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Abstract

Dispersion of electromagnetic waves is usually described in terms of an integro-differential equation. In this paper we show that whenever a differential operator can be found that annihilates the susceptibility kernel of the medium, then dispersion can be modeled by a partial differential equation without nonlocal operators.

1 Introduction

It is well-known [7] that dispersion can be modeled by using local partial differential equations of the form

$$\left(\frac{\partial}{\partial t} + c_1 \frac{\partial}{\partial x}\right) \cdots \left(\frac{\partial}{\partial t} + c_n \frac{\partial}{\partial x}\right) \phi + \left(\frac{\partial}{\partial t} + a_1 \frac{\partial}{\partial x}\right) \cdots \left(\frac{\partial}{\partial t} + a_m \frac{\partial}{\partial x}\right) \phi = 0$$

where $m < n$. The solutions of such equations are decomposed into a hierachy of waves with the "higher order waves" characterized by the phase velocities $c_1, \ldots, c_n$ and the "lower order waves" by the velocities $a_1, \ldots, a_m$.

The dispersion of electromagnetic waves in dielectric media is described in most of the texts by an integro-differential equation of the form [1]

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) \phi + \int_0^t \chi(t - t')\phi(x, t') \, dt' = 0$$

which involves an integral operator (with χ being proportional to the so-called susceptibility kernel) that is nonlocal in time. While this is not inordinately difficult to deal with, it is frequently simpler in numerical and analytical computations to use equations containing only local partial differential operators, as in the first paragraph.

Recently a paper [6] has appeared which uses just such a local differential equation to describe wave propagation in a Debye medium. The Debye model is an effective description of electromagnetic wave propagation in polar liquids, such as water, for a wide range of frequencies.

In this paper, we show that whenever one is able to find a differential operator $\mathbf{P}$ that annihilates the susceptibility kernel χ appearing in the above integro-differential equation, then it is possible to derive a local differential operator to replace the nonlocal integro-differential operator. This enables us to derive partial differential equations describing wave propagation in Lorentz and chiral media in addition to the Debye medium mentioned above. We then use well-known asymptotic methods to derive the behavior of both the higher and lower order waves for each of these media.

Section 2 contains a general description of the derivation of local differential equations appropriate for Debye and Lorentz media. The hyperbolicity of the resulting differential operators is proven in Section 3. Sections 4 and 5 contain symptotic analyses for Debye and Lorentz media, respectively. Finally, a matrix extension of the procedure used in Section 2 is presented in Section 6 and applied to the Condon model of a chiral medium.

2   Derivation of Local Differential Operators

Consider a homogeneous, dispersive dielectric medium occupying the half-space $z > 0$. The medium is assumed to be quiescent for time $t < 0$ and is characterized by the constitutive relations

$$\begin{cases} \boldsymbol{D}(\boldsymbol{r}, t) = \epsilon_0 \left[\boldsymbol{E}(\boldsymbol{r}, t) + \int_0^t \boldsymbol{E}(\boldsymbol{r}, t - t')\chi(t')\, dt' \right] \\ \boldsymbol{H}(\boldsymbol{r}, t) = \dfrac{1}{\mu_0} \boldsymbol{B}(\boldsymbol{r}, t) \end{cases} \tag{2.1}$$

We consider two specific models of dispersion.

In the relaxation or Debye model ($\alpha \geq 0$, $\tau > 0$),

$$\chi(t) = \alpha e^{-t/\tau}$$

Here χ is the unique solution of the problem

$$\begin{cases} \{L_1(\chi)\}(t) := \tau \chi_t(t) + \chi(t) = 0, \quad t > 0 \\ \chi(0) = \alpha \end{cases} \tag{2.2}$$

In the resonance or Lorentz model ($\omega_p \geq 0$, $\omega_0 \geq 0$, $\nu \geq 0$),

$$\chi(t) = \omega_p^2 e^{-\frac{\nu t}{2}} \frac{\sin \nu_0 t}{\nu_0}, \quad \nu_0^2 = \omega_0^2 - \frac{\nu^2}{4}$$

In this case χ is the unique solution of the problem

$$\begin{cases} \{L_2(\chi)\}(t) := \chi_{tt}(t) + \nu \chi_t(t) + \omega_0^2 \chi(t) = 0, \quad t > 0 \\ \chi(0) = 0 \\ \chi_t(0) = \omega_p^2 \end{cases} \tag{2.3}$$

We now assume that the dielectric is illuminated by a normally incident plane wave which we assume, without loss of generality, to be linearly polarized. Then using Maxwell's equations and the constitutive relation (2.1) it can be shown that the electric field is the solution of the following problem:

$$\begin{cases} \{M(E)\}(z, t) := E_{zz}(z, t) - \dfrac{1}{c^2} \left[E_{tt}(z, t) + \int_0^t \chi(t - t') E_{tt}(z, t')\, dt' \right] \\ \qquad\qquad = 0, \quad z > 0, \quad t > 0 \\ E(z, 0) = 0, \quad E_t(z, 0) = 0, \quad z > 0 \\ E(0, t) = f(t), \quad t > 0 \end{cases} \tag{2.4}$$

Applying the operator L_1 defined in problem (2.2) to $M(E)$ defined in problem (2.4) for a Debye medium yields an operator without a time convolution and consisting entirely of local space and time derivatives:

$$\{L_1(M(E))\}(z,t) = \tau\frac{\partial}{\partial t}\left[E_{zz}(z,t) - \frac{1}{c^2}E_{tt}(z,t)\right] + E_{zz}(z,t) - \frac{1}{a^2}E_{tt}(z,t)$$

where

$$a^2 := \frac{c^2}{\alpha\tau + 1} < c^2$$

A local operator such as this arises naturally because L_1 annihilates the susceptibility kernel χ.

Since the solution to problem (2.2) is unique, the conditions

$$\begin{cases} \{L_1(M(E))\}(z,t) = 0, & z > 0, \quad t > 0 \\ \{M(E)\}(z,0) = 0, & z > 0 \end{cases}$$

imply that

$$\{M(E)\}(z,t) = 0, \quad z > 0, \quad t > 0$$

Of course

$$\{M(E)\}(z,0) = E_{zz}(z,0) - \frac{1}{c^2}E_{tt}(z,0) = 0, \quad z > 0$$

Since the medium is assumed to be quiescent for $z > 0$, $t \le 0$, we have

$$E(z,0) = 0, \quad z > 0$$

so

$$E_{tt}(z,0) = 0, \quad z > 0$$

Hence, for a Debye medium, problem (2.4) is equivalent to the problem

$$\begin{cases} \tau\frac{\partial}{\partial t}\left[E_{zz}(z,t) - \frac{1}{c^2}E_{tt}(z,t)\right] + E_{zz}(z,t) - \frac{1}{a^2}E_{tt}(z,t) = 0, & z > 0,\ t > 0 \\ E(z,0) = 0, \quad E_t(z,0) = 0, \quad E_{tt}(z,0) = 0, \quad z > 0 \\ E(0,t) = f(t), \quad t > 0 \end{cases} \qquad (2.5)$$

Applying the operator L_2 defined in problem (2.3) to $M(E)$ for a Lorentz medium yields another local differential operator:

$$\{L_2(M(E))\}(z,t) = \frac{\partial^2}{\partial t^2}\left[E_{zz}(z,t) - \frac{1}{c^2}E_{tt}(z,t)\right] + \nu\frac{\partial}{\partial t}\left[E_{zz}(z,t) - \frac{1}{c^2}E_{tt}(z,t)\right]$$
$$+ \omega_0^2\left[E_{zz}(z,t) - \frac{1}{a^2}E_{tt}(z,t)\right]$$

where

$$a^2 := \frac{c^2\omega_0^2}{\omega_p^2 + \omega_0^2} < c^2$$

One can then show in the same way as above that, for a Lorentz medium, problem (2.4) is equivalent to the problem

$$\begin{cases} \frac{\partial^2}{\partial t^2}\left[E_{zz}(z,t) - \frac{1}{c^2}\,E_{tt}(z,t)\right] + \nu\frac{\partial}{\partial t}\left[E_{zz}(z,t) - \frac{1}{c^2}\,E_{tt}(z,t)\right] \\ \qquad +\ \omega_0^2\left[E_{zz}(z,t) - \frac{1}{a^2}\,E_{tt}(z,t)\right] = 0, \quad z > 0, \quad t > 0 \\ E(z,0) = 0, \quad E_t(z,0) = 0, \quad E_{tt}(z,0) = 0, \quad E_{ttt}(z,0) = 0, \quad z > 0 \\ E(0,t) = f(t), \quad t > 0 \end{cases} \qquad (2.6)$$

In the following sections we investigate the properties of solutions of problems (2.5) and (2.6) and the operators appearing in these problems.

3   Hyperbolicity of Differential Operators

Define the differential operator K_1 by

$$\begin{aligned} \{K_1(E)\}(z,t) &:= \{L_1(M(E))\}(z,t) \\ &= \tau\frac{\partial}{\partial t}\left[E_{zz}(z,t) - \frac{1}{c^2}E_{tt}(z,t)\right] + E_{zz}(z,t) - \frac{1}{a^2}E_{tt}(z,t) \end{aligned}$$

Replace the $z-$ and $t-$ derivatives by ix and iy, respectively, to get the polynomial (the symbol for the PDE)

$$P(x,y) := i\tau y(-x^2 + \frac{y^2}{c^2}) + (-x^2 + \frac{y^2}{a^2})$$

The polynomial corresponding to the principal part of K_1 is

$$P_3(x,y) := i\tau y(-x^2 + \frac{y^2}{c^2})$$

The normal $\hat{n} = n_x\hat{i} + n_y\hat{j}$ to the characteristic curves of K_1 satisfies $P_3(n_x, n_y) = 0$, i.e.,

$$n_y = 0 \quad \text{and} \quad n_x = \pm\frac{n_y}{c}$$

corresponding, respectively, to the characteristics

$$z = \text{constant} \quad \text{and} \quad t = \pm\frac{z}{c} + \text{constant}$$

The differential operator K_1 is said to be hyperbolic [2] in the $\hat{n}$ direction if:

1. the complex roots of $P(x + n_x\lambda, y + n_y\lambda) = 0$ satisfy $\text{Im}\,\lambda \geq \gamma$ for all real x and y and some constant γ,

2. $P_3(n_x, n_y) \neq 0$.

For $\hat{n} = \hat{i} = (1, 0)$, corresponding to Cauchy data given on the line $z = \text{constant}$, $P_3(1, 0) = 0$. So K_1 is not hyperbolic in the $(1,0)$ direction. For $\hat{n} = \hat{j} = (0, 1)$, corresponding to Cauchy data given on the line $t = \text{constant}$, $P_3(0, 1) = \frac{i\tau}{c^2} \neq 0$. Moreover

$$P(x, y + \lambda) = \frac{i\tau}{c^2}(y + \lambda)^3 + \frac{1}{a^2}(y + \lambda)^2 - i\tau x^2(y + \lambda) - x^2 = 0 \qquad (3.1)$$

Let $z = i(y + \lambda) + \gamma$, where $\gamma \in \mathbb{R}$, in (3.1) to get the real polynomial

$$p(z) := \frac{\tau}{c^2}(z - \gamma)^3 + \frac{1}{a^2}(z - \gamma)^2 + \tau x^2(z - \gamma) + x^2 = 0 \qquad (3.2)$$

We show that the roots of (3.1) satisfy $\text{Im}\,\lambda \geq 0$ by investigating the stability of the polynomial $p(z)$ defined in (3.2).

Recall that a real polynomial $p(z)$ is stable [3] if all of its zeros have negative real part. It can be shown [3] that the real polynomial $p(z) = \alpha_3 z^3 + \alpha_2 z^2 + \alpha_1 z + \alpha_0$, where $\alpha_3 > 0$, is stable if and only if $\alpha_2 > 0, \alpha_0 > 0$ and $\alpha_1\alpha_2 - \alpha_0\alpha_3 > 0$. For our polynomial $p(z)$

$$\alpha_3 = \frac{\tau}{c^2} \qquad\qquad \alpha_2 = -3\frac{\tau}{c^2}\gamma + \frac{1}{a^2}$$

$$\alpha_1 = 3\frac{\tau}{c^2}\gamma^2 - 2\frac{\gamma}{a^2} + \tau x^2 \qquad \alpha_0 = -\frac{\tau}{c^2}\gamma^3 + \frac{\gamma^2}{a^2} - \tau x^2\gamma + x^2$$

Clearly, if $\gamma < 0$, then

$$\alpha_3 > 0, \quad \alpha_2 > 0 \quad\text{and}\quad \alpha_0 > 0$$

Moreover,

$$\alpha_1\alpha_2 - \alpha_0\alpha_3 = -8\frac{\tau^2}{c^4}\gamma^3 + 8\frac{\tau}{c^2 a^2}\gamma^2 - \left(\frac{2}{a^4} + \frac{2\tau^2 x^2}{c^2}\right)\gamma + \frac{a\tau^2}{c^2}x^2 > 0$$

if $\gamma < 0$. Thus $p(z)$ is stable if $\gamma < 0$. So the roots of (3.1) satisfy

$$\text{Im}\,\lambda = \text{Im}(y + \lambda) = -\,\text{Re}\,z + \gamma > \gamma$$

for all real x and y and any negative constant γ. So K_1 is hyperbolic in the $\hat{n} = (0, 1)$ direction.

We can prove the hyperbolicity of the differential operator K_2 defined by

$$\{K_2(E)\}(z, t) := \{L_2(M(E))\}(z, t) = \frac{\partial^2}{\partial t^2}\left[E_{zz}(z, t) - \frac{1}{c^2}E_{tt}(z, t)\right]$$

$$+ \nu\frac{\partial}{\partial t}\left[E_{zz}(z, t) - \frac{1}{c^2}E_{tt}(z, t)\right] + \omega_0^2\left[E_{zz}(z, t) - \frac{1}{a^2}E_{tt}(z, t)\right]$$

in the (0,1) direction in the same way by considering the polynomials

$$P(x,y) = -y^2 \left(-x^2 + \frac{y^2}{c^2} \right) + i\nu y \left(-x^2 + \frac{y^2}{c^2} \right) + \omega_0^2 \left(-x^2 + \frac{y^2}{a^2} \right)$$

$$P_4(x,y) = -y^2 \left(-x^2 + \frac{y^2}{c^2} \right)$$

The characteristics of K_2 are

$$z = \text{ constant} \qquad \text{and} \qquad t = \pm \frac{z}{c} + \text{ constant}$$

So the differential operator K_2 is not hyperbolic in the (1,0) direction. To show hyperbolicity in the (0,1) direction, we again let $z = i(y + \lambda) + \gamma$, $\gamma \in \mathbb{R}$, in $P(x, y + \lambda) = 0$ to get the real polynomial

$$p(z) := (z - \gamma)^4 + \nu(z - \gamma)^3 + c^2 \left(x^2 + \frac{\omega_0^2}{a^2} \right) (z - \gamma)^2 + \nu c^2 x^2 (z - \gamma) + \omega_0^2 c^2 x^2 = 0$$

$$(3.3)$$

By using the result [3] that a real polynomial $p(z) = \alpha_4 z^4 + \alpha_3 z^3 + \alpha_2 z^2 + \alpha_1 z + \alpha_0$, where $\alpha_4 > 0$, is stable if and only if $\alpha_3 > 0$, $\alpha_0 > 0$, $\alpha_3 \alpha_2 - \alpha_4 \alpha_1 > 0$, $\alpha_1(\alpha_3 \alpha_2 - \alpha_4 \alpha_1) - \alpha_0 \alpha_3^2 > 0$, we can show that $p(z)$ in (3.3) is stable if $\gamma < 0$. So K_2 is hyperbolic in the $\hat{n} = (0, 1)$ direction.

4 Asymptotic Analysis—Debye Model

The equations describing dispersion expressed in terms of local differential operators can be used as a launching point for asymptotic analysis [6]. We follow the analysis of Whitham [7] for a different differential equation.

The partial differential equation in problem (2.5) appropriate for a Debye medium can be rewritten in the form

$$\eta \left(\frac{\partial}{\partial t} + c \frac{\partial}{\partial z} \right) \frac{\partial}{\partial t} \left(\frac{\partial}{\partial t} - c \frac{\partial}{\partial z} \right) E(z,t) + \left(\frac{\partial}{\partial t} + a \frac{\partial}{\partial z} \right) \left(\frac{\partial}{\partial t} - a \frac{\partial}{\partial z} \right) E(z,t) = 0$$

$$(4.1)$$

where

$$\eta = \frac{a^2}{c^2} \tau = \frac{\tau}{\alpha \tau + 1}$$

For water at microwave frequencies, $\tau \approx 8\,p$ sec. and $\alpha \tau \approx 80$ [6]. Equation (4.1) can be solved using Laplace transforms in the t variable. Write $E(z,t)$ in terms of its Laplace transform $\widetilde{E}(z,p)$,

$$E(z,t) = \frac{1}{2\pi i} \int_{\mathcal{B}} \widetilde{E}(z,p) e^{pt} dp, \quad t > 0 \tag{4.2}$$

where $\mathcal{B}$ is a Bromwich contour $Re\,p = $ constant. Substitution of (4.2) into (4.1) yields an equation for $\widetilde{E}$ which has solution

$$\widetilde{E}(z,p) = F(p)e^{zP_1(p)} + G(p)e^{zP_2(p)}$$

where

$$P_1(p) = -\frac{p}{c}\sqrt{\frac{1+\eta p}{\eta p + a^2/c^2}} \quad \text{and} \quad P_2(p) = \frac{p}{c}\sqrt{\frac{1+\eta p}{\eta p + a^2/c^2}}$$

As $Re\,p \to \infty$, P_1 and P_2 behave as

$$P_1(p) = -\frac{p}{c} - \frac{\alpha}{2c} + 0(p^{-1}) \quad \text{and} \quad P_2(p) = \frac{p}{c} + \frac{\alpha}{2c} + 0(p^{-1}) \tag{4.3}$$

Hence we must choose $G(p) = 0$. So

$$E(z,t) = \frac{1}{2\pi i}\int_{\mathcal{B}} F(p)e^{pt + P_1(p)z}dp \tag{4.4}$$

At $z = 0$,

$$f(t) = E(0,t) = \frac{1}{2\pi i}\int_{\mathcal{B}} F(p)e^{pt}dp$$

so

$$F(p) = \int_0^\infty f(t)e^{-pt}dt \tag{4.5}$$

By choosing $\mathcal{B}$ far enough to the right in the complex p-plane, we can substitute (4.3) in (4.4) to conclude that asymptotically

$$E(z,t) \sim f(t - \frac{z}{c})\exp(\frac{-\alpha z}{2c})$$

along the characteristic $t = z/c$.

A perturbation procedure can be used to deduce this result. By substituting

$$E(z,t) = \sum_{n=0}^\infty \eta^n E_n(\xi,\sigma)$$

where

$$\xi = \eta^{-2}(z - ct) \quad \text{and} \quad \sigma = \eta^{-1}t$$

into problem (2.5), we find that E_0 is the solution of the problem

$$\begin{cases} \left(\dfrac{\partial}{\partial t} + c\dfrac{\partial}{\partial t}\right)E_0(z,t) + \dfrac{1}{\eta}\dfrac{c^2 - a^2}{2c^2}E_0(z,t) = 0, & z > 0, \quad t > 0 \\ E_0(z,0) = 0, & z > 0 \\ E_0(0,t) = f(t), & t > 0 \end{cases}$$

This problem is easily solved to yield

$$E_0(z,t) = f(t - \frac{z}{c}) \exp\left(-\frac{\alpha z}{2c}\right)$$

The behavior of the electric field along the line $z = at$ can be deduced by applying the method of steepest descent to the representation of E given in (4.4) and (4.5). The saddle points of $g(p) = pt + P_1(p)z$ occur at those $p = \hat{p}$ for which

$$0 = \frac{d}{dp}[pt + P_1(p)z] = t + zP_1'(\hat{p}) \tag{4.6}$$

This equation determines $\hat{p}$ as a function of z and t. To get the first term in the asymptotic expansion, we expand $g(p)$ up to quadratic terms in $p - \hat{p}$ and deform the contour of integration into the curve of steepest descent $\mathcal{C}$ through $p = \hat{p}$. Hence

$$E(z,t) \sim \exp[t\hat{p} + zP_1(\hat{p})]\frac{1}{2\pi i}\int_{\mathcal{C}} F(p) \exp\left[\frac{1}{2}zP_1''(\hat{p})(p - \hat{p})^2\right] dp$$

To simplify the analysis we assume that $\int_0^{\infty} |f(t)|dt < \infty$ so that $F(p)$ has no singularity at $p = 0$.

The exponential outside the integral is stationary in z when $P_1(\hat{p}) = 0$, that is

$$\hat{p} = 0 \;\; \text{or} \;\; \hat{p} = -\frac{1}{\eta}$$

The point $p = -1/\eta$ is a branch point of P_1, and P_1 has another singularity at $p = -a^2/c^2\eta$. So $\hat{p} = 0$ is the appropriate point through which to pass the contour $\mathcal{C}$. Returning to (4.6), we find that those z and t for which $\hat{p} = 0$ is a saddle point satisfy $z = at$. Moreover the exponential factor is easily shown to be maximized at $\hat{p} = 0$, *i.e.*, along $z = at$.

Near $p = 0$,

$$P_1(p) = -\frac{p}{a} + \frac{\eta}{2}\frac{c^2 - a^2}{a^3}p^2 + 0(p^3)$$

so, upon substitution in (4.4),

$$E(z,t) \sim \frac{1}{2\pi i}\int_{\mathcal{C}} F(p) \exp\left[p(t - \frac{z}{a}) + \frac{p^2\eta(c^2 - a^2)}{2a^3}z\right] dp \tag{4.7}$$

To a first approximation

$$E(z,t) \sim f(t - z/a)$$

So the electric field is exponentially small except in the neightborhood of $z = at$ along which the main part of the signal travels [7]. The effect of the quadratic term

in (4.7) can be evaluated by noting that (4.7) is a solution, at least symbolically, of the ill-posed problem

$$
\begin{cases}
\left(\dfrac{\partial}{\partial t} + a\dfrac{\partial}{\partial z}\right)\phi(z,t) = \dfrac{\eta(c^2 - a^2)}{2a^2}\dfrac{\partial^2}{\partial t^2}\phi(z,t), & z > 0, \quad t > 0 \\[2mm]
\phi(z,0) = 0, \quad \phi_t(z,0) = 0, \quad z > 0 \\[2mm]
\phi(0,t) = f(t), \quad t > 0
\end{cases}
$$

Noting that $\frac{\eta(c^2-a^2)}{2a^2}\phi_{tt}$ is of order η/T compared with ϕ_t and $a\phi_z$, where T is a time characteristic of variations in $f(t)$, it is consistent to replace $\frac{\partial}{\partial t}$ by $a\frac{\partial}{\partial z}$ in the second derivative term of the above partial differential equation to get the well-posed problem

$$
\begin{cases}
\left(\dfrac{\partial}{\partial t} + a\dfrac{\partial}{\partial z}\right)\phi(z,t) = \dfrac{\eta}{2}(c^2 - a^2)\dfrac{\partial^2}{\partial z^2}\phi(z,t), & z > 0, \ t > 0 \\[2mm]
\phi(z,0) = 0, \ z > 0 \\[2mm]
\phi(0,t) = f(t), \ t > 0
\end{cases}
\tag{4.8}
$$

This problem has the solution

$$
\phi(z,t) = \frac{z}{\sqrt{4\pi d}} \int_0^t \frac{f(\tau)}{\sqrt{(t-\tau)^3}} \exp\left(-\frac{1}{4d}\frac{[z - a(t-\tau)]^2}{t-\tau}\right) d\tau \sim f\left(t - \frac{z}{a}\right)
$$

where

$$
d = \frac{\eta}{2}(c^2 - a^2)
$$

Contrary to results presented in reference [6], we do not replace $f(t) = E(0,t)$ in this equation for $\phi(z,t)$ by

$$
f(t - z_0/c)\exp\left(-\frac{\alpha z_0}{2c}\right)
$$

for $z_0 \sim 2c/\alpha$ since the asymptotic result obtained immediately after (4.5) is valid only along the characteristic $t = z/c$. This is indicated by the Tauberian theorems for the Laplace transform.

Again, this result can be derived by using a perturbation procedure. By substituting

$$
E = \sum_{n=0}^{\infty} \eta^{n/2} E_n(\xi, t)
$$

where

$$
\xi = \eta^{-1/2}(z - at)
$$

into problem (2.5), one finds that E_0 solves the problem (4.8) above. So

$$
E_0(z,t) = \frac{z}{\sqrt{4\pi d}} \int_0^t \frac{f(\tau)}{\sqrt{(t-\tau)^3}} \exp\left(-\frac{1}{4d}\frac{[z - a(t-\tau)]^2}{t-\tau}\right) d\tau
$$

5 Asymptotic Analysis—Lorentz Model

A similar analysis can be performed for a Lorentz medium via the partial differential equation appearing in problem (2.6), which we rewrite as

$$\eta \left(\frac{\partial^2}{\partial t^2} + \nu \frac{\partial}{\partial t} \right) \left(\frac{\partial}{\partial t} + c \frac{\partial}{\partial z} \right) \left(\frac{\partial}{\partial t} - c \frac{\partial}{\partial z} \right) E(z,t)$$
$$+ \left(\frac{\partial}{\partial t} + a \frac{\partial}{\partial t} \right) \left(\frac{\partial}{\partial t} - a \frac{\partial}{\partial z} \right) E(z,t) = 0 \tag{5.1}$$

where

$$\eta = \frac{a^2}{c^2 \omega_0^2} = \frac{1}{\omega_p^2 + \omega_0^2}$$

For valence electrons $\omega_0 \approx 10^{15}$ sec.$^{-1}$, and in a liquid or solid $\omega_p \approx \omega_0$ [4].

Writing $E(z,t)$ in terms of its Laplace transform $\widetilde{E}(z,p)$ as in (4.2) and substituting into (5.1) yields

$$\widetilde{E}(z,p) = F(p)e^{zP_1(p)} + G(p)e^{zP_2(p)}$$

where

$$P_1(p) = -\frac{p}{c} \sqrt{\frac{1 + \eta \nu p + \eta p^2}{a^2/c^2 + \eta \nu p + \eta p^2}} \quad \text{and} \quad P_2(p) = \frac{p}{c} \sqrt{\frac{1 + \eta \nu p + \eta p^2}{a^2/c^2 + \eta \nu p + \eta p^2}}$$

As Re $p \to \infty$, P_1 and P_2 behave as

$$P_1(p) = -\frac{p}{c} - \left(\frac{c^2 - a^2}{2\eta c^3} \right) \frac{1}{p} + 0(p^{-2}) \quad \text{and} \quad P_2(p) = \frac{p}{c} + \left(\frac{c^2 - a^2}{2\eta c^3} \right) \frac{1}{p} + 0(p^{-2}) \tag{5.2}$$

Again, we must choose $G(p) = 0$. So

$$E(z,t) = \frac{1}{2\pi i} \int_{\mathcal{B}} F(p)e^{pt + P_1(p)z} dp \tag{5.3}$$

where $F(p)$ is the Laplace transform of $f(t)$, the boundary condition in problem (2.6).

By choosing $\mathcal{B}$ far enough to the right, we can substitute the expansion (5.2) into (5.3) and conclude that asymptotically, along the characteristic $z = ct$,

$$E(z,t) \sim \frac{1}{2\pi i} \int_{\mathcal{B}} F(p) \exp \left[p(t - z/c) - \left(\frac{c^2 - a^2}{2\eta c^3} \right) \frac{z}{p} \right] dp$$
$$= f(t - z/c) - \sqrt{\left(\frac{c^2 - a^2}{2\eta c^3} \right) z} \int_0^t \frac{f(t - s - z/c)}{\sqrt{s}} J_1 \left(2\sqrt{\left(\frac{c^2 - a^2}{2\eta c^3} \right) zs} \right) ds \tag{5.4}$$

Letting

$$f(t) = (\sin \omega t)H(t) \approx \omega t H(t) \text{ for } 0 < t \ll \frac{1}{\omega}$$

we get

$$E(z,t) \sim \omega \sqrt{\frac{t - z/c}{\gamma z}} \, J_1\left(2\sqrt{\gamma z(t - z/c)}\right) \tag{5.5}$$

where

$$\gamma = \frac{c^2 - a^2}{2\eta c^3} = \frac{\omega_p^2}{2c}$$

Equation (5.5) is known as Sommerfeld's first precursor.

A perturbation procedure can be used to deduce (5.4). By substituting

$$E(z,t) = \sum_{n=0}^{\infty} \eta^n E_n(\xi, t)$$

where

$$\xi = \eta^{-1}(z - ct)$$

into problem (2.6), it can be shown that E_0 is the solution of the problem

$$\begin{cases} \dfrac{\partial}{\partial z}\left(\dfrac{\partial}{\partial t} + c\dfrac{\partial}{\partial z}\right) E_0(z,t) - \dfrac{c^2 - a^2}{2\eta c^3} E_0(z,t) = 0, & z > 0, \quad t > 0 \\ E_0(z,0) = 0, \quad z > 0 \\ E_0(0,t) = f(t), \quad t > 0 \end{cases}$$

Moreover,

$$E_0(z,t) \sim \frac{1}{2\pi i} \int_{\mathcal{B}} F(p) \exp\left[p(t - z/c) - \left(\frac{c^2 - a^2}{2\eta c^3}\right)\frac{z}{p}\right] dp$$

Applying the method of steepest descent yields the behavior of $E(z,t)$ along the line $z = at$. Equation (4.6) determines the saddle points $\hat{p}$ of $g(p) = pt + P_1(p)z$ as functions of z and t, and

$$E(z,t) \sim \exp[t\hat{p} + zP_1(\hat{p})]\frac{1}{2\pi i} \int_{\mathcal{C}} F(p) \exp\left[\frac{1}{2}zP_1''(\hat{p})(p - \hat{p})^2\right] dp$$

where $\mathcal{C}$ is the curve of steepest descent through $\hat{p}$. So that $F(p)$ has no singularity at $p = 0$, we assume that $\int_0^{\infty} |f(t)| \, dt < \infty$.

The exponential factor outside the integral is stationary in z when $P_1(\hat{p}) = 0$, that is

$$\hat{p} = 0 \text{ or } \hat{p} = -\frac{\nu}{2} \pm i\sqrt{\frac{1}{\eta} - \frac{\nu^2}{4}}$$

As with the Debye case, $\hat{p} = 0$ is the appropriate point through which to pass the contour $\mathcal{C}$. Using (4.6), we find that those z and t for which $\hat{p} = 0$ is a saddle point lie along the line $z = at$. The exponential factor is again easily shown to be maximized.

Near $p = 0$,

$$P_1(p) = -\frac{p}{a} + \frac{\eta\nu}{2}\left(\frac{c^2 - a^2}{a^3}\right)p^2 + 0(p^3)$$

so

$$E(z,t) \sim \frac{1}{2\pi i}\int_{\mathcal{C}} F(p) \exp\left[p(t - z/a) + \frac{p^2\eta\nu(c^2 - a^2)}{2a^3}z\right]dp$$

Similarly to the Debye case this will be asymptotic to the solution of the problem

$$\begin{cases} \left(\frac{\partial}{\partial t} + a\frac{\partial}{\partial z}\right)\phi(a,t) = \frac{\eta\nu}{2}(c^2 - a^2)\frac{\partial^2}{\partial z^2}\phi(z,t), & z > 0, \quad t > 0 \\ \phi(z,0) = 0, \quad t > 0 \\ \phi(0,t) = f(t), \quad t > 0 \end{cases}$$

Letting $d = \eta\nu(c^2 - a^2)/2$, this problem has solution

$$\phi(z,t) = \frac{z}{\sqrt{4\pi d}}\int_0^t \frac{f(\tau)}{\sqrt{(t-\tau)^3}} \exp\left(-\frac{1}{4d}\frac{[z - a(t-\tau)]^2}{t - \tau}\right)d\tau \sim f(t - z/a)$$

So, as with a Debye medium, the main signal arrives with speed a and appears asymptotically as

$$E(z,t) \sim f(t - z/a) \tag{5.6}$$

with broadening due to dispersion.

The result (5.6) can be derived using a perturbation procedure by substituting

$$E(z,t) = \sum_{n=0}^{\infty} \eta^\nu E_n(z,t)$$

into problem (2.6). One readily finds that

$$E_0(z,t) = f(t - z/a)$$

6 Matrix Extension—Condon Model

The procedure described in section II for deriving equivalent local differential equations for dispersion has a matrix extension that allows one to deduce local differential equations describing chiral media. Consider a homogeneous, dissipative, reciprocal,

bi-isotropic medium [5] filling the half-space $z > 0$ that is assumed to be quiescent for time $t < 0$. Such a medium is characterized by constitutive relations for the form

$$\begin{cases} \boldsymbol{D}(\boldsymbol{r}, t) = \epsilon_0 \left[\boldsymbol{E}(\boldsymbol{r}, t) + \int_0^t \boldsymbol{E}(\boldsymbol{r}, t - t')G(t')\, dt' + c \int_0^t \boldsymbol{B}(\boldsymbol{r}, t - t')K(t')\, dt' \right] \\ \boldsymbol{H}(\boldsymbol{r}, t) = \epsilon_0 \left[c \int_0^t \boldsymbol{E}(\boldsymbol{r}, t - t')K(t')\, dt' + c^2 \boldsymbol{B}(\boldsymbol{r}, t') \right], \quad c^2 = \dfrac{1}{\epsilon_0 \mu_o} \end{cases}$$

(6.1)

where, in the Condon model,

$$\begin{cases} G(t) = \omega_p^2 e^{-\frac{\nu t}{2}} \dfrac{\sin \nu_0 t}{\nu_0}, \quad \nu_0^2 = \omega_0^2 - \dfrac{\nu^2}{4} \\ K(t) = \dfrac{\omega_p^2}{f \nu_0} e^{-\frac{\nu t}{2}} \left(\nu_0 \cos \nu_0 t - \dfrac{\nu}{2} \sin \nu_0 t \right) \end{cases}$$

The functions G and K are the unique solutions of the problems

$$\begin{cases} \{L_2(G)\}(t) := G_{tt}(t) + \nu\, G_t(t) + \omega_0^2 G(t) = 0, \quad t > 0 \\ G(0) = 0 \\ G_t(0) = \omega_p^2 \end{cases}$$

(6.2)

and

$$\begin{cases} \{L_2(K)\}(t) = 0, \quad t > 0 \\ K(0) = \omega_p^2 / f \\ K_t(0) = -\omega_p^2 \nu / f \end{cases}$$

Assume that the medium is illuminated by a normally incident plane wave. Letting $\hat{x}_1$ and $\hat{x}_2$ be orthogonal unit vectors perpendicular to the z-axis, we write the electric field in column matrix form as

$$\boldsymbol{E}(z, t) = \hat{x}_1 E_1(z, t) + \hat{x}_2 E_2(z, t) = \begin{pmatrix} E_1(z, t) \\ E_2(z, t) \end{pmatrix}$$

Using the functions G and K above, form the matrices

$$\mathbf{G}(t) = \begin{pmatrix} G(t) & 0 \\ 0 & G(t) \end{pmatrix}, \qquad \mathbf{K}(t) = \begin{pmatrix} 0 & -K(t) \\ K(t) & 0 \end{pmatrix}$$

Applying Maxwell's equations and the constitutive relations (6.1), we find that the electric field $\boldsymbol{E}$ solves the following problem:

$$\begin{cases} \{T(\boldsymbol{E})\}(z, t) := \boldsymbol{E}_{tt}(z, t) - \dfrac{1}{c^2} \boldsymbol{E}_{tt} z, t - \dfrac{1}{c^2} \int_0^t \mathbf{G}(t - t') \boldsymbol{E}_{tt}(z, t') dt' \\ \quad + \dfrac{2}{c} \int_0^t \mathbf{K}(t - t') \boldsymbol{E}_{zt}(z, t') dt' = \boldsymbol{0}, \quad z > 0, \quad t > 0 \\ \boldsymbol{E}(z, 0) = \boldsymbol{0}, \quad z > 0 \\ \boldsymbol{E}_t(z, 0) = \boldsymbol{0}, \quad z > 0 \\ \boldsymbol{E}(0, t) = \boldsymbol{f}(t), \quad t > 0 \end{cases}$$

(6.3)

As before, we now apply the operator L_2 of problem (6.2) to $T(\boldsymbol{E})$ defined in problem (6.3) to get an operator consisting entirely of space and time derivatives:

$$\{L_2(T(\boldsymbol{E}))\}(z,t) = \frac{\partial^2}{\partial t^2}\left[\boldsymbol{E}_{zz}(z,t) - \frac{1}{c^2}\boldsymbol{E}_{tt}(z,t)\right] + \nu\frac{\partial}{\partial t}\left[\boldsymbol{E}_{zz}(z,t) - \frac{1}{c^2}\boldsymbol{E}_{tt}(z,t)\right]$$
$$+ \omega_0^2\left[\boldsymbol{E}_{zz}(z,t) - \frac{1}{a^2}\boldsymbol{E}_{tt}(z,t)\right] - \frac{2}{c}\frac{\omega_p^2}{f}\boldsymbol{J}\boldsymbol{E}_{ztt}(z,t)$$

where

$$a^2 = \frac{c^2\omega_0^2}{\omega_p^2 + \omega_0^2} \quad\text{and}\quad \boldsymbol{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Since the solution to problem (6.2) is unique, the conditions

$$\begin{cases} \{L_2(T(\boldsymbol{E}))\}(z,t) = \boldsymbol{0}, & z > 0,\ t > 0 \\ \{T(\boldsymbol{E})\}(z,0) = \boldsymbol{0}, & z > 0 \\ \dfrac{\partial}{\partial t}\{T(\boldsymbol{E})\}(z,0) = \boldsymbol{0}, & z > 0 \end{cases}$$

imply that

$$\{T(\boldsymbol{E})\}(z,t) = \boldsymbol{0},\ z > 0,\ t > 0$$

Here

$$\{T(\boldsymbol{E})\}(z,0) = \boldsymbol{E}_{zz}(z,0) - \frac{1}{c^2}\boldsymbol{E}_{tt}(z,0) = \boldsymbol{0},\ z > 0$$

and

$$\frac{\partial}{\partial t}\{T(\boldsymbol{E})\}(z,0) = \boldsymbol{E}_{zzt}(z,0) - \frac{1}{c^2}\boldsymbol{E}_{ttt}(z,0) + \frac{2}{c}\boldsymbol{K}(0)\boldsymbol{E}_{zt}(z,0) = \boldsymbol{0},\ z > 0$$

If

$$\boldsymbol{E}(z,0) = \boldsymbol{0} \quad\text{and}\quad \boldsymbol{E}_t(z,0),\ z > 0$$

then clearly

$$\boldsymbol{E}_{tt}(z,0) = \boldsymbol{0} \quad\text{and}\quad \boldsymbol{E}_{ttt}(z,0) = \boldsymbol{0},\ z > 0$$

Hence problem (6.3) is equivalent to

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}\left[\boldsymbol{E}_{zz}(z,t) - \dfrac{1}{c^2}\boldsymbol{E}_{tt}(z,t)\right] + \nu\dfrac{\partial}{\partial t}\left[\boldsymbol{E}_{zz}(z,t) - \dfrac{1}{c^2}\boldsymbol{E}_{tt}(z,t)\right] \\ + \omega_0^2\left[\boldsymbol{E}_{zz}(z,t) - \dfrac{1}{a^2}\boldsymbol{E}_{tt}(z,t)\right] + \dfrac{2}{c}\dfrac{\omega_p^2}{f}\boldsymbol{J}\boldsymbol{E}_{ztt}(z,t) = \boldsymbol{0}, & z > 0,\quad t > 0 \\ \boldsymbol{E}(z,0) = \boldsymbol{0}, \quad \boldsymbol{E}_t(z,0) = \boldsymbol{0}, \quad \boldsymbol{E}_{tt}(z,0) = \boldsymbol{0}, \quad \boldsymbol{E}_{ttt}(z,0) = \boldsymbol{0}, & z > 0 \\ \boldsymbol{E}(0,t) = \boldsymbol{f}(t), \quad t > 0 \end{cases}$$

We have assumed for simplicity that the medium is homogeneous, but there is no obstacle to prevent one from deriving more complicated local differential operators in the case when the material parameters depend on position.

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