mathrm{i}\xi h_0^{(1)}(\xi)$$

has the solution

$$\begin{aligned} I_l(x) &= -\xi h_{l+1}^{(1)}(\xi)P_l(x) \\ &\quad + \sum_{k=0}^{[l/2]}(-1)^k\left(\xi h_{l+1-2k}^{(1)}(\xi) + \xi h_{l-1-2k}^{(1)}(\xi)\right)P_{l-2k}(x), \quad l = 0, 1, 2, \ldots \end{aligned} \tag{3.2}$$

[2]Outside the interval $z \in [-a, a]$ the recursion relation reads

$$I_{l+1}(z) = \frac{2l+1}{l+1}DI_l(z) + \frac{l}{l+1}I_{l-1}(z), \quad I_0(z) = \mathrm{e}^{\mathrm{i}kz} \quad z \geq a$$

which is easily solved by induction over the integer l. The result is

$$I_l(z) = \mathrm{i}^{-l}\mathrm{e}^{\mathrm{i}kz}, \quad z \geq a$$

in agreement with the result above.

Proof. We prove the proposition by induction over the integer l. The recursion relation (3.2) is true for $l = 0$, due to the properties of the spherical Hankel functions [15, 10.16.1]. We have from (3.2)

$$I_0(x) = \xi h_{-1}^{(1)}(\xi) = \xi \left(\frac{\pi}{2\xi}\right)^{1/2} H_{-1/2}^{(1)}(\xi) = \mathrm{i}\xi \left(\frac{\pi}{2\xi}\right)^{1/2} H_{1/2}^{(1)}(\xi) = \mathrm{i}\xi h_0^{(1)}(\xi)$$

Now assume the solution (3.2) holds for all integers less than or equal to l, and we want to prove that it holds for $l+1$. We have from (3.1) and the induction assumption

$$\begin{aligned}
I_{l+1}(x) &= \frac{2l+1}{l+1}\xi h_l^{(1)}(\xi) x P_l(x) - \frac{2l+1}{\xi(l+1)} I_l'(x) + \frac{l}{l+1} I_{l-1}(x) \\
&= \xi h_l^{(1)}(\xi) P_{l+1}(x) + \frac{2l+1}{\xi(l+1)} \xi h_{l+1}^{(1)}(\xi) P_l'(x) \\
&- \frac{2l+1}{\xi(l+1)} \sum_{k=0}^{[l/2]} (-1)^k \left(\xi h_{l+1-2k}^{(1)}(\xi) + \xi h_{l-1-2k}^{(1)}(\xi)\right) P_{l-2k}'(x) \\
&+ \frac{l}{l+1} \sum_{k=0}^{[(l-1)/2]} (-1)^k \left(\xi h_{l-2k}^{(1)}(\xi) + \xi h_{l-2-2k}^{(1)}(\xi)\right) P_{l-1-2k}(x)
\end{aligned}$$

where we used the following recursion relation for the Legendre polynomials:

$$(2l+1) x P_l(x) = (l+1) P_{l+1}(x) + l P_{l-1}(x)$$

We conclude that $I_{l+1}(x)$ is a polynomial in x of the order $l+1$, and therefore can be expanded in a series of Legendre polynomials. The form is

$$I_{l+1}(x) = \sum_{n=0}^{[(l+1)/2]} a_n P_{l+1-2n}(x)$$

where a_n depends on l and ξ. The coefficients a_n are determined by orthogonality of the Legendre polynomials.

$$a_n = \frac{2l+3-4n}{2} \int_{-1}^{1} I_{l+1}(x) P_{l+1-2n}(x) \, \mathrm{d}x$$

The first coefficient is special.

$$a_0 = \xi h_l^{(1)}(\xi) = -\xi h_{l+2}^{(1)}(\xi) + \left(\xi h_{l+2}^{(1)}(\xi) + \xi h_l^{(1)}(\xi)\right)$$

Proceed in the same way with the remaining coefficients, $n = 1, 2, \ldots, [(l+1)/2]$.

$$\begin{aligned}
a_n &= \frac{2l+1}{l+1} \frac{2l+3-4n}{2} h_{l+1}^{(1)}(\xi) I_{l,l+1-2n} \\
&- \frac{2l+1}{l+1} \frac{2l+3-4n}{2} \sum_{k=0}^{[l/2]} (-1)^k \left(h_{l+1-2k}^{(1)}(\xi) + h_{l-1-2k}^{(1)}(\xi)\right) I_{l-2k,l+1-2n} \\
&+ \frac{l}{l+1} \sum_{k=0}^{[(l-1)/2]} (-1)^k \left(\xi h_{l-2k}^{(1)}(\xi) + \xi h_{l-2-2k}^{(1)}(\xi)\right) \delta_{k,n-1}
\end{aligned}$$

where we used the notion

$$I_{k,n} = \int_{-1}^{1} P_k'(x)P_n(x)\ \mathrm{d}x = \begin{cases} 0, & 0 \le k \le n \\ 1-(-1)^{k+n}, & 0 \le n < k \end{cases}$$

Use this result, and the following recursion relation for the spherical Hankel functions:

$$(2l+1)h_l^{(1)}(\xi) = \xi h_{l+1}^{(1)}(\xi) + \xi h_{l-1}^{(1)}(\xi) \tag{3.3}$$

We get

$$\begin{aligned} a_n &= \frac{2l+1}{l+1}(2l+3-4n)\left(h_{l+1}^{(1)}(\xi) - \underbrace{\sum_{k=0}^{n-1}(-1)^k\left(h_{l+1-2k}^{(1)}(\xi)+h_{l-1-2k}^{(1)}(\xi)\right)}_{=h_{l+1}^{(1)}(\xi)+(-1)^{n-1}h_{l+1-2n}^{(1)}(\xi)}\right) \\ &+ \frac{l}{l+1}(-1)^{n-1}\left(\xi h_{l+2-2n}^{(1)}(\xi)+\xi h_{l-2n}^{(1)}(\xi)\right) \\ &= \frac{2l+1}{l+1}(-1)^n(2l+3-4n)h_{l+1-2n}^{(1)}(\xi) - \frac{l}{l+1}(-1)^n\left(\xi h_{l+2-2n}^{(1)}(\xi)+\xi h_{l-2n}^{(1)}(\xi)\right) \\ &= (-1)^n\left(\xi h_{l+2-2n}^{(1)}(\xi)+\xi h_{l-2n}^{(1)}(\xi)\right) \end{aligned}$$

Collecting the results gives

$$\begin{aligned} I_{l+1}(x) &= \sum_{n=0}^{[(l+1)/2]} a_n P_{l+1-2n}(x) \\ &= -\xi h_{l+2}^{(1)}(\xi)P_{l+1}(x) + \sum_{n=0}^{[(l+1)/2]}(-1)^n\left(\xi h_{l+2-2n}^{(1)}(\xi)+\xi h_{l-2n}^{(1)}(\xi)\right)P_{l+1-2n}(x) \end{aligned}$$

which is the statement (3.2) for $l+1$, and the proposition is proved. □

Alternative expressions of the integral $I(z)$ in the interval $z \in [-a, a]$ can be found. The following corollary shows some.

Corollary 3.1. *The integral $I(z)$ in Proposition 3.1 has the following alternative expressions:*

$$I_l(x) = -\xi h_{l+1}^{(1)}(\xi)P_l(x) + \sum_{k=0}^{[l/2]}(-1)^k(2l-4k+1)h_{l-2k}^{(1)}(\xi)P_{l-2k}(x), \quad l=0,1,2,\ldots \tag{3.4}$$

and

$$\begin{aligned} I_l(x) &= \mathrm{i}^{1-l}\xi h_0^{(1)}(\xi)P_{l-2[l/2]}(x) \\ &+ \sum_{k=0}^{[l/2]-1}(-1)^k\xi h_{l-2k-1}^{(1)}(\xi)\left(P_{l-2k}(x)-P_{l-2k-2}(x)\right), \quad l=0,1,2,\ldots \end{aligned} \tag{3.5}$$

and

$$
\begin{aligned}
I_l(x) &= \mathrm{i}^{1-l}\xi h_0^{(1)}(\xi)P_{l-2[l/2]}(x) \\
&\quad - \sum_{k=0}^{[l/2]-1}(-1)^k\xi h_{l-2k-1}^{(1)}(\xi)\frac{2l-4k-1}{(l-2k-1)(l-2k)}P'_{l-2k-1}(x), \quad l=0,1,2,\ldots
\end{aligned} \tag{3.6}
$$

where the two last sums are zero for $l=0,1$.

Proof. The solution in (3.4) is equivalent to (3.2), which is easily seen since the spherical Hankel functions $h_l^{(1)}(\xi)$ satisfy the recursion relation (3.3). The representation in (3.5) is simply a rearrangement of the sum in (3.2). We obtain from (3.2) $(l=0,1,2,\ldots)$

$$
\begin{aligned}
I_l(x) &= \sum_{k=0}^{[l/2]-1}(-1)^k\xi h_{l-1-2k}^{(1)}(\xi)\left(P_{l-2k}(x)-P_{l-2-2k}(x)\right) \\
&\qquad + (-1)^{[l/2]}\xi h_{l-1-2[l/2]}^{(1)}(\xi)P_{l-2[l/2]}(x) \\
&= \sum_{k=0}^{[l/2]-1}(-1)^k\xi h_{l-1-2k}^{(1)}(\xi)\left(P_{l-2k}(x)-P_{l-2-2k}(x)\right) \\
&\qquad + \mathrm{i}^{1-l}\xi h_0^{(1)}(\xi)P_{l-2[l/2]}(x)
\end{aligned}
$$

where we used [15, 10.16.1]

$$
h_{-1}^{(1)}(\xi) = \mathrm{i}h_0^{(1)}(\xi)
$$

Finally, the relation (3.6) from (3.5) with the use of the recursion relation

$$
l(l+1)\left(P_{l+1}(x)-P_{l-1}(x)\right) = -(2l+1)(1-x^2)P'_l(x)
$$

□

In the original variables z and a, we have

$$
\begin{aligned}
I_l(z) &= \mathrm{i}^{1-l}kah_0^{(1)}(ka)P_{l-2[l/2]}(z/a) \\
&\quad + \sum_{n=0}^{[l/2]-1}(-1)^n kah_{l-2n-1}^{(1)}(ka)\left(P_{l-2n}(z/a)-P_{l-2n-2}(z/a)\right), \quad l=0,1,2,\ldots
\end{aligned}
$$

or

$$
\begin{aligned}
I_l(z) &= -kah_{l+1}^{(1)}(ka)P_l(z/a) \\
&\qquad + \sum_{k=0}^{[l/2]}(-1)^k(2l-4k+1)h_{l-2k}^{(1)}(ka)P_{l-2k}(z/a), \quad l=0,1,2,\ldots
\end{aligned}
$$

and we see that the integral $I_l(z)$ can be written as a finite sum of spherical waves (except the first term). The most singular term in powers of ka is of the order $(ka)^{1-l}$ (order $O(1)$ if $l=0$), which is most easily seen from the representation in (3.5).

4 Fourier transform of $I_l(z)$

The indefinite Fourier transform of the function $I_l(z)$ has also importance in the analysis of [9]. More specifically, our goal in this section is to compute

$$\widehat{I}_l^{\pm}(z) = k \int_{z_0}^{z} I_l(t) \mathrm{e}^{\pm \mathrm{i}kt} \, \mathrm{d}t, \quad z \geq z_0, \quad l = 0, 1, 2, \ldots \tag{4.1}$$

where z_0 is a fixed number such that $z_0 < -a$.

The function $I_l(t)$ has explicit forms in the three intervals $[z_0, -a]$, $(-a, a)$, and $[a, \infty)$. The explicit forms are:

$$I_l(t) = \mathrm{i}^l \mathrm{e}^{-\mathrm{i}kt}, \quad t \leq -a$$

and in the interval $t \in (-a, a)$ as a finite sum of spherical waves

$$\begin{aligned} I_l(t) &= \mathrm{i}^{1-l} ka h_0^{(1)}(ka) P_{l-2[l/2]}(t/a) \\ &+ \sum_{n=0}^{[l/2]-1} (-1)^n ka h_{l-2n-1}^{(1)}(ka) \left(P_{l-2n}(t/a) - P_{l-2n-2}(t/a) \right) \end{aligned}$$

In the interval $t \geq a$

$$I_l(t) = \mathrm{i}^{-l} \mathrm{e}^{\mathrm{i}kt}$$

To compute the indefinite Fourier transform we need to calculate the function

$$h_l^{\pm}(z) = k \int_{-a}^{z} P_l(t/a) \mathrm{e}^{\pm \mathrm{i}kt} \, \mathrm{d}t = ka \int_{-1}^{z/a} P_l(t) \mathrm{e}^{\pm \mathrm{i}kat} \, \mathrm{d}t, \quad |z| \leq a \tag{4.2}$$

For $z = a$ the integral is a spherical Bessel function, *viz.*,

$$h_l^{\pm}(a) = k \int_{-a}^{a} P_l(t/a) \mathrm{e}^{\pm \mathrm{i}kt} \, \mathrm{d}t = ka \int_{-1}^{1} P_l(t) \mathrm{e}^{\pm \mathrm{i}kat} \, \mathrm{d}t = 2ka(\pm \mathrm{i})^l j_l(ka)$$

We divide the interval $[z_0, z]$ in three parts. In the interval $z_0 \leq z < -a$, we have

$$\widehat{I}_l^{\pm}(z) = \mathrm{i}^l k \int_{z_0}^{z} \mathrm{e}^{\mathrm{i}(\pm 1 - 1)kt} \, \mathrm{d}t = \mathrm{i}^l \begin{cases} k(z - z_0) \\ \frac{1}{2\mathrm{i}} \left(\mathrm{e}^{-2\mathrm{i}kz_0} - \mathrm{e}^{-2\mathrm{i}kz} \right) \end{cases}$$

and in the interval $-a < z < a$, we have

$$\begin{aligned} \widehat{I}_l^{\pm}(z) &= \mathrm{i}^l \begin{cases} k(-a - z_0) \\ \frac{1}{2\mathrm{i}} \left(\mathrm{e}^{-2\mathrm{i}kz_0} - \mathrm{e}^{2\mathrm{i}ka} \right) \end{cases} + \mathrm{i}^{1-l} ka h_0^{(1)}(ka) h_{l-2[l/2]}^{\pm}(z) \\ &+ \sum_{n=0}^{[l/2]-1} (-1)^n ka h_{l-2n-1}^{(1)}(ka) \left(h_{l-2n}^{\pm}(z) - h_{l-2n-2}^{\pm}(z) \right) \end{aligned}$$

and in the interval $a < z$, we have

$$\begin{aligned}\widehat{I}_l^{\pm}(z) = \mathrm{i}^l &\begin{cases} k(-a - z_0) \\ \frac{1}{2\mathrm{i}}\left(\mathrm{e}^{-2\mathrm{i}kz_0} - \mathrm{e}^{2\mathrm{i}ka}\right)\end{cases} + \mathrm{i}^{1-l} 2(ka)^2 h_0^{(1)}(ka)(\pm\mathrm{i})^{l-2[l/2]} j_{l-2[l/2]}(ka) \\ &+ 2(ka)^2(\pm\mathrm{i})^l \sum_{n=0}^{[l/2]-1} h_{l-2n-1}^{(1)}(ka)\left(j_{l-2n}(ka) + j_{l-2n-2}(ka)\right) \\ &+ \mathrm{i}^{-l}\begin{cases} \frac{1}{2\mathrm{i}}\left(\mathrm{e}^{2\mathrm{i}kz} - \mathrm{e}^{2\mathrm{i}ka}\right) \\ k(z-a)\end{cases}\end{aligned}$$

5 Indefinite integral of Legendre polynomials

It remains to find an effective method to compute the functions $h_l^{\pm}(z)$ in (4.2). To this end, define

$$h_l(\eta,\zeta) = \int_{-1}^{\eta} P_l(t)\mathrm{e}^{\mathrm{i}\zeta t}\,\mathrm{d}t, \quad |\eta| \leq 1 \tag{5.1}$$

We see that $h_l(1,\zeta) = 2\mathrm{i}^l j_l(\zeta)$. In terms of the functions $h_l(\eta,\zeta)$, the functions $h_l^{\pm}(z)$ are

$$h_l^{\pm}(z) = ka h_l(z/a, \pm ka)$$

Our ambition in this section is to find an efficient method to compute the integrals in (5.1). We express the function $h_l(\eta,\zeta)$ as a recursion relation.

5.1 Solution by recursion

The following recursion relation of Legendre polynomials is useful:

$$P_l(t) = \frac{1}{2l+1}\left(P'_{l+1}(t) - P'_{l-1}(t)\right)$$

Integration by parts then implies ($P_l(-1) = (-1)^l$)

$$\begin{aligned}h_l(\eta,\zeta) &= \int_{-1}^{\eta} P_l(t)\mathrm{e}^{\mathrm{i}\zeta t}\,\mathrm{d}t = \frac{1}{2l+1}\int_{-1}^{\eta}\left(P'_{l+1}(t) - P'_{l-1}(t)\right)\mathrm{e}^{\mathrm{i}\zeta t}\,\mathrm{d}t \\ &= \frac{1}{2l+1}\left(P_{l+1}(\eta) - P_{l-1}(\eta)\right)\mathrm{e}^{\mathrm{i}\zeta\eta} - \frac{\mathrm{i}\zeta}{2l+1}\left(h_{l+1}(\eta,\zeta) - h_{l-1}(\eta,\zeta)\right)\end{aligned}$$

or solving for $h_{l+1}(\eta,\zeta)$

$$h_{l+1}(\eta,\zeta) = \frac{1}{\mathrm{i}\zeta}\left(P_{l+1}(\eta) - P_{l-1}(\eta)\right)\mathrm{e}^{\mathrm{i}\zeta\eta} - \frac{2l+1}{\mathrm{i}\zeta} h_l(\eta,\zeta) + h_{l-1}(\eta,\zeta), \quad l = 1,2,3,\ldots$$

The functions $h_l(\eta,\zeta)$ can therefore be found by iteration with starting values

$$h_0(\eta,\zeta) = \frac{1}{\mathrm{i}\zeta}\left(\mathrm{e}^{\mathrm{i}\zeta\eta} - \mathrm{e}^{-\mathrm{i}\zeta}\right) = \frac{1}{\mathrm{i}\zeta}P_0(\eta)\mathrm{e}^{\mathrm{i}\zeta\eta} + h_0^{(2)}(\zeta) = \eta h_0^{(1)}(\zeta\eta) + h_0^{(2)}(\zeta)$$

and

$$h_1(\eta,\zeta) = \frac{1}{\mathrm{i}\zeta}\left(\eta \mathrm{e}^{\mathrm{i}\zeta\eta} + \mathrm{e}^{-\mathrm{i}\zeta}\right) + \frac{1}{\zeta^2}\left(\mathrm{e}^{\mathrm{i}\zeta\eta} - \mathrm{e}^{-\mathrm{i}\zeta}\right)$$
$$= \frac{1}{\mathrm{i}\zeta}\left(P_1(\eta) - \frac{1}{\mathrm{i}\zeta}P_0(\eta)\right)\mathrm{e}^{\mathrm{i}\zeta\eta} + \mathrm{i}h_1^{(2)}(\zeta) = \mathrm{i}\eta^2 h_1^{(1)}(\zeta\eta) + \mathrm{i}h_1^{(2)}(\zeta)$$

To find the general solution to this recursion scheme, we start by solving the homogeneous difference equation.

Lemma 5.1. *The solution to the homogeneous difference equation*

$$a_{l+1} + \frac{2l+1}{\mathrm{i}\zeta}a_l - a_{l-1} = 0, \quad l = 1,2,3,\dots$$

given the initial values a_0 and a_1 is

$$a_l = -\frac{\zeta^2}{2\mathrm{i}}\left(a_0 h_0^{(2)\prime}(\zeta) - \mathrm{i}a_1 h_0^{(2)}(\zeta)\right)\mathrm{i}^l h_l^{(1)}(\zeta)$$
$$+ \frac{\zeta^2}{2\mathrm{i}}\left(a_0 h_0^{(1)\prime}(\zeta) - \mathrm{i}a_1 h_0^{(1)}(\zeta)\right)\mathrm{i}^l h_l^{(2)}(\zeta), \quad l = 2,3,4,\dots$$

Proof. Two linearly independent solutions to the homogeneous difference equation in the lemma are $\mathrm{i}^l h_l^{(1)}(\zeta)$ and $\mathrm{i}^l h_l^{(2)}(\zeta)$, which is easily proved by the recursion relation $f_{l+1}(z) - (2l+1)f_l(z)/z + f_{l-1}(z) = 0$, where $f_l(z)$ is any spherical Bessel or Hankel function. The general solution therefore is

$$a_l = c_1\mathrm{i}^l h_l^{(1)}(\zeta) + c_2\mathrm{i}^l h_l^{(2)}(\zeta), \quad l = 2,3,4,\dots$$

where c_1 and c_2 are constants determined by the starting values a_0 and a_1. Explicitly, we get

$$\begin{cases} c_1 h_0^{(1)}(\zeta) + c_2 h_0^{(2)}(\zeta) = a_0 \\ c_1\mathrm{i}h_1^{(1)}(\zeta) + c_2\mathrm{i}h_1^{(2)}(\zeta) = a_1 \end{cases}$$

with solution

$$\begin{cases} c_1 = -\dfrac{\zeta^2}{2\mathrm{i}}\left(a_0 h_0^{(2)\prime}(\zeta) - \mathrm{i}a_1 h_0^{(2)}(\zeta)\right) \\ c_2 = \dfrac{\zeta^2}{2\mathrm{i}}\left(a_0 h_0^{(1)\prime}(\zeta) - \mathrm{i}a_1 h_0^{(1)}(\zeta)\right) \end{cases}$$

where we used the Wronskian of the spherical Hankel functions.

$$h_n^{(2)}(z)h_n^{(1)\prime}(z) - h_n^{(2)\prime}(z)h_n^{(1)}(z) = \frac{2\mathrm{i}}{z^2}$$

and $h_0^{(1,2)\prime}(z) = -h_1^{(1,2)}(z)$. This completes the proof of the lemma. □

We are now ready for the solution to the inhomogeneous difference equation in $h_l(\eta,\zeta)$ above. We formulate this as a lemma.

Lemma 5.2. *Define an iteration scheme by*

$$h_{l+1}(\eta,\zeta) = \frac{1}{\mathrm{i}\zeta}\left(P_{l+1}(\eta) - P_{l-1}(\eta)\right)\mathrm{e}^{\mathrm{i}\zeta\eta} - \frac{2l+1}{\mathrm{i}\zeta}h_l(\eta,\zeta) + h_{l-1}(\eta,\zeta), \quad l = 1,2,3,\ldots$$

with starting values

$$h_0(\eta,\zeta) = \eta h_0^{(1)}(\zeta\eta) + h_0^{(2)}(\zeta)$$

and

$$h_1(\eta,\zeta) = \mathrm{i}\left(\eta^2 h_1^{(1)}(\zeta\eta) + h_1^{(2)}(\zeta)\right)$$

The solution is

$$h_l(\eta,\zeta) = f_l(\eta,\zeta)\mathrm{e}^{\mathrm{i}\zeta\eta} + \mathrm{i}^l h_l^{(2)}(\zeta), \quad l = 0,1,2,3,\ldots$$

where

$$\begin{aligned} f_l(\eta,\zeta) = \mathrm{i}^l h_l^{(1)}(\zeta)\Bigg\{\sum_{k=1}^{l} \frac{1}{\zeta h_{k-1}^{(1)}(\zeta) h_k^{(1)}(\zeta)}\Bigg(&-\sum_{n=0}^{k} \mathrm{i}^{-n+1}(2n+1)\frac{h_n^{(1)}(\zeta)}{\zeta}P_n(\eta) \\ &+ \mathrm{i}^{-k+2}h_k^{(1)}(\zeta)P_{k-1}(\eta) + \mathrm{i}^{-k+1}h_{k+1}^{(1)}(\zeta)P_k(\eta)\Bigg) - \mathrm{i}\frac{P_0(\eta)}{\zeta h_0^{(1)}(\zeta)}\Bigg\}, \quad l = 0,1,2,\ldots \end{aligned}$$

Proof. We first subtract the part of the solution that contains the spherical Hankel function of the second kind $h_l^{(2)}(\zeta)$ and the exponential function $\mathrm{e}^{\mathrm{i}\zeta\eta}$. To this end, let $h_l(\eta,\zeta) = f_l(\eta,\zeta)\mathrm{e}^{\mathrm{i}\zeta\eta} + \mathrm{i}^l h_l^{(2)}(\zeta)$. The recursion relation for $f_l(\eta,\zeta)$ is easily found by the use of the recursion relation $h_{l+1}^{(2)}(z) = (2l+1)h_l^{(2)}(z)/z - h_{l-1}^{(2)}(z)$. We get the new difference equation

$$f_{l+1}(\eta,\zeta) = \frac{1}{\mathrm{i}\zeta}\left(P_{l+1}(\eta) - P_{l-1}(\eta)\right) - \frac{2l+1}{\mathrm{i}\zeta}f_l(\eta,\zeta) + f_{l-1}(\eta,\zeta), \quad l = 1,2,3,\ldots$$

with starting values

$$f_0(\eta,\zeta) = \frac{1}{\mathrm{i}\zeta}P_0(\eta)$$

and

$$f_1(\eta,\zeta) = \frac{1}{\mathrm{i}\zeta}\left(P_1(\eta) - \frac{1}{\mathrm{i}\zeta}P_0(\eta)\right)$$

To simplify the notation, we put the difference equation in a standard form [1].

$$a_{n+2} + p_1(n)a_{n+1} + p_0(n)a_n = q(n), \quad n = 1,2,\ldots$$

where

$$\begin{cases} a_n = f_{n-1}(\eta,\zeta) \\ p_1(n) = \dfrac{2n+1}{\mathrm{i}\zeta} \\ p_0(n) = -1 \\ q(n) = \dfrac{1}{\mathrm{i}\zeta}\left(P_{n+1}(\eta) - P_{n-1}(\eta)\right) \end{cases}$$

with initial values

$$\begin{cases} a_1 = \dfrac{1}{\mathrm{i}\zeta} P_0(\eta) \\ a_2 = \dfrac{1}{\mathrm{i}\zeta} \left(P_1(\eta) - \dfrac{1}{\mathrm{i}\zeta} P_0(\eta) \right) \end{cases}$$

A solution to the homogeneous difference equation is (see Lemma 5.1)

$$y_l = \mathrm{i}^{l-1} h_{l-1}^{(1)}(\zeta)$$

The final solution then is [1], $(n = 3, 4, \ldots)$

$$a_n = \left(\sum_{k=1}^{n-1} \prod_{j=1}^{k-1} \frac{p_0(j) y_j}{y_{j+2}} \left(\sum_{l=1}^{k-1} \frac{q(l)}{y_{l+2}} \left[\prod_{m=1}^{l} \frac{p_0(m) y_m}{y_{m+2}} \right]^{-1} + \frac{a_2}{y_2} - \frac{a_1}{y_1} \right) + \frac{a_1}{y_1} \right) y_n$$

Insert the explicit values, and we obtain

$$\begin{aligned} f_l(\eta, \zeta) = \Bigg\{ \sum_{k=1}^{l} \frac{1}{\zeta h_{k-1}^{(1)}(\zeta) h_k^{(1)}(\zeta)} \Bigg(&- \sum_{n=1}^{k-1} h_n^{(1)}(\zeta) \frac{P_{n+1}(\eta) - P_{n-1}(\eta)}{\mathrm{i}^n} + \mathrm{i} h_1^{(1)}(\zeta) P_0(\eta) \\ &- h_0^{(1)}(\zeta) \left(P_1(\eta) + \mathrm{i} \frac{1}{\zeta} P_0(\eta) \right) \Bigg) - \mathrm{i} \frac{P_0(\eta)}{\zeta h_0^{(1)}(\zeta)} \Bigg\} \mathrm{i}^l h_l^{(1)}(\zeta), \quad l = 2, 3, 4, \ldots \end{aligned}$$

This relation holds also for $l = 0, 1$, provided the sums with upper limit smaller than the lower limit are interpreted as zero.

We now simplify the sum in this expression.

$$\begin{aligned} S &= -\sum_{n=1}^{k-1} \mathrm{i}^{-n} h_n^{(1)}(\zeta) \left(P_{n+1}(\eta) - P_{n-1}(\eta) \right) + \mathrm{i} h_1^{(1)}(\zeta) P_0(\eta) - h_0^{(1)}(\zeta) P_1(\eta) \\ &= \mathrm{i} h_1^{(1)}(\zeta) \left(P_2(\eta) - P_0(\eta) \right) + h_2^{(1)}(\zeta) \left(P_3(\eta) - P_1(\eta) \right) - \mathrm{i} h_3^{(1)}(\zeta) \left(P_4(\eta) - P_2(\eta) \right) \\ &\quad + \ldots - \mathrm{i}^{-k+2} h_{k-2}^{(1)}(\zeta) \left(P_{k-1}(\eta) - P_{k-3}(\eta) \right) - \mathrm{i}^{-k+1} h_{k-1}^{(1)}(\zeta) \left(P_k(\eta) - P_{k-2}(\eta) \right) \\ &\quad + \mathrm{i} h_1^{(1)}(\zeta) P_0(\eta) - h_0^{(1)}(\zeta) P_1(\eta) \\ &= - \left(h_0^{(1)}(\zeta) + h_2^{(1)}(\zeta) \right) P_1(\eta) + \mathrm{i} \left(h_1^{(1)}(\zeta) + h_3^{(1)}(\zeta) \right) P_2(\eta) \\ &\quad + \left(h_2^{(1)}(\zeta) + h_4^{(1)}(\zeta) \right) P_3(\eta) - \mathrm{i} \left(h_3^{(1)}(\zeta) + h_5^{(1)}(\zeta) \right) P_4(\eta) + \ldots \\ &\quad - \mathrm{i}^{-k+2} \left(h_{k-2}^{(1)}(\zeta) + h_k^{(1)}(\zeta) \right) P_{k-1}(\eta) - \mathrm{i}^{-k+1} \left(h_{k-1}^{(1)}(\zeta) + h_{k+1}^{(1)}(\zeta) \right) P_k(\eta) \\ &\quad + \mathrm{i}^{-k+2} h_k^{(1)}(\zeta) P_{k-1}(\eta) + \mathrm{i}^{-k+1} h_{k+1}^{(1)}(\zeta) P_k(\eta) \end{aligned}$$

The recursion relation $h_{l+1}^{(1)}(z) + h_{l-1}^{(1)}(z) = (2l+1) h_l^{(1)}(z)/z$ implies

$$\begin{aligned} S &= -3 \frac{h_1^{(1)}(\zeta)}{\zeta} P_1(\eta) + 5\mathrm{i} \frac{h_2^{(1)}(\zeta)}{\zeta} P_2(\eta) + 7 \frac{h_3^{(1)}(\zeta)}{\zeta} P_3(\eta) - 9\mathrm{i} \frac{h_4^{(1)}(\zeta)}{\zeta} P_4(\eta) + \ldots \\ &\quad - \mathrm{i}^{-k+1} (2k+1) \frac{h_k^{(1)}(\zeta)}{\zeta} P_k(\eta) + \mathrm{i}^{-k+2} h_k^{(1)}(\zeta) P_{k-1}(\eta) + \mathrm{i}^{-k+1} h_{k+1}^{(1)}(\zeta) P_k(\eta) \\ &= - \sum_{n=1}^{k} \mathrm{i}^{-n+1} (2n+1) \frac{h_n^{(1)}(\zeta)}{\zeta} P_n(\eta) + \mathrm{i}^{-k+2} h_k^{(1)}(\zeta) P_{k-1}(\eta) + \mathrm{i}^{-k+1} h_{k+1}^{(1)}(\zeta) P_k(\eta) \end{aligned}$$

which gives

$$
\begin{aligned}
f_l(\eta,\zeta) = \Bigg\{ \sum_{k=1}^{l} \frac{1}{\zeta h_{k-1}^{(1)}(\zeta) h_k^{(1)}(\zeta)} \Bigg(& - \sum_{n=1}^{k} \mathrm{i}^{-n+1}(2n+1) \frac{h_n^{(1)}(\zeta)}{\zeta} P_n(\eta) \\
& + \mathrm{i}^{-k+2} h_k^{(1)}(\zeta) P_{k-1}(\eta) + \mathrm{i}^{-k+1} h_{k+1}^{(1)}(\zeta) P_k(\eta) - \mathrm{i}\frac{h_0^{(1)}(\zeta)}{\zeta} P_0(\eta) \Bigg) \\
& - \mathrm{i}\frac{P_0(\eta)}{\zeta h_0^{(1)}(\zeta)} \Bigg\} \mathrm{i}^l h_l^{(1)}(\zeta)
\end{aligned}
$$

or

$$
\begin{aligned}
f_l(\eta,\zeta) = \Bigg\{ \sum_{k=1}^{l} \frac{1}{\zeta h_{k-1}^{(1)}(\zeta) h_k^{(1)}(\zeta)} \Bigg(& - \sum_{n=0}^{k} \mathrm{i}^{-n+1}(2n+1) \frac{h_n^{(1)}(\zeta)}{\zeta} P_n(\eta) \\
& + \mathrm{i}^{-k+2} h_k^{(1)}(\zeta) P_{k-1}(\eta) + \mathrm{i}^{-k+1} h_{k+1}^{(1)}(\zeta) P_k(\eta) \Bigg) - \mathrm{i}\frac{P_0(\eta)}{\zeta h_0^{(1)}(\zeta)} \Bigg\} \mathrm{i}^l h_l^{(1)}(\zeta)
\end{aligned}
$$

This completes the lemma. □

In conclusion, the functions $h_l^{\pm}(z)$ defined in (4.2) can be expressed in the function $h(\eta,\zeta)$ in (5.1). Specifically, we have

$$
h_l^{\pm}(z) = ka h_l(z/a, \pm ka)
$$

6 Summary and explicit terms

This paper contains an evaluation of a non-trivial integral that occurs in the formulation of scattering by randomly distributed obstacles.

To summarize, the integral $I_l(z)$ in (1.1) has been solved and the solution outside the interval $[-a,a]$ is a simple exponential function in kz, while inside the interval $[-a,a]$, the solution can be found in a finite series of spherical waves. The finite sum of spherical waves depends on the two parameters kz and ka, or, more precisely, the parameter ka and a polynomial of the order l in the parameter z/a. Several equivalent solutions are presented in the paper, one of them is ($l = 0, 1, 2, \ldots$)

$$
I_l(z) = \begin{cases}
\mathrm{i}^l \mathrm{e}^{-\mathrm{i}kz}, \quad z \le -a \\
\mathrm{i}^{1-l} ka h_0^{(1)}(ka) P_{l-2[l/2]}(z/a) \\
\quad + \displaystyle\sum_{n=0}^{[l/2]-1} (-1)^n ka h_{l-2n-1}^{(1)}(ka) \left(P_{l-2n}(z/a) - P_{l-2n-2}(z/a) \right), \quad z \in [-a,a] \\
\mathrm{i}^{-l} \mathrm{e}^{\mathrm{i}kz}, \quad z \ge a
\end{cases}
$$

The first integrals, $l = 0, 1, 2$, are of interest for low-frequency expansions. For $l = 0$ the integral is

$$
I_0(z) = \begin{cases}
\mathrm{e}^{-\mathrm{i}kz}, & z \le -a \\
\mathrm{e}^{\mathrm{i}ka}, & z \in [-a,a] \\
\mathrm{e}^{\mathrm{i}kz}, & z \ge a
\end{cases}
$$

and for $l = 1$ the result is

$$I_1(z) = \begin{cases} \mathrm{i}\mathrm{e}^{-\mathrm{i}kz}, & z \le -a \\ -\mathrm{i}\mathrm{e}^{\mathrm{i}ka}\dfrac{z}{a}, & z \in [-a, a] \\ -\mathrm{i}\mathrm{e}^{\mathrm{i}kz}, & z \ge a \end{cases}$$

For $l = 2$ the result is

$$I_2(z) = \begin{cases} -\mathrm{e}^{-\mathrm{i}kz}, & z \le -a \\ \mathrm{e}^{\mathrm{i}ka}\dfrac{(ka)^2(3\mathrm{i} + ka) - 3(\mathrm{i} + ka)(kz)^2}{2(ka)^3}, & z \in [-a, a] \\ -\mathrm{e}^{\mathrm{i}kz}, & z \ge a \end{cases}$$

and we notice that the integral contains a polynomial in z/a of order l.

Moreover, the indefinite Fourier transform of $I_l(z)$ has also been investigated. More precisely, the integral, see (4.1)

$$\widehat{I}_l^{\pm}(z) = k \int_{z_0}^{z} I_l(t)\mathrm{e}^{\pm\mathrm{i}kt}\, \mathrm{d}t, \quad z \ge z_0, \quad l = 0, 1, 2, \ldots$$

is shown to have a solution expressed in spherical waves.

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