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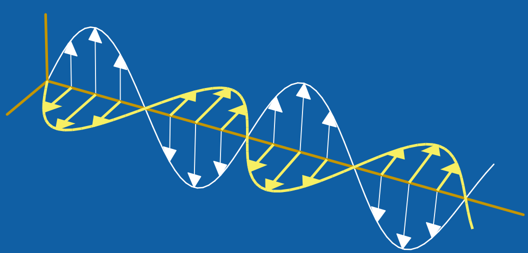

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Abstract

The time-domain version of the optical theorem is discussed. The theorem relates the sum of the scattered and absorbed energies, for a plane pulse that is scattered from a bounded obstacle, to the far-field amplitude in the forward direction. Several fundamental results concerning the scattered and absorbed energies that follow from the theorem are discussed.

1 Introduction

The history of the optical theorem extends over more than a century, cf [8]. It is used in all areas of physics where waves are involved and one can find derivations of it in elementary textbooks in quantum mechanics, acoustics and electromagnetics, cf [1], [4] and [7]. In the frequency domain the theorem gives a relation between the sum of the scattered and absorbed powers and the far-field amplitude in the forward direction, for a plane wave that is scattered from a bounded object. The theorem gives a fast way to calculate the sum of the scattered and absorbed powers, but even more important is that it can be used as a test for analytic and numerical methods. The test is that the optical theorem and the brute force method of integrating the power flow densities over a closed surface that circumscribes the scattering object, should give the same value of the sum of the scattered and absorbed powers.

The time-domain version of the optical theorem is less known than its frequency-domain counterpart. It relates the sum of the scattered and absorbed energies to the far-field amplitude for a transient plane wave that is scattered from a bounded object. One would think that the time-domain optical theorem is discussed in a number of papers and books. However, only two papers were found on this matter in the most common data bases. They are written by A. T. de Hoop, where the first one [2] presents the electromagnetic case and the second one [3] the acoustic case. There is a vast literature where the optical theorem is discussed and it is not unlikely that there are other places where the time-domain theorem is analyzed.

In the papers by de Hoop the incident wave is a transient wave with semi-infinite extent. If instead the transient wave is a pulse with finite width the theorem exposes some fundamental relations for the scattered energy. The relations might be regarded as curiosities but are nevertheless fundamental and quite interesting. Some of them were presented in [5], and they are further discussed in this paper. In addition two other relations are presented. The first one is based on the time-domain optical theorem and reciprocity and the other one is an energy relation for linearly polarized plane waves.

There are several connections from this paper to Staffan's research. Thus in subsection 3.1 scattering from several objects is discussed. This is an area to which Staffan and his coworker Bo Peterson gave important contributions in the mid-seventies. They developed a method to obtain the transition matrix (T-matrix) for two or more objects if the T-matrix for each object was known, cf [9], [12] and [10]. A similar method was developed for layered objects, cf [11], and this links to subsection 3.2. It was mentioned above that the optical theorem offers a condition that can be used as a test for numerical methods. In the T-matrix method this condition reads $T^{\dagger}T = -\mathrm{Re}T$, where T is the T-matrix and † denotes Hermite conjugate. This condition was invaluable for all, including the author of this paper, that wrote codes for different types of T-matrices in Staffan's research group during the seventies and eighties. Finally, in subsection 3.5 the connection between the optical theorem, reciprocity and Staffan's sailboat is established.

In appendix A a non-rigorous derivation of the time-domain optical theorem is given. The derivation starts from the assumption that there is an extinction of the incident field behind the scattering object and that the corresponding reduction of the energy in the incident field equals the sum of the scattered and absorbed energies. For more rigorous proofs the papers by de Hoop are recommended. In appendix B it is shown that the transformation from the time-domain optical theorem to the frequency-domain theorem can be done by the Parseval's relation.

2 The optical theorem in time domain

Assume a bounded scattering object with volume V. The electromagnetic properties of the object are arbitrary. The volume outside the object is denoted V' and for simplicity it is assumed to be vacuum there. Hence the wave speed in the object is less than or equal to the wave speed in V'. The incident wave is an electromagnetic plane wave pulse that propagates in the positive $z-$direction

$$\boldsymbol{E}^i(z,t) = \boldsymbol{E}_0\left(t - z/c_0\right), \tag{2.1}$$

where c_0 is the speed of light in vacuum. The vector $\boldsymbol{E}^i(z,t)$ lies in the xy-plane since the wave is transverse. It is assumed that the incident pulse has finite length, which implies that $\boldsymbol{E}_0(t)$ is zero outside some time interval $t_0 < t < t_1$. The leading edge of the pulse then arrives at $z = 0$ when $t = t_0$ and the trailing edge when $t = t_1$. Outside the object the total field is decomposed into the incident field and the scattered field as

$$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}^i(z,t) + \boldsymbol{E}^s(\boldsymbol{r},t). \tag{2.2}$$

In the far zone the scattered field is a spherical wave [4]

$$\boldsymbol{E}^s(\boldsymbol{r},t) = \frac{\boldsymbol{F}(\hat{\boldsymbol{r}}, t - r/c_0)}{r} \qquad \text{as } r \to \infty, \tag{2.3}$$

where the far-field amplitude $\boldsymbol{F}(\hat{\boldsymbol{r}}, t-r/c_0)$ is a transverse field moving in the radial direction with speed c_0. Causality implies that in the forward direction no signal can arrive prior to the incident pulse and thus $\boldsymbol{F}(\hat{\boldsymbol{z}},t) = 0$ for times $t < t_0$.

The optical theorem in the time domain states that when the incident field has passed the scattering object, the sum of the scattered and absorbed energies is obtained from

$$W_T = W_s(t) + W_a(t) = -\frac{4\pi}{\mu_0}\int_{t_0}^{t_1} \boldsymbol{E}_0(t') \cdot \int_{t_0}^{t'} \boldsymbol{F}(\hat{\boldsymbol{z}}, t'')dt''dt'. \tag{2.4}$$

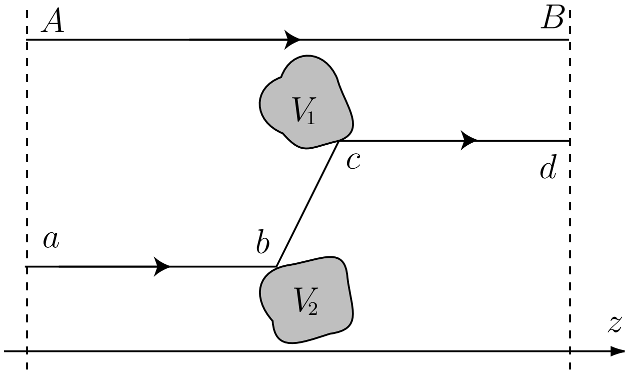

Figure 1: Scattering of a plane wave pulse from two objects. If the difference between the shortest travel time for a trip $a \to b \to c \to d$, or $a \to c \to b \to d$, and the travel time for the trip $A \to B$ is larger than the length of the pulse, i.e. $t_1 - t_0$, then $W_T = W_{T1} + W_{T2}$ where W_{T1} is the sum of the scattered and absorbed energies for object V_1 with object V_2 not present, and vice versa for W_{T2}.

Here $W_s(t)$ is the scattered energy and $W_a(t)$ is the energy absorbed in the object. The sum of the scattered and absorbed energies $W_T = W_s(t) + W_a(t)$ is independent of time although each of the two terms can be time dependent. ¿From Eq. (2.4) it is seen that the sum of the scattered and absorbed energies is determined by the far-field amplitude in the forward direction, $\boldsymbol{F}(\hat{\boldsymbol{z}}, t)$, during the time interval $[t_0, t_1]$.

3 Implications of the theorem

In this section some fundamental results that follow from causality and the optical theorem are discussed. The results presented in subsections 3.1–3.4 have been discussed in [5], where they also were verified numerically. Those results have no correspondence in the frequency domain since they are based on the assumption that the incident pulses are of finite length. The results in the two last subsections 3.5 and 3.6 have not been presented before. They are valid also in the frequency domain since they do not relay on an assumption that the incident pulse has support in a finite time interval.

3.1 Scattering from several objects

Let the incident field be the plane wave pulse given by Eq. (2.1), where $\boldsymbol{E}_0(t) = 0$ for $t < t_0$ and $t > t_1$. Consider two scattering objects with volumes V_1 and V_2, as in Figure 1, that are separated in the xy-plane. The objects give rise to a scattered field $\boldsymbol{E}^s$. The sum of the scattered and absorbed energies for the two objects is W_T. If the objects are close to each other then, in general,

$$W_T \neq W_{T1} + W_{T2}, \tag{3.1}$$

where W_{T1} is the sum of the scattered and absorbed energies from object 1 when object 2 is not present, and vice versa for W_{T2}. The inequality is due to multiple

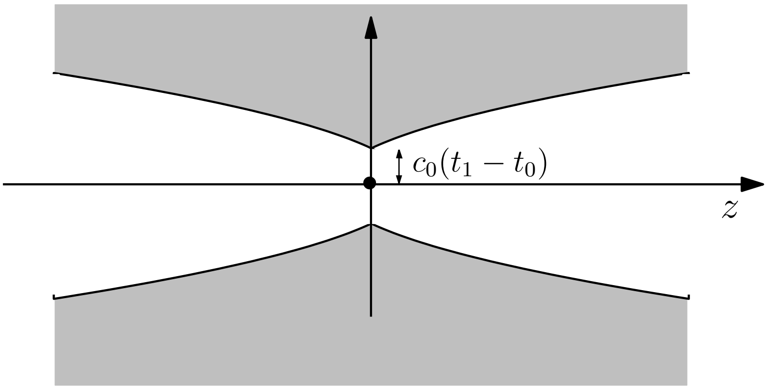

Figure 2: A point like object is at the origin. The relation (3.2) holds if the other object is entirely in the shaded region.

scattering. If the pulse is short enough then the inequality in Eq. 3.1 is turned into an equality. This occurs if the length $t_1 - t_0$ of the incident pulse is shorter than the difference between the travel time for the shortest trip $a \to b \to c \to d$ (or $a \to c \to b \to d$) and the travel time for the straight line $A \to B$ in Figure 1. Every multiple scattered wave is then delayed at least a time $t_1 - t_0$ compared to the incident wave. In the time interval $t_0 < t < t_1$ the scattering amplitude $\boldsymbol{F}(\hat{\boldsymbol{z}}, t)$ is consequently independent of the multiple scattered waves and reads $\boldsymbol{F}(\hat{\boldsymbol{z}}, t) = \boldsymbol{F}_1(\hat{\boldsymbol{z}}, t) + \boldsymbol{F}_2(\hat{\boldsymbol{z}}, t)$, where $\boldsymbol{F}_i$, $i = 1, 2$ is the scattering amplitude from object i if only that object is present. Thus from Eq. (2.4)

$$W_T = W_{T1} + W_{T2}. \tag{3.2}$$

Notice that it is not because multiple scattering effects are small that the equality holds. It is straightforward to generalize the relation in Eq. 3.2 to an arbitrary number of scattering objects.

Example: Consider two scattering objects where one is point like. If the point like object is located at the origin the other one has to be outside the two paraboloidal surfaces defined by $z = (x^2 + y^2 - c_0^2(t_1 - t_0)^2)/(2c_0(t_1 - t_0))$ and $z = -(x^2 + y^2 - c_0^2(t_1 - t_0)^2)/(2c_0(t_1 - t_0))$, see Fig. 2, in order for Eq. (3.2) to be valid.

3.2 Scattering from layered objects

The same arguments that were used in scattering from several objects can be used for a layered object with an inner volume V_2 and an enclosing layer V_1, as in Figure 3. The incident field has finite length such that $\boldsymbol{E}_0(t) = 0$ when $t < t_0$ and $t > t_1$. Assume that the layer V_1 is thick enough and its wave speed is small enough to ensure that the shortest travel time for the trip $a \to b \to c$ in Figure 3 is at least a time $t_1 - t_0$ longer than the travel time for the straight line $A \to B$. In that case the reflections from the inner volume give no contribution to the far-field amplitude $\boldsymbol{F}(\hat{\boldsymbol{z}}, t)$ for times $t_0 < t < t_1$, or phrased differently, $\boldsymbol{F}(\hat{\boldsymbol{z}}, t)$ is independent of the inner volume V_2 for $t_0 < t < t_1$. According to Eq. 2.4 the sum of the scattered and

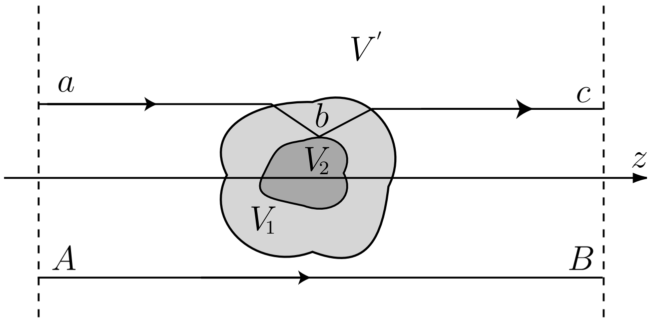

Figure 3: Scattering of a plane wave pulse from a layered object. If the difference between the shortest travel time for a trip $a \to b \to c$ and the travel time for the trip $A \to B$ is larger than the length of the pulse, i.e. $t_1 - t_0$, then the sum of the scattered and absorbed energies W_T is independent of the inner object V_2.

absorbed energies is then independent of the inner object. The conclusion is that for a short enough incident plane wave pulse, only the outermost layer of a layered object contributes to the sum of the scattered and absorbed energies.

3.3 Scattering of discontinuous pulses

It is very hard to numerically calculate the scattered field from bounded objects when the incident pulse is discontinuous. The numerical methods that are available for scattering problems are limited to certain frequency bands. Thus there are methods that can be applied for low frequencies, e.g. finite element methods, the method of moments, the T-matrix method, or Mie scattering. For higher frequencies approximate methods such as the geometrical theory of diffraction, geometrical optics, or physical optics methods can be used. Often the frequency bands for these methods do not overlap and this is a problem since the frequency spectrum of a discontinuous pulse ranges from very low to very high frequencies. The frequency bands where the scattered field is inaccurately calculated make the Fourier transformation to the time domain inaccurate. Purely time-domain methods suffer from similar problems. In this section it is shown that even though the scattered field can not be calculated for discontinuous incident pulses, the sum of the scattered and absorbed energies can be obtained with high accuracy.

Consider the incident pulse

$$\boldsymbol{E}^i(z,t) = \boldsymbol{E}_0\left(t - z/c_0\right), \tag{3.3}$$

where $\boldsymbol{E}_0(t)$ is a smooth function that is approximately zero for $t < t_0$ and $t > t_1$. It is assumed that the corresponding far-field amplitude, $\boldsymbol{F}(\hat{\boldsymbol{r}}, t)$, can be accurately calculated for any t. It is then possible to accurately calculate the integral

$$-\frac{4\pi}{\mu_0}\int_{t_0}^{t} \boldsymbol{E}_0(t')\cdot\int_{t_0}^{t'} \boldsymbol{F}(\hat{\boldsymbol{z}}, t'')dt''dt' \tag{3.4}$$

for any t. Then consider an incident pulse as in Eq. (3.3) but with discontinuous time-dependence

$$\boldsymbol{E}_1(t) = \begin{cases} \boldsymbol{E}_0(t) & t_0 < t < t_m < t_1 \\ \boldsymbol{0} & t > t_m \end{cases} \tag{3.5}$$

The corresponding far-field amplitude is denoted $\boldsymbol{F}_1(\hat{\boldsymbol{r}}, t)$. The sum of the scattered and absorbed energies for this pulse is

$$W_{T1} = W_{a1}(t) + W_{s1}(t) = -\frac{4\pi}{\mu_0} \int_{t_0}^{t_m} \boldsymbol{E}_1(t') \cdot \int_{t_0}^{t'} \boldsymbol{F}_1(\hat{\boldsymbol{z}}, t'') dt'' dt'. \tag{3.6}$$

Causality implies that

$$\boldsymbol{F}_1(\hat{\boldsymbol{z}}, t) = \boldsymbol{F}(\hat{\boldsymbol{z}}, t) \quad t < t_m, \tag{3.7}$$

since the incident pulses $\boldsymbol{E}_0(t)$ and $\boldsymbol{E}_1(t)$ are identical for $t < t_m$. Hence

$$W_{T1} = -\frac{4\pi}{\mu_0} \int_{t_0}^{t_m} \boldsymbol{E}_0(t') \cdot \int_{t_0}^{t'} \boldsymbol{F}(\hat{\boldsymbol{z}}, t'') dt'' dt'. \tag{3.8}$$

Thus the sum of the scattered and absorbed energies for the discontinuous pulse $\boldsymbol{E}_1(t)$ can be obtained from the far field of the continuous pulse $\boldsymbol{E}_0(t)$. Simsalabim, the time-domain optical theorem has turned a numerical nightmare into a piece of cake (well, at least into a simpler numerical problem).

3.3.1 Quasi-static approximation

If the pulse in Eq. (3.3) is slowly varying the scattered energy from the discontinuous pulse in Eq. (3.5) can be obtained by a quasi-static calculation that corresponds to a time-domain version of Rayleigh scattering. For simplicity only the case when the scattering objects are lossless is discussed. Since the electric field changes slowly for $t < t_m$ the electromagnetic energy absorbed in the scattering object is approximately given by the quasi-static expression, cf. [4],

$$W_{a1}(t) = \frac{1}{2} \boldsymbol{p} \cdot \boldsymbol{E}^i(0, t), \qquad t < t_m, \tag{3.9}$$

where it is assumed that the origin is located inside the object. The vector $\boldsymbol{p}$ is the induced dipole moment of the object, and the absorbed energy is then equal to the potential energy of an induced dipole $\boldsymbol{p}$ in a homogeneous electric field. The dipole moment can be obtained by solving the boundary value problem of a dielectric object in a static homogeneous electric field. When the incident wave is slowly varying for $t < t_m$, the scattered energy becomes very small and can be neglected compared to $W_a(t)$ for $t < t_m$. At $t = t_m$ the incident pulse $\boldsymbol{E}_0(t)$ is shut off and the absorbed energy radiates and transforms into scattered energy. Thus the final value of the scattered energy from a slowly varying incident pulse with a discontinuity at $t = t_m$ is approximately given by

$$W_{s1} = W_{a1}(t_m) = \frac{1}{2}\boldsymbol{p} \cdot \boldsymbol{E}^i(0, t_m). \tag{3.10}$$

As an example it is seen that the induced dipole moment for a homogeneous dielectric sphere with radius a in an electric field $\boldsymbol{E}^i(0, t_m)$ is given by

$$\boldsymbol{p} = 4\pi a^3 \varepsilon_0 \frac{\varepsilon_r - 1}{\varepsilon_r + 2}\boldsymbol{E}^i(0, t_m). \tag{3.11}$$

The corresponding scattered energy is given by

$$W_{s1} = 2\pi a^3 \varepsilon_0 \frac{\varepsilon_r - 1}{\varepsilon_r + 2}|\boldsymbol{E}^i(0, t_m)|^2 \tag{3.12}$$

An expression that can be numerically evaluated by paper and pen in a minute.

3.4 Scattering from dispersive objects

Simple linear dispersive materials are in the time domain characterized by the electric and magnetic susceptibility kernels, $\chi_e(t)$ and $\chi_m(t)$. Consider the case where the material of the scattering object is characterized by the constitutive relations, cf [4],

$$\begin{cases} \boldsymbol{B}(\boldsymbol{r}, t) = \mu \boldsymbol{H}(\boldsymbol{r}, t) + \mu_0 \displaystyle\int_{-\infty}^{t} \chi_m(t - t')\boldsymbol{H}(\boldsymbol{r}, t')dt' \\ \boldsymbol{D}(\boldsymbol{r}, t) = \varepsilon \boldsymbol{E}(\boldsymbol{r}, t) + \varepsilon_0 \displaystyle\int_{-\infty}^{t} \chi_e(t - t')\boldsymbol{E}(\boldsymbol{r}, t')dt'. \end{cases} \tag{3.13}$$

From the optical theorem it follows that the sum of the scattered and absorbed energies only depends on $\chi_m(t)$ and $\chi_e(t)$ in the time interval $0 < t < t_1 - t_0$.

3.5 Reciprocity

The following definition of a reciprocal medium is given in [6]:

In the time domain a medium is defined to be reciprocal at a point $\boldsymbol{r}$ *in a region* V *if and only if*

$$\iint_{S_{\boldsymbol{r}}} \left\{ \left(\boldsymbol{E}^a \otimes \boldsymbol{H}^b\right)(t) + \left(\boldsymbol{H}^a \otimes \boldsymbol{E}^b\right)(t) \right\} \cdot \hat{\boldsymbol{n}} dS = 0 \tag{3.14}$$

holds for all times t*, for all electromagnetic fields* $\{\boldsymbol{E}^a, \boldsymbol{B}^a\}$ *and* $\{\boldsymbol{E}^b, \boldsymbol{B}^b\}$*, and for every closed surface* $S_{\boldsymbol{r}} \subset V$ *around the point* $\boldsymbol{r}$*. The medium in a volume* V *is reciprocal in* V *if and only if it is reciprocal at all points in* V*.*

The operator $\otimes$ is defined by

$$\left(\boldsymbol{E}^a \otimes \boldsymbol{H}^b\right)(t) = \int_{-\infty}^{\infty} \boldsymbol{E}^a(t-t') \times \boldsymbol{H}^b(t')dt' \tag{3.15}$$

Let the fields $\{\boldsymbol{E}^a, \boldsymbol{B}^a\}$ be the total fields from an incident plane wave pulse traveling in the positive $z-$direction, i.e.

$$\boldsymbol{E}^{ia}(z,t) = \hat{\boldsymbol{x}} E_0\left(t - z/c_0\right), \tag{3.16}$$

and let $\left\{\boldsymbol{E}^b, \boldsymbol{B}^b\right\}$ be the total fields when the incident plane wave pulse travels in the negative $z-$direction, i.e. when

$$\boldsymbol{E}^{ib}(z,t) = \hat{\boldsymbol{x}} E_0\left(t + z/c_0\right). \tag{3.17}$$

If the scattering object is reciprocal it follows that

$$\hat{\boldsymbol{x}} \cdot \boldsymbol{F}^a(\hat{\boldsymbol{z}},t) = \hat{\boldsymbol{x}} \cdot \boldsymbol{F}^b(-\hat{\boldsymbol{z}},t) \tag{3.18}$$

for all times. Here $\boldsymbol{F}^a(\hat{\boldsymbol{z}},t)$ and $\boldsymbol{F}^b(-\hat{\boldsymbol{z}},t)$ are the far-field amplitudes in the forward direction of the field $\boldsymbol{E}^a$ and the field $\boldsymbol{E}^b$, respectively. The derivation of Eq. (3.18) relies on Eq. (3.14) and causality. The sum of the scattered and absorbed energies is given by Eq. (2.4). From Eq. (3.18) it is then seen that

$$W_T^a = \int_{t_0}^{t_1} E_0(t')\hat{\boldsymbol{x}} \cdot \int_{t_0}^{t'} \boldsymbol{F}^a(\hat{\boldsymbol{z}},t'')dt''dt' = \int_{t_0}^{t_1} E_0(t')\hat{\boldsymbol{x}} \cdot \int_{t_0}^{t'} \boldsymbol{F}^b(-\hat{\boldsymbol{z}},t'')dt''dt' = W_T^b, \tag{3.19}$$

where W_T^a and W_T^b are the sum of the scattered and absorbed energies for the incident pulses $\boldsymbol{E}^{ia}$ and $\boldsymbol{E}^{ib}$, respectively. Thus the incident waves $\boldsymbol{E}^{ia}(z,t)$ and $\boldsymbol{E}^{ib}(z,t)$ give the same sum of the scattered and absorbed energies. This is illustrated in Fig. 4.

3.6 Polarization

Consider a linearly polarized plane wave that is scattered from a bounded object with linear constitutive relations. If the incident wave is polarized in the direction $(\hat{\boldsymbol{x}}\cos\phi + \hat{\boldsymbol{y}}\sin\phi)$, i.e.

$$\boldsymbol{E}_1^i(z,t) = (\hat{\boldsymbol{x}}\cos\phi + \hat{\boldsymbol{y}}\sin\phi)E_0\left(t - z/c_0\right), \tag{3.20}$$

the sum of the scattered and absorbed energies is denoted $W(\phi)$. It is assumed that $E_0(t)$ has support for $t_0 < t < t_1$, where is possible to have $t_1 = \infty$. If the incident wave instead has a polarization perpendicular to $\boldsymbol{E}_1^i$, e.g.

$$\boldsymbol{E}_2^i(z,t) = \hat{\boldsymbol{z}} \times \boldsymbol{E}_2^i(z,t) = (-\hat{\boldsymbol{x}}\sin\phi + \hat{\boldsymbol{y}}\cos\phi)E_0\left(t - z/c_0\right), \tag{3.21}$$

the sum of the scattered and absorbed energies is $W(\phi+\pi/2)$. The optical theorem implies that the sum $W(\phi) + W(\phi+\pi/2)$ is independent of the angle ϕ. To see this

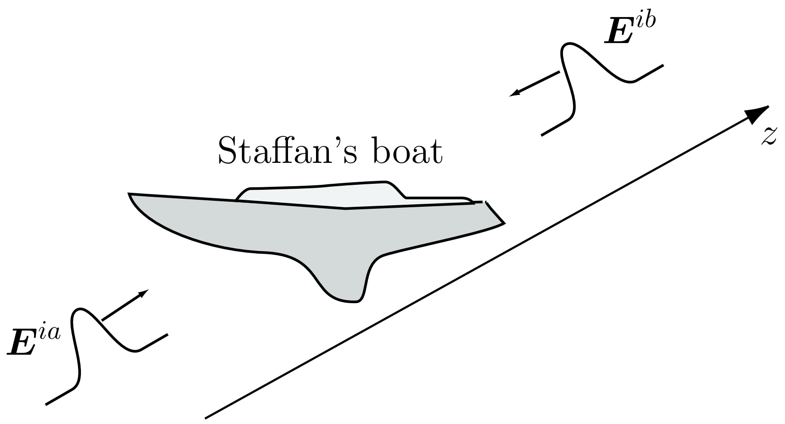

Figure 4: Launching of Staffan's sailboat an early morning in May. Since the boat is reciprocal the sum of the scattered and absorbed energies for the incident plane wave $\boldsymbol{E}^{ia}(z,t)$ equals the sum of the scattered and absorbed energies for the incident plane wave $\boldsymbol{E}^{ib}(z,t)$. The influence on the fields from the surroundings, including all of the assisting graduate students, is neglected.

it is convenient to introduce a scattering matrix that relates the far-field amplitude to the incident plane wave. In the case of an incident wave propagating in the $z-$direction the far-field amplitude in the forward direction, $\boldsymbol{F}(\hat{\boldsymbol{z}},t) = \hat{\boldsymbol{x}}F_x(\hat{\boldsymbol{z}},t) + \hat{\boldsymbol{y}}F_y(\hat{\boldsymbol{z}},t)$, is given by

$$\begin{pmatrix} F_x(\hat{\boldsymbol{z}},t) \\ F_y(\hat{\boldsymbol{z}},t) \end{pmatrix} = \begin{pmatrix} S_{xx}(t) & S_{xy}(t) \\ S_{yx}(t) & S_{yy}(t) \end{pmatrix} * \begin{pmatrix} \hat{\boldsymbol{x}}\cdot\boldsymbol{E}^i(z,t) \\ \hat{\boldsymbol{y}}\cdot\boldsymbol{E}^i(z,t) \end{pmatrix} = \mathbf{S}(t) * \begin{pmatrix} \hat{\boldsymbol{x}}\cdot\boldsymbol{E}^i(z,t) \\ \hat{\boldsymbol{y}}\cdot\boldsymbol{E}^i(z,t) \end{pmatrix} \tag{3.22}$$

where $\boldsymbol{E}^i = \hat{\boldsymbol{x}}E^i_x(\hat{\boldsymbol{r}}\, t) + \hat{\boldsymbol{y}}E^i_y(\hat{\boldsymbol{r}},t)$ is the incident field. The asterisk $*$ denotes convolution in time. The scattering matrix $\mathbf{S}(t)$ is independent of the incident field. Its matrix elements are non-classical functions, but if $E_0(t)$ in Eqs. (3.20) and (3.21) is a smooth classical function then the matrix elements of $[\mathbf{S} * E_0](t)$ are smooth classical functions. In the case of the incident wave in Eq. (3.20) it is seen that the sum of the scattered and absorbed energies is given by

$$\begin{aligned} W(\phi) &= -\frac{\mu_0}{4\pi}\int_{t_0}^{t_1} E_0(t')\int_{t_0}^{t'} (\cos\phi\hat{\boldsymbol{x}}\cdot\boldsymbol{F}_1(\hat{\boldsymbol{z}},t'') + \sin\phi\hat{\boldsymbol{y}}\cdot\boldsymbol{F}_1(\hat{\boldsymbol{z}},t''))\, dt''dt' \\ &= -\frac{\mu_0}{4\pi}\int_{t_0}^{t_1} E_0(t')\int_{t_0}^{t'} \big([S_{xx} * E_0](t'')\cos^2\phi + [S_{yy} * E_0](t'')\sin^2\phi \\ &\quad +[(S_{xy}+S_{yx}) * E_0](t'')\cos\phi\sin\phi\big)\, dt''dt'. \end{aligned} \tag{3.23}$$

Thus

$$W(\phi) + W(\phi \pm \pi/2) = -\frac{\mu_0}{4\pi}\int_{t_0}^{t_1} E_0(t')\int_{t_0}^{t'} ([S_{xx} * E_0](t'') + [S_{yy} * E_0](t''))\, dt''dt' \tag{3.24}$$

which is a independent of the angle ϕ.

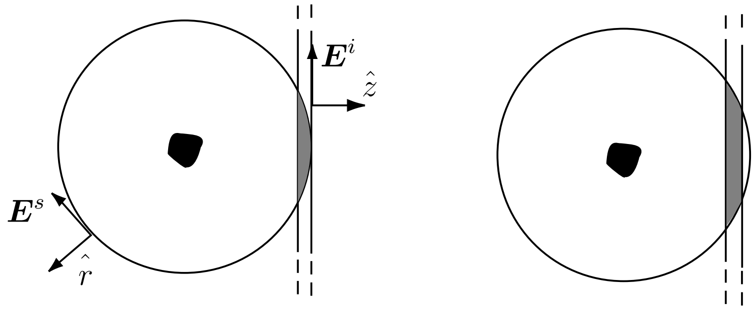

Figure 5: The scattering object (black region), the incident wave and the scattered wave. The wave front of the scattered wave is the circle and the incident pulse is between the two straight vertical lines. The shaded region is the region of extinction of the incident field, i.e. the region that contributes to the integral in Eq. (A.4). The figure to the right shows a case when the wave speed inside the scattering object is larger than for the surrounding medium.

Appendix A Derivation of the theorem

Most of the techniques that are used to derive the optical theorem in the frequency domain can be generalized to the time-domain. In [2] a derivation of the time-domain version of the optical theorem is given in detail for the electromagnetic case. The proof is based upon a surface integral representation of the scattered field. ¿From the proof it is clearly seen that the theorem holds regardless of the electromagnetic properties of the scattering object. In this appendix a simpler, but less general and rigorous, derivation is given. In the derivation it is not assumed that the medium surrounding the scattering object is vacuum. Thus the wave speed in the scattering object is not restricted to be less than the wave speed of the surrounding volume.

The general expression of the sum of the electric and magnetic energies in a volume V_0 is given by

$$W_e(t) + W_m(t) = \frac{1}{2}\int_{V_0} \varepsilon|\boldsymbol{E}(\boldsymbol{r},t)|^2 + \mu|\boldsymbol{H}(\boldsymbol{r},t)|^2 dv. \tag{A.1}$$

Assume that time is large enough for the incident field to have passed the object and be in the far zone. The absorbed energy in the volume of the scattering object, V, is the difference between the energy of the incident field, W_i, and the electromagnetic energy in the region outside the object, i.e. V', at time t, thus

$$\begin{aligned} W_a(t) &= W_i - \frac{1}{2}\int_{V'} \varepsilon|\boldsymbol{E}^i(\boldsymbol{r},t) + \boldsymbol{E}^s(\boldsymbol{r},t)|^2 + \mu|\boldsymbol{H}^i(\boldsymbol{r},t) + \boldsymbol{H}^s(\boldsymbol{r},t)|^2 dv \\ &= -W_s(t) - \int_{V'} \varepsilon\boldsymbol{E}^i(\boldsymbol{r},t)\cdot\boldsymbol{E}^s(\boldsymbol{r},t) + \mu\boldsymbol{H}^i(\boldsymbol{r},t)\cdot\boldsymbol{H}^s(\boldsymbol{r},t) dv \end{aligned} \tag{A.2}$$

where

$$W_s(t) = \frac{1}{2}\int_{V'} \varepsilon\boldsymbol{E}^s(\boldsymbol{r},t)\cdot\boldsymbol{E}^s(\boldsymbol{r},t) + \mu\boldsymbol{H}^s(\boldsymbol{r},t)\cdot\boldsymbol{H}^s(\boldsymbol{r},t) dv \tag{A.3}$$

is the energy stored in the scattered field, or equivalently, the scattered energy. Thus

$$W_s(t) + W_a(t) = -\int_{V'} \varepsilon \boldsymbol{E}^i(\boldsymbol{r},t)\cdot \boldsymbol{E}^s(\boldsymbol{r},t) + \mu \boldsymbol{H}^i(\boldsymbol{r},t)\cdot \boldsymbol{H}^s(\boldsymbol{r},t)dv \tag{A.4}$$

In the last integral the domain of integration is reduced to the intersection of the supports of $\boldsymbol{E}^i$ and $\boldsymbol{E}^s$, as depicted in figure 5. For the plane incident wave the magnetic field is related to the electric field by $\boldsymbol{H}^i = \sqrt{\frac{\varepsilon}{\mu}}\hat{\boldsymbol{z}} \times \boldsymbol{E}^i$ and since the intersection is in the far zone $\boldsymbol{H}^s = \sqrt{\frac{\varepsilon}{\mu}}\hat{\boldsymbol{r}} \times \boldsymbol{E}^s$. Introduce cylindrical coordinates ρ, ϕ, z and use the far-field expression for $\boldsymbol{E}^s$. The integral is then reduced to

$$\begin{aligned} W_s(t) + W_a(t) &= -4\pi\varepsilon \int\limits_{c(t-t_1)}^{c(t-t_0)} \boldsymbol{E}_0(t-z/c)\cdot \int\limits_0^{\infty} \frac{\rho}{\sqrt{\rho^2+z^2}} \boldsymbol{F}(\hat{\boldsymbol{z}}, t-\sqrt{\rho^2+z^2}/c)d\rho dz \\ &= -4\pi\varepsilon c \int\limits_{c(t-t_1)}^{c(t-t_0)} \boldsymbol{E}_0(t-z/c)\cdot \int\limits_{-\infty}^{t-z/c} \boldsymbol{F}(\hat{\boldsymbol{z}}, t'')dt''dz. \end{aligned} \tag{A.5}$$

Now $\boldsymbol{F}(t'')$ is zero for $t'' < t_f$ and then the substitution $t' = t - z/c$ transforms the integral to

$$W_s(t) + W_a(t) = -\frac{4\pi}{\mu}\int_{t_0}^{t_1} \boldsymbol{E}_0(t')\cdot \int_{t_f}^{t'} \boldsymbol{F}(\hat{\boldsymbol{z}}, t'')dt''dt', \tag{A.6}$$

which is the optical theorem in the time-domain[1]. Notice that the sum $W_s(t)+W_a(t)$ is independent of time although each of the two terms can be time-dependent.

Appendix B Transformation to frequency domain

It is possible to transform the time-domain optical theorem to the frequency-domain theorem, and vice versa, by Parseval's relation. In this appendix it is shown how the transformation from time domain to frequency domain is done.

If $f(t)$ and $g(t)$ are two square integrable functions with Fourier transforms $\tilde{f}(\omega)$ and $\tilde{g}(\omega)$ then Parseval's relation reads

$$\int_{-\infty}^{\infty} f(t)g^*(t)dt = \int_{-\infty}^{\infty} \tilde{f}(\omega)\tilde{g}^*(\omega)d\omega \tag{B.1}$$

[1]There is a minor discrepancy between the formula (A.6) and the corresponding formula in [2]. In [2] the lower integration limit in the second integral in Eq. (A.6) is t_0 instead of t_f, which seems to be valid only if the wave speed of the object is less than the wave speed for the medium surrounding the object.

where the asterisk denotes complex conjugate. Before Parseval's relation is applied to Eq. (2.4) it is appropriate to do an integration by parts of the right hand side of the equation. Thus

$$\begin{aligned} W_T &= \frac{4\pi}{\mu_0}\int_{t_0}^{t_1}\left(\int_{t_0}^{t}\boldsymbol{E}_0(t')dt' - \int_{t_0}^{t_1}\boldsymbol{E}_0(t')dt'\right)\cdot\boldsymbol{F}(\hat{\boldsymbol{z}},t)dt \\ &= \frac{4\pi}{\mu_0}\int_{-\infty}^{\infty}\left(\int_{t_0}^{t}\boldsymbol{E}_0(t')dt' - \int_{t_0}^{t_1}\boldsymbol{E}_0(t')dt'\right)\cdot\boldsymbol{F}(\hat{\boldsymbol{z}},t)dt \end{aligned} \tag{B.2}$$

where the integration limits were set to $-\infty$ and ∞ since the integrand is zero for $t < t_0$ and $t > t_1$. A reasonable assumption is that $\boldsymbol{E}_0(t)$ is square integrable. Then if t_1 is assumed to be finite one can prove that $\int_{-\infty}^{t}\boldsymbol{E}_0(t')dt' - \int_{-\infty}^{t_1}\boldsymbol{E}_0(t')dt'$ and $\boldsymbol{F}(t)$ are square integrable and Parseval's relation is applicable. The Fourier transform of $\int_{-\infty}^{t}\boldsymbol{E}_0(t')dt' - \int_{-\infty}^{t_1}\boldsymbol{E}_0(t')dt'$ is $i/\omega\tilde{\boldsymbol{E}}_0(\omega)$ where

$$\tilde{\boldsymbol{E}}_0(\omega) = \int_{-\infty}^{\infty}\boldsymbol{E}_0(t)e^{i\omega t}d\omega \tag{B.3}$$

is the Fourier transform of $\boldsymbol{E}_0(t)$. Thus Parseval's relation gives

$$W_T = -\int_{-\infty}^{\infty}\frac{4\pi i}{\mu_0\omega}\tilde{\boldsymbol{E}}_0^*(\omega)\cdot\tilde{\boldsymbol{F}}(\hat{\boldsymbol{z}},\omega)d\omega \tag{B.4}$$

where $\tilde{\boldsymbol{F}}(\hat{\boldsymbol{z}},\omega)$ is the Fourier transform of $\boldsymbol{F}(\hat{\boldsymbol{z}},t)$. The interpretation of Eq. (B.4) is that $-4\pi i/(\mu_0\omega)\tilde{\boldsymbol{E}}_0^*(\omega)\cdot\tilde{\boldsymbol{F}}(\hat{\boldsymbol{z}},\omega)$ is the energy spectrum of the sum of the scattered and absorbed energies. To obtain the sum of the time average scattered and absorbed powers for an incident time-harmonic wave $\boldsymbol{E}^i(z,\omega) = \boldsymbol{E}_0(\omega)e^{i\omega z/c_0}$ one takes the real part of the energy spectrum and multiplies with one half. This gives the well-known formula for the optical theorem in the frequency domain

$$W_s + W_a = -\frac{2\pi}{\mu_0\omega}\mathrm{Re}\left\{i\tilde{\boldsymbol{E}}_0^*(\omega)\cdot\tilde{\boldsymbol{F}}(\hat{\boldsymbol{z}},\omega)\right\}. \tag{B.5}$$

References

[1] C. F. Bohren and D. R. Huffman. *Absorption and Scattering of Light by Small Particles.* John Wiley & Sons, New York, 1983.

[2] A. T. de Hoop. A time domain energy theorem for scattering of plane electromagnetic waves. *Radio Sci.*, **19**, 1179–1184, 1984.

[3] A. T. de Hoop. A time–domain energy theorem for scattering of plane acoustic waves in fluids. *J. Acoust. Soc. Am.*, **77**, 11–14, 1985.

[4] J. D. Jackson. *Classical Electrodynamics.* John Wiley & Sons, New York, second edition, 1975.

[5] A. Karlsson. On the time-domain version of the optical theorem. Technical Report LUTEDX/(TEAT-7077)/1–10/(1998), Lund Institute of Technology, Department of Electromagnetic Theory, P.O. Box 118, S-211 00 Lund, Sweden, 1998.

[6] A. Karlsson and G. Kristensson. Constitutive relations, dissipation and reciprocity for the Maxwell equations in the time domain. *J. Electro. Waves Applic.*, **6**(5/6), 537–551, 1992.

[7] L. D. Landau and E. M. Lifshitz. *Quantum Mechanics.* Pergamon, Oxford, third edition, 1977.

[8] R. Newton. Optical theorem and beyond. *Am. J. Phys*, **44**, 639–642, 1976.

[9] B. Peterson and S. Ström. T-matrix for electromagnetic scattering from an arbitrary number of scatterers and representations of E(3). *Phys. Rev. D*, **8**, 3661–3678, 1973.

[10] B. Peterson and S. Ström. Matrix formulation of acoustic scattering from an arbitrary number of scatterers. *J. Acoust. Soc. Am.*, **56**, 771–780, 1974.

[11] B. Peterson and S. Ström. T-matrix formulation of electromagnetic scattering from multilayered scatterers. *Phys. Rev. D*, **10**(8), 2670–2684, 1974.

[12] S. Ström. T matrix for electromagnetic scattering from an arbitrary number of scatterers with continuously varying electromagnetic properties. *Phys. Rev. D*, **10**(8), 2685–2690, 1974.