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# On the propagation problem in a metallic homogeneous bi-isotropic waveguide

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## Abstract

We investigate the problem of defining propagating constants and modes in metallic waveguides of an arbitrary cross section, filled with a homogeneous bi-isotropic material. The approach follows the guidelines of the classical theory for the isotropic, homogeneous, lossless waveguide: starting with the Maxwell system, we formulate a spectral problem where the (square of the) propagation constant shows up as the eigenvalue and the corresponding mode as the eigenvector. The difficulty that arises, and this is a feature of chirality, is that the eigenvalue is involved in the boundary conditions. The main result is that the problem is solvable whenever the Dirichlet problem for the Helmholtz equation in the cross section is solvable. The analysis also confirms the splitting in left and circularly polarized waves.

## 1 Introduction

Wave propagation in bi-isotropic materials is well documented in the literature. The monographs [15] and [17] develop the relevant theory in a systematic way, and they also give an extensive list of references of the accomplishments in the field. As a branch of mathematical physics, bi-isotropic materials have also been studied quite well, see e.g. [25] and the references therein. An investigation of the transmission line problem with bi-isotropic material is given by Olyslager [19].

On the other hand, the theory of a guided wave seems to go back in the history to the years of Lord Rayleigh [22], and a systematic account of it can be found in [6]. We also refer to [27] for some early discussions about the nomenclature and the problem of the mode ordering and classification in an arbitrary waveguide. By a waveguide we here mean a cylinder with a bounded cross section modeled as a cylindrical domain with an axis parallel to the  $z$ -axis in an  $Oxyz$  realization of the 3-space. The cylindrical geometry provides a screened environment, which results in high signal to noise ratio (SNR) in applications involving measurements.

The electromagnetic propagation in a waveguide is principally characterized by three items:

**geometry:** the shape of the cross section of the cylinder or waveguide

**walls:** the properties of the medium from which the cylinder is manufactured

**material:** the properties of the medium enclosed by the walls of the cylinder

In this paper, we adopt a waveguide with perfectly conducting walls and an arbitrary cross section that encloses a homogeneous bi-isotropic medium. From the mathematical point of view, geometry describes the domain, walls provide the boundary conditions and the material defines the appropriate differential equation.

The circular chiral waveguide, referred as “chirowaveguide”, drew some attention in the 90’s. In fact, the topic was already discussed a few years earlier [11]. In [12] a complete solution was provided and a dispersion relation was obtained. The (numerical) solution of this equation gives the propagation constants. Reinert *et al.* [23]

discussed the theory of the inverse problem. A theory concerning the chirowaveguide with an arbitrary cross section was presented in Ref. 20. For the bi-isotropic waveguide a similar treatment was made in Ref. 17. However, the eigenvalue problem (and the operator it concerns) is not very clear in neither of these two references. Early numerical treatments are published by Svedin [26], who implements a FEM method for the propagation problem in a chirowaveguide.

The main aim of this article is to formulate the eigenvalue problem for the metallic homogeneous bi-isotropic waveguide and discuss its solvability, providing a solid mathematical foundation for future reference. The approach follows the guidelines of the classical theory for the metallic isotropic, homogeneous, lossless waveguide, as it is described in [5, Ch. 4], [10, Ch. IX.4], [13, Ch. 8]. Actually, we give two possible forms for the eigenvalue problem; roughly speaking one complicates the operator and the other complicates the boundary conditions. This reveals clearly that the existence of chirality leads always the problem to a non-standard form.

Taking these complicated boundary conditions as the starting point, we reformulate the problem as a functional differential equation (FDE), which involves the Dirichlet-to-Neumann operator for the 2D Helmholtz equation of the cross section. The (implicit) dispersion equation can be realized as the condition for the FDE to admit non-trivial solutions. The solutions of the dispersion equation (expected to consist of a complex sequence with no accumulation points) provide the propagation constants. Going back to the FDE, its corresponding solutions serve as boundary data for the Dirichlet problem for the 2D Helmholtz equation in the cross section. Thus, if we know the corresponding Dirichlet Green function, we can construct the modes of the waveguide.

As an intermediate result, we show that there exists a natural splitting of the field in left and right circularly polarized components. This fact confirms the Bohren transformation, proposed already in 1974 [2]. Actually, the Bohren transformation has been used extensively in the study of chiral media described by the Drude–Born–Fedorov constitutive relations, see [1] and references therein. Actually, what is new here is that we do not assume *a priori* this splitting as in the majority of the existent work, but we confirm that this is an essential step for the solution.

The paper is organized as follows: in Section 2, the notation and the basic equations are introduced. The waveguide geometry and the perfect conductor boundary condition are discussed in Section 3. In Section 4 we justify the quest for propagation constants via a separation of variables technique, and in Section 5 we formulate the spectral problem in an operator pencil form. Section 6 contains a transformation to a more standard form. One method of solution, utilizing the Dirichlet-to-Neumann (DtN) mapping is discussed in Section 7. In Section 8 we reconsider the problem employing the Null-field approach. Section 10 contains a summary of our results and gives directions for further investigations. In the final appendices we survey various mathematical facts that are used in the text.

## 2 Prerequisites

In this section we formally determine the equation that is used in this paper.

### 2.1 The Maxwell system and material modeling

In the absence of sources the Maxwell equations are

$$\begin{cases} i\omega \mathbf{D} = -\nabla \times \mathbf{H} \\ i\omega \mathbf{B} = \nabla \times \mathbf{E} \end{cases} \quad (\text{M})$$

We have here assumed time harmonic waves with (angular) frequency  $\omega$  (the time convention is taken to be  $e^{-i\omega t}$ ). The bi-isotropic material is modeled by the constitutive relations

$$\begin{cases} \mathbf{D} = \varepsilon \mathbf{E} + \xi \mathbf{H} \\ \mathbf{B} = \zeta \mathbf{E} + \mu \mathbf{H} \end{cases} \quad (\text{CR})$$

or equivalently,

$$\begin{cases} \mathbf{D} = \frac{\varepsilon\mu - \xi\zeta}{\mu} \mathbf{E} + \frac{\xi}{\mu} \mathbf{B} \\ \mathbf{H} = -\frac{\zeta}{\mu} \mathbf{E} + \frac{1}{\mu} \mathbf{B} \end{cases} \quad (\text{CR}')$$

This is the Lindell *et al.* model [17], which is widely used as a model for bi-isotropic materials. Other, equivalent formulations are also frequently used, see Ref. 17. The parameters  $\varepsilon$ ,  $\xi$ ,  $\zeta$ ,  $\mu$  are complex numbers, which, in general, are dependent on the spatial variables and the frequency  $\omega$ . We pose now two postulates concerning these parameters.

**Assumption 1** (Spatial homogeneity).  $\varepsilon$ ,  $\xi$ ,  $\zeta$ ,  $\mu$  are independent of the spatial variables  $\mathbf{r} := x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \in \Omega \subset \mathbb{R}^3$ .

**Assumption 2** (Invertibility of the constitutive matrix).  $d := \varepsilon\mu - \zeta\xi \neq 0$ .

We substitute (CR) into (M) to obtain

$$\begin{cases} i\omega(\varepsilon \mathbf{E} + \xi \mathbf{H}) = -\nabla \times \mathbf{H} \\ i\omega(\zeta \mathbf{E} + \mu \mathbf{H}) = \nabla \times \mathbf{E} \end{cases} \quad (2.1)$$

and if we substitute (CR') into (M),

$$\begin{cases} i\omega(d\mathbf{E} + \xi\mathbf{B}) = \zeta\nabla \times \mathbf{E} - \nabla \times \mathbf{B} \\ i\omega\mathbf{B} = \nabla \times \mathbf{E} \end{cases} \quad (2.1')$$

One crucial characteristic of the time harmonic Maxwell system is that we can calculate the one field as a function of the other (this is known as the impedance operator and this fact is not so evident in the time domain). If we choose to work with the electric field here, then the knowledge of  $\mathbf{E}$  implies the knowledge of  $\mathbf{H}$  also, via the equation

$$\mathbf{H} = \frac{1}{i\omega\mu} (-i\omega\zeta \mathbf{E} + \nabla \times \mathbf{E}) \quad (2.2)$$

If, alternatively, we choose to work with the magnetic flux density, the electric field is calculated as

$$\mathbf{E} = \frac{1}{i\omega d} (i\omega(\zeta - \xi)\mathbf{B} - \nabla \times \mathbf{B}) \quad (2.2')$$

In other words, the time harmonic Maxwell system can be reduced to an equation involving only three components.

In order to eliminate  $\mathbf{H}$  and  $\mathbf{E}$  from (2.1) and (2.1'), respectively, we use (2.2) and (2.2'), respectively. After some straightforward calculations, we find that both  $\mathbf{E}$  and  $\mathbf{B}$  satisfy the vectorial PDE

$$\nabla \times \nabla \times \mathbf{A} - 2\omega\chi\nabla \times \mathbf{A} - \omega^2 d\mathbf{A} = 0 \quad (*)$$

where  $\mathbf{A}$  is either  $\mathbf{E}$  or  $\mathbf{B}$ , and the chirality parameter  $\chi$  is denoted by

$$\chi := i\frac{\zeta - \xi}{2}$$

This is in general a complex number and it serves as a measure of the chirality of the material. The equation (\*) describes the propagation of the electric field or the magnetic flux density inside an arbitrary domain  $\Omega$  which is occupied by a bi-isotropic material.

*Remark 1.* The material parameters enter only via the chirality  $\chi$  and the determinant  $d$ .

*Remark 2.* The reason we choose to work either with the electric field or the magnetic flux is because the boundary conditions is simply expressed in these fields.

## 2.2 The Gauss equations

Due to (M), the fields  $\mathbf{D}$ ,  $\mathbf{B}$  are incompressible, *i.e.*,

$$\nabla \cdot \mathbf{D} = \nabla \cdot \mathbf{B} = 0 \quad (G)$$

and, by using (CR)

$$\begin{cases} \nabla \cdot (\varepsilon\mathbf{E} + \xi\mathbf{H}) = 0 \\ \nabla \cdot (\zeta\mathbf{E} + \mu\mathbf{H}) = 0 \end{cases} \quad (2.3)$$

However, the two assumptions (of homogeneity and invertibility) lead to incompressible fields  $\mathbf{E}$ ,  $\mathbf{H}$ , that is

$$\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{H} = 0 \quad (2.4)$$

The well known vector identity  $\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$  and (G) or (2.4) turn (\*) into

$$\nabla^2\mathbf{A} + 2\omega\chi\nabla \times \mathbf{A} + \omega^2 d\mathbf{A} = 0 \quad (\star)$$

The analysis of equation ( $\star$ ) in a waveguide geometry is the subject of our paper.

### 3 Waveguide geometry and boundary conditions

We now turn our attention to cylindrical domains which represent waveguides. Especially, we will consider an infinite waveguide described by

$$\Omega := \Omega_{\perp} \times \mathbb{R}$$

where  $\Omega_{\perp} \subset \mathbb{R}^2$  is a bounded, connected domain with a sufficiently smooth (Lipschitz continuity is usually sufficient) boundary  $\Gamma_{\perp} := \partial\Omega_{\perp}$ . We will refer to  $\mathbb{R}^2$  as the transverse plane and to  $\Omega_{\perp}$  as the cross section of the waveguide. The axis  $\mathbb{R}$  of the waveguide is often referred as the longitudinal line. The wall of the waveguide is the boundary  $\Gamma := \Gamma_{\perp} \times \mathbb{R}$ . We denote by  $\hat{\boldsymbol{\nu}} = \nu_x \hat{\boldsymbol{x}} + \nu_y \hat{\boldsymbol{y}} \in \mathbb{R}^2$  the outward normal to the curve  $\Gamma_{\perp}$  in the transverse plane. Then as a vector in  $\mathbb{R}^3$ ,  $\hat{\boldsymbol{n}} = \nu_x \hat{\boldsymbol{x}} + \nu_y \hat{\boldsymbol{y}}$  is the outward normal to the wall  $\Gamma$ .

**Assumption 3** (PEC). The wall  $\Gamma$  is a perfect conductor.

This is applicable, for instance, if the waveguide is a metallic wall. The PEC wall is modeled by the boundary condition

$$\hat{\boldsymbol{n}} \times \boldsymbol{E} = 0 \quad \text{on } \Gamma \quad (\text{BC})$$

A simple calculation gives

$$\hat{\boldsymbol{n}} \times \boldsymbol{E} = \nu_y E_z \hat{\boldsymbol{x}} - \nu_x E_z \hat{\boldsymbol{y}} + (\nu_x E_y - \nu_y E_x) \hat{\boldsymbol{z}}$$

Since  $\nu_x, \nu_y$  cannot vanish simultaneously, we have

$$E_z = 0 \quad \text{on } \Gamma \quad (\text{BC}_z)$$

Consider, in the sequel, the transverse field

$$\boldsymbol{E}_{\perp} := E_x \hat{\boldsymbol{x}} + E_y \hat{\boldsymbol{y}}$$

and the  $2 \times 2$  matrix

$$V := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Seen as an operator in  $\mathbb{R}^2$ ,  $V$  acts as a clockwise  $\pi/2$  rotation. Furthermore,  $V$  is antisymmetric and  $V^2 = -I$ . Now  $\hat{\boldsymbol{\tau}} = -V\hat{\boldsymbol{\nu}}$  is a tangential unit vector on  $\Gamma_{\perp}$  and we deduce that

$$\hat{\boldsymbol{\tau}} \cdot \boldsymbol{E}_{\perp} = 0 \quad \text{on } \Gamma \quad (\text{BC}_{\perp})$$

*Remark 3.* The boundary condition (BC) implies that the normal component of  $\boldsymbol{B}$  vanishes on  $\Gamma$ . To see this, use the following equality

$$i\omega \hat{\boldsymbol{n}} \cdot \boldsymbol{B} = \hat{\boldsymbol{n}} \cdot \nabla \times \boldsymbol{E} = -\nabla_{\Gamma} \cdot (\hat{\boldsymbol{n}} \times \boldsymbol{E}) \quad \text{on } \Gamma \quad (3.1)$$

where  $\nabla_{\Gamma}$  is the surface gradient defined by

$$\nabla_{\Gamma} f := \nabla f - \frac{\partial f}{\partial \hat{\boldsymbol{n}}} \hat{\boldsymbol{n}},$$

where the right-hand side is evaluated on the surface  $\Gamma$ , see [18, Sec. 3.3.4]. Actually (3.1), which is valid for a Lipschitz boundary, and (BC), show that

$$\hat{\boldsymbol{n}} \cdot \boldsymbol{B} = \hat{\boldsymbol{\nu}} \cdot \boldsymbol{B}_{\perp} = 0 \quad \text{on } \Gamma \quad (\text{BC}'_{\perp})$$



## 4 Separation of the longitudinal variable

The structure of the waveguide problem allows us to look at  $(\star)$  as an evolution problem of second order with respect to the longitudinal variable  $z \in \mathbb{R}$ . First of all, we can formally write

$$\nabla^2 = \nabla_{\perp}^2 + \partial_{zz}^2$$

We decompose the field  $\mathbf{A}$  into a transverse and a longitudinal part as:

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{\perp} \\ A_z \end{pmatrix}$$

The *curl* operator can then formally be expressed in a matrix notation as

$$\nabla \times \mathbf{A} = \begin{bmatrix} -\partial_z V & V \nabla_{\perp} \\ \nabla_{\perp} \cdot V & 0 \end{bmatrix} \begin{pmatrix} \mathbf{A}_{\perp} \\ A_z \end{pmatrix}$$

The transverse gradient  $\nabla_{\perp}$  (transverse gradient  $\nabla_{\perp} \cdot$ ) is understood as a column (row) vector. Equation  $(\star)$  is then written

$$\partial_{zz}^2 \begin{pmatrix} \mathbf{A}_{\perp} \\ A_z \end{pmatrix} - 2\omega\chi\partial_z \begin{bmatrix} V & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \mathbf{A}_{\perp} \\ A_z \end{pmatrix} + (\nabla_{\perp}^2 + 2\omega\chi C_0 + \omega^2 dI) \begin{pmatrix} \mathbf{A}_{\perp} \\ A_z \end{pmatrix} = 0 \quad (4.1)$$

where

$$C_0 := \begin{bmatrix} 0 & V \nabla_{\perp} \\ \nabla_{\perp} \cdot V & 0 \end{bmatrix}$$

Consider  $\begin{pmatrix} \mathbf{A}_{\perp} \\ A_z \end{pmatrix}$  as a function of  $z \in \mathbb{R}$  with values in a product function space over  $\Omega_{\perp}$ . Equation (4.1) can be viewed as a second order evolution problem (with respect to the variable  $z$ ) and having as state space the aforementioned function space. Augmented with initial conditions, the equation becomes a Cauchy problem. However, in general, it is not expected that such conditions are available. In fact, we proceed by applying a separation of variables techniques, *i.e.*, we search for solutions of the special form

$$\mathbf{A}(\mathbf{r}) = \varphi(z)\mathbf{u}(\boldsymbol{\rho}), \quad z \in \mathbb{R} \quad \boldsymbol{\rho} := x\hat{\mathbf{x}} + y\hat{\mathbf{y}} \in \Omega_{\perp}$$

We impose the initial condition  $\varphi(0) = 1$  for normalization. Substituting to (4.1) we obtain the system

$$\begin{cases} \varphi''(z)\mathbf{u}_{\perp} - 2\omega\chi\varphi'(z)V\mathbf{u}_{\perp} & +\varphi(z)(\nabla_{\perp}^2\mathbf{u}_{\perp} + 2\omega\chi\nabla_{\perp}u_z + \omega^2d\mathbf{u}_{\perp}) = 0 \\ \varphi''(z)u_z & +\varphi(z)(\nabla_{\perp}^2u_z + 2\omega\chi\nabla_{\perp} \cdot \mathbf{u}_{\perp} + \omega^2du_z) = 0 \end{cases} \quad (4.2)$$

The second equation in (4.2) implies that

$$\frac{\varphi''}{\varphi} = \text{constant}$$

Using this fact, the first equation in (4.2) gives also

$$\frac{\varphi'}{\varphi} = \text{constant} \quad (4.3)$$

Denoting the (generally complex) constant  $\gamma$  in (4.3), and using the initial condition for  $\varphi$  we obtain that  $\varphi(z) = e^{\gamma z}$ . Thus, all special solutions have the form

$$\mathbf{A}(\mathbf{r}) = e^{\gamma z} \mathbf{u}(\boldsymbol{\rho}), \quad z \in \mathbb{R}, \quad \boldsymbol{\rho} \in \Omega_{\perp} \quad (\text{SS})$$

## 5 Outlook of the problem

We are now in a position to pose the propagation problem to a metallic bi-isotropic waveguide. The *curl* operator, restricted to the subspace of functions of the form given in (SS), is

$$C_{\gamma} := \begin{bmatrix} -\gamma V & V \nabla_{\perp} \\ \nabla_{\perp} \cdot V & 0 \end{bmatrix}$$

and it acts on functions of the transverse variables  $(x, y) \in \Omega_{\perp}$ , *i.e.*,

$$\mathbf{u} := \begin{pmatrix} \mathbf{u}_{\perp} \\ u_z \end{pmatrix}$$

In this context,  $\gamma$  is a fixed parameter. Equation (2.4), *i.e.*, the Gauss equation, reads

$$\nabla_{\perp} \cdot \mathbf{u}_{\perp} + \gamma u_z = 0 \quad (5.1)$$

The boundary conditions (BC $_{\perp}$ ) and (BC $_z$ ) are written

$$\hat{\boldsymbol{\tau}} \cdot \mathbf{u}_{\perp} = 0 \quad \text{and} \quad u_z = 0 \quad \text{on} \quad \Gamma_{\perp} \quad (5.2)$$

or, equivalently, (BC' $_{\perp}$ ) turns into

$$\hat{\mathbf{n}} \cdot \mathbf{u}_{\perp} = 0 \quad \text{on} \quad \Gamma_{\perp} \quad (5.3)$$

We will use (5.2) in this paper.

The formal square of the operator  $C_{\gamma}$  is the transverse Helmholtz operator, *i.e.*,  $C_{\gamma}^2 = -\gamma^2 I - \nabla_{\perp}^2$ , where the Laplacian  $\nabla_{\perp}^2$  applies to each of the Cartesian components of the field. The details of this calculation is given in Appendix B. Equation (4.1) becomes, in accordance with ( $\star$ ),

$$C_{\gamma}^2 \mathbf{u} - 2\omega \chi C_{\gamma} \mathbf{u} - \omega^2 d\mathbf{u} = 0 \quad (5.4)$$

We agree to call a pair  $\gamma \in \mathbb{C}$ ,  $\mathbf{u} : \Omega_{\perp} \rightarrow \mathbb{C}^3$  (non-zero function) a *solution* of (5.4) if it turns the equation into an identity. The complex number  $\gamma$  is called an *eigenvalue* or a *propagation constant* and the function  $\mathbf{u}$  a corresponding *mode* of the waveguide. According to this terminology, the bi-isotropic waveguide propagation problem is called *well-posed* if the propagation constants of the waveguide form a sequence  $(\gamma_n) \subset \mathbb{C}$  with the following properties:

- it has no accumulation points other than  $\infty$ , and
- it can be ordered in such a way that

$$|\gamma_1| \leq \dots \leq |\gamma_n| \leq \dots \rightarrow \infty$$

We will now reformulate (5.4) as an eigenvalue problem. Define formally the operator

$$P(\gamma) = C_\gamma^2 - 2\omega\chi C_\gamma - \omega^2 dI$$

To simplify the notation we denominate

$$f(\omega) := 2\omega\chi(\omega) \quad , \quad g(\omega) := \omega^2 d(\omega)$$

and thus we obtain an operator pencil with parameter  $\gamma$

$$P(\gamma) = C_\gamma^2 - f(\omega)C_\gamma - g(\omega)I$$

which is quadratic with respect to  $\gamma$ .

We say that  $\gamma$  is an *eigenvalue* of  $P(\cdot)$  if there is a non-zero field  $\mathbf{u}$  (corresponding *eigenvector*) such that  $P(\gamma)\mathbf{u} = 0$ . Using this terminology,  $\gamma, \mathbf{u}$ , is a solution of (5.4) (*i.e.*,  $\gamma$  is a propagation constant and  $\mathbf{u}$  a corresponding mode) if and only if  $\gamma$  is an eigenvalue of  $P(\cdot)$  with corresponding eigenvector  $\mathbf{u}$ .

## 6 Reduction to a standard form

In Section 5, we transformed our waveguide propagation problem into an eigenvalue problem for a quadratic operator pencil. The special form of this pencil (it is a trinomial of  $C_\gamma$ ) allows us to simplify the situation even more. We will factorize the pencil, namely, by applying an elementary “completing the square” method. Actually, we can write

$$\begin{aligned} P_\gamma(\lambda) &= C_\gamma^2 - f(\omega)C_\gamma - g(\omega)I = \\ &= C_\gamma^2 - 2\frac{f(\omega)}{2}C_\gamma + \left(\frac{f(\omega)}{2}I\right)^2 - \frac{f(\omega)^2}{4}I - g(\omega)I = \\ &= \left(C_\gamma - \frac{f(\omega)}{2}I\right)^2 - \left(\frac{1}{2}\sqrt{f(\omega)^2 + 4g(\omega)}I\right)^2 \end{aligned}$$

In the last equation we can apply the “difference of squares” identity since the operators commute trivially. Thus we obtain

$$\begin{aligned} P_\gamma(\omega) &= \left(C_\gamma - \frac{f(\omega) - \sqrt{f(\omega)^2 + 4g(\omega)}}{2}I\right) \left(C_\gamma - \frac{f(\omega) + \sqrt{f(\omega)^2 + 4g(\omega)}}{2}I\right) \\ &= \left(C_\gamma - \frac{f(\omega) + \sqrt{f(\omega)^2 + 4g(\omega)}}{2}I\right) \left(C_\gamma - \frac{f(\omega) - \sqrt{f(\omega)^2 + 4g(\omega)}}{2}I\right) \end{aligned} \quad (6.1)$$

For notational convenience we introduce

$$\kappa_\pm = \kappa_\pm(\omega) := \frac{f(\omega) \pm \sqrt{f(\omega)^2 + 4g(\omega)}}{2} = \omega(\chi \pm \sqrt{\chi^2 + d}) \quad (6.2)$$

and we then have

$$P(\gamma) = (C_\gamma - \kappa_+ I)(C_\gamma - \kappa_- I) = (C_\gamma - \kappa_- I)(C_\gamma - \kappa_+ I) \quad (6.3)$$

Actually  $\kappa_\pm$  are the roots of the quadratic equation

$$x^2 - f(\omega)x - g(\omega) = 0 \quad (6.4)$$

*Remark 4.* The left-hand side of (6.4) we recognize the characteristic polynomial of the  $2 \times 2$  matrix

$$M := i\omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon & \xi \\ \zeta & \mu \end{bmatrix} = i\omega \begin{bmatrix} -\zeta & -\mu \\ \varepsilon & \xi \end{bmatrix}$$

and  $\kappa_+$ ,  $\kappa_-$  are exactly the eigenvalues of this matrix. This is because the solution of equation (\*) is a Beltrami field; for a further discussion, see [16]. Moreover, due to invertibility hypothesis  $d \neq 0$ ,  $\kappa_+$ ,  $\kappa_-$  are non-zero.

*Remark 5.* The operations concerning  $C_\gamma$  must be handled with care since it is a matrix with unbounded differential operator entries. Actually, as we have already noted,  $C_\gamma$  is formally the *curl* operator restricted to functions of the form (SS).

The following result is evident.

**Proposition 6.1.** *If either  $\kappa_+$  or  $\kappa_-$  is an eigenvalue of  $C_\gamma$ , with a corresponding eigenvector  $\mathbf{u}$ , then  $\gamma$  is an eigenvalue of  $P(\cdot)$ , with corresponding eigenvector  $\mathbf{u}$ .*

We are really interested in the opposite statement, *viz.*,

**Proposition 6.2.** *Let  $\gamma$  be an eigenvalue of  $P(\cdot)$ , with a corresponding eigenvector  $\mathbf{u}$ . Then there are three possibilities:*

- a)  $\kappa_+$  is an eigenvalue of  $C_\gamma$ , with a corresponding eigenvector  $\mathbf{u}$ , or
- b)  $\kappa_-$  is an eigenvalue of  $C_\gamma$ , with a corresponding eigenvector  $\mathbf{u}$ , or
- c)  $\kappa_+$  is an eigenvalue of  $C_\gamma$ , with a corresponding eigenvector  $(C_\gamma - \kappa_- I)\mathbf{u}$  and  $\kappa_-$  is an eigenvalue of  $C_\gamma$ , with a corresponding eigenvector  $(C_\gamma - \kappa_+ I)\mathbf{u}$ .

We investigate the consequences of this proposition in the next three subsections.

## 6.1 First case, a) or b)

Suppose that  $\mathbf{u}$  is an eigenvector of  $C_\gamma$  corresponding to  $\kappa_+$  or  $\kappa_-$ . By (A.2) we have that  $u_z$  is then an eigenfunction of the (minus) Laplacian, corresponding to the eigenvalue  $k_+^2 := \kappa_+^2 + \gamma^2$  or  $k_-^2 := \kappa_-^2 + \gamma^2$ . The condition for vanishing  $k_\pm = 0$  is

$$\begin{cases} \gamma = \pm i\kappa_+ = \pm i\omega(\chi + \sqrt{\chi^2 + d}) \\ \gamma = \pm i\kappa_- = \pm i\omega(\chi - \sqrt{\chi^2 + d}) \end{cases}$$

respectively.

Taking (5.2) into account, we have that  $u_z$  is an eigenfunction of the (minus) Dirichlet Laplacian. By the first equation in (A.1) and the fact that  $u_z = 0$  on the boundary, we find that  $\hat{\mathbf{n}} \cdot \mathbf{u}_\perp = 0$  on  $\Gamma_\perp$ . Finally, (A.3) gives that  $u_z$  also satisfies the Neumann boundary condition  $\partial u_z / \partial \hat{\mathbf{n}} = 0$  on  $\Gamma_z$ . So,  $u_z$  is also an eigenfunction of the (minus) Neumann Laplacian. This forces  $u_z$  to vanish and we obtain the trivial solution  $u_z \equiv 0$ .

## 6.2 Second case, c)

We suppose that  $\mathbf{v}_\pm := C_\gamma \mathbf{u} - \kappa_\mp \mathbf{u}$  is an eigenfunction of  $C_\gamma$  corresponding to  $\kappa_\pm$ . The initial field  $\mathbf{u}$  can be recovered by

$$\mathbf{u} = \frac{\mathbf{v}_+ - \mathbf{v}_-}{\kappa_+ - \kappa_-} \quad (6.5)$$

The longitudinal field  $v_{\pm z}$  is an eigenfunction of the (minus) Laplacian corresponding to the eigenvalue  $k_\pm^2 := \kappa_\pm^2 + \gamma^2$ , *i.e.*, we obtain the Helmholtz equations

$$\nabla_\perp^2 v_{\pm z} + k_\pm^2 v_{\pm z} = 0 \quad (6.6)$$

Equivalently, we formulate the formal eigenvalue problem for a matrix operator

$$\begin{bmatrix} -\nabla_\perp^2 - \kappa_+^2 I & 0 \\ 0 & -\nabla_\perp^2 - \kappa_-^2 I \end{bmatrix} \begin{pmatrix} v_{+z} \\ v_{-z} \end{pmatrix} = \gamma^2 \begin{pmatrix} v_{+z} \\ v_{-z} \end{pmatrix} \quad (6.7)$$

The boundary conditions are coupled and can be found by using (6.5); actually,

$$v_{+z} - v_{-z} = (\kappa_+ - \kappa_-)u_z, \quad \hat{\boldsymbol{\tau}} \cdot \mathbf{v}_{+\perp} - \hat{\boldsymbol{\tau}} \cdot \mathbf{v}_{-\perp} = (\kappa_+ - \kappa_-)\hat{\boldsymbol{\tau}} \cdot \mathbf{u}_\perp$$

The first equation gives

$$v_{+z} = v_{-z} \quad \text{on } \Gamma_\perp \quad (6.8)$$

whereas, by using (A.3), we calculate

$$\mathbf{v}_{\pm\perp} = \frac{\gamma}{k_\pm^2} \nabla_\perp v_{\pm z} + \frac{\kappa_\pm}{k_\pm^2} V \nabla_\perp v_{\pm z}$$

and by taking the scalar product with  $\hat{\boldsymbol{\tau}}$ , we find

$$\frac{\gamma}{k_+^2} \frac{\partial v_{+z}}{\partial \hat{\boldsymbol{\tau}}} - \frac{\kappa_+}{k_+^2} \frac{\partial v_{+z}}{\partial \hat{\boldsymbol{n}}} = \frac{\gamma}{k_-^2} \frac{\partial v_{-z}}{\partial \hat{\boldsymbol{\tau}}} - \frac{\kappa_-}{k_-^2} \frac{\partial v_{-z}}{\partial \hat{\boldsymbol{n}}} \quad \text{on } \Gamma_\perp \quad (6.9)$$

The problem (6.7), (6.8), (6.9) is of interior transmission problem type, which arises in the inverse scattering theory, and the problem has initiated great research efforts during the last years, see [8]. The important fact is that the well-posedness of the waveguide propagation problem has now been transformed to a problem of pure mathematical nature. The big difficulty here is that the eigenvalue is involved in the boundary condition (6.9).

## 7 Operator solution of the problem — DtN mapping

The purpose of this section is to give an outline of a possible way to solve the problem employing the Dirichlet-to-Neumann (DtN) mapping.

To start with, let  $\boldsymbol{\rho} = \boldsymbol{\rho}(\theta) = x(\theta)\hat{\mathbf{x}} + y(\theta)\hat{\mathbf{y}}$ ,  $0 \leq \theta \leq 2\pi$  be a parametrization of the curve  $\Gamma_{\perp}$ . We assume that the curve is oriented in the positive direction. The tangential vector is given by

$$\hat{\boldsymbol{\tau}}(\theta) = \frac{\dot{x}(\theta)\hat{\mathbf{x}} + \dot{y}(\theta)\hat{\mathbf{y}}}{\dot{r}(\theta)}$$

where

$$\dot{r}(\theta) := \sqrt{|\dot{x}(\theta)|^2 + |\dot{y}(\theta)|^2}$$

Introduce now an auxiliary function  $\varphi : [0, 2\pi] \rightarrow \mathbb{C}$  that prescribes the common values of  $v_{+z}$ ,  $v_{-z}$  on  $\Gamma_{\perp}$  according to (6.8):

$$v_{+z}(\boldsymbol{\rho}(\theta)) = \varphi(\theta) = v_{-z}(\boldsymbol{\rho}(\theta))$$

We have then

$$\frac{\partial v_{+z}}{\partial \hat{\boldsymbol{\tau}}(\theta)} = \frac{\dot{\varphi}(\theta)}{\dot{r}(\theta)} = \frac{\partial v_{-z}}{\partial \hat{\boldsymbol{\tau}}(\theta)}$$

Let now  $\Lambda_{\pm} := \Lambda_{k_{\pm}}$  be the formal Dirichlet-to-Neumann mapping for the equation (6.6), see appendix D. Then the boundary condition (6.9) is rewritten as follows

$$\frac{\gamma}{k_{+}^2} \frac{\dot{\varphi}(\theta)}{\dot{r}(\theta)} - \frac{\kappa_{+}}{k_{+}^2} (\Lambda_{+}\varphi)(\theta) = \frac{\gamma}{k_{-}^2} \frac{\dot{\varphi}(\theta)}{\dot{r}(\theta)} - \frac{\kappa_{-}}{k_{-}^2} (\Lambda_{-}\varphi)(\theta) \quad \text{for } 0 \leq \theta \leq 2\pi \quad (7.1)$$

Equation (7.1) is, in general, a functional differential equation (FDE) with respect to  $\varphi$ . The problem actually reads

$$\begin{cases} \gamma(k_{-}^2 - k_{+}^2)\dot{\varphi}(\theta) = (Q\varphi)(\theta) & \text{for } 0 \leq \theta \leq 2\pi \\ \varphi(0) = \varphi(2\pi) \end{cases} \quad (\text{FDE})$$

We leave the term  $\gamma(k_{+}^2 - k_{-}^2)$  in the left-hand side of the equation in order to allow it to vanish; this is the case of non-chiral isotropic media. The operator  $Q$  is defined formally as

$$(Q\varphi)(\theta) := \dot{r}(\theta)[\kappa_{+}k_{-}^2(\Lambda_{+}\varphi)(\theta) - \kappa_{-}k_{+}^2(\Lambda_{-}\varphi)(\theta)]$$

and it is expected to be unbounded. After this, the (implicit) dispersion equation should be seen as a condition for (FDE) to have non-trivial solutions.

When  $\kappa_{\pm}$  are known, the dispersion relation is an equation with respect to  $\gamma$ ; from this one can calculate the propagation constants. Once a propagation constant is known, we can solve (FDE) to obtain (a non-zero)  $\varphi$ . For this, we need the Dirichlet Green function  $G_{\pm} := G_{k_{\pm}}$  and then we can reconstruct the original mode  $\mathbf{u}$  by employing (6.5). This means that we can calculate the electric field  $\mathbf{E}$  and, consequently, the magnetic field  $\mathbf{H}$ .

*Remark 6.* In the case of isotropic media,  $\xi = \zeta = 0$ , we have that

$$\kappa_{\pm} = \pm\omega\sqrt{\varepsilon\mu}$$

Consequently  $k_+^2 = k_-^2 = \omega^2 \epsilon \mu := k^2$  and (FDE) degenerates to

$$\Lambda_k \varphi = 0$$

which admits only the trivial solution. Then both  $v_{+z}, v_{-z}$  are eigenfunctions of the (minus) Dirichlet Laplacian corresponding to the eigenvalue  $k^2$ . If we choose them to have opposite signs then we actually retrieve the TM modes.

## 7.1 Circular cross section

In this section, we illustrate how the operator solution is directly applicable to the known case of the circular cross section, see [12] and [17, Section 4.3]. We start with the Helmholtz equation (C.1) and we take  $\Omega_\perp$  to be the interior of the circle of radius  $a$ . The standard parametrization for the circle is  $x = a \cos \theta$ ,  $y = a \sin \theta$ ,  $0 \leq \theta \leq 2\pi$ . It is well known that the (unique) solution of the Dirichlet problem for the Helmholtz equation, with Dirichlet data  $\varphi$ , is represented in polar coordinates as ( $k$  is the wave number in the Helmholtz equation)

$$v(\rho, \theta) = \sum_{m=-\infty}^{\infty} \frac{J_m(k\rho)}{J_m(ka)} \hat{\varphi}(m) e^{im\theta}, \quad 0 \leq \rho \leq a, \quad 0 \leq \theta < 2\pi \quad (7.2)$$

Here  $J_m$  is the Bessel function of first kind and order  $m$  and  $\hat{\varphi}(m)$  is the  $m^{\text{th}}$  Fourier coefficient of  $\varphi$ , *i.e.*,

$$\hat{\varphi}(m) := \frac{1}{2\pi} \int_0^{2\pi} \varphi(\theta') e^{-im\theta'} d\theta'$$

We know also that the normal derivation at angle  $\theta$  on the circle coincides with the partial derivation with respect to  $\rho$  at the point  $(a, \theta)$ . This remark gives immediately the DtN operator from (7.2) as follows.

$$(\Lambda_k \varphi)(\theta) = \sum_{m=-\infty}^{\infty} k \frac{J'_m(ka)}{J_m(ka)} \hat{\varphi}(m) e^{im\theta} \quad (7.3)$$

From (7.3) we get

$$(\Lambda_\pm \varphi)(\theta) = \sum_{m=-\infty}^{\infty} k_\pm \frac{J'_m(k_\pm a)}{J_m(k_\pm a)} \hat{\varphi}(m) e^{im\theta}$$

and, by substituting into the equation in (FDE), we obtain

$$\begin{aligned} \gamma(k_-^2 - k_+^2) \sum_{m=-\infty}^{\infty} im \hat{\varphi}(m) e^{im\theta} = a \left[ \kappa_+ k_-^2 \sum_{m=-\infty}^{\infty} k_+ \frac{J'_m(k_+ a)}{J_m(k_+ a)} \hat{\varphi}(m) e^{im\theta} - \right. \\ \left. - \kappa_- k_+^2 \sum_{m=-\infty}^{\infty} k_- \frac{J'_m(k_- a)}{J_m(k_- a)} \hat{\varphi}(m) e^{im\theta} \right] \quad (7.4) \end{aligned}$$

By equating the Fourier coefficients, we obtain after a rearrangement of the terms

$$k_+^2 J_m(k_+ a) [im\gamma J_m(k_- a) - a\kappa_- k_- J'_m(k_- a)] - k_-^2 J_m(k_- a) [im\gamma J_m(k_+ a) - a\kappa_+ k_+ J'_m(k_+ a)] = 0 \quad (7.5)$$

This is exactly the (implicit with respect to  $\gamma$ ) dispersion equation obtained in [12]. Equation (7.5) can be solved numerically, and it is observed (but is not proved) that for every  $m = 0, \pm 1, \pm 2, \dots$  there exist a countably infinite number of roots. As a consequence, we expect a double sequence  $(\gamma_{mn})$  of propagation constants with no finite accumulation point and diverging to infinity when either  $m \rightarrow \infty$  or  $n \rightarrow \infty$ . This confirms the well-posedness of the problem. Graphs of  $\gamma_{mn}$  as a function of  $\omega$  are found in [12].

Let us now fix a propagation constant  $\gamma_{mn}$ . Then in (7.5) only the coefficient  $\hat{\varphi}(m)$  is allowed to be nonzero and thus we find as Dirichlet data the function

$$\phi_m(\theta) = A_m e^{im\theta}$$

$A_m$  is an arbitrary constant (it can be taken equal to 1). By putting

$$k_{\pm}^{mn} = \sqrt{\kappa_{\pm}^2 + \gamma_{mn}^2}$$

we calculate the corresponding circularly polarized longitudinal fields

$$v_{z\pm}^{mn}(\rho, \theta) = \frac{J_m(k_{\pm}^{mn} \rho)}{J_m(k_{\pm}^{mn} a)} e^{im\theta}$$

from which the electric field is obtained.

## 8 Matrix solution of the problem — Null-field approach

We revisit the problem of finding the propagation constant  $\gamma$ . This time we employ the Null-field approach. This approach originates from pioneer work done by Peter Waterman in the 1960's and 70's [28, 29]. The method has later been generalized by several authors, and also been applied to the waveguide problem, see *e.g.*, [4].

The goal is to solve the Helmholtz equation (6.6) with the boundary conditions given in (6.8) and (6.9). We represent the solutions  $v_{\pm z}$  by

$$\oint_{\Gamma_{\perp}} (\Phi_{k_{\pm}}(\boldsymbol{\rho}; \boldsymbol{\rho}') \hat{\boldsymbol{\nu}}(\boldsymbol{\rho}') \cdot \nabla'_{\perp} v_{\pm z}(\boldsymbol{\rho}') - v_{\pm z}(\boldsymbol{\rho}') \hat{\boldsymbol{\nu}}(\boldsymbol{r}') \cdot \nabla'_{\perp} \Phi_{k_{\pm}}(\boldsymbol{\rho}; \boldsymbol{\rho}')) d\ell' = \begin{cases} v_{\pm z}(\boldsymbol{\rho}), & \boldsymbol{\rho} \in \Omega_{\perp} \\ 0, & \boldsymbol{\rho} \in \mathbb{R}^3 \setminus \bar{\Omega}_{\perp} \end{cases} \quad (8.1)$$

where the line element on the bounding curve is denoted  $d\ell$  and where the free space Green's function is

$$\Phi_k(\boldsymbol{\rho}; \boldsymbol{\rho}') = \frac{i}{4} H_0^{(1)}(k|\boldsymbol{\rho} - \boldsymbol{\rho}'|)$$



The lower part of the representation (8.1) is often called the extinction part, and this part is utilized in the solution of the propagation problem in this section.

The Green function has an expansion in cylindrical wave functions [3]

$$\Phi_k(\boldsymbol{\rho}; \boldsymbol{\rho}') = \frac{i\pi}{2} \sum_m u_m(k\boldsymbol{\rho}_>) v_m^\dagger(k\boldsymbol{\rho}_<) = \frac{i\pi}{2} \sum_m u_m^\dagger(k\boldsymbol{\rho}_>) v_m(k\boldsymbol{\rho}_<) \quad (8.2)$$

where  $\boldsymbol{\rho}_>(\boldsymbol{\rho}_<)$  denotes the largest(smallest) vector of the  $\boldsymbol{\rho}$  and  $\boldsymbol{\rho}'$ , and where the cylindrical wave functions,  $u_m$  and  $v_m$ , are defined in terms of Hankel and Bessel functions as

$$\begin{cases} u_m(k\boldsymbol{\rho}) = \sqrt{\frac{1}{2\pi}} H_m^{(1)}(k\rho) e^{im\phi} \\ v_m(k\boldsymbol{\rho}) = \sqrt{\frac{1}{2\pi}} J_m(k\rho) e^{im\phi} \end{cases} \quad (8.3)$$

where  $m = 0, \pm 1, \pm 2, \dots$ , and the superscript  $\dagger$  indicates that  $i \rightarrow -i$  in the exponential. Note that the symbol  $\dagger$  does not correspond to the complex conjugate of the functions.

This expansion of the Green function in (8.2) implies for an  $\boldsymbol{\rho}$  outside the circumscribing circle of the waveguide that the extinction part of the representation (8.1) in a two-dimensional system becomes

$$\begin{aligned} & \oint_{\Gamma_\perp} \left( \begin{pmatrix} v_m^\dagger(k_+\boldsymbol{\rho}') & 0 \\ 0 & v_m^\dagger(k_-\boldsymbol{\rho}') \end{pmatrix} \begin{pmatrix} \frac{\partial v_{+z}(\boldsymbol{\rho}')}{\partial \nu} \\ \frac{\partial v_{-z}(\boldsymbol{\rho}')}{\partial \nu} \end{pmatrix} - \begin{pmatrix} \frac{\partial v_m^\dagger(k_+\boldsymbol{\rho}')}{\partial \nu} & 0 \\ 0 & \frac{\partial v_m^\dagger(k_-\boldsymbol{\rho}')}{\partial \nu} \end{pmatrix} \begin{pmatrix} v_{+z}(\boldsymbol{\rho}') \\ v_{-z}(\boldsymbol{\rho}') \end{pmatrix} \right) d\ell' \\ & = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \forall m \end{aligned}$$

Eliminating the surface fields  $\frac{\partial v_-(\boldsymbol{\rho})}{\partial \nu}$  and  $v_-(\boldsymbol{\rho})$  by the use of the boundary conditions in (6.8) and (6.9), we get

$$\begin{aligned} & \oint_{\Gamma_\perp} \left( \begin{pmatrix} v_m^\dagger(k_+\boldsymbol{\rho}) & 0 \\ 0 & v_m^\dagger(k_-\boldsymbol{\rho}) \end{pmatrix} \begin{pmatrix} \frac{\partial v_{+z}(\boldsymbol{\rho})}{\partial \nu} \\ \frac{\gamma}{\kappa_-} \frac{\partial v_{+z}(\boldsymbol{\rho})}{\partial \tau} - \frac{\gamma}{k_+^2 \kappa_-} \frac{\partial v_{+z}(\boldsymbol{\rho})}{\partial \tau} + \frac{\kappa_+}{k_+^2 \kappa_-} \frac{\partial v_{+z}(\boldsymbol{\rho})}{\partial \nu} \end{pmatrix} \right. \\ & \left. - \begin{pmatrix} \frac{\partial v_m^\dagger(k_+\boldsymbol{\rho})}{\partial \nu} & 0 \\ 0 & \frac{\partial v_m^\dagger(k_-\boldsymbol{\rho})}{\partial \nu} \end{pmatrix} \begin{pmatrix} v_{+z}(\boldsymbol{\rho}) \\ v_{+z}(\boldsymbol{\rho}) \end{pmatrix} \right) d\ell = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \forall m \end{aligned} \quad (8.4)$$

These are the Null-field equations.

The surface fields  $v_{+z}(\boldsymbol{\rho})$  and  $\frac{\partial}{\partial \nu} v_{+z}(\boldsymbol{\rho})$  are unknown, and we expand them in terms of the arbitrary bases  $\{f_m(\boldsymbol{\rho})\}_{m \in I}$  and  $\{g_m(\boldsymbol{\rho})\}_{m \in I}$ , respectively, for some index set  $I$ , *i.e.*,

$$\begin{cases} v_{+z}(\boldsymbol{\rho}) = \sum_{m \in I} \alpha_m f_m(\boldsymbol{\rho}) \\ \frac{\partial v_{+z}(\boldsymbol{\rho})}{\partial \tau} = \sum_{m \in I} \alpha_m \frac{\partial f_m(\boldsymbol{\rho})}{\partial \tau} \end{cases} \quad \frac{\partial v_{+z}(\boldsymbol{\rho})}{\partial \nu} = \sum_{m \in I} \beta_m g_m(\boldsymbol{\rho}), \quad \boldsymbol{\rho} \in \Gamma_\perp \quad (8.5)$$

Note that the expansion of the field  $v_{+z}(\boldsymbol{\rho})$  and its tangential derivative has the same expansion coefficients. By insertion of the surface field expansions in (8.4) we get

$$\sum_{m' \in I_{\Gamma_{\perp}}} \oint \left( \begin{pmatrix} v_m^{\dagger}(k_+\boldsymbol{\rho}) & 0 \\ 0 & v_m^{\dagger}(k_-\boldsymbol{\rho}) \end{pmatrix} \begin{pmatrix} \beta_{m'} g_{m'}(\boldsymbol{\rho}) \\ \alpha_{m'} \frac{\gamma}{\kappa_-} \left(1 - \frac{k_-^2}{k_+^2}\right) \frac{\partial f_{m'}(\boldsymbol{\rho})}{\partial \tau} + \beta_{m'} \frac{\kappa_+ k_-^2}{\kappa_- k_+^2} g_{m'}(\boldsymbol{\rho}) \end{pmatrix} \right. \\ \left. - \alpha_{m'} \begin{pmatrix} \frac{\partial v_m^{\dagger}(k_+\boldsymbol{\rho})}{\partial \nu} & 0 \\ 0 & \frac{\partial v_m^{\dagger}(k_-\boldsymbol{\rho})}{\partial \nu} \end{pmatrix} \begin{pmatrix} f_{m'}(\boldsymbol{\rho}) \\ f_{m'}(\boldsymbol{\rho}) \end{pmatrix} \right) d\ell = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \forall m$$

This system of equations are most conveniently written as

$$\sum_{m'} \begin{pmatrix} Q_{mm'}^1 & Q_{mm'}^2 \\ Q_{mm'}^3 & Q_{mm'}^4 \end{pmatrix} \begin{pmatrix} \alpha_{m'} \\ \beta_{m'} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \forall m \quad (8.6)$$

where the infinite-dimensional matrices  $Q^{1,2,3,4}$  are defined as

$$\left\{ \begin{aligned} Q_{mm'}^1 &= - \oint_{\Gamma_{\perp}} \frac{\partial v_m^{\dagger}(k_+\boldsymbol{\rho})}{\partial \nu} f_{m'}(\boldsymbol{\rho}) d\ell \\ Q_{mm'}^2 &= \oint_{\Gamma_{\perp}} v_m^{\dagger}(k_+\boldsymbol{\rho}) g_{m'}(\boldsymbol{\rho}) d\ell \\ Q_{mm'}^3 &= - \oint_{\Gamma_{\perp}} \frac{\partial v_m^{\dagger}(k_-\boldsymbol{\rho})}{\partial \nu} f_{m'}(\boldsymbol{\rho}) + \frac{\gamma}{\kappa_-} \left( \frac{k_-^2}{k_+^2} - 1 \right) v_m^{\dagger}(k_-\boldsymbol{\rho}) \frac{\partial f_{m'}(\boldsymbol{\rho})}{\partial \tau} d\ell \\ Q_{mm'}^4 &= \frac{\kappa_+ k_-^2}{\kappa_- k_+^2} \oint_{\Gamma_{\perp}} v_m^{\dagger}(k_-\boldsymbol{\rho}) g_{m'}(\boldsymbol{\rho}) d\ell \end{aligned} \right. \quad (8.7)$$

To have a non-trivial solution to the propagation problem we require

$$\det \begin{pmatrix} Q_{mm'}^1 & Q_{mm'}^2 \\ Q_{mm'}^3 & Q_{mm'}^4 \end{pmatrix} = 0 \quad (8.8)$$

This determinant equation determines the propagation constant  $\gamma$  of the waveguide.

## 8.1 Cylindrical expansion functions

The sets of basis,  $\{f_m(\boldsymbol{\rho})\}_{m \in I}$  and  $\{g_m(\boldsymbol{\rho})\}_{m \in I}$ , are arbitrary. Interestingly, there is a specific set of expansion functions that results in a more compact form of determinant equation. These are the regular cylindrical functions  $v_m$  defined in (8.3). In (8.5) choose

$$\begin{cases} f_m(\boldsymbol{\rho}) = v_m(k_+\boldsymbol{\rho}) \\ g_m(\boldsymbol{\rho}) = \frac{\partial v_m(k_+\boldsymbol{\rho})}{\partial \nu} \end{cases} \quad m = 0, \pm 1, \pm 2, \dots \quad (8.9)$$

These sets of expansion functions can be proven to be linearly independent and complete<sup>1</sup> for any  $k$  in  $L^2(\Gamma_\perp)$ , but not necessarily a (Schauder) basis [14]. With this choice, the first row in (8.6) reads

$$\sum_{m'} \oint_{\Gamma_\perp} \left( \beta_{m'} v_m^\dagger(k_+\boldsymbol{\rho}) \frac{\partial v_{m'}(k_+\boldsymbol{\rho})}{\partial \nu} - \alpha_{m'} \frac{\partial v_m^\dagger(k_+\boldsymbol{\rho})}{\partial \nu} v_{m'}(k_+\boldsymbol{\rho}) \right) d\ell = 0, \quad \forall m$$

Now use the identity

$$\oint_{\Gamma_\perp} \left( v_m^\dagger(k_+\boldsymbol{\rho}) \frac{\partial v_{m'}(k_+\boldsymbol{\rho})}{\partial \nu} - \frac{\partial v_m^\dagger(k_+\boldsymbol{\rho})}{\partial \nu} v_{m'}(k_+\boldsymbol{\rho}) \right) d\ell = 0, \quad \forall m, m'$$

and we obtain

$$\sum_{m'} (\beta_{m'} - \alpha_{m'}) \oint_{\Gamma_\perp} v_m^\dagger(k_+\boldsymbol{\rho}) \frac{\partial v_{m'}(k_+\boldsymbol{\rho})}{\partial \nu} d\ell = 0, \quad \forall m$$

Under the assumption that the cylindrical functions in (8.9) are a basis, the matrix

$$A_{mm'} = \oint_{\Gamma_\perp} v_m^\dagger(k_+\boldsymbol{\rho}) \frac{\partial v_{m'}(k_+\boldsymbol{\rho})}{\partial \nu} d\ell$$

is invertible [14], implying  $\beta_m = \alpha_m$ , and the second row in (8.6) becomes

$$\sum_{m'} Q_{mm'} \alpha_{m'} = 0$$

where the infinite-dimensional matrix  $Q$  is defined as

$$Q_{mm'} = \oint_{\Gamma_\perp} \left( v_m^\dagger(k_-\boldsymbol{\rho}) \left( \frac{\gamma}{\kappa_-} \left( 1 - \frac{k_-^2}{k_+^2} \right) \frac{\partial v_{m'}(k_+\boldsymbol{\rho})}{\partial \tau} + \frac{\kappa_+ k_-^2}{\kappa_- k_+^2} \frac{\partial v_{m'}(k_+\boldsymbol{\rho})}{\partial \nu} \right) - \frac{\partial v_m^\dagger(k_-\boldsymbol{\rho})}{\partial \nu} v_{m'}(k_+\boldsymbol{\rho}) \right) d\ell' \quad (8.10)$$

Again, to have a non-trivial solution to the problem we need  $\det Q = 0$ .

## 8.2 Implementation

Let  $\boldsymbol{\rho} = \rho(\phi) \hat{\boldsymbol{\rho}}$  be a parametrization of the contour  $\Gamma_\perp$  w.r.t. the azimuth angle  $\phi \in (0, 2\pi)$ . Then

$$\hat{\boldsymbol{\tau}} = \frac{\rho'(\phi) \hat{\boldsymbol{\rho}} + \rho(\phi) \hat{\boldsymbol{\phi}}}{\sqrt{(\rho'(\phi))^2 + (\rho(\phi))^2}},$$

and

$$\hat{\boldsymbol{\nu}} = \frac{\rho(\phi) \hat{\boldsymbol{\rho}} - \rho'(\phi) \hat{\boldsymbol{\phi}}}{\sqrt{(\rho'(\phi))^2 + (\rho(\phi))^2}}, \quad d\ell = \left| \frac{d\boldsymbol{\rho}(\phi)}{d\phi} \right| d\phi = \sqrt{(\rho'(\phi))^2 + (\rho(\phi))^2} d\phi$$

---

<sup>1</sup>Carefully avoiding internal resonances.

**Trigonometric functions** Use the Fourier system

$$f_m(\boldsymbol{\rho}) = g_m(\boldsymbol{\rho}) = \sqrt{\frac{1}{2\pi}} e^{im\phi}$$

This system is orthonormal over  $[0, 2\pi)$ , *i.e.*,

$$\int_0^{2\pi} f_m(\boldsymbol{\rho}(\phi)) f_{m'}(\boldsymbol{\rho}(\phi)) d\phi = \delta_{mm'}$$

and it forms a (Riesz) basis for a star-shaped domain, see [21] for the result on the spherical harmonics.

With this set of expansion functions, the  $Q$  matrices in (8.7) are

$$\left\{ \begin{array}{l} Q_{mm'}^1 = -\frac{1}{2\pi} \int_0^{2\pi} J'_m(k_+\rho(\phi)) k_+\rho(\phi) e^{i(m'-m)\phi} d\phi \\ \quad - \frac{im}{2\pi} \int_0^{2\pi} J_m(k_+\rho(\phi)) \frac{\rho'(\phi)}{\rho(\phi)} e^{i(m'-m)\phi} d\phi \\ Q_{mm'}^2 = \frac{1}{2\pi} \int_0^{2\pi} J_m(k_+\rho(\phi)) \sqrt{(\rho'(\phi))^2 + (\rho(\phi))^2} e^{i(m'-m)\phi} d\phi \\ Q_{mm'}^3 = -\frac{1}{2\pi} \int_0^{2\pi} J'_m(k_-\rho(\phi)) k_-\rho(\phi) e^{i(m'-m)\phi} d\phi \\ \quad - \frac{im}{2\pi} \int_0^{2\pi} J_m(k_-\rho(\phi)) \frac{\rho'(\phi)}{\rho(\phi)} e^{i(m'-m)\phi} d\phi \\ \quad + \frac{im'\gamma}{\kappa_-} \left(1 - \frac{k_-^2}{k_+^2}\right) \frac{1}{2\pi} \int_0^{2\pi} J_m(k_-\rho(\phi)) e^{i(m'-m)\phi} d\phi \\ Q_{mm'}^4 = \frac{1}{2\pi} \frac{\kappa_+ k_-^2}{\kappa_- k_+^2} \int_0^{2\pi} J_m(k_-\rho(\phi)) \sqrt{(\rho'(\phi))^2 + (\rho(\phi))^2} e^{i(m'-m)\phi} d\phi \end{array} \right.$$

### 8.2.1 Circular cross section

For a circular cross section  $\rho = a$ , these  $Q$ -matrices are diagonal in the  $m$  index with entries

$$\left\{ \begin{array}{l} Q_{mm'}^1 = -\delta_{mm'} J'_m(k_+a) k_+a \\ Q_{mm'}^2 = \delta_{mm'} J_m(k_+a) a \\ Q_{mm'}^3 = -\delta_{mm'} \left( J'_m(k_-a) k_-a - \frac{im\gamma}{\kappa_-} \left(1 - \frac{k_-^2}{k_+^2}\right) J_m(k_-a) \right) \\ Q_{mm'}^4 = \delta_{mm'} \frac{\kappa_+ k_-^2}{\kappa_- k_+^2} J_m(k_-a) a \end{array} \right.$$

and the condition that the determinant in (8.8) vanishes becomes for each  $m = m'$  index

$$Q_{mm}^1 Q_{mm}^4 - Q_{mm}^2 Q_{mm}^3 = 0$$

The explicit form of this expression is

$$\begin{aligned} & k_+ a k_-^2 a^2 \kappa_+ a J'_m(k_+ a) J_m(k_- a) - k_+^2 a^2 k_- a \kappa_- a J_m(k_+ a) J'_m(k_- a) \\ & + i m \gamma a (k_+^2 a^2 - k_-^2 a^2) J_m(k_+ a) J_m(k_- a) = 0 \end{aligned} \quad (8.11)$$

which is identical to (7.5), and for an isotropic material ( $k = k_- = k_+$ ) reduces to

$$J'_m(ka) J_m(ka) = 0$$

Similarly, using the  $Q$ -matrix in (8.10) gives

$$\begin{aligned} Q_{mm} = & J_m(k_- a) \left( \frac{\gamma}{\kappa_-} \left( 1 - \frac{k_-^2}{k_+^2} \right) \frac{i m}{a} J_m(k_+ a) + \frac{\kappa_+}{\kappa_-} \frac{k_-^2}{k_+^2} k_+ J'_m(k_+ a) \right) a \\ & - k_- J'_m(k_- a) J_m(k_+ a) a \end{aligned}$$

and the condition of vanishing determinant again gives the same result as above.

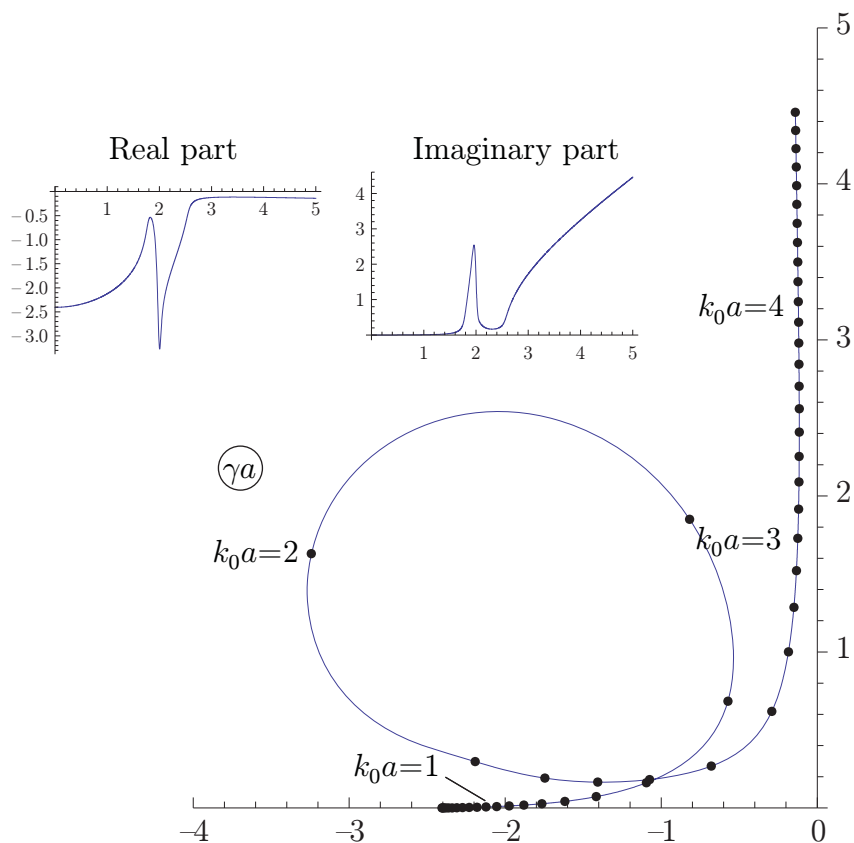
## 9 Numerical illustration

In this section, we illustrate the analysis presented in this paper by a numerical example, and, for simplicity, we adopt a waveguide with a circular cross section. The dispersion relation for this geometry is given as the roots to a transcendental equation explicitly given in Section 7.1 and 8.2.1, see (7.5) and (8.11), respectively. Dispersion relations for this geometry and constant material parameters have previously been published in the literature, see *e.g.*, [11, 12]. These materials are all lossless, and therefore the propagating modes show no damping. However, dispersion effects are always present at least to some extent, and here we illustrate the theory with a more realistic material model that includes dispersion and losses.

All realistic material shows dispersion effects — at least in some frequency interval, and, moreover, the chirality parameter  $\chi$  cannot assume a non-zero value at zero frequency. To our knowledge, the only known dispersion model for bi-isotropic effects is Condon's model [9, 17]. In order to model a passive material, the proper models for the permittivity and the permeability have to be adopted. A dispersive model, that is passive, is the following combined Lorentz-Debye model for the permittivity and permeability and Condon's model for the chirality parameter, *i.e.*,

$$\begin{cases} \epsilon(\omega) = 1 - \frac{\omega_{pe}^2}{\omega^2 - \omega_0^2 + i\omega\nu_e} + \frac{\alpha\tau}{1 - i\omega\tau} \\ \mu(\omega) = 1 - \frac{\omega_{pm}^2}{\omega^2 - \omega_0^2 + i\omega\nu_m} \\ \xi(\omega) = -\zeta(\omega) = \frac{i\omega\omega_c}{\omega^2 - \omega_0^2 + i\omega\nu_c} \end{cases}$$

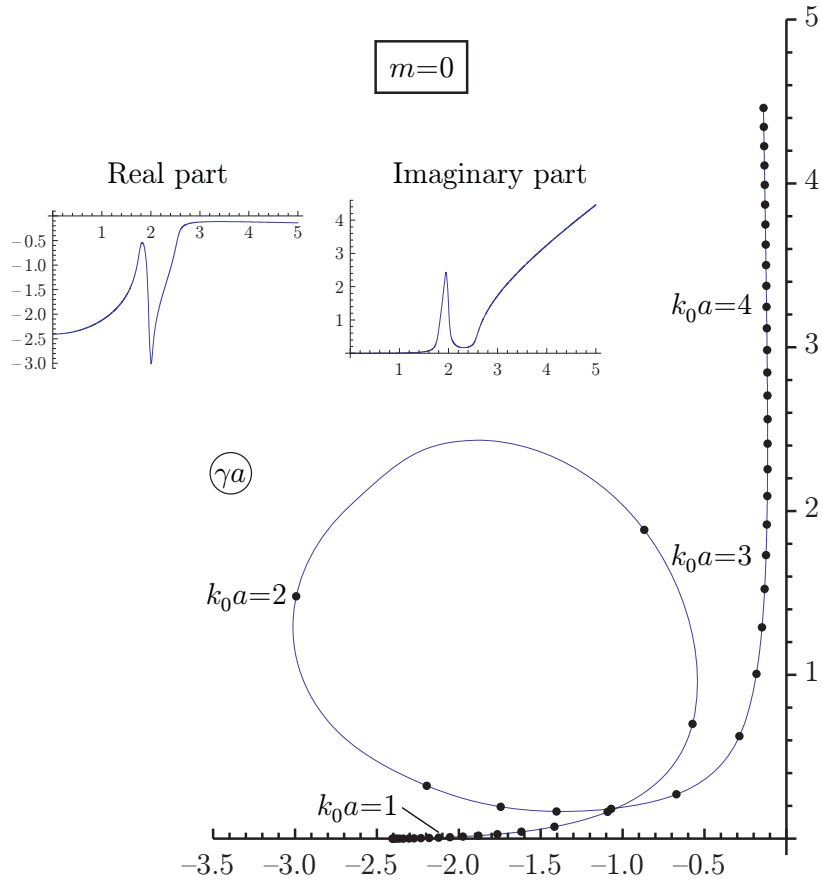
The dispersion curve, *i.e.*, the curve of  $\gamma a$  in the complex  $\gamma a$ -plane as a function of the normalized frequency  $k_0 a = \omega a / c_0$ , for the lowest order mode is displayed



**Figure 1:** The dispersion relation of the lowest order mode for non-chiral material data in Table 2 in the complex  $\gamma a$ -plane, *i.e.*,  $\omega_c = 0$ . The curve shows the propagation constant  $\gamma a$  as a function of the normalized frequency  $k_0 a$ . The dots along the curve show the normalized frequency  $k_0 a = 0, 0.1, 0.2, \dots, 5$ . The curve starts at  $\gamma a = -2.405$  at zero frequency. The inserts show the real and imaginary parts of  $\gamma a$  as a function of  $k_0 a$ .

in Figures 1–4 for three different values of the azimuthal index  $m = 0, \pm 1$ . The lowest order mode for  $m = 0$  starts at  $\gamma a = -2.405$  (TM) at zero frequency, and for  $m = \pm 1$  the curves start at  $\gamma a = -1.841$  (TE) at zero frequency. The data of the dispersive material is explicitly displayed in Table 2. For reference, we display the non-chiral,  $\omega_c = 0$ , dispersion curve in Figure 1. The loop is inherent with the resonance characteristic of the material, and notice also that the propagation constant is no longer purely real or imaginary, but in general a complex number due to losses. The loop is traversed very quickly — a fact that is due to the small losses of the material.

The Figures 2–4 show the distinctive difference between the different modes corresponding to  $m = 0$ ,  $m = +1$ , and  $m = -1$  — the loop is more narrow in  $m = +1$  than in  $m = -1$ , and  $m = 0$  lies somewhere in between. This is due to the sign of our chirality parameter. Notice that for  $m = \pm 1$ , the points corresponding to zero frequency,  $k_0 a = 0$ , are different from the starting points of the non-chiral

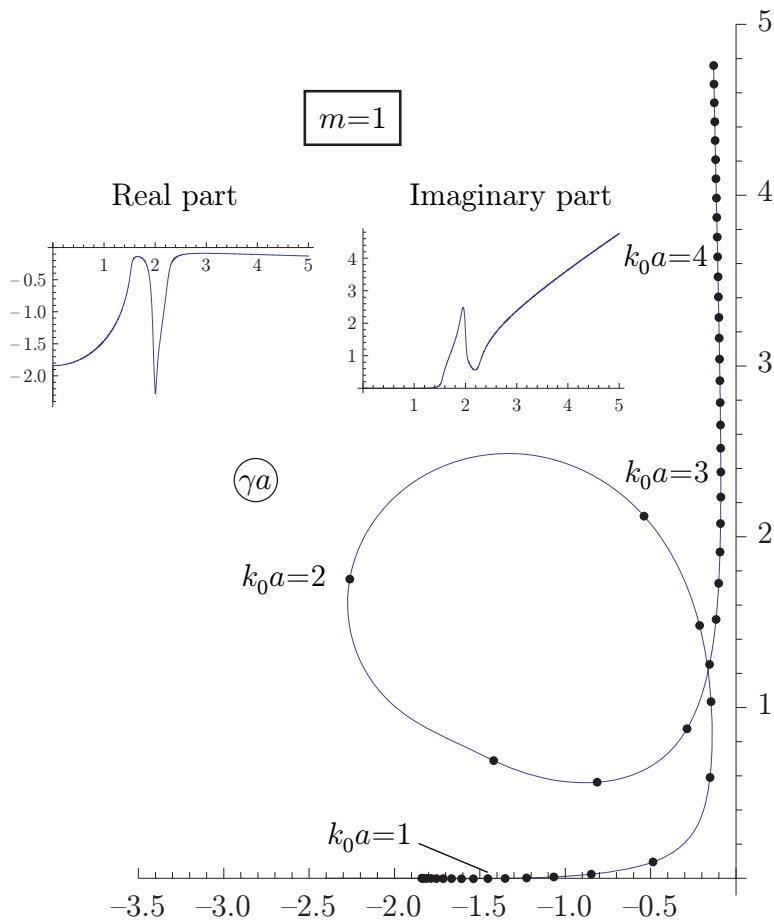


**Figure 2:** The dispersion relation of the lowest order mode for material data in Table 2 in the complex  $\gamma a$ -plane. The curve shows the propagation constant  $\gamma a$  as a function of the normalized frequency  $k_0 a$  for  $m = 0$ . The dots along the curve show the normalized frequency  $k_0 a = 0, 0.1, 0.2, \dots, 5$ . The curve starts at  $\gamma a = -2.405$  at zero frequency. The inserts show the real and imaginary parts of  $\gamma a$  as a function of  $k_0 a$ .

and the  $m = 0$  curves.

The asymptotic behavior of the curve  $\gamma a$  as  $k_0 a \rightarrow \infty$  approaches  $\text{Re } \gamma a \approx -0.1$ , which corresponds well to the order of magnitude of the losses in the material, *i.e.*,  $\alpha\tau = 0.1$  and  $\tau c_0/a = 0.2$ . In fact, under the assumption that the frequency is high enough so that only the Debye term in  $\epsilon(\omega)$  contributes to its imaginary part, and  $\text{Re } \epsilon \approx 1$ ,  $\mu \approx 1$ , and  $\xi \approx 0$ , we have

$$\gamma a \approx -\frac{k_0 a \text{Im } \epsilon}{2} + i k_0 a \quad \Rightarrow \quad \gamma a \approx -\frac{1}{2} \frac{k_0^2 a \alpha \tau^2 c_0}{1 + k_0^2 \tau^2 c_0^2} + i k_0 a \quad \Rightarrow \quad \gamma a \approx -0.125 + 5i$$



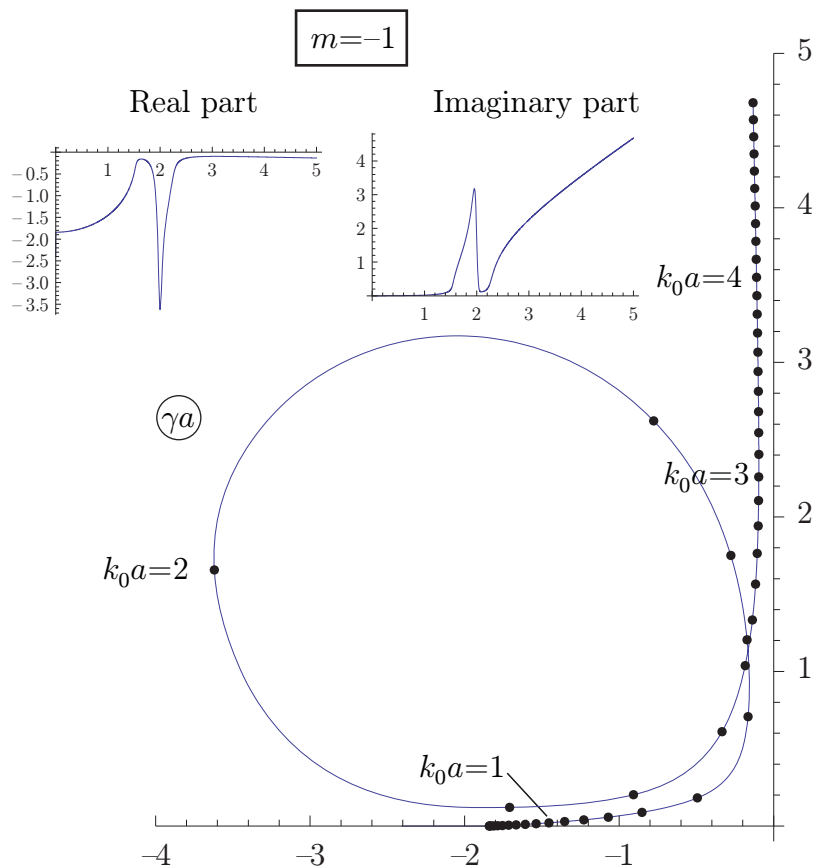
**Figure 3:** The dispersion relation of the lowest order mode for material data in Table 2 in the complex  $\gamma a$ -plane. The curve shows the propagation constant  $\gamma a$  as a function of the normalized frequency  $k_0 a$  for  $m = 1$ . The dots along the curve show the normalized frequency  $k_0 a = 0, 0.1, 0.2, \dots, 5$ . The curve starts at  $\gamma a = -1.841$  at zero frequency. The inserts show the real and imaginary parts of  $\gamma a$  as a function of  $k_0 a$ .

## 10 Conclusions and further research

In this paper, we have considered the propagation problem inside a metallic waveguide filled with a homogeneous, bi-isotropic medium. We define propagation constants and modes and their spectral problems. We present this problem in two approaches; roughly speaking, one form has a complicated operator but simple boundary conditions, whereas the other has a simple operator but complicated boundary conditions. Following the second approach, we reduce the problem into a functional differential equation on the line, which is directly subject to mathematical analysis.

Actually, we have a solution to the bi-isotropic waveguide problem whenever we can solve the Dirichlet problem for the 2D Helmholtz in the cross section. Two different methods of solving the problem are suggested — one employing the Dirichlet-





**Figure 4:** The dispersion relation of the lowest order mode for material data in Table 2 in the complex  $\gamma a$ -plane. The curve shows the propagation constant  $\gamma a$  as a function of the normalized frequency  $k_0 a$  for  $m = -1$ . The dots along the curve show the normalized frequency  $k_0 a = 0, 0.1, 0.2, \dots, 5$ . The curve starts at  $\gamma a = -1.841$  at zero frequency. The inserts show the real and imaginary parts of  $\gamma a$  as a function of  $k_0 a$ .

to-Neumann mapping, and the other utilizing the integral representation of the solutions to the Helmholtz equation, *i.e.*, the Null-field approach. Both these methods have potential to solve the propagation problem for a more general cross section geometry than the circular one, which, to our knowledge, is the only case treated in full detail.

We finish with some suggestions for further investigations that can extend and enrich the present work.

**Solvability of (FDE).** To prove in a rigorous mathematical manner that (FDE) has a discrete spectrum.

**Other geometries.** To obtain analytical or numerical solutions for geometries different than the circular one. The rectangular waveguide is of special interest.

**More complicated media.** To consider mediums described by more parameters

$\omega_{pe}a/c_0 = \omega_{pm}a/c_0 = 0.5$
$\omega_0a/c_0 = 2$
$\nu_ea/c_0 = \nu_ma/c_0 = \nu_ca/c_0 = 0.1$
$\omega_c a/c_0 = 0.04$
$\alpha\tau = 0.1$
$\tau c_0/a = 0.2$

**Table 2:** The data of the dispersive bi-isotropic material in Figures 1–4.

with the full  $6 \times 6$  material matrix as the final goal.

**The inverse problem.** To determine the parameters of the bi-isotropic medium by applying measurements of the material slab in the waveguide. We are particularly interested in this problem and in [24] a method to calculate the propagation constants of an arbitrary linear material is developed. The basic question is if the knowledge of the propagation constants is sufficient to determine the material.

## Acknowledgements

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## Appendix A Eigenvalues and eigenvectors of $C_\gamma$

Here, we consider briefly the formal eigenvalue–eigenvector problem for the operator  $C_\gamma$ , *i.e.*, the equation  $C_\gamma \mathbf{v} = \lambda \mathbf{v}$ . This is equivalent to the system

$$\begin{cases} -\gamma V \mathbf{v}_\perp + V \nabla_\perp v_z & = \lambda \mathbf{v}_\perp \\ \nabla_\perp \cdot V \mathbf{v}_\perp & = \lambda v_z \end{cases} \quad (\text{A.1})$$

Solving with respect to the longitudinal field  $v_z$  we find

$$-\nabla_\perp^2 v_z = (\lambda^2 + \gamma^2) v_z \quad (\text{A.2})$$

Thereby,  $v_z$  is an eigenvector of the minus transverse Laplacian corresponding to the eigenvalue  $\lambda^2 + \gamma^2$ . The transverse field is then calculated by the first equation in (A.1) and it is given by

$$\mathbf{v}_\perp = \frac{\gamma}{\lambda^2 + \gamma^2} \nabla_\perp v_z + \frac{\lambda}{\lambda^2 + \gamma^2} V \nabla_\perp v_z \quad (\text{A.3})$$

As a consequence,  $\mathbf{v}$  is completely determined by its longitudinal part. More precisely, (A.2) reduces the situation to an eigenvalue–eigenvector problem for the Laplacian. To study the same problem rigorously in domains, one has just to assign the boundary condition for  $v_z$ .

## Appendix B The operator $C_\gamma^2$

As a first step we formally calculate

$$C_\gamma^2 = \begin{bmatrix} -\gamma V & V\nabla_\perp \\ \nabla_\perp \cdot V & 0 \end{bmatrix} \begin{bmatrix} -\gamma V & V\nabla_\perp \\ \nabla_\perp \cdot V & 0 \end{bmatrix} = \begin{bmatrix} -\gamma^2 I_2 + V\nabla_\perp \nabla_\perp \cdot V & \gamma \nabla_\perp \\ \gamma \nabla_\perp \cdot & -\nabla_\perp \cdot \nabla_\perp \end{bmatrix}$$

We have now

$$V\nabla_\perp \nabla_\perp \cdot V = \begin{pmatrix} \partial_y \\ -\partial_x \end{pmatrix} (-\partial_y \quad \partial_x) = \begin{bmatrix} -\partial_{yy}^2 & \partial_y \partial_x \\ \partial_x \partial_y & -\partial_{xx}^2 \end{bmatrix}$$

On the right-hand side of this equation we add and subtract the matrix

$$\begin{bmatrix} \partial_{xx}^2 & 0 \\ 0 & \partial_{yy}^2 \end{bmatrix}$$

This way, we obtain

$$V\nabla_\perp \nabla_\perp \cdot V = \begin{bmatrix} -\nabla_\perp^2 & \\ 0 & -\nabla_\perp^2 \end{bmatrix} + \begin{bmatrix} \partial_{xx}^2 & \partial_{xy}^2 \\ \partial_{xy}^2 & \partial_{yy}^2 \end{bmatrix} = -\nabla_\perp^2 I_2 + \nabla_\perp \nabla_\perp \cdot$$

Thus

$$C_\gamma^2 = \begin{bmatrix} (-\gamma^2 - \nabla_\perp^2) I_2 + \nabla_\perp \nabla_\perp \cdot & \gamma \nabla_\perp \\ \gamma \nabla_\perp \cdot & -\nabla_\perp^2 \end{bmatrix}$$

Apply  $C_\gamma^2$  to a field satisfying Gauss equation (5.1)

$$\begin{aligned} C_\gamma^2 \begin{pmatrix} \mathbf{u}_\perp \\ u_z \end{pmatrix} &= \begin{pmatrix} (-\gamma^2 - \nabla_\perp^2) \mathbf{u}_\perp + \nabla_\perp \nabla_\perp \cdot \mathbf{u}_\perp + \gamma \nabla_\perp u_z \\ \gamma \nabla_\perp \cdot \mathbf{u}_\perp - \nabla_\perp^2 u_z \end{pmatrix} \\ &= \begin{pmatrix} (-\gamma^2 - \nabla_\perp^2) \mathbf{u}_\perp - \gamma \nabla_\perp u_z + \gamma \nabla_\perp u_z \\ -\gamma^2 u_z - \nabla_\perp^2 u_z \end{pmatrix} = (-\gamma^2 - \nabla_\perp^2) \begin{pmatrix} \mathbf{u}_\perp \\ u_z \end{pmatrix} \end{aligned}$$

## Appendix C The 2D Helmholtz equation

We concentrate some facts concerning the equation

$$\nabla_\perp^2 v + k^2 v = 0 \quad \text{in } \Omega_\perp \quad (\text{C.1})$$

where  $\Omega_\perp \subset \mathbb{R}^2$  is a bounded domain with boundary  $\Gamma_\perp$  (a closed curve) and  $k \in \mathbb{C}$  is a constant. The relevant material is exposed in full detail in [7]. Let  $\hat{\nu}$  be the exterior normal on  $\Gamma_\perp$ . It is known that the function

$$\Phi_k(\boldsymbol{\rho}; \boldsymbol{\rho}') = \frac{i}{4} H_0^{(1)}(k\varrho),$$

where  $\varrho := |\boldsymbol{\rho} - \boldsymbol{\rho}'|$  defines a solution of the Helmholtz equation for  $\boldsymbol{\rho} \neq \boldsymbol{\rho}'$ . Actually,  $\Phi_k$  defines a fundamental solution with a pole at  $\boldsymbol{\rho}'$ . If  $v$  solves (C.1), then we have the Green's representation formula

$$\oint_{\Gamma_\perp} \left( \Phi_k(\boldsymbol{\rho}; \boldsymbol{\rho}') \frac{\partial v}{\partial \hat{\nu}}(\boldsymbol{\rho}') - \frac{\partial \Phi_k(\boldsymbol{\rho}; \boldsymbol{\rho}')}{\partial \hat{\nu}(\boldsymbol{\rho}')} v(\boldsymbol{\rho}') \right) d\ell(\boldsymbol{\rho}') = \begin{cases} v(\boldsymbol{\rho}) & , \text{ if } \boldsymbol{\rho} \in \Omega_\perp \\ 0 & , \text{ if } \boldsymbol{\rho} \in \Omega_\perp^{\text{ext}} \end{cases} \quad (\text{C.2})$$

Here by  $\Omega_{\perp}^{\text{ext}}$  we denote the set  $\mathbb{R}^2 \setminus \overline{\Omega}_{\perp}$ ,  $\overline{\Omega}_{\perp} := \Omega_{\perp} \cup \Gamma_{\perp}$ .

Now consider a function  $h(\boldsymbol{\rho}; \boldsymbol{\rho}')$ , defined for  $\boldsymbol{\rho} \in \Omega_{\perp}$ ,  $\boldsymbol{\rho}' \in \overline{\Omega}_{\perp}$ , which solves the Dirichlet problem

$$\begin{cases} \nabla_{\perp, \boldsymbol{\rho}'}^2 h(\boldsymbol{\rho}; \boldsymbol{\rho}') = 0 & \text{for } \boldsymbol{\rho} \in \Omega_{\perp} \\ h(\boldsymbol{\rho}; \boldsymbol{\rho}') = -\Phi_k(\boldsymbol{\rho}; \boldsymbol{\rho}') & \text{for } \boldsymbol{\rho}' \in \Gamma_{\perp} \end{cases} \quad (\text{C.3})$$

Then the function

$$G_k(\boldsymbol{\rho}; \boldsymbol{\rho}') := -\frac{\partial}{\partial \hat{\nu}(\boldsymbol{\rho}')} [\Phi_k(\boldsymbol{\rho}; \boldsymbol{\rho}') + h(\boldsymbol{\rho}; \boldsymbol{\rho}')] \quad (\text{C.4})$$

is called the Dirichlet Green function and solves actually the Dirichlet problem for (C.1); the function

$$v(\boldsymbol{\rho}) := \oint_{\Gamma_{\perp}} G_k(\boldsymbol{\rho}; \boldsymbol{\rho}') \varphi(\boldsymbol{\rho}') d\ell(\boldsymbol{\rho}') \quad , \quad \boldsymbol{\rho} \in \Omega_{\perp} \quad (\text{C.5})$$

satisfies (C.1) and  $v = \varphi$  on  $\Gamma_{\perp}$ .

## Appendix D The Dirichlet-to-Neumann mapping

Let  $v$  be a solution of (C.1). The Dirichlet-to-Neumann (DtN) mapping (or operator)  $\Lambda_k$  is defined formally as

$$\Lambda_k : v|_{\Gamma_{\perp}} \mapsto \frac{\partial v}{\partial \hat{\nu}}$$

Suppose that the Dirichlet problem for (C.1) is uniquely solvable (incidentally, this has to do with the shape of  $\Gamma_{\perp}$  and whether  $k^2$  is not an eigenvalue of the Dirichlet Laplacian). In that case, we have a list of consequences.

1. The problem (C.3) has a unique solution.
2. We can construct the Dirichlet Green function  $G_k$ .
3. We have the integral representation (C.5) for the solution of the Dirichlet problem.
4. The DtN operator is a well defined univalued function and is given by the formula

$$(\Lambda_k f)(\boldsymbol{\rho}) = \frac{\partial}{\partial \hat{\nu}(\boldsymbol{\rho})} \oint_{\Gamma_{\perp}} G_k(\boldsymbol{\rho}; \boldsymbol{\rho}') f(\boldsymbol{\rho}') d\ell(\boldsymbol{\rho}') \quad , \quad \boldsymbol{\rho} \in \Gamma_{\perp} \quad (\text{D.1})$$

We can clearly see that the constructions of the solution to the Dirichlet problem, of the Dirichlet Green function and of the DtN mapping are, more or less, equivalent processes. Actually, if we know the Dirichlet data and  $\Lambda_k$ , then we can construct the solution by using (C.2). The knowledge of the Green function gives us  $\Lambda_k$  via (D.1), but the differentiation with respect to the exterior normal, *i.e.*, the partial differentiation, under the integral sign needs, in general, a careful interpretation. Since differentiation is involved,  $\Lambda_k$  is expected to be an unbounded linear operator.

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