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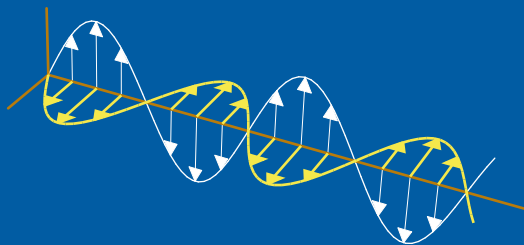
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# Homogenization of the Maxwell equations at fixed frequency

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## Abstract

The homogenization of the Maxwell equations at fixed frequency is addressed in this paper. The bulk (homogenized) electric and magnetic properties of a material with a periodic microstructure are found from the solution of a local problem on the unit cell by suitable averages. The material can be anisotropic, and satisfies a coercivity condition. The exciting field is generated by an incident field from sources outside the material under investigation. A suitable sesquilinear form is defined for the interior problem, and the exterior Calderón operator is used to solve the exterior radiating fields. The concept of two-scale convergence is employed to solve the homogenization problem. A new a priori estimate is proved as well as a new result on the correctors.

## 1 Introduction

The concept of two-scale convergence is a well established tool in the theory of homogenization of elliptic equations with rapidly oscillating coefficients, see *e.g.*, [2, 3, 7, 10, 12, 14, 18, 22, 24, 29–31]. The results apply to several types of partial differential equations that are used in the engineering sciences, such as heat conduction, elastic deformation, porous media, and acoustics. The situation is, however, different with the Maxwell equations, and the few results that exist adopt boundary conditions that are of less importance in applications. Specifically, the boundary conditions employed in the literature, see *e.g.*, [4, 6, 16, 19, 23, 29–31] are that of perfectly conducting walls. This situation applies to the case of a resonator filled with a heterogeneous material, but for other situations these boundary conditions are less applicable. Moreover, there is a need for a better understanding of how a microscopic structure alters the macroscopic electric and magnetic behavior of the material if the sources of the electromagnetic fields are located outside the heterogeneous material. In fact, most applications in the engineering sciences use external excitations, and to find the homogenized parameters of a heterogeneous material other boundary conditions, such as the penetrable boundary conditions, must be used.

The engineering literature is dominated by the simple mixture formulae, which are derived using physical arguments. For an excellent overview and history of the mixture formulae, see [25].

The object of this paper is to thoroughly analyze the homogenization of the Maxwell equations for a bounded object with penetrable boundary conditions. This homogenization problem seems not to have been published before in the literature. Moreover, the boundary condition implies that the excitation must be due to external sources. This situation is very important in many engineering applications, such as antenna applications. The two-scale convergence of the Maxwell equations depends on an a priori estimate of the fields. The external sources alters the traditional way of homogenization with two-scale convergence. In fact, in addition to the interior homogenization problem, there is an exterior scattering problem that couples via the boundary conditions to the interior problem. We solve this problem by introducing the Calderón operators, which map the tangential electric field to the

tangential magnetic field on the bounding surface. In order to apply the boundary conditions and the Calderón operators a new a priori estimate has been derived. The paper also includes new results on the correctors.

The paper is organized in the following way. Section 2 contains the prerequisites of the paper. The existence of solutions is proved in Section 3, and the homogenization of the Maxwell equations is derived in Section 4. We illustrate the exterior Calderón operator with two examples in Section 5. The paper is concluded with a series of appendices that contain definitions of function spaces, Appendix B, and some important theorems, Appendix C. An alternative proof of the a priori estimate for plane wave incidence, based on an energy integral, is given in Appendix A. In Appendix D, the vector spherical waves used in Section 5 are defined.

## 2 Formulation of the problem

### 2.1 Domain and incident fields

Assume  $\Omega$  is a bounded, open, simply connected set in  $\mathbb{R}^3$  with  $C^{1,1}$  boundary,  $\partial\Omega$ . The outward pointing unit normal is  $\hat{\nu}$ . The exterior of the volume  $\Omega$  is denoted  $\Omega_e = \mathbb{R}^3 \setminus \overline{\Omega}$ , which is assumed vacuous. See Figure 1 for a typical geometry.

The incident field,  $\mathbf{E}_i$  and  $\mathbf{H}_i$ , is assumed to have its sources outside  $\Omega$  in a bounded region  $\Omega_i$ , *i.e.*,  $\Omega \cap \Omega_i = \emptyset$ . It is assumed to be a fixed field throughout this paper. Outside this region the fields satisfy the time-harmonic Maxwell equations in vacuum (time convention  $e^{-i\omega t}$ ), *i.e.*, they satisfy<sup>1</sup>

$$\begin{cases} \nabla \times \mathbf{E}_i(\mathbf{x}) = ik_0 \mathbf{H}_i(\mathbf{x}) \\ \nabla \times \mathbf{H}_i(\mathbf{x}) = -ik_0 \mathbf{E}_i(\mathbf{x}) \end{cases} \quad \mathbf{x} \in \mathbb{R}^3$$

The wave number in vacuum is  $k_0 = \omega/c_0$ , where  $\omega$  is the angular frequency of the fields, and  $c_0$  is the speed of light in vacuum. The incident fields  $\mathbf{E}_i$  and  $\mathbf{H}_i$  are assumed to have traces on  $\partial\Omega$  belonging to  $H^{-\frac{1}{2}}(\text{div}, \partial\Omega)$ , *i.e.*,  $(\hat{\nu} \times \mathbf{E}_i, \hat{\nu} \times \mathbf{H}_i) \in H^{-\frac{1}{2}}(\text{div}, \partial\Omega) \times H^{-\frac{1}{2}}(\text{div}, \partial\Omega)$ , see Appendix B for definitions of the function spaces. Otherwise, the incident fields are arbitrary.

A particular incident field, which is commonly used in the literature, is the plane wave, *i.e.*,

$$\begin{cases} \mathbf{E}_i(\mathbf{x}) = \mathcal{E}_0 e^{ik_0 \hat{\mathbf{k}}_i \cdot \mathbf{x}} \\ \mathbf{H}_i(\mathbf{x}) = \hat{\mathbf{k}}_i \times \mathcal{E}_0 e^{ik_0 \hat{\mathbf{k}}_i \cdot \mathbf{x}} \end{cases} \quad \mathbf{x} \in \mathbb{R}^3$$

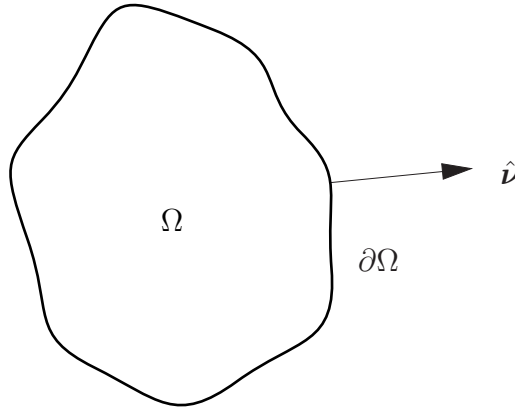
where  $\mathcal{E}_0$  is a constant complex vector, and the unit vector of incidence  $\hat{\mathbf{k}}_i$  satisfies  $\hat{\mathbf{k}}_i \cdot \mathcal{E}_0 = 0$ . The power flow of the incident plane wave in the direction of propagation

---

<sup>1</sup>We use scaled electric and magnetic fields in this paper, *i.e.*, the SI-unit fields  $\mathbf{E}_{\text{SI}}$  and  $\mathbf{H}_{\text{SI}}$  are related to the fields  $\mathbf{E}$  and  $\mathbf{H}$  used in this paper by

$$\mathbf{E}_{\text{SI}}(\mathbf{x}) = \frac{\mathbf{E}(\mathbf{x})}{\sqrt{\epsilon_0}}, \quad \mathbf{H}_{\text{SI}}(\mathbf{x}) = \frac{\mathbf{H}(\mathbf{x})}{\sqrt{\mu_0}}$$

where the permittivity and permeability of vacuum are denoted  $\epsilon_0$  and  $\mu_0$ , respectively.



**Figure 1:** Typical geometry of the scattering problem in this paper.

is

$$P_i = \frac{1}{2} \operatorname{Re} \hat{\mathbf{k}}_i \cdot (\mathbf{E}_i(\mathbf{x}) \times \mathbf{H}_i(\mathbf{x})^*) = \frac{|\mathcal{E}_0|^2}{2} \quad (2.1)$$

## 2.2 Interior problem

In  $\Omega$  we assume there is a material modelled by the permittivity dyadic  $\boldsymbol{\epsilon}(\mathbf{x})$  and the permeability dyadic  $\boldsymbol{\mu}(\mathbf{x})$ . The permittivity dyadic is assumed to satisfy

$$-ik_0 \boldsymbol{\xi} \cdot (\boldsymbol{\epsilon}(\mathbf{x}) - \boldsymbol{\epsilon}(\mathbf{x})^\dagger) \cdot \boldsymbol{\xi}^* \geq C_1 |\boldsymbol{\xi}|^2 \quad \text{for all } \boldsymbol{\xi} \in \mathbb{C}^3 \text{ and a.e. } \mathbf{x} \in \Omega \quad (2.2)$$

and

$$|\boldsymbol{\epsilon}(\mathbf{x}) \cdot \boldsymbol{\xi}| \leq C_2 |\boldsymbol{\xi}| \quad \text{for all } \boldsymbol{\xi} \in \mathbb{C}^3 \text{ and a.e. } \mathbf{x} \in \Omega \quad (2.3)$$

where  $^\dagger$  denotes the Hermitian of the dyadic  $\boldsymbol{\epsilon}$  and where  $C_i > 0$ ,  $i = 1, 2$ . The condition in (2.2) corresponds physically to a passive material, *i.e.*, a material that show dissipation. The entries of  $\boldsymbol{\epsilon}(\mathbf{x})$  are assumed to belong to  $L^\infty(\Omega)$ , which implies (2.3). Similar assumptions hold for the permeability  $\boldsymbol{\mu}$ . We note that it follows that  $\boldsymbol{\epsilon}$  and  $\boldsymbol{\mu}$  are invertible and that the inverses have the same kind of properties [9, p. 22].

In  $\Omega$  the electric field  $\mathbf{E}$  and the magnetic field  $\mathbf{H}$  satisfy the Maxwell equations

$$\begin{cases} \nabla \times \mathbf{E}(\mathbf{x}) = ik_0 \boldsymbol{\mu}(\mathbf{x}) \cdot \mathbf{H}(\mathbf{x}) \\ \nabla \times \mathbf{H}(\mathbf{x}) = -ik_0 \boldsymbol{\epsilon}(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x}) \end{cases} \quad \mathbf{x} \in \Omega \quad (2.4)$$

We are looking for solutions  $\mathbf{E}$  and  $\mathbf{H}$  of these equations in the space  $H(\operatorname{rot}, \Omega)$ . A weak formulation of the solution to this problem is found in Section 3.2.1.

## 2.3 Exterior problem

The presence of the material in the domain  $\Omega$  distorts the incident field  $\mathbf{E}_i$  and  $\mathbf{H}_i$ . This distortion is denoted the scattered field,  $\mathbf{E}_s$  and  $\mathbf{H}_s$ . They belong to

$H_{\text{loc}}(\text{rot}, \overline{\Omega}_e)$  and satisfy

$$\begin{cases} \nabla \times \mathbf{E}_s(\mathbf{x}) = ik_0 \mathbf{H}_s(\mathbf{x}) \\ \nabla \times \mathbf{H}_s(\mathbf{x}) = -ik_0 \mathbf{E}_s(\mathbf{x}) \end{cases} \quad \mathbf{x} \in \Omega_e \quad (2.5)$$

Moreover, the scattered fields satisfy the Silver-Müller radiation condition at infinity, *i.e.*, one of the following conditions [11]

$$\begin{cases} \hat{\mathbf{x}} \times \mathbf{E}_s(\mathbf{x}) - \mathbf{H}_s(\mathbf{x}) = o(1/x) \\ \hat{\mathbf{x}} \times \mathbf{H}_s(\mathbf{x}) + \mathbf{E}_s(\mathbf{x}) = o(1/x) \end{cases} \quad \text{as } x \rightarrow \infty \quad (2.6)$$

uniformly in all directions  $\hat{\mathbf{x}}$ .

In  $\Omega_e$  the sum of the incident and the scattered field is denoted the total field, *i.e.*,

$$\begin{cases} \mathbf{E}_t(\mathbf{x}) = \mathbf{E}_i(\mathbf{x}) + \mathbf{E}_s(\mathbf{x}) \\ \mathbf{H}_t(\mathbf{x}) = \mathbf{H}_i(\mathbf{x}) + \mathbf{H}_s(\mathbf{x}) \end{cases} \quad \mathbf{x} \in \Omega_e$$

The boundary conditions on  $\partial\Omega$  are

$$\begin{cases} \hat{\boldsymbol{\nu}} \times \mathbf{E}_i|_{\partial\Omega} + \hat{\boldsymbol{\nu}} \times \mathbf{E}_s|_{\partial\Omega} = \hat{\boldsymbol{\nu}} \times \mathbf{E}|_{\partial\Omega} \\ \hat{\boldsymbol{\nu}} \times \mathbf{H}_i|_{\partial\Omega} + \hat{\boldsymbol{\nu}} \times \mathbf{H}_s|_{\partial\Omega} = \hat{\boldsymbol{\nu}} \times \mathbf{H}|_{\partial\Omega} \end{cases} \quad (2.7)$$

where the traces of the fields are taken from the outside (inside) in the left (right) hand side of the equations and belongs to  $H^{-\frac{1}{2}}(\text{div}, \partial\Omega)$ .

## 2.4 Calderón operators

The Calderón operator  $C^e$  utilizes the solution of a specific exterior problem. In fact, the following exterior problem, based upon (2.5) and (2.6) and given  $\mathbf{m} \in H^{-\frac{1}{2}}(\text{div}, \partial\Omega)$ , is fundamental:

$$\begin{aligned} & 1) (\mathbf{E}_s, \mathbf{H}_s) \in H_{\text{loc}}(\text{rot}, \overline{\Omega}_e) \times H_{\text{loc}}(\text{rot}, \overline{\Omega}_e) \\ & 2) \begin{cases} \nabla \times \mathbf{E}_s(\mathbf{x}) = ik_0 \mathbf{H}_s(\mathbf{x}) \\ \nabla \times \mathbf{H}_s(\mathbf{x}) = -ik_0 \mathbf{E}_s(\mathbf{x}) \end{cases} \quad \mathbf{x} \in \Omega_e \\ & 3) \begin{cases} \hat{\mathbf{x}} \times \mathbf{E}_s(\mathbf{x}) - \mathbf{H}_s(\mathbf{x}) = o(1/x) \\ \text{or} \\ \hat{\mathbf{x}} \times \mathbf{H}_s(\mathbf{x}) + \mathbf{E}_s(\mathbf{x}) = o(1/x) \end{cases} \quad \text{as } x \rightarrow \infty \quad \text{Problem R)} \\ & 4) \hat{\boldsymbol{\nu}} \times \mathbf{E}_s|_{\partial\Omega} = \mathbf{m} \in H^{-\frac{1}{2}}(\text{div}, \partial\Omega) \end{aligned} \quad (2.8)$$

This problem has a unique solution [4, 9], see also Section 3.1.

We have the following results proved in [9, p. 35]:

**Theorem 2.1.** *With the boundary  $\partial\Omega$  of regularity  $C^{1,1}$ , the mapping*

$$\gamma_\tau : \mathbf{u} \rightarrow \hat{\boldsymbol{\nu}} \times \mathbf{u}|_{\partial\Omega}$$

*is a continuous mapping from  $H_{\text{loc}}(\text{rot}, \overline{\Omega}_e)$  onto  $H^{-\frac{1}{2}}(\text{div}, \partial\Omega)$ .*



The trace theorem is a local property of the field at the boundary, and the theorem shows that the field loses regularity on the boundary. We note that a similar results holds when the trace is taken from the inside of the boundary, see Section 3.2.

The linear mapping of the electric field to the corresponding magnetic field on the boundary for a solution of the exterior problem is denoted the exterior Calderón operator. The following definition makes this definition precise:

**Definition 2.1.** *The exterior Calderón operator  $C^e$  is defined as*

$$C^e : \mathbf{m} \rightarrow \hat{\nu} \times \mathbf{H}_s|_{\partial\Omega}, \quad H^{-\frac{1}{2}}(\text{div}, \partial\Omega) \rightarrow H^{-\frac{1}{2}}(\text{div}, \partial\Omega)$$

where  $\mathbf{m} = \hat{\nu} \times \mathbf{E}_s|_{\partial\Omega}$  and the fields  $\mathbf{E}_s$  and  $\mathbf{H}_s$  satisfy Problem R) in (2.8).

Notice that the exterior Calderón operator  $C^e$  is uniquely defined for all  $\mathbf{m} \in H^{-\frac{1}{2}}(\text{div}, \partial\Omega)$ , since Problem R) has a unique solution. Two explicit examples of the exterior Calderón operator are given in Section 5.

**Theorem 2.2.** *The exterior Calderón operator defined in Definition 2.1 has the following properties:*

1. *The exterior Calderón operator satisfies the positivity condition*

$$\text{Re} \iint_{\partial\Omega} C^e(\mathbf{m}) \cdot (\hat{\nu} \times \mathbf{m}^*) dS \geq 0, \quad \forall \mathbf{m} \in H^{-\frac{1}{2}}(\text{div}, \partial\Omega) \quad (2.9)$$

2. *The exterior Calderón operator satisfies*

$$(C^e)^2 = -\mathbf{I} \text{ on } H^{-\frac{1}{2}}(\text{div}, \partial\Omega)$$

*which implies that  $C^e$  is bounded on  $H^{-\frac{1}{2}}(\text{div}, \partial\Omega)$ .*

3. *The exterior Calderón operator is independent of the material properties inside  $\Omega$ .*

Here  $dS$  denotes the surface measure of  $\partial\Omega$ .

**Proof of Theorem 2.2:** Property 1. is a simple consequence of the radiation condition and proved in, *e.g.*, [9]. Specifically, the radiation conditions, (2.6), imply

$$\text{Re} \iint_{\partial\Omega} \hat{\nu} \cdot (\mathbf{E}_s \times \mathbf{H}_s^*) dS = \text{Re} \iint_{|\mathbf{x}|=R} \hat{\mathbf{x}} \cdot (\mathbf{E}_s \times \mathbf{H}_s^*) dS = \iint_{|\mathbf{x}|=R} |\mathbf{E}_s|^2 dS + o(1)$$

as  $R \rightarrow \infty$ , which implies (2.9), since  $\hat{\nu} \cdot (\mathbf{E}_s^* \times \mathbf{H}_s) = -C^e(\hat{\nu} \times \mathbf{E}_s) \cdot \mathbf{E}_s^*$ .

Moreover, to prove property 2. we utilize the symmetry  $\{\mathbf{E}_s, \mathbf{H}_s\} \rightarrow \{\mathbf{H}_s, -\mathbf{E}_s\}$  in the equation (2.5) and the uniqueness of the exterior problem.

Property 3. is a consequence of the uniqueness of the exterior problem. ■

An immediate consequence of the positivity property of  $C^e$  is that

$$-\text{Re} \iint_{\partial\Omega} C^e(\hat{\nu} \times \mathbf{E}_s) \cdot \mathbf{E}_s^* dS \geq 0, \quad \forall \hat{\nu} \times \mathbf{E}_s \in H^{-\frac{1}{2}}(\text{div}, \partial\Omega) \quad (2.10)$$

### 3 Existence of solutions

The existence of exterior and interior solutions is addressed in this section.

#### 3.1 Exterior problem

The system (2.5) with the radiation condition (2.6) supplied with the boundary condition:

$$\hat{\nu} \times \mathbf{E}_s|_{\partial\Omega} = \mathbf{m} \in H^{-\frac{1}{2}}(\text{div}, \partial\Omega)$$

i.e., Problem R) in (2.8), has a unique solution in  $(\mathbf{E}_s, \mathbf{H}_s) \in H_{\text{loc}}(\text{rot}, \overline{\Omega}_e) \times H_{\text{loc}}(\text{rot}, \overline{\Omega}_e)$  for any  $\mathbf{m} \in H^{-\frac{1}{2}}(\text{div}, \partial\Omega)$  [9, p. 107].

#### 3.2 Interior problem

We have the interior trace result, similar to Theorem 2.1.

**Theorem 3.1.** *With the boundary  $\partial\Omega$  of regularity  $C^{1,1}$ , the mapping*

$$\gamma_\tau : \mathbf{u} \rightarrow \hat{\nu} \times \mathbf{u}|_{\partial\Omega}$$

*is a continuous mapping from  $H(\text{rot}, \Omega)$  onto  $H^{-\frac{1}{2}}(\text{div}, \partial\Omega)$ .*

##### 3.2.1 Sesquilinear form and weak solutions

Using Theorem 3.1 we can now define the sesquilinear form [9]

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) = & - \iiint_{\Omega} \left\{ \frac{1}{ik_0} (\nabla \times \mathbf{v}^*) \cdot \boldsymbol{\mu}^{-1} \cdot (\nabla \times \mathbf{u}) + ik_0 \mathbf{v}^* \cdot \boldsymbol{\epsilon} \cdot \mathbf{u} \right\} dv \\ & - \iint_{\partial\Omega} C^e(\hat{\nu} \times \mathbf{u}) \cdot \mathbf{v}^* dS \end{aligned}$$

for  $\mathbf{u}$  and  $\mathbf{v}$  in  $H(\text{rot}, \Omega)$ . We denote the volume measure in  $\mathbb{R}^3$  by  $dv$  in this paper.

A weak formulation of the original problem is then to find  $\mathbf{E} \in H(\text{rot}, \Omega)$  such that

$$a(\mathbf{E}, \mathbf{v}) = \iint_{\partial\Omega} (\hat{\nu} \times \mathbf{H}_i - C^e(\hat{\nu} \times \mathbf{E}_i)) \cdot \mathbf{v}^* dS, \quad \forall \mathbf{v} \in H(\text{rot}, \Omega) \quad (3.1)$$

This solution satisfies the boundary conditions (2.7), and couples to an exterior, outwardly radiation solution, and implicitly the radiation condition by the exterior

Calderón operator. The corresponding magnetic field  $\mathbf{H}$  is then constructed as<sup>2</sup>

$$\begin{cases} \mathbf{H}(\mathbf{x}) = -\frac{i}{k_0} \boldsymbol{\mu}^{-1}(\mathbf{x}) \cdot (\nabla \times \mathbf{E}(\mathbf{x})) \\ \nabla \times \mathbf{H}(\mathbf{x}) = -ik_0 \boldsymbol{\epsilon}(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x}) \end{cases} \quad \mathbf{x} \in \Omega$$

To see this, let  $\mathbf{E}$  be a sufficiently regular solution to the Maxwell equations, (2.4). Then (3.1) is equivalent to the Maxwell equations with a coupling to an exterior solution since

$$\begin{aligned} a(\mathbf{E}, \mathbf{v}) &= - \iiint_{\Omega} \{(\nabla \times \mathbf{v}^*) \cdot \mathbf{H} - \mathbf{v}^* \cdot (\nabla \times \mathbf{H})\} dv - \iint_{\partial\Omega} C^e(\hat{\nu} \times \mathbf{E}) \cdot \mathbf{v}^* dS \\ &= \iint_{\partial\Omega} \{(\hat{\nu} \times \mathbf{H}) \cdot \mathbf{v}^* - C^e(\hat{\nu} \times \mathbf{E}) \cdot \mathbf{v}^*\} dS \end{aligned}$$

which is identical to (3.1) by the use of the boundary conditions on  $\partial\Omega$  and by definition

$$\iint_{\partial\Omega} C^e(\hat{\nu} \times \mathbf{E}_s) \cdot \mathbf{v}^* dS = \iint_{\partial\Omega} (\hat{\nu} \times \mathbf{H}_s) \cdot \mathbf{v}^* dS$$

Moreover, the sesquilinear form  $a$  is coercive, *i.e.*,

$$\begin{aligned} \operatorname{Re} a(\mathbf{u}, \mathbf{u}) &= - \iiint_{\Omega} \frac{1}{ik_0} (\nabla \times \mathbf{u}^*) \cdot (\boldsymbol{\mu}^{-1} - \boldsymbol{\mu}^{-1\dagger}) \cdot (\nabla \times \mathbf{u}) dv \\ &\quad - \iiint_{\Omega} ik_0 \mathbf{u}^* \cdot (\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^\dagger) \cdot \mathbf{u} dv \\ &\quad - \operatorname{Re} \iint_{\partial\Omega} C^e(\hat{\nu} \times \mathbf{u}) \cdot \mathbf{u}^* dS \geq C \|\mathbf{u}\|_{H(\operatorname{rot}, \Omega)}^2 \end{aligned} \tag{3.2}$$

---

<sup>2</sup>This construction is consistent since  $-ik_0 \boldsymbol{\epsilon}(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x})$  is the weak curl of  $\mathbf{H}(\mathbf{x}) = -\frac{i}{k_0} \boldsymbol{\mu}^{-1}(\mathbf{x}) \cdot (\nabla \times \mathbf{E}(\mathbf{x}))$ . In fact, we have

$$(\mathbf{H}, \nabla \times \phi) + ik_0 (\boldsymbol{\epsilon} \cdot \mathbf{E}, \phi) = 0 \quad \forall \phi \in D(\Omega; \mathbb{C}^3)$$

since  $a(\mathbf{E}, \phi) = 0, \forall \phi \in D(\Omega; \mathbb{C}^3)$ .

since from (2.2) we get <sup>3</sup>

$$\begin{cases} -ik_0 \boldsymbol{\xi} \cdot (\boldsymbol{\epsilon}(\mathbf{x}) - \boldsymbol{\epsilon}(\mathbf{x})^\dagger) \cdot \boldsymbol{\xi}^* \geq C_1 |\boldsymbol{\xi}|^2 \\ \frac{i}{k_0} \boldsymbol{\xi} \cdot (\boldsymbol{\mu}^{-1}(\mathbf{x}) - \boldsymbol{\mu}^{-1\dagger}(\mathbf{x})) \cdot \boldsymbol{\xi}^* \geq C_2 |\boldsymbol{\xi}|^2 \end{cases} \quad \forall \boldsymbol{\xi} \in \mathbb{C}^3 \text{ and a.e. } \mathbf{x} \in \Omega$$

and we have also used (2.10).

### 3.2.2 Existence of unique solution

Equation (3.1) has a unique solution in  $H(\text{rot}, \Omega)$  due to the Lax-Milgram theorem, see Theorem B.1, since the sesquilinear form  $a(\mathbf{u}, \mathbf{v})$  is continuous, bounded, and coercive, and the right hand side of (3.1) is continuous on  $H(\text{rot}, \Omega)$ . In fact,

$$\begin{aligned} & \left| \iint_{\partial\Omega} (\hat{\mathbf{v}} \times \mathbf{H}_i - C^e(\hat{\mathbf{v}} \times \mathbf{E}_i)) \cdot \mathbf{v}^* dS \right| \\ & \leq \left( \|\hat{\mathbf{v}} \times \mathbf{H}_i\|_{H^{-\frac{1}{2}}(\text{div}, \partial\Omega)} + \|C^e(\hat{\mathbf{v}} \times \mathbf{E}_i)\|_{H^{-\frac{1}{2}}(\text{div}, \partial\Omega)} \right) \|\mathbf{v}\|_{H^{-\frac{1}{2}}(\text{rot}, \partial\Omega)} \\ & \leq C' \left( \|\hat{\mathbf{v}} \times \mathbf{H}_i\|_{H^{-\frac{1}{2}}(\text{div}, \partial\Omega)} + \|(\hat{\mathbf{v}} \times \mathbf{E}_i)\|_{H^{-\frac{1}{2}}(\text{div}, \partial\Omega)} \right) \|\mathbf{v}\|_{H(\text{rot}, \Omega)} \end{aligned}$$

by Minkowski's inequality, duality [9, p. 38], and the continuous dependence of the trace norm on the norm of the corresponding function space.

## 4 Homogenization

So far we have considered a general heterogeneous scattering problem with a unique solution in  $H(\text{rot}, \Omega)$  for given incident electromagnetic field. But if the heterogeneous material in  $\Omega$  has a typical spatial scale which is much smaller than the size of the domain then one runs into severe numerical problems if one tries to apply some standard numerical code, *e.g.*, a finite element method (FEM). The principal obstacle is that the fine scales requires a very fine numerical mesh which generates a far too large linear system of equations for any computer to solve. However, if the wavelength of the incident field is much larger than the fine scale then the field can not resolve the fine scale and the solution of the Maxwell equations can be approximated by the solution of a scattering problem with constant coefficients, *i.e.*, the

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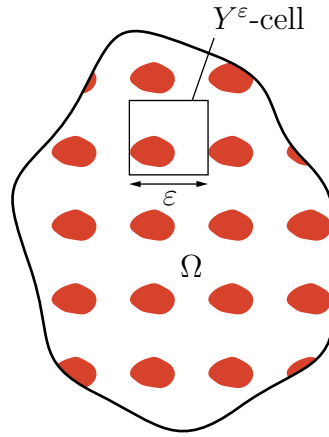
<sup>3</sup>With (2.2) we get

$$\frac{i}{k_0} (\boldsymbol{\mu}^t \cdot \boldsymbol{\zeta}) \cdot (\boldsymbol{\mu}^{-1} - \boldsymbol{\mu}^{-1\dagger}) \cdot (\boldsymbol{\mu}^t \cdot \boldsymbol{\zeta})^* = \frac{i}{k_0} \boldsymbol{\zeta} \cdot (\boldsymbol{\mu}^\dagger - \boldsymbol{\mu}) \cdot \boldsymbol{\zeta}^* \geq \frac{C_2}{k_0^2} |\boldsymbol{\zeta}|^2$$

Apply this result with  $\boldsymbol{\zeta} = \boldsymbol{\mu}^{t-1} \cdot \boldsymbol{\xi}$  and we get

$$\frac{i}{k_0} \boldsymbol{\xi} \cdot (\boldsymbol{\mu}^{-1} - \boldsymbol{\mu}^{-1\dagger}) \cdot \boldsymbol{\xi}^* \geq \frac{C_2}{k_0^2} |\boldsymbol{\mu}^{t-1} \cdot \boldsymbol{\xi}|^2 \geq C |\boldsymbol{\xi}|^2$$

since  $\boldsymbol{\mu}$  is invertible.



**Figure 2:** Typical periodic geometry of the material parameters.

heterogeneous material in  $\Omega$  has been replaced by a homogeneous material with the same effective material properties. The procedure to find these effective properties of the heterogeneous material is called homogenization.

#### 4.1 Heterogeneous problem

Let us begin with the definition of a  $Y$ -cell which is the open unit cube in  $\mathbb{R}^3$ , *i.e.*,  $Y = ]0, 1[^3$ . Further, from now on we assume that  $\epsilon$  and  $\mu$  are  $Y$ -periodic which is defined as  $\epsilon(\mathbf{x} + \hat{\mathbf{e}}_k) = \epsilon(\mathbf{x})$  for every  $k = 1, 2, 3$ , where  $\hat{\mathbf{e}}_k$ ,  $k = 1, 2, 3$ , is the canonical basis in  $\mathbb{R}^3$ .

In the following, we assume that the material in the domain  $\Omega$  is periodic with period  $\epsilon$  in the three Cartesian coordinate directions, *i.e.*, it is the union of a collection of disjoint, open identical cubes<sup>4</sup> with side length  $\epsilon$  ( $Y^\epsilon$ -cells), see Figure 2. It is easily verified that the scaled permeability and permittivity,  $\epsilon(\mathbf{x}/\epsilon)$  and  $\mu(\mathbf{x}/\epsilon)$ , are periodic with period  $\epsilon$ .

In  $\Omega$  the fields satisfy the source free Maxwell equations<sup>5</sup>

$$\begin{cases} \nabla \times \mathbf{E}^\epsilon(\mathbf{x}) = ik_0 \mathbf{B}^\epsilon(\mathbf{x}) \\ \nabla \times \mathbf{H}^\epsilon(\mathbf{x}) = -ik_0 \mathbf{D}^\epsilon(\mathbf{x}) \\ \nabla \cdot \mathbf{B}^\epsilon(\mathbf{x}) = 0 \\ \nabla \cdot \mathbf{D}^\epsilon(\mathbf{x}) = 0 \end{cases} \quad \mathbf{x} \in \Omega$$

<sup>4</sup>More general would be  $Y = (0, a_1) \times (0, a_2) \times (0, a_3)$ , where  $a_i > 0$ ,  $i = 1, 2, 3$ , and  $\epsilon(\mathbf{x} + a_k \hat{\mathbf{e}}_k) = \epsilon(\mathbf{x})$  for every  $\mathbf{x} \in \mathbb{R}^3$  and for every  $k = 1, 2, 3$ . Similar result holds for the permeability  $\mu$ .

<sup>5</sup>The electric and magnetic fields are scaled as above, see Footnote 1, and the SI-unit flux densities  $\mathbf{D}_{\text{SI}}$  and  $\mathbf{B}_{\text{SI}}$  are related to the fields  $\mathbf{D}$  and  $\mathbf{B}$  used in this paper by

$$\mathbf{D}_{\text{SI}}(\mathbf{x}) = \sqrt{\epsilon_0} \mathbf{D}(\mathbf{x}), \quad \mathbf{B}_{\text{SI}}(\mathbf{x}) = \sqrt{\mu_0} \mathbf{B}(\mathbf{x})$$

almost everywhere, with boundary conditions given by (2.7). By using the constitutive relations for the periodic material,

$$\begin{cases} \mathbf{D}^\varepsilon(\mathbf{x}) = \boldsymbol{\epsilon}(\mathbf{x}/\varepsilon) \cdot \mathbf{E}^\varepsilon(\mathbf{x}) \\ \mathbf{B}^\varepsilon(\mathbf{x}) = \boldsymbol{\mu}(\mathbf{x}/\varepsilon) \cdot \mathbf{H}^\varepsilon(\mathbf{x}) \end{cases} \quad \mathbf{x} \in \Omega$$

we eliminate  $\mathbf{D}^\varepsilon$ ,  $\mathbf{B}^\varepsilon$  and obtain

$$\begin{cases} \nabla \times \mathbf{E}^\varepsilon(\mathbf{x}) = ik_0 \boldsymbol{\mu}(\mathbf{x}/\varepsilon) \cdot \mathbf{H}^\varepsilon(\mathbf{x}) \\ \nabla \times \mathbf{H}^\varepsilon(\mathbf{x}) = -ik_0 \boldsymbol{\epsilon}(\mathbf{x}/\varepsilon) \cdot \mathbf{E}^\varepsilon(\mathbf{x}) \\ \nabla \cdot \{\boldsymbol{\epsilon}(\mathbf{x}/\varepsilon) \cdot \mathbf{E}^\varepsilon(\mathbf{x})\} = 0 \\ \nabla \cdot \{\boldsymbol{\mu}(\mathbf{x}/\varepsilon) \cdot \mathbf{H}^\varepsilon(\mathbf{x})\} = 0 \end{cases} \quad \mathbf{x} \in \Omega \quad (4.1)$$

where the solution  $(\mathbf{E}^\varepsilon, \mathbf{H}^\varepsilon)$  is in  $H(\text{rot}, \Omega) \times H(\text{rot}, \Omega)$  and belongs to a family of solutions, one for each  $\varepsilon$ . In the homogenization procedure we identify the limit of the fields  $\mathbf{E}^\varepsilon, \mathbf{H}^\varepsilon$  when  $\varepsilon \rightarrow 0$ . This limit satisfies the homogenized system with constant coefficients which is a model of a homogeneous material.

#### 4.1.1 A priori estimate

We note that the heterogeneous system in (4.1) is of the same form as (2.4) and that the constitutive relations satisfy the same assumptions as in Section 2.2. A weak formulation of the two first equations in (4.1) supplied with boundary conditions (2.7) reads

$$a^\varepsilon(\mathbf{E}^\varepsilon, \mathbf{v}) = \iint_{\partial\Omega} (\hat{\nu} \times \mathbf{H}_i - C^e(\hat{\nu} \times \mathbf{E}_i)) \cdot \mathbf{v}^* dS, \quad \forall \mathbf{v} \in H(\text{rot}, \Omega) \quad (4.2)$$

where

$$\begin{aligned} a^\varepsilon(\mathbf{u}, \mathbf{v}) = & - \iiint_{\Omega} \left\{ \frac{1}{ik_0} (\nabla \times \mathbf{v}^*) \cdot \boldsymbol{\mu}^{-1}(\mathbf{x}/\varepsilon) \cdot (\nabla \times \mathbf{u}) \right. \\ & \left. + ik_0 \mathbf{v}^* \cdot \boldsymbol{\epsilon}(\mathbf{x}/\varepsilon) \cdot \mathbf{u} \right\} dv - \iint_{\partial\Omega} C^e(\hat{\nu} \times \mathbf{u}) \cdot \mathbf{v}^* dS \end{aligned} \quad (4.3)$$

We have the following a priori estimate.

**Theorem 4.1.** *Let  $\mathbf{E}^\varepsilon, \mathbf{H}^\varepsilon$  be a solution of (4.2), then*

$$\|\mathbf{E}^\varepsilon\|_{H(\text{rot}, \Omega)} + \|\mathbf{H}^\varepsilon\|_{H(\text{rot}, \Omega)} \leq C$$

where the constant  $C$  depends only on the domain  $\Omega$ , the material parameters in  $\Omega$ , and the strength of the incident field.

**Proof of Theorem 4.1:** The sesquilinear form  $a^\varepsilon(\mathbf{u}, \mathbf{v})$  is coercive, cf. (3.2), and the weak formulation (4.2) gives:

$$\begin{aligned} C\|\mathbf{E}^\varepsilon\|_{H(\text{rot}, \Omega)}^2 &\leq \text{Re } a^\varepsilon(\mathbf{E}^\varepsilon, \mathbf{E}^\varepsilon) \leq |a^\varepsilon(\mathbf{E}^\varepsilon, \mathbf{E}^\varepsilon)| \\ &= \left| \iint_{\partial\Omega} (\hat{\nu} \times \mathbf{H}_i - C^e(\hat{\nu} \times \mathbf{E}_i)) \cdot (\mathbf{E}^\varepsilon)^* dS \right| \\ &\leq \left( \|\hat{\nu} \times \mathbf{H}_i\|_{H^{-\frac{1}{2}}(\text{div}, \partial\Omega)} + \|C^e(\hat{\nu} \times \mathbf{E}_i)\|_{H^{-\frac{1}{2}}(\text{div}, \partial\Omega)} \right) \|\mathbf{E}^\varepsilon\|_{H^{-\frac{1}{2}}(\text{rot}, \partial\Omega)} \\ &\leq C' \left( \|\hat{\nu} \times \mathbf{H}_i\|_{H^{-\frac{1}{2}}(\text{div}, \partial\Omega)} + \|(\hat{\nu} \times \mathbf{E}_i)\|_{H^{-\frac{1}{2}}(\text{div}, \partial\Omega)} \right) \|\mathbf{E}^\varepsilon\|_{H(\text{rot}, \Omega)} \end{aligned}$$

by Minkowski's inequality, duality [9, p. 38], and the continuous dependence of the trace norm on the norm of the corresponding function space. It follows now that

$$\|\mathbf{E}^\varepsilon\|_{H(\text{rot}, \Omega)} \leq C' \left( \|\hat{\nu} \times \mathbf{H}_i\|_{H^{-\frac{1}{2}}(\text{div}, \partial\Omega)} + \|(\hat{\nu} \times \mathbf{E}_i)\|_{H^{-\frac{1}{2}}(\text{div}, \partial\Omega)} \right) \leq C$$

by the assumption of the incident field. The bound of  $\mathbf{E}^\varepsilon$  can now be used in (4.1) to get the estimate of  $\mathbf{H}^\varepsilon$ .  $\blacksquare$

## 4.2 Homogenized problem

**Theorem 4.2.** *The sequence of solutions  $(\mathbf{E}^\varepsilon, \mathbf{H}^\varepsilon)$  of (4.1) converge weakly in  $H(\text{rot}, \Omega) \times H(\text{rot}, \Omega)$  to  $(\mathbf{E}, \mathbf{H}) \in H(\text{rot}, \Omega) \times H(\text{rot}, \Omega)$  the unique solution of the homogenized Maxwell equations*

$$\begin{cases} \nabla \times \mathbf{E}(\mathbf{x}) = ik_0 \boldsymbol{\mu}^h \cdot \mathbf{H}(\mathbf{x}) \\ \nabla \times \mathbf{H}(\mathbf{x}) = -ik_0 \boldsymbol{\epsilon}^h \cdot \mathbf{E}(\mathbf{x}) \\ \nabla \cdot \mathbf{B}(\mathbf{x}) = 0 \\ \nabla \cdot \mathbf{D}(\mathbf{x}) = 0 \end{cases} \quad (4.4)$$

which is coupled to the exterior problem (2.5)–(2.6) via the boundary conditions (2.7). The homogenized permeability and permittivity  $\boldsymbol{\epsilon}^h$  and  $\boldsymbol{\mu}^h$  are defined by

$$\begin{cases} \boldsymbol{\epsilon}^h = \iiint_Y \boldsymbol{\epsilon}(\mathbf{y}) \cdot (\mathbf{I}_3 - \nabla_{\mathbf{y}} \chi_e(\mathbf{y})) d\mathbf{v}_{\mathbf{y}} \\ \boldsymbol{\mu}^h = \iiint_Y \boldsymbol{\mu}(\mathbf{y}) \cdot (\mathbf{I}_3 - \nabla_{\mathbf{y}} \chi_h(\mathbf{y})) d\mathbf{v}_{\mathbf{y}} \end{cases}$$

$$\chi_e(\mathbf{y}) = \sum_{i=1}^3 \chi_e^i(\mathbf{y}) \hat{\mathbf{e}}_i \quad \chi_h(\mathbf{y}) = \sum_{i=1}^3 \chi_h^i(\mathbf{y}) \hat{\mathbf{e}}_i \quad (4.5)$$

and where  $\chi_e^i(\mathbf{y})$  and  $\chi_h^i(\mathbf{y})$ ,  $i = 1, 2, 3$ , in  $H_\#^1(Y)/\mathbb{C}$  solve the local elliptic problems

$$\begin{cases} \iiint_Y \nabla_{\mathbf{y}} w(\mathbf{y}) \cdot \boldsymbol{\epsilon}(\mathbf{y}) \cdot (\hat{\mathbf{e}}_i - \nabla_{\mathbf{y}} \chi_e^i(\mathbf{y})) d\mathbf{v}_{\mathbf{y}} = 0 \\ \iiint_Y \nabla_{\mathbf{y}} w(\mathbf{y}) \cdot \boldsymbol{\mu}(\mathbf{y}) \cdot (\hat{\mathbf{e}}_i - \nabla_{\mathbf{y}} \chi_h^i(\mathbf{y})) d\mathbf{v}_{\mathbf{y}} = 0 \end{cases} \quad (4.6)$$

for all  $w \in H_{\#}^1(Y)$ .

We note that the weak convergence is sharp in the sense that it never converges strongly in  $H(\text{rot}, \Omega)$  except in the electrostatic case (see note after Theorem C.1). However, we can get strong convergence by the use of corrector functions, see Section 4.2.2. These functions contain the fine scale information in the problem and yield strong convergence when scaled and added to the homogenized solution.

**Proof of Theorem 4.2:** We use the concept of two-scale convergence, see Appendix C. Due to the a priori estimates there exists a subsequence which converges in the two-scale sense. We will keep the index  $\varepsilon$  for this subsequence. In the end we conclude that the whole original sequence converges due to the fact that the homogenized system has a unique solution. Let  $\phi(\mathbf{x}) = \varepsilon w(\mathbf{x}/\varepsilon) \mathbf{v}(\mathbf{x})$ , where  $w \in H_{\#}^1(Y)$  and  $\mathbf{v} \in C_0^\infty(\Omega; \mathbb{C}^3)$ . Then  $\phi \in H(\text{rot}, \Omega)$  and is an admissible test function. We get in (4.1)

$$\left\{ \begin{array}{l} \iint_{\Omega} \mathbf{E}^\varepsilon(\mathbf{x}) \cdot \{ \varepsilon w(\mathbf{x}/\varepsilon) \nabla_{\mathbf{x}} \times \mathbf{v}(\mathbf{x}) + \nabla_{\mathbf{y}} w(\mathbf{x}/\varepsilon) \times \mathbf{v}(\mathbf{x}) \} dv \\ - ik_0 \iint_{\Omega} \varepsilon w(\mathbf{x}/\varepsilon) \mathbf{v}(\mathbf{x}) \cdot \{ \boldsymbol{\mu}(\mathbf{x}/\varepsilon) \cdot \mathbf{H}^\varepsilon(\mathbf{x}) \} dv = 0 \\ \iint_{\Omega} \mathbf{H}^\varepsilon(\mathbf{x}) \cdot \{ \varepsilon w(\mathbf{x}/\varepsilon) \nabla_{\mathbf{x}} \times \mathbf{v}(\mathbf{x}) + \nabla_{\mathbf{y}} w(\mathbf{x}/\varepsilon) \times \mathbf{v}(\mathbf{x}) \} dv \\ + ik_0 \iint_{\Omega} \varepsilon w(\mathbf{x}/\varepsilon) \mathbf{v}(\mathbf{x}) \cdot \{ \boldsymbol{\epsilon}(\mathbf{x}/\varepsilon) \cdot \mathbf{E}^\varepsilon(\mathbf{x}) \} dv = 0 \end{array} \right.$$

In the limit  $\varepsilon \searrow 0$  we get

$$\left\{ \begin{array}{l} \iint_{\Omega} \mathbf{E}^\varepsilon(\mathbf{x}) \cdot (\nabla_{\mathbf{y}} w(\mathbf{x}/\varepsilon) \times \mathbf{v}(\mathbf{x})) dv \rightarrow 0 \\ \iint_{\Omega} \mathbf{H}^\varepsilon(\mathbf{x}) \cdot (\nabla_{\mathbf{y}} w(\mathbf{x}/\varepsilon) \times \mathbf{v}(\mathbf{x})) dv \rightarrow 0 \end{array} \right.$$

since  $\mathbf{E}^\varepsilon$  and  $\mathbf{H}^\varepsilon$  are uniformly bounded in  $\varepsilon$  in the  $L^2(\Omega; \mathbb{C}^3)$ -norm. By the use of Theorem C.3, we get

$$\left\{ \begin{array}{l} \iint_{\Omega} \iint_Y \mathbf{E}_0(\mathbf{x}, \mathbf{y}) \cdot (\nabla_{\mathbf{y}} w(\mathbf{y}) \times \mathbf{v}(\mathbf{x})) dv_{\mathbf{y}} dv_{\mathbf{x}} = 0 \\ \iint_{\Omega} \iint_Y \mathbf{H}_0(\mathbf{x}, \mathbf{y}) \cdot (\nabla_{\mathbf{y}} w(\mathbf{y}) \times \mathbf{v}(\mathbf{x})) dv_{\mathbf{y}} dv_{\mathbf{x}} = 0 \end{array} \right.$$



which implies after cyclic permutation that

$$\begin{cases} \iiint_Y \mathbf{E}_0(\mathbf{x}, \mathbf{y}) \times \nabla_{\mathbf{y}} w(\mathbf{y}) dv_{\mathbf{y}} = 0 \\ \iiint_Y \mathbf{H}_0(\mathbf{x}, \mathbf{y}) \times \nabla_{\mathbf{y}} w(\mathbf{y}) dv_{\mathbf{y}} = 0 \end{cases} \quad \mathbf{x} \in \Omega \text{ a.e.}$$

for all  $w \in H_{\#}^1(Y)$ . The functions  $\mathbf{E}_0(\mathbf{x}, \mathbf{y})$  and  $\mathbf{H}_0(\mathbf{x}, \mathbf{y})$  both belong to the space  $L^2(\Omega; L_{\#}^2(Y; \mathbb{C}^3))$ . From Lemma C.1 we conclude that the fields  $\mathbf{E}_0(\mathbf{x}, \mathbf{y})$  and  $\mathbf{H}_0(\mathbf{x}, \mathbf{y})$  can be decomposed as

$$\begin{cases} \mathbf{E}_0(\mathbf{x}, \mathbf{y}) = \mathbf{E}(\mathbf{x}) + \nabla_{\mathbf{y}} \Phi_1(\mathbf{x}, \mathbf{y}) \\ \mathbf{H}_0(\mathbf{x}, \mathbf{y}) = \mathbf{H}(\mathbf{x}) + \nabla_{\mathbf{y}} \Psi_1(\mathbf{x}, \mathbf{y}) \end{cases}$$

where

$$\mathbf{E}(\mathbf{x}) = \langle \mathbf{E}_0(\mathbf{x}, \mathbf{y}) \rangle = \iiint_Y \mathbf{E}_0(\mathbf{x}, \mathbf{y}) dv_{\mathbf{y}}$$

and similarly for the field  $\mathbf{H}_0(\mathbf{x}, \mathbf{y})$ . In summary,

$$\begin{cases} \mathbf{E}^\varepsilon(\mathbf{x}) \xrightarrow{2-\text{s}} \mathbf{E}(\mathbf{x}) + \nabla_{\mathbf{y}} \Phi_1(\mathbf{x}, \mathbf{y}) \\ \mathbf{H}^\varepsilon(\mathbf{x}) \xrightarrow{2-\text{s}} \mathbf{H}(\mathbf{x}) + \nabla_{\mathbf{y}} \Psi_1(\mathbf{x}, \mathbf{y}) \end{cases}$$

Multiplication of (4.1) by the admissible test functions  $\phi \in C_0^\infty(\Omega; \mathbb{C}^3)$  gives

$$\begin{cases} \iiint_{\Omega} \nabla_{\mathbf{x}} \times \mathbf{E}^\varepsilon(\mathbf{x}) \cdot \phi(\mathbf{x}) dv - ik_0 \iiint_{\Omega} \phi(\mathbf{x}) \cdot \{\boldsymbol{\mu}(\mathbf{x}/\varepsilon) \cdot \mathbf{H}^\varepsilon(\mathbf{x})\} dv = 0 \\ \iiint_{\Omega} \nabla_{\mathbf{x}} \times \mathbf{H}^\varepsilon(\mathbf{x}) \cdot \phi(\mathbf{x}) dv + ik_0 \iiint_{\Omega} \phi(\mathbf{x}) \cdot \{\boldsymbol{\epsilon}(\mathbf{x}/\varepsilon) \cdot \mathbf{E}^\varepsilon(\mathbf{x})\} dv = 0 \end{cases}$$

In the limit  $\varepsilon \searrow 0$  we get

$$\begin{cases} \iiint_{\Omega} \nabla_{\mathbf{x}} \times \mathbf{E}(\mathbf{x}) \cdot \phi(\mathbf{x}) dv_{\mathbf{x}} \\ \quad - ik_0 \iiint_{\Omega} \iiint_Y \phi(\mathbf{x}) \cdot \boldsymbol{\mu}(\mathbf{y}) \cdot (\mathbf{H}(\mathbf{x}) + \nabla_{\mathbf{y}} \Psi_1(\mathbf{x}, \mathbf{y})) dv_{\mathbf{y}} dv_{\mathbf{x}} = 0 \\ \iiint_{\Omega} \nabla_{\mathbf{x}} \times \mathbf{H}(\mathbf{x}) \cdot \phi(\mathbf{x}) dv_{\mathbf{x}} \\ \quad + ik_0 \iiint_{\Omega} \iiint_Y \phi(\mathbf{x}) \cdot \boldsymbol{\epsilon}(\mathbf{y}) \cdot (\mathbf{E}(\mathbf{x}) + \nabla_{\mathbf{y}} \Phi_1(\mathbf{x}, \mathbf{y})) dv_{\mathbf{y}} dv_{\mathbf{x}} = 0 \end{cases} \quad (4.7)$$

Here we have used Theorem C.5 which states that

$$\nabla \times \mathbf{E}^\varepsilon \xrightarrow{2-\text{s}} \nabla_{\mathbf{x}} \times \mathbf{E}_0(\mathbf{x}, \mathbf{y}) + \nabla_{\mathbf{y}} \times \mathbf{E}_1(\mathbf{x}, \mathbf{y})$$

which gives the weak limit  $\nabla_{\mathbf{x}} \times \mathbf{E}(\mathbf{x})$  since the admissible test function  $\phi$  does not depend on  $\mathbf{y}$ .

The divergence equations are multiplied with  $v(\mathbf{x}) = \varepsilon\psi(\mathbf{x})\phi(\mathbf{x}/\varepsilon)$  where  $\psi \in C_0^\infty(\Omega)$ ,  $\phi \in H_\#^1(Y)$ . We note that  $w_\epsilon(\mathbf{y}) = \hat{\mathbf{e}}_i \cdot \boldsymbol{\epsilon}(\mathbf{y}) \cdot \hat{\mathbf{e}}_j \in L_\#^\infty(Y)$ , and  $w_\mu(\mathbf{y}) = \hat{\mathbf{e}}_i \cdot \boldsymbol{\mu}(\mathbf{y}) \cdot \hat{\mathbf{e}}_j \in L_\#^\infty(Y)$ , which implies that  $w_\epsilon(\mathbf{y})\nabla_{\mathbf{y}}\phi$  and  $w_\mu(\mathbf{y})\nabla_{\mathbf{y}}\phi \in L_\#^2(Y; \mathbb{C}^3)$ . Theorem C.1 and an integration by parts give

$$\begin{aligned} & \lim_{\varepsilon \searrow 0} \iiint_{\Omega} \nabla \cdot \{ \boldsymbol{\epsilon}(\mathbf{x}/\varepsilon) \cdot \mathbf{E}^\varepsilon(\mathbf{x}) \} \varepsilon\psi(\mathbf{x})\phi(\mathbf{x}/\varepsilon) dv_{\mathbf{x}} \\ &= - \lim_{\varepsilon \searrow 0} \iiint_{\Omega} \{ \varepsilon \nabla \psi(\mathbf{x})\phi(\mathbf{x}/\varepsilon) + \psi(\mathbf{x})\nabla_{\mathbf{y}}\phi(\mathbf{x}/\varepsilon) \} \cdot \{ \boldsymbol{\epsilon}(\mathbf{x}/\varepsilon) \cdot \mathbf{E}^\varepsilon(\mathbf{x}) \} dv_{\mathbf{x}} \\ &= - \iiint_{\Omega} \iiint_Y \psi(\mathbf{x})\nabla_{\mathbf{y}}\phi(\mathbf{y}) \cdot \boldsymbol{\epsilon}(\mathbf{y}) \cdot \{ \mathbf{E}(\mathbf{x}) + \nabla_{\mathbf{y}}\Phi_1(\mathbf{x}, \mathbf{y}) \} dv_{\mathbf{y}} dv_{\mathbf{x}} = 0 \end{aligned}$$

for all  $\phi \in H_\#^1(Y)$  and all  $\psi \in H_0^1(\Omega)$ . Using similar arguments for the magnetic field we get the local equations,

$$\begin{cases} \iiint_Y \nabla_{\mathbf{y}}\phi(\mathbf{y}) \cdot \boldsymbol{\epsilon}(\mathbf{y}) \cdot \{ \mathbf{E}(\mathbf{x}) + \nabla_{\mathbf{y}}\Phi_1(\mathbf{x}, \mathbf{y}) \} dv_{\mathbf{y}} = 0 \\ \iiint_Y \nabla_{\mathbf{y}}\phi(\mathbf{y}) \cdot \boldsymbol{\mu}(\mathbf{y}) \cdot \{ \mathbf{H}(\mathbf{x}) + \nabla_{\mathbf{y}}\Psi_1(\mathbf{x}, \mathbf{y}) \} dv_{\mathbf{y}} = 0 \end{cases} \quad \mathbf{x} \in \Omega \text{ a.e.} \quad (4.8)$$

Define the vector fields

$$\boldsymbol{\chi}_e(\mathbf{y}) = \sum_{i=1}^3 \chi_e^i(\mathbf{y}) \hat{\mathbf{e}}_i \quad \boldsymbol{\chi}_h(\mathbf{y}) = \sum_{i=1}^3 \chi_h^i(\mathbf{y}) \hat{\mathbf{e}}_i$$

The variables can be separated by using the Ansatz

$$\begin{cases} \nabla_{\mathbf{y}}\Phi_1(\mathbf{x}, \mathbf{y}) = -\nabla_{\mathbf{y}}\boldsymbol{\chi}_e(\mathbf{y}) \cdot \mathbf{E}(\mathbf{x}) \\ \nabla_{\mathbf{y}}\Psi_1(\mathbf{x}, \mathbf{y}) = -\nabla_{\mathbf{y}}\boldsymbol{\chi}_h(\mathbf{y}) \cdot \mathbf{H}(\mathbf{x}) \end{cases}$$

inserted into equation (4.8) which gives

$$\begin{cases} \langle \nabla_{\mathbf{y}}\phi(\mathbf{y}) \cdot (\boldsymbol{\epsilon}(\mathbf{y}) - \boldsymbol{\epsilon}(\mathbf{y}) \cdot \nabla_{\mathbf{y}}\boldsymbol{\chi}_e(\mathbf{y})) \rangle \cdot \mathbf{E}(\mathbf{x}) = 0 \\ \langle \nabla_{\mathbf{y}}\phi(\mathbf{y}) \cdot (\boldsymbol{\mu}(\mathbf{y}) - \boldsymbol{\mu}(\mathbf{y}) \cdot \nabla_{\mathbf{y}}\boldsymbol{\chi}_h(\mathbf{y})) \rangle \cdot \mathbf{H}(\mathbf{x}) = 0 \end{cases}$$

for all  $\phi \in H_\#^1(Y)$ . *i.e.*,

$$\begin{cases} \nabla_{\mathbf{y}} \cdot (\boldsymbol{\epsilon}(\mathbf{y}) - \boldsymbol{\epsilon}(\mathbf{y}) \cdot \nabla_{\mathbf{y}}\boldsymbol{\chi}_e(\mathbf{y})) = 0 \\ \nabla_{\mathbf{y}} \cdot (\boldsymbol{\mu}(\mathbf{y}) - \boldsymbol{\mu}(\mathbf{y}) \cdot \nabla_{\mathbf{y}}\boldsymbol{\chi}_h(\mathbf{y})) = 0 \end{cases}$$

a.e. in  $\Omega \times Y$ . Inserting the solutions of the local equations into (4.7) yields the macroscopic homogenized equations

$$\left\{ \begin{array}{l} \iint\limits_{\Omega} \nabla_{\mathbf{x}} \times \mathbf{E}(\mathbf{x}) \cdot \boldsymbol{\phi}(\mathbf{x}) dv_{\mathbf{x}} \\ \quad - ik_0 \iint\limits_{\Omega} \iint\limits_Y \boldsymbol{\phi}(\mathbf{x}) \cdot (\boldsymbol{\mu}(\mathbf{y}) - \boldsymbol{\mu}(\mathbf{y}) \cdot \nabla_{\mathbf{y}} \chi_h(\mathbf{y})) dv_{\mathbf{y}} \cdot \mathbf{H}(\mathbf{x}) dv_{\mathbf{x}} = 0 \\ \iint\limits_{\Omega} \nabla_{\mathbf{x}} \times \mathbf{H}(\mathbf{x}) \cdot \boldsymbol{\phi}(\mathbf{x}) dv_{\mathbf{x}} \\ \quad + ik_0 \iint\limits_{\Omega} \iint\limits_Y \boldsymbol{\phi}(\mathbf{x}) \cdot (\boldsymbol{\epsilon}(\mathbf{y}) - \boldsymbol{\epsilon}(\mathbf{y}) \cdot \nabla_{\mathbf{y}} \chi_e(\mathbf{y})) dv_{\mathbf{y}} \cdot \mathbf{E}(\mathbf{x}) dv_{\mathbf{x}} = 0 \end{array} \right.$$

and

$$\left\{ \begin{array}{l} \nabla \cdot \mathbf{B}(\mathbf{x}) = 0 \\ \nabla \cdot \mathbf{D}(\mathbf{x}) = 0 \end{array} \right.$$

which defines the homogenized permeability and permittivity as

$$\left\{ \begin{array}{l} \boldsymbol{\epsilon}^h = \iint\limits_Y \boldsymbol{\epsilon}(\mathbf{y}) \cdot (\mathbf{I}_3 - \nabla_{\mathbf{y}} \chi_e(\mathbf{y})) dv_{\mathbf{y}} \\ \boldsymbol{\mu}^h = \iint\limits_Y \boldsymbol{\mu}(\mathbf{y}) \cdot (\mathbf{I}_3 - \nabla_{\mathbf{y}} \chi_h(\mathbf{y})) dv_{\mathbf{y}} \end{array} \right.$$

*i.e.*,  $\mathbf{B} = \boldsymbol{\mu}^h \cdot \mathbf{H}$  and  $\mathbf{D} = \boldsymbol{\epsilon}^h \cdot \mathbf{E}$ . The existence of a unique solution of the homogenized system follows from the properties of the homogenized permeability and permittivity,  $\boldsymbol{\mu}^h$  and  $\boldsymbol{\epsilon}^h$ , respectively (see section 4.2.1), which satisfies the same assumptions as the material properties for the heterogeneous system.  $\blacksquare$

#### 4.2.1 The properties of the homogenized parameters

An immediate consequence of Theorem 4.2 is that the homogenized parameters are independent of the properties of the domain  $\Omega$  and of the properties of the incident field. Moreover, the homogenized material properties satisfy the same assumptions as the heterogeneous parameters do, *i.e.*, they are coercive and bounded. Coercivity and boundedness follow from the fact that the homogenized parameters are bounded from below and above by the harmonic and arithmetic averages of the heterogeneous parameters, hence the homogenized parameters are bounded from below and above (*e.g.*, see [5] or [27]). If the heterogeneous material parameters are symmetric (reciprocal material), then the homogenized parameters are also symmetric as proved below.

**Proposition 4.1.** *The homogenized permeability and permittivity are symmetric provided the heterogeneous parameters are symmetric.*

**Proof of Proposition 4.1:** We restrict ourselves to the electric parameters since the arguments for the permeability are the same. By assumption the material parameters are symmetrical, *i.e.*,  $\epsilon(\mathbf{y}) = \epsilon^t(\mathbf{y})$  and  $\mu(\mathbf{y}) = \mu^t(\mathbf{y})$ .

We define the average over the  $Y$ -cell by

$$\langle f \rangle = \int \int \int_Y f(\mathbf{y}) dv_{\mathbf{y}}$$

The local problem, (4.6), can be written as, ( $i = 1, 2, 3$ )

$$\langle \nabla_{\mathbf{y}} w(\mathbf{y}) \cdot \epsilon(\mathbf{y}) \cdot \hat{\mathbf{e}}_i \rangle = \langle \nabla_{\mathbf{y}} w(\mathbf{y}) \cdot \epsilon(\mathbf{y}) \cdot \nabla_{\mathbf{y}} \chi_e^i(\mathbf{y}) \rangle$$

for all  $w \in H_{\#}^1(Y)$ . We rewrite these equations in one set of equations, see (4.5)

$$\langle \nabla_{\mathbf{y}} w(\mathbf{y}) \cdot \epsilon(\mathbf{y}) \rangle = \langle \nabla_{\mathbf{y}} w(\mathbf{y}) \cdot \epsilon(\mathbf{y}) \cdot \nabla_{\mathbf{y}} \chi_e(\mathbf{y}) \rangle$$

for all  $w \in H_{\#}^1(Y)$ . Due to the symmetry in  $\epsilon$  we get

$$\langle \epsilon(\mathbf{y}) \cdot \nabla_{\mathbf{y}} \chi(\mathbf{y}) \rangle = \langle (\nabla_{\mathbf{y}} \chi_e(\mathbf{y}))^t \cdot \epsilon(\mathbf{y}) \cdot \nabla_{\mathbf{y}} \chi_e(\mathbf{y}) \rangle$$

if we choose  $w = \chi_e^i$ .

The homogenized parameters in (4.4) are

$$\begin{aligned} \epsilon^h &= \langle \epsilon(\mathbf{y}) \rangle - \langle \epsilon(\mathbf{y}) \cdot \nabla_{\mathbf{y}} \chi_e(\mathbf{y}) \rangle \\ &= \langle \epsilon(\mathbf{y}) \rangle - \langle (\nabla_{\mathbf{y}} \chi_e(\mathbf{y}))^t \cdot \epsilon(\mathbf{y}) \cdot \nabla_{\mathbf{y}} \chi_e(\mathbf{y}) \rangle \end{aligned}$$

which proves that  $\epsilon^h$  is symmetric. ■

#### 4.2.2 Correctors

This section is concluded by the proof of a new result on correctors.

We begin with the two-scale limit of the heterogeneous system (4.1) which is given by

$$\left\{ \begin{aligned} & \int_{\Omega} \int \int \int_Y (\nabla_{\mathbf{x}} \times \mathbf{E}_0(\mathbf{x}, \mathbf{y}) + \nabla_{\mathbf{y}} \times \mathbf{E}_1(\mathbf{x}, \mathbf{y})) \cdot \phi(\mathbf{x}, \mathbf{y}) dv_{\mathbf{y}} dv_{\mathbf{x}} \\ &= ik_0 \int_{\Omega} \int \int \int_Y \phi(\mathbf{x}, \mathbf{y}) \cdot \mu(\mathbf{y}) \cdot \mathbf{H}_0(\mathbf{x}, \mathbf{y}) dv_{\mathbf{y}} dv_{\mathbf{x}} \\ & \int_{\Omega} \int \int \int_Y (\nabla_{\mathbf{x}} \times \mathbf{H}_0(\mathbf{x}, \mathbf{y}) + \nabla_{\mathbf{y}} \times \mathbf{H}_1(\mathbf{x}, \mathbf{y})) \cdot \phi(\mathbf{x}, \mathbf{y}) dv_{\mathbf{y}} dv_{\mathbf{x}} \\ &= -ik_0 \int_{\Omega} \int \int \int_Y \phi(\mathbf{x}, \mathbf{y}) \cdot \epsilon(\mathbf{y}) \cdot \mathbf{E}_0(\mathbf{x}, \mathbf{y}) dv_{\mathbf{y}} dv_{\mathbf{x}}, \end{aligned} \right. \quad (4.9)$$

for all  $\phi \in D(\Omega; C_{\#}^{\infty}(Y; \mathbb{C}^3))$ . These equations follow from the fact that, see Appendix C,

$$\mathbf{E}^{\varepsilon}(\mathbf{x}) \xrightarrow{2-\text{s}} \mathbf{E}_0(\mathbf{x}, \mathbf{y})$$

and

$$\nabla \times \mathbf{E}^\varepsilon(\mathbf{x}) \xrightarrow{2-s} \nabla_{\mathbf{x}} \times \mathbf{E}_0(\mathbf{x}, \mathbf{y}) + \nabla_{\mathbf{y}} \times \mathbf{E}_1(\mathbf{x}, \mathbf{y}).$$

where

$$\begin{cases} \mathbf{E}_0 \in L^2(\Omega; L^2_\#(Y; \mathbb{C}^3)) \\ \nabla_{\mathbf{x}} \times \mathbf{E}_0 \in L^2(\Omega; L^2_\#(Y; \mathbb{C}^3)) \\ \mathbf{E}_1 \in L^2(\Omega; H_\#(\text{rot}, Y)/\mathbb{C}) \end{cases}$$

The system (4.9) contains macroscopic and microscopic information which gives the homogenized system when averaged over the local scale. The local equations and the two-scale limit system (4.9) provide us with the following correctors in the case when the solution of the homogenized system is smooth enough.

**Theorem 4.3.** *Let  $\mathbf{E}^\varepsilon, \mathbf{H}^\varepsilon$  be the solution of (4.1) and let  $\mathbf{E}, \mathbf{H}$  be the solution of the homogenized Maxwell equations (4.4) and  $\mathbf{E}_1, \mathbf{H}_1$  solve the two-scale limit system (4.9). If  $\mathbf{E}_0, \mathbf{H}_0, \mathbf{E}_1, \mathbf{H}_1, \nabla_{\mathbf{x}} \times \mathbf{E}_0, \nabla_{\mathbf{x}} \times \mathbf{H}_0, \nabla_{\mathbf{x}} \times \mathbf{E}_1, \nabla_{\mathbf{x}} \times \mathbf{H}_1, \nabla_{\mathbf{y}} \times \mathbf{E}_1$  and  $\nabla_{\mathbf{y}} \times \mathbf{H}_1$ , are admissible test functions then,*

$$\begin{cases} \lim_{\varepsilon \rightarrow 0} \|\mathbf{E}^\varepsilon(\mathbf{x}) - \mathbf{E}_0(\mathbf{x}, \mathbf{x}/\varepsilon) - \varepsilon \mathbf{E}_1(\mathbf{x}, \mathbf{x}/\varepsilon)\|_{H(\text{rot}, \Omega)} = 0 \\ \lim_{\varepsilon \rightarrow 0} \|\mathbf{H}^\varepsilon(\mathbf{x}) - \mathbf{H}_0(\mathbf{x}, \mathbf{x}/\varepsilon) - \varepsilon \mathbf{H}_1(\mathbf{x}, \mathbf{x}/\varepsilon)\|_{H(\text{rot}, \Omega)} = 0 \end{cases}$$

where

$$\begin{cases} \mathbf{E}_0(\mathbf{x}, \mathbf{y}) = \mathbf{E}(\mathbf{x}) - \nabla_{\mathbf{y}} \chi_e(\mathbf{y}) \cdot \mathbf{E}(\mathbf{x}) \\ \mathbf{H}_0(\mathbf{x}, \mathbf{y}) = \mathbf{H}(\mathbf{x}) - \nabla_{\mathbf{y}} \chi_h(\mathbf{y}) \cdot \mathbf{H}(\mathbf{x}) \end{cases}$$

$$\chi_e(\mathbf{y}) = \sum_{i=1}^3 \chi_e^i(\mathbf{y}) \hat{e}_i \quad \chi_h(\mathbf{y}) = \sum_{i=1}^3 \chi_h^i(\mathbf{y}) \hat{e}_i$$

and where  $\chi_e^i(\mathbf{y})$  and  $\chi_h^i(\mathbf{y})$ ,  $i = 1, 2, 3$ , in  $H^1_\#(Y)$  solve the local problems (4.6).

**Proof:** The assumptions imply that, see Theorem C.5

$$\begin{cases} \mathbf{E}^\varepsilon \xrightarrow{2-s} \mathbf{E}_0(\mathbf{x}, \mathbf{y}) \\ \nabla \times \mathbf{E}^\varepsilon \xrightarrow{2-s} \nabla_{\mathbf{x}} \times \mathbf{E}_0(\mathbf{x}, \mathbf{y}) + \nabla_{\mathbf{y}} \times \mathbf{E}_1(\mathbf{x}, \mathbf{y}) \end{cases}$$

and  $\nabla_{\mathbf{y}} \times \mathbf{E}_0(\mathbf{x}, \mathbf{y}) = \mathbf{0}$ .

The proof is carried out using the sesquilinear form

$$Q^\varepsilon(\mathbf{u}, \mathbf{v}) = - \int \int \int_{\Omega} \left\{ \frac{1}{ik_0} (\nabla \times \mathbf{v}^*) \cdot \boldsymbol{\mu}^{-1}(\mathbf{x}/\varepsilon) \cdot (\nabla \times \mathbf{u}) + ik_0 \mathbf{v}^* \cdot \boldsymbol{\epsilon}(\mathbf{x}/\varepsilon) \cdot \mathbf{u} \right\} dv$$

which is identical to (4.3), but without the surface integral term.

The coercivity assumption, (2.2), implies

$$C \|\mathbf{u}(\mathbf{x})\|_{H(\text{rot}, \Omega)}^2 \leq \text{Re } Q^\varepsilon(\mathbf{u}, \mathbf{u}).$$

We get

$$C\|\mathbf{E}^\varepsilon(\mathbf{x}) - \mathbf{E}_0(\mathbf{x}, \mathbf{x}/\varepsilon) - \varepsilon\mathbf{E}_1(\mathbf{x}, \mathbf{x}/\varepsilon)\|_{H(\text{rot}, \Omega)}^2 \leq I_1^\varepsilon + I_2^\varepsilon$$

where

$$\begin{cases} I_1^\varepsilon = \text{Re } Q^\varepsilon(\mathbf{E}^\varepsilon(\mathbf{x}), \mathbf{A}_\varepsilon(\mathbf{x})) \\ I_2^\varepsilon = -\text{Re } Q^\varepsilon(\mathbf{E}_0(\mathbf{x}, \mathbf{x}/\varepsilon) + \varepsilon\mathbf{E}_1(\mathbf{x}, \mathbf{x}/\varepsilon), \mathbf{A}_\varepsilon(\mathbf{x})) \end{cases}$$

where, for short, we denote  $\mathbf{A}_\varepsilon(\mathbf{x}) = \mathbf{E}^\varepsilon(\mathbf{x}) - \mathbf{E}_0(\mathbf{x}, \mathbf{x}/\varepsilon) - \varepsilon\mathbf{E}_1(\mathbf{x}, \mathbf{x}/\varepsilon)$ . Due to the assumptions of the fields in  $\mathbf{A}_\varepsilon(\mathbf{x})$ , we have

$$\begin{cases} \mathbf{A}_\varepsilon \xrightarrow{2-s} \mathbf{0} \\ \nabla \times \mathbf{A}_\varepsilon \xrightarrow{2-s} \mathbf{0} \end{cases}$$

since

$$\begin{cases} \mathbf{E}^\varepsilon \xrightarrow{2-s} \mathbf{E}_0(\mathbf{x}, \mathbf{y}) \\ \nabla \times \mathbf{E}^\varepsilon \xrightarrow{2-s} \nabla_{\mathbf{x}} \times \mathbf{E}_0(\mathbf{x}, \mathbf{y}) + \nabla_{\mathbf{y}} \times \mathbf{E}_1(\mathbf{x}, \mathbf{y}) \end{cases}$$

and  $\nabla_{\mathbf{y}} \times \mathbf{E}_0(\mathbf{x}, \mathbf{y}) = \mathbf{0}$ .

We start by analyzing the first integral  $I_1^\varepsilon$ . Since  $\mathbf{E}^\varepsilon$  satisfies the Maxwell equations, (4.1), we get

$$I_1^\varepsilon = \text{Re} \iiint_{\Omega} (\nabla \times \mathbf{H}^\varepsilon(\mathbf{x})) \cdot \mathbf{A}_\varepsilon(\mathbf{x})^* dv - \text{Re} \iiint_{\Omega} \mathbf{H}^\varepsilon(\mathbf{x}) \cdot (\nabla \times \mathbf{A}_\varepsilon(\mathbf{x}))^* dv$$

We now use  $\nabla \cdot (\nabla \times \mathbf{H}^\varepsilon) = 0$  and  $\nabla \cdot (\nabla \times \mathbf{A}_\varepsilon) = 0$ , and, moreover, the fact that  $\nabla \times \mathbf{H}^\varepsilon \in L^2(\Omega; \mathbb{C}^3)$  and  $\nabla \times \mathbf{A}_\varepsilon \in L^2(\Omega; \mathbb{C}^3)$ . The div-curl lemma, see [27, 28], can be used and the limit is zero, since

$$\mathbf{A}_\varepsilon(\mathbf{x}) \rightharpoonup \mathbf{0}, \text{ and } \nabla \times \mathbf{A}_\varepsilon(\mathbf{x}) \rightharpoonup \mathbf{0}$$

weakly in  $L^2(\Omega; \mathbb{C}^3)$ .

The second integral is now analyzed.

$$\begin{aligned} I_2^\varepsilon = -\text{Re} \iiint_{\Omega} \left\{ \frac{1}{ik_0} (\nabla \times \mathbf{A}_\varepsilon(\mathbf{x}))^* \cdot \boldsymbol{\mu}^{-1}(\mathbf{x}/\varepsilon) \right. \\ \cdot (\nabla_{\mathbf{x}} \times \mathbf{E}_0(\mathbf{x}, \mathbf{x}/\varepsilon) + \varepsilon \nabla_{\mathbf{x}} \times \mathbf{E}_1(\mathbf{x}, \mathbf{x}/\varepsilon) + \nabla_{\mathbf{y}} \times \mathbf{E}_1(\mathbf{x}, \mathbf{x}/\varepsilon)) \\ \left. + ik_0 \mathbf{A}_\varepsilon(\mathbf{x})^* \cdot \boldsymbol{\epsilon}(\mathbf{x}/\varepsilon) \cdot (\mathbf{E}_0(\mathbf{x}, \mathbf{x}/\varepsilon) + \varepsilon \mathbf{E}_1(\mathbf{x}, \mathbf{x}/\varepsilon)) \right\} dv_{\mathbf{x}} \end{aligned}$$

We pass to the limit,  $\varepsilon \searrow 0$ , and use that  $\boldsymbol{\mu}^{-1}(\mathbf{x}/\varepsilon) \cdot (\nabla_{\mathbf{x}} \times \mathbf{E}_0(\mathbf{x}, \mathbf{x}/\varepsilon) + \varepsilon \nabla_{\mathbf{x}} \times \mathbf{E}_1(\mathbf{x}, \mathbf{x}/\varepsilon) + \nabla_{\mathbf{y}} \times \mathbf{E}_1(\mathbf{x}, \mathbf{x}/\varepsilon))$  and  $\boldsymbol{\epsilon}(\mathbf{x}/\varepsilon) \cdot (\mathbf{E}_0(\mathbf{x}, \mathbf{x}/\varepsilon) + \varepsilon \mathbf{E}_1(\mathbf{x}, \mathbf{x}/\varepsilon))$  are admissible test functions and obtain

$$\lim_{\varepsilon \searrow 0} I_2^\varepsilon = 0$$

and the theorem is proved. ■

**Remark 4.1.** *It is still an open question how irregular a function can be and still be an admissible test function. However, if the homogenized solution  $\mathbf{E} \in C(\bar{\Omega}; \mathbb{C}^3)$  then  $\mathbf{E}_0 \in L^2_{\#}(Y; C(\bar{\Omega}; \mathbb{C}^3))$  is admissible (see Appendix C). Further, if  $\mathbf{E} \in C(\bar{\Omega}; \mathbb{C}^3)$ , then  $\mathbf{H} \in C(\bar{\Omega}; \mathbb{C}^3)$  by symmetry, and via (4.9) we find that  $\nabla_{\mathbf{x}} \times \mathbf{E}_0 + \nabla_{\mathbf{y}} \times \mathbf{E}_1$  is smooth in  $x$ , and for sufficient smoothness  $\nabla_{\mathbf{x}} \times \mathbf{E}_1$  is also an admissible test function. To the knowledge of the authors there exist no results about regularity of the solutions of the Maxwell equations in the anisotropic, constant coefficient case. However, we believe that for sufficient regular boundary and incident fields, the solutions are admissible test functions.*

## 5 Examples

In this section, we give two explicit examples of the exterior Calderón operator.

### 5.1 Plane boundary

The general representation of the solution to Problem R) in (2.8) in a region  $x_3 > c$  (plane interface  $\Omega$ ,  $x_3 = c$ ) is found by a Fourier transform in the lateral coordinates  $x_1$  and  $x_2$ .

The Fourier transform  $\mathbf{E}(\boldsymbol{\xi}, x_3)$  of the electric field  $\mathbf{E}(\mathbf{x})$ ,  $\mathbf{x} = \hat{\mathbf{e}}_1 x_1 + \hat{\mathbf{e}}_2 x_2 + \hat{\mathbf{e}}_3 x_3$ , with respect to the lateral position vector  $\boldsymbol{\rho} = \hat{\mathbf{e}}_1 x_1 + \hat{\mathbf{e}}_2 x_2$  is defined by

$$\mathbf{E}(\boldsymbol{\xi}, x_3) = \iint_{\mathbb{R}^2} \mathbf{E}(\mathbf{x}) e^{-i\boldsymbol{\xi} \cdot \boldsymbol{\rho}} d\boldsymbol{\rho}$$

where the Fourier variable  $\boldsymbol{\xi}$  is

$$\boldsymbol{\xi} = \hat{\mathbf{e}}_1 \xi_1 + \hat{\mathbf{e}}_2 \xi_2$$

and  $d\boldsymbol{\rho} = dx_1 dx_2$ . The modulus of this vector is denoted  $\xi$ , *i.e.*,

$$\xi = \sqrt{\xi_1^2 + \xi_2^2}$$

By the Fourier inversion formula,

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi^2} \iint_{\mathbb{R}^2} \mathbf{E}(\boldsymbol{\xi}, x_3) e^{i\boldsymbol{\xi} \cdot \boldsymbol{\rho}} d\boldsymbol{\xi}$$

where  $d\boldsymbol{\xi} = d\xi_1 d\xi_2$ . Specifically, the tangential electric field on the surface  $\partial\Omega$  is

$$-\hat{\mathbf{e}}_3 \times (\hat{\mathbf{e}}_3 \times \mathbf{E}(\mathbf{x}))|_{\partial\Omega} = \frac{1}{4\pi^2} \iint_{\mathbb{R}^2} \mathbf{A}(\boldsymbol{\xi}) e^{i\boldsymbol{\xi} \cdot \boldsymbol{\rho}} d\boldsymbol{\xi}$$

where  $\mathbf{A}(\boldsymbol{\xi})$  is the Fourier transform of the trace of the tangential electric field.

The general solution of the solution to Problem R) in (2.8) in a region  $x_3 > c$  is [17]

$$\begin{cases} \mathbf{E}(\mathbf{x}) = \frac{1}{4\pi^2} \iint_{\mathbb{R}^2} \left( \mathbf{I}_2 - \frac{\xi}{\xi_3} \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_{\parallel} \right) \cdot \mathbf{A}(\boldsymbol{\xi}) e^{i\boldsymbol{\xi} \cdot \boldsymbol{\rho} + i\xi_3(x_3 - c)} d\boldsymbol{\xi} \\ \mathbf{H}(\mathbf{x}) = \frac{1}{4\pi^2} \iint_{\mathbb{R}^2} \left( \frac{\boldsymbol{\xi}}{k_0} + \frac{\xi_3}{k_0} \hat{\mathbf{e}}_3 \right) \times \left( \mathbf{I}_2 - \frac{\xi}{\xi_3} \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_{\parallel} \right) \cdot \mathbf{A}(\boldsymbol{\xi}) e^{i\boldsymbol{\xi} \cdot \boldsymbol{\rho} + i\xi_3(x_3 - c)} d\boldsymbol{\xi} \end{cases}$$

where  $\mathbf{I}_2$  is the identity dyadic in  $\mathbb{R}^2$ , and a pertinent orthogonal basis in  $\mathbb{R}^2$  is  $\{\hat{\mathbf{e}}_{\parallel}, \hat{\mathbf{e}}_{\perp}\}$ , defined by

$$\hat{\mathbf{e}}_{\parallel} = \boldsymbol{\xi}/\xi, \quad \hat{\mathbf{e}}_{\perp} = \hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_{\parallel}$$

and where

$$\xi_3 = (k_0^2 - \xi^2)^{1/2} = \begin{cases} \sqrt{k_0^2 - \xi^2} & \text{for } \xi < k_0 \\ i\sqrt{\xi^2 - k_0^2} & \text{for } \xi > k_0 \end{cases}$$

and the standard convention of the square root of a non-negative argument is intended.

The representation of the fields can be simplified using dyadic calculus

$$\begin{cases} \mathbf{E}(\mathbf{x}) = \frac{1}{4\pi^2} \iint_{\mathbb{R}^2} \left( \mathbf{I}_2 - \frac{\xi}{\xi_3} \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_{\parallel} \right) \cdot \mathbf{A}(\boldsymbol{\xi}) e^{i\boldsymbol{\xi} \cdot \boldsymbol{\rho} + i\xi_3(x_3 - c)} d\boldsymbol{\xi} \\ \mathbf{H}(\mathbf{x}) = \frac{1}{4\pi^2} \iint_{\mathbb{R}^2} \left( \frac{\xi}{k_0} \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_{\perp} + \frac{k_0}{\xi_3} \hat{\mathbf{e}}_{\perp} \hat{\mathbf{e}}_{\parallel} - \frac{\xi_3}{k_0} \hat{\mathbf{e}}_{\parallel} \hat{\mathbf{e}}_{\perp} \right) \cdot \mathbf{A}(\boldsymbol{\xi}) e^{i\boldsymbol{\xi} \cdot \boldsymbol{\rho} + i\xi_3(x_3 - c)} d\boldsymbol{\xi} \end{cases}$$

From these relations the exterior Calderón operator is the transformation from

$$\hat{\mathbf{e}}_3 \times \mathbf{E}(\mathbf{x})|_{\partial\Omega} = \frac{1}{4\pi^2} \iint_{\mathbb{R}^2} \hat{\mathbf{e}}_3 \times \mathbf{A}(\boldsymbol{\xi}) e^{i\boldsymbol{\xi} \cdot \boldsymbol{\rho}} d\boldsymbol{\xi}$$

to

$$\hat{\mathbf{e}}_3 \times \mathbf{H}(\mathbf{x})|_{\partial\Omega} = -\frac{1}{4\pi^2} \iint_{\mathbb{R}^2} \left( \frac{k_0}{\xi_3} \hat{\mathbf{e}}_{\parallel} \hat{\mathbf{e}}_{\parallel} - \frac{\xi_3}{k_0} \hat{\mathbf{e}}_{\perp} \hat{\mathbf{e}}_{\perp} \right) \cdot \mathbf{A}(\boldsymbol{\xi}) e^{i\boldsymbol{\xi} \cdot \boldsymbol{\rho}} d\boldsymbol{\xi}$$

where the vector field  $\mathbf{A}(\boldsymbol{\xi})$  is determined from  $\hat{\mathbf{e}}_3 \times \mathbf{E}(\mathbf{x})|_{\partial\Omega}$  by

$$\mathbf{A}(\boldsymbol{\xi}) = - \iint_{\mathbb{R}^2} \hat{\mathbf{e}}_3 \times (\hat{\mathbf{e}}_3 \times \mathbf{E}(\mathbf{x}))|_{\partial\Omega} e^{-i\boldsymbol{\xi} \cdot \boldsymbol{\rho}} d\boldsymbol{\rho}$$

We note that in this example the domain and the boundary are unbounded which yields other function spaces for the traces. We refer to [9] for the details.



## 5.2 Spherical boundary

For a spherical boundary,  $x = a$ , the exterior Calderón operator can be represented in a series of vector spherical waves, see Appendix D.

The general solution of the solution to Problem R) in (2.8) in a region  $x > a$  is, see (D.2) and (D.3)

$$\begin{cases} \mathbf{E}(\mathbf{x}) = \sum_{\tau n} a_{\tau n} \mathbf{u}_{\tau n}(k_0 \mathbf{x}) \\ \mathbf{H}(\mathbf{x}) = -i \sum_{\tau n} a_{\tau n} \mathbf{u}_{\bar{\tau} n}(k_0 \mathbf{x}) \end{cases}$$

where the index  $\bar{\tau}$  is the dual index of  $\tau$ , defined by  $\bar{1} = 2$  and  $\bar{2} = 1$ .

The traces of the electric and the magnetic fields are ( $\kappa = k_0 a$ )

$$\begin{cases} \hat{\mathbf{x}} \times \mathbf{E}(\mathbf{x})|_{\partial\Omega} = \sum_n \left( a_{1n} h_l^{(1)}(\kappa) \mathbf{A}_{2n}(\hat{\mathbf{x}}) - a_{2n} \frac{(\kappa h_l^{(1)}(\kappa))'}{\kappa} \mathbf{A}_{1n}(\hat{\mathbf{x}}) \right) \\ \hat{\mathbf{x}} \times \mathbf{H}(\mathbf{x})|_{\partial\Omega} = -i \sum_n \left( a_{2n} h_l^{(1)}(\kappa) \mathbf{A}_{2n}(\hat{\mathbf{x}}) - a_{1n} \frac{(\kappa h_l^{(1)}(\kappa))'}{\kappa} \mathbf{A}_{1n}(\hat{\mathbf{x}}) \right) \end{cases}$$

For given tangential field  $\hat{\mathbf{x}} \times \mathbf{E}(\mathbf{x})|_{\partial\Omega}$ , the expansion coefficients  $a_{\tau n}$  are found by the orthogonality relation, see (D.1).

$$\begin{cases} a_{1n} = \frac{1}{h_l^{(1)}(\kappa)} \iint_{\gamma} \mathbf{A}_{2n}(\hat{\mathbf{x}}) \cdot (\hat{\mathbf{x}} \times \mathbf{E}(\mathbf{x})|_{\partial\Omega}) \\ a_{2n} = -\frac{\kappa}{(\kappa h_l^{(1)}(\kappa))'} \iint_{\gamma} \mathbf{A}_{1n}(\hat{\mathbf{x}}) \cdot (\hat{\mathbf{x}} \times \mathbf{E}(\mathbf{x})|_{\partial\Omega}) \end{cases}$$

The exterior Calderón mapping is the mapping from  $\hat{\mathbf{x}} \times \mathbf{E}(\mathbf{x})|_{\partial\Omega}$  (which determines the expansion coefficients  $a_{\tau n}$  uniquely) to  $\hat{\mathbf{x}} \times \mathbf{H}(\mathbf{x})|_{\partial\Omega}$ .

## 6 Conclusions

This paper analyzes the homogenization of the Maxwell equations for a material with periodic microscale. The material can be anisotropic, and satisfies a coercivity condition (passive material), and the sources of the excitation are located in the region outside the heterogeneous material in  $\Omega$ . We utilize the concept of two-scale convergence. A new a priori estimate is established and a proof of strong convergence of the corrector fields is presented. The homogenized parameters are shown to be independent of the properties of the domain  $\Omega$  and of the properties of the incident field.

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## Appendix A Energy estimates

In this section, we derive the energy estimate used in this paper, and an alternative proof of the a priori estimate for an incident plane wave.

**Theorem A.1.** *Let  $\mathbf{E}$  and  $\mathbf{H}$  in  $H(\text{rot}, \Omega)$  satisfy*

$$\begin{cases} \nabla \times \mathbf{E}(\mathbf{x}) = ik_0 \boldsymbol{\mu}(\mathbf{x}) \cdot \mathbf{H}(\mathbf{x}) \\ \nabla \times \mathbf{H}(\mathbf{x}) = -ik_0 \boldsymbol{\epsilon}(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x}) \end{cases} \quad \mathbf{x} \in \Omega$$

where the wave number in vacuum is  $k_0 = \omega/c_0$ . Define the energy integral

$$\mathcal{E} = \frac{k_0}{2} \iiint_{\Omega} \{ -\mathbf{E}^* \cdot \boldsymbol{\epsilon}^\dagger \cdot \mathbf{E} + \mathbf{H}^* \cdot \boldsymbol{\mu} \cdot \mathbf{H} \} dv$$

Then the energy integral satisfies

$$\iint_{\partial\Omega} \hat{\boldsymbol{\nu}} \cdot (\mathbf{E} \times \mathbf{H}^*) \, dS = 2i\mathcal{E}$$

and, moreover, for the permittivity and permeability dyadics defined in Section 2.2 the energy integral satisfies

$$\text{Im } \mathcal{E} \geq 0$$

and

$$\text{Re} \iint_{\partial\Omega} \hat{\boldsymbol{\nu}} \cdot (\mathbf{E} \times \mathbf{H}^*) \, dS \leq 0$$

**Proof of Theorem A.1:** The Maxwell equations imply (the Poynting's theorem)

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}^*) = ik_0 (-\mathbf{E}^* \cdot \boldsymbol{\epsilon}^\dagger \cdot \mathbf{E} + \mathbf{H}^* \cdot \boldsymbol{\mu} \cdot \mathbf{H}) \quad (\text{A.1})$$

Integrate both sides of (A.1) over the volume  $\Omega$  and we get

$$\iint_{\partial\Omega} \hat{\boldsymbol{\nu}} \cdot (\mathbf{E} \times \mathbf{H}^*) \, dS = 2i\mathcal{E}$$

where we used the definition of the energy integral  $\mathcal{E}$ . The real and the imaginary parts of this expression are

$$\begin{cases} \text{Re} \iint_{\partial\Omega} \hat{\boldsymbol{\nu}} \cdot (\mathbf{E} \times \mathbf{H}^*) \, dS = -2 \text{Im } \mathcal{E} \\ \text{Im} \iint_{\partial\Omega} \hat{\boldsymbol{\nu}} \cdot (\mathbf{E} \times \mathbf{H}^*) \, dS = 2 \text{Re } \mathcal{E} \end{cases}$$

Here, the real and the imaginary part of the energy integral are

$$\begin{cases} \operatorname{Re} \mathcal{E} = \frac{k_0}{4} \iiint_{\Omega} \{ -\mathbf{E}^* \cdot (\boldsymbol{\epsilon} + \boldsymbol{\epsilon}^\dagger) \cdot \mathbf{E} + \mathbf{H}^* \cdot (\boldsymbol{\mu} + \boldsymbol{\mu}^\dagger) \cdot \mathbf{H} \} dv \\ \operatorname{Im} \mathcal{E} = \frac{k_0}{4i} \iiint_{\Omega} \{ \mathbf{E}^* \cdot (\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^\dagger) \cdot \mathbf{E} + \mathbf{H}^* \cdot (\boldsymbol{\mu} - \boldsymbol{\mu}^\dagger) \cdot \mathbf{H} \} dv \end{cases}$$

From the passive conditions, see (2.2), *i.e.*,

$$\begin{cases} -ik_0 (\boldsymbol{\epsilon}(\mathbf{x}) - \boldsymbol{\epsilon}^\dagger(\mathbf{x})) \\ -ik_0 (\boldsymbol{\mu}(\mathbf{x}) - \boldsymbol{\mu}^\dagger(\mathbf{x})) \end{cases} \text{ are non-negative definite for all } \mathbf{x} \in \Omega$$

we deduce

$$\operatorname{Im} \mathcal{E} \geq 0$$

and the theorem is proved. ■

An alternative proof of the a priori estimate for an incident plane wave concludes this appendix.

**Alternative proof of the a priori estimate:** From Theorem A.1, any solution in  $H(\operatorname{rot}, \Omega)$  of the Maxwell equations, (2.4), satisfy

$$\operatorname{Re} \iint_{\partial\Omega} \hat{\boldsymbol{\nu}} \cdot (\mathbf{E} \times \mathbf{H}^*) dS = -2 \operatorname{Im} \mathcal{E}$$

where the fields in the surface integral are the traces of the fields taken from the inside of  $\Omega$ . Here, the imaginary part of the energy integral is

$$\operatorname{Im} \mathcal{E} = -i \frac{k_0}{4} \iiint_{\Omega} \{ \mathbf{E}^* \cdot (\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^\dagger) \cdot \mathbf{E} + \mathbf{H}^* \cdot (\boldsymbol{\mu} - \boldsymbol{\mu}^\dagger) \cdot \mathbf{H} \} dv$$

From the passive conditions, see (2.2), *i.e.*,

$$\begin{cases} -ik_0 \boldsymbol{\xi} \cdot (\boldsymbol{\epsilon}(\mathbf{x}) - \boldsymbol{\epsilon}(\mathbf{x})^\dagger) \cdot \boldsymbol{\xi}^* \geq C_1 |\boldsymbol{\xi}|^2 \\ -ik_0 \boldsymbol{\xi} \cdot (\boldsymbol{\mu}(\mathbf{x}) - \boldsymbol{\mu}(\mathbf{x})^\dagger) \cdot \boldsymbol{\xi}^* \geq C_2 |\boldsymbol{\xi}|^2 \end{cases} \quad \text{for all } \boldsymbol{\xi} \in \mathbb{C}^3 \text{ and all } \mathbf{x} \in \Omega$$

we have

$$2 \operatorname{Im} \mathcal{E} \geq C \left( \|\mathbf{E}\|_{L^2(\Omega; \mathbb{C}^3)}^2 + \|\mathbf{H}\|_{L^2(\Omega; \mathbb{C}^3)}^2 \right)$$

where  $C > 0$  depends only on the domain  $\Omega$  and the norm of the material. Therefore,

$$\|\mathbf{E}\|_{L^2(\Omega; \mathbb{C}^3)}^2 + \|\mathbf{H}\|_{L^2(\Omega; \mathbb{C}^3)}^2 \leq -C \operatorname{Re} \iint_{\partial\Omega} \hat{\boldsymbol{\nu}} \cdot (\mathbf{E} \times \mathbf{H}^*) dS \quad (\text{A.2})$$

With the use of the boundary conditions, (2.7), we can rewrite the surface integral in terms of the traces of the fields from the outside, The result is

$$\operatorname{Re} \iint_{\partial\Omega} \hat{\boldsymbol{\nu}} \cdot (\mathbf{E} \times \mathbf{H}^*) dS = \operatorname{Re} \iint_{\partial\Omega} \hat{\boldsymbol{\nu}} \cdot (\mathbf{E}_s \times \mathbf{H}_s^* + \mathbf{E}_i \times \mathbf{H}_s^* + \mathbf{E}_s \times \mathbf{H}_i^*) dS \quad (\text{A.3})$$

since

$$\operatorname{Re} \iint_{\partial\Omega} \hat{\boldsymbol{\nu}} \cdot (\mathbf{E}_i \times \mathbf{H}_i^*) dS = 0$$

for any solution that satisfy the Maxwell equation (in vacuum) inside  $\Omega$ . Moreover, the radiation conditions, (2.6), imply

$$\operatorname{Re} \iint_{\partial\Omega} \hat{\boldsymbol{\nu}} \cdot (\mathbf{E}_s \times \mathbf{H}_s^*) dS = \operatorname{Re} \iint_{|\mathbf{x}|=R} \hat{\mathbf{x}} \cdot (\mathbf{E}_s \times \mathbf{H}_s^*) dS = \iint_{|\mathbf{x}|=R} |\mathbf{E}_s|^2 dS + o(1) \quad (\text{A.4})$$

as  $R \rightarrow \infty$ . We introduce the the far field amplitude  $\mathbf{F}(\hat{\mathbf{x}})$  defined by [11]

$$\mathbf{F}(\hat{\mathbf{x}}) = \lim_{x \rightarrow \infty} k_0 x \mathbf{E}_s(\mathbf{x}) e^{-ik_0 x}$$

This limit is well-defined and an explicit expression of it is [11]

$$\mathbf{F}(\hat{\mathbf{x}}) = i \frac{k_0^2}{4\pi} \hat{\mathbf{x}} \times \iint_{\partial\Omega} \left[ \hat{\boldsymbol{\nu}}(\mathbf{x}') \times \mathbf{E}_s(\mathbf{x}') - \hat{\mathbf{x}} \times (\hat{\boldsymbol{\nu}}(\mathbf{x}') \times \mathbf{H}_s(\mathbf{x}')) \right] e^{-ik_0 \hat{\mathbf{x}} \cdot \mathbf{x}'} dS_{\mathbf{x}'} \quad (\text{A.5})$$

With this definition we easily get from (A.4) above

$$k_0^2 \operatorname{Re} \iint_{\partial\Omega} \hat{\boldsymbol{\nu}} \cdot (\mathbf{E}_s \times \mathbf{H}_s^*) dS = \|\mathbf{F}\|_\gamma^2$$

where  $\|\cdot\|_\gamma$  is the square norm on the unit sphere  $\gamma$  in  $\mathbb{R}^3$ , see Appendix B.

We now use this in (A.3), and we arrive at

$$\begin{aligned} \operatorname{Re} \iint_{\partial\Omega} \hat{\boldsymbol{\nu}} \cdot (\mathbf{E} \times \mathbf{H}^*) dS &= \frac{\|\mathbf{F}\|_\gamma^2}{k_0^2} \\ &+ \operatorname{Re} \iint_{\partial\Omega} \hat{\boldsymbol{\nu}} \cdot \left( \mathbf{E}_0^* \times \mathbf{H}_s + \mathbf{E}_s \times (\hat{\mathbf{k}}_i \times \mathcal{E}_0^*) \right) e^{-ik_0 \hat{\mathbf{k}}_i \cdot \mathbf{x}} dS \end{aligned}$$

A cyclic permutations of the vectors in the integrand

$$\begin{aligned} \hat{\boldsymbol{\nu}} \cdot \left[ \mathbf{E}_s \times (\hat{\mathbf{k}}_i \times \mathcal{E}_0^*) \right] &= (\hat{\boldsymbol{\nu}} \times \mathbf{E}_s) \cdot (\hat{\mathbf{k}}_i \times \mathcal{E}_0^*) = -\mathcal{E}_0^* \cdot \left[ \hat{\mathbf{k}}_i \times (\hat{\boldsymbol{\nu}} \times \mathbf{E}_s) \right] \\ \hat{\boldsymbol{\nu}} \cdot (\mathcal{E}_0^* \times \mathbf{H}_s) &= -\mathcal{E}_0^* \cdot (\hat{\boldsymbol{\nu}} \times \mathbf{H}_s) = \mathcal{E}_0^* \cdot \left\{ \hat{\mathbf{k}}_i \times \left[ \hat{\mathbf{k}}_i \times (\hat{\boldsymbol{\nu}} \times \mathbf{H}_s) \right] \right\} \end{aligned}$$

where we have used  $\mathcal{E}_0 \cdot \hat{\mathbf{k}}_i = 0$ , are helpful in the identification of the above surface integral as an expression in the far field amplitude (A.5).<sup>6</sup> The result is

$$\operatorname{Re} \iint_{\partial\Omega} \hat{\boldsymbol{\nu}} \cdot (\mathbf{E} \times \mathbf{H}^*) dS = \frac{\|\mathbf{F}\|_\gamma^2}{k_0^2} + \operatorname{Re} \left\{ \frac{4\pi i}{k_0^2} \mathcal{E}_0^* \cdot \mathbf{F}(\hat{\mathbf{k}}_i) \right\}$$

This implies that our estimate in (A.2) becomes

$$\|\mathbf{E}\|_{L^2(\Omega; \mathbb{C}^3)}^2 + \|\mathbf{H}\|_{L^2(\Omega; \mathbb{C}^3)}^2 \leq -C\|\mathbf{F}\|_\gamma^2 + C|\mathcal{E}_0| \|\mathbf{F}\|_\infty$$

for some constant  $C > 0$  that depends only on the domain  $\Omega$  and the norm of the material. The norm  $\|\cdot\|_\infty$  denotes the supremum norm on the unit sphere  $\gamma$  in  $\mathbb{R}^3$ .

We now assume there exists an estimate

$$\|\mathbf{F}\|_\infty^2 \leq C \left( \|\mathbf{E}\|_{L^2(\Omega; \mathbb{C}^3)}^2 + \|\mathbf{H}\|_{L^2(\Omega; \mathbb{C}^3)}^2 \right) \quad (\text{A.6})$$

for some constant  $C > 0$  that depends only on the domain  $\Omega$  and the norm of the material. This estimate implies that

$$\begin{aligned} \|\mathbf{E}\|_{L^2(\Omega; \mathbb{C}^3)}^2 + \|\mathbf{H}\|_{L^2(\Omega; \mathbb{C}^3)}^2 &\leq -C\|\mathbf{F}\|_\gamma^2 + C\frac{\delta}{2}|\mathcal{E}_0|^2 + \frac{C}{2\delta}\|\mathbf{F}\|_\infty^2 \\ &\leq C\frac{\delta}{2}|\mathcal{E}_0|^2 + \frac{C^2}{2\delta} \left( \|\mathbf{E}\|_{L^2(\Omega; \mathbb{C}^3)}^2 + \|\mathbf{H}\|_{L^2(\Omega; \mathbb{C}^3)}^2 \right) \end{aligned}$$

where  $\delta > 0$  is arbitrary. Choose  $\delta$  such that

$$\delta > C^2/2$$

and we get

$$\|\mathbf{E}\|_{L^2(\Omega; \mathbb{C}^3)}^2 + \|\mathbf{H}\|_{L^2(\Omega; \mathbb{C}^3)}^2 \leq C'|\mathcal{E}_0|^2 \leq C$$

for some constants  $C', C > 0$  that depend only on the domain  $\Omega$  and the norm of the material. The right hand side is proportional to the power flow of the incident field, see (2.1), and independent of the material parameters.

Still, we have to prove (A.6). This is accomplished by using an alternative version of (A.5)<sup>7</sup>

$$\mathbf{F}(\hat{\mathbf{x}}) = i\frac{k_0^2}{4\pi}\hat{\mathbf{x}} \times \iint_{\partial\Omega} \left[ \hat{\boldsymbol{\nu}}(\mathbf{x}') \times \mathbf{E}(\mathbf{x}') - \hat{\mathbf{x}} \times (\hat{\boldsymbol{\nu}}(\mathbf{x}') \times \mathbf{H}(\mathbf{x}')) \right] e^{-ik_0\hat{\mathbf{x}} \cdot \mathbf{x}'} dS_{\mathbf{x}'}$$

<sup>6</sup>This result is closely related to the optical theorem for electromagnetic waves [15, 20, 21].

<sup>7</sup>This is an alternative version of the far field amplitude in (A.5) which involves the total field. An easy way of proving it is to start with (A.5) and then to add the following integral to both sides:

$$\hat{\mathbf{x}} \times \iint_{\partial\Omega} \left[ \hat{\boldsymbol{\nu}}(\mathbf{x}') \times \mathbf{E}_i(\mathbf{x}') - \hat{\mathbf{x}} \times (\hat{\boldsymbol{\nu}}(\mathbf{x}') \times \mathbf{H}_i(\mathbf{x}')) \right] e^{-ik_0\hat{\mathbf{x}} \cdot \mathbf{x}'} dS_{\mathbf{x}'} = \mathbf{0}$$

Apply the Green's theorem<sup>8</sup> and use the Maxwell equations, (2.4), in  $\Omega$ . We get

$$\begin{aligned}
\mathbf{F}(\hat{\mathbf{x}}) &= \frac{ik_0^2}{4\pi} \hat{\mathbf{x}} \times \iiint_{\Omega} \left[ \nabla' \times \left( \mathbf{E}(\mathbf{x}') e^{-ik_0 \hat{\mathbf{x}} \cdot \mathbf{x}'} \right) \right. \\
&\quad \left. - \hat{\mathbf{x}} \times \left( \nabla' \times \left( \mathbf{H}(\mathbf{x}') e^{-ik_0 \hat{\mathbf{x}} \cdot \mathbf{x}'} \right) \right) \right] dv_{\mathbf{x}'} \\
&= -\frac{k_0^3}{4\pi} \hat{\mathbf{x}} \times \iiint_{\Omega} \left[ \boldsymbol{\mu}(\mathbf{x}') \cdot \mathbf{H}(\mathbf{x}') + \hat{\mathbf{x}} \times (\boldsymbol{\epsilon}(\mathbf{x}') \cdot \mathbf{E}(\mathbf{x}')) \right] e^{-ik_0 \hat{\mathbf{x}} \cdot \mathbf{x}'} dv_{\mathbf{x}'} \\
&\quad + \frac{k_0^3}{4\pi} \hat{\mathbf{x}} \times \iiint_{\Omega} \left[ \hat{\mathbf{x}} \times \mathbf{E}(\mathbf{x}') - \hat{\mathbf{x}} \times (\hat{\mathbf{x}} \times \mathbf{H}(\mathbf{x}')) \right] e^{-ik_0 \hat{\mathbf{x}} \cdot \mathbf{x}'} dv_{\mathbf{x}'} \\
&= -\frac{k_0^3}{4\pi} \hat{\mathbf{x}} \times \iiint_{\Omega} \left[ (\boldsymbol{\mu}(\mathbf{x}') - \mathbf{I}) \cdot \mathbf{H}(\mathbf{x}') \right. \\
&\quad \left. + \hat{\mathbf{x}} \times (\boldsymbol{\epsilon}(\mathbf{x}') - \mathbf{I}) \cdot \mathbf{E}(\mathbf{x}') \right] e^{-ik_0 \hat{\mathbf{x}} \cdot \mathbf{x}'} dv_{\mathbf{x}'}
\end{aligned}$$

From this expression and we easily arrive at (A.6)

$$\|\mathbf{F}\|_{\infty}^2 \leq C \left( \|\mathbf{E}\|_{L^2(\Omega; \mathbb{C}^3)}^2 + \|\mathbf{H}\|_{L^2(\Omega; \mathbb{C}^3)}^2 \right)$$

with the use of (2.3), *i.e.*,

$$\begin{cases} |\boldsymbol{\epsilon}(\mathbf{x}) \cdot \boldsymbol{\xi}| \leq C_1 |\boldsymbol{\xi}| \\ |\boldsymbol{\mu}(\mathbf{x}) \cdot \boldsymbol{\xi}| \leq C_2 |\boldsymbol{\xi}| \end{cases} \quad \text{for all } \boldsymbol{\xi} \in \mathbb{C}^3 \text{ and all } \mathbf{x} \in \Omega$$

We have proved

$$\|\mathbf{E}\|_{L^2(\Omega; \mathbb{C}^3)}^2 + \|\mathbf{H}\|_{L^2(\Omega; \mathbb{C}^3)}^2 \leq C$$

for some constant  $C > 0$  that depends only on the domain  $\Omega$ , the norm of the material and the amplitude of the incident field.

The corresponding estimates on, *i.e.*,  $\|\nabla \times \mathbf{E}\|_{L^2(\Omega; \mathbb{C}^3)}^2 + \|\nabla \times \mathbf{H}\|_{L^2(\Omega; \mathbb{C}^3)}^2 \leq C$ , follow at once by the use of (2.4). ■

## Appendix B Function spaces

In this appendix, we list the various function spaces used in this paper. Let  $\Omega$  be a bounded, open, simply connected set in  $\mathbb{R}^3$  with Lipschitz boundary  $\partial\Omega$ . A  $Y$ -periodic function,  $f$ , is defined as  $f(\mathbf{x} + \hat{\mathbf{e}}_k) = f(\mathbf{x})$  for every  $k = 1, 2, 3$ , where  $\hat{\mathbf{e}}_k$ ,  $k = 1, 2, 3$ , is the canonical basis in  $\mathbb{R}^3$ .

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<sup>8</sup>We use an alternative version of the Green's theorem

$$\iint_{\partial\Omega} \hat{\boldsymbol{\nu}}(\mathbf{x}) \times \mathbf{E}(\mathbf{x}) dS_{\mathbf{x}} = \iiint_{\Omega} \nabla \times \mathbf{E}(\mathbf{x}) dv_{\mathbf{x}}$$

The space  $C(\Omega)$  is the space of continuous functions in  $\Omega$ . We also use  $C_0(\overline{\Omega})$  which consists of all uniformly continuous functions which are zero at the boundary. The space  $C^\infty(\Omega)$  is the space of infinitely continuously differentiable functions in  $\Omega$ , and  $C_0^\infty(\Omega)$  are the functions in this space with compact support in  $\Omega$ , which we also denote  $D(\Omega)$ . Moreover,

$$C_\#^\infty(Y) = \{\phi \in C^\infty(\mathbb{R}^3), \phi \text{ } Y\text{-periodic}\}$$

Several function spaces with square integrable functions are used in this paper. The basic space is

$$L^2(\Omega) \stackrel{\text{def}}{=} \left\{ u(\mathbf{x}) : u \text{ Lebesgue integrable, } \iiint_{\Omega} |u(\mathbf{x})|^2 dv_{\mathbf{x}} < \infty \right\}$$

with norm

$$\|u\|_{L^2(\Omega)} = \left\{ \iiint_{\Omega} |u(\mathbf{x})|^2 dv_{\mathbf{x}} \right\}^{1/2}$$

Similarly for vector-valued spaces we have the norm

$$\|\mathbf{u}\|_{L^2(\Omega; \mathbb{C}^3)} = \left\{ \iiint_{\Omega} |\mathbf{u}(\mathbf{x})|^2 dv_{\mathbf{x}} \right\}^{1/2}$$

We also define two function spaces of periodic functions.

$$L_\#^2(Y) \stackrel{\text{def}}{=} \{\text{the completion of } C_\#^\infty(Y) \text{ in the } L^2(Y)\text{-norm}\}$$

and

$$\begin{aligned} L_\#^\infty(Y) &\stackrel{\text{def}}{=} \{\phi \in L^\infty(\mathbb{R}^3), \phi \text{ } Y\text{-periodic}\} \\ \begin{cases} H(\text{div}, \Omega) &\stackrel{\text{def}}{=} \{\mathbf{u} \in L^2(\Omega; \mathbb{C}^3) : \nabla \cdot \mathbf{u} \in L^2(\Omega)\} \\ H(\text{rot}, \Omega) &\stackrel{\text{def}}{=} \{\mathbf{u} \in L^2(\Omega; \mathbb{C}^3) : \nabla \times \mathbf{u} \in L^2(\Omega; \mathbb{C}^3)\} \end{cases} \end{aligned}$$

which are Hilbert spaces with norms

$$\begin{cases} \|\mathbf{u}\|_{H(\text{div}, \Omega)} = \left( \|\mathbf{u}\|_{L^2(\Omega; \mathbb{C}^3)}^2 + \|\nabla \cdot \mathbf{u}\|_{L^2(\Omega)}^2 \right)^{1/2} \\ \|\mathbf{u}\|_{H(\text{rot}, \Omega)} = \left( \|\mathbf{u}\|_{L^2(\Omega; \mathbb{C}^3)}^2 + \|\nabla \times \mathbf{u}\|_{L^2(\Omega; \mathbb{C}^3)}^2 \right)^{1/2} \end{cases}$$

The curl and the divergence are defined in the weak sense as

$$\begin{cases} (\nabla \times \mathbf{u}, \phi) = (\mathbf{u}, \nabla \times \phi), & \forall \phi \in D(\Omega; \mathbb{C}^3) \\ (\nabla \cdot \mathbf{u}, \phi) = -(\mathbf{u}, \nabla \phi), & \forall \phi \in D(\Omega) \end{cases}$$

In the exterior region, we define spaces of locally integrable functions as

$$\begin{cases} H_{\text{loc}}(\text{div}, \overline{\Omega}_e) \stackrel{\text{def}}{=} \{ \mathbf{u} \in D'(\Omega_e; \mathbb{C}^3) : \xi \mathbf{u} \in H(\text{div}, \Omega_e), \forall \xi \in D(\mathbb{R}^3) \} \\ H_{\text{loc}}(\text{rot}, \overline{\Omega}_e) \stackrel{\text{def}}{=} \{ \mathbf{u} \in D'(\Omega_e; \mathbb{C}^3) : \xi \nabla \times \mathbf{u} \in H(\text{rot}, \Omega_e), \forall \xi \in D(\mathbb{R}^3) \} \end{cases}$$

where  $\Omega_e = \mathbb{R}^3 \setminus \overline{\Omega}$  and  $D'(\Omega_e)$  is the space of distributions. The appropriate trace spaces used in this paper are  $H^{-\frac{1}{2}}(\text{div}, \partial\Omega)$  and  $H^{-\frac{1}{2}}(\text{rot}, \partial\Omega)$  defined by

$$\begin{cases} H^{-\frac{1}{2}}(\text{div}, \partial\Omega) \stackrel{\text{def}}{=} \{ \mathbf{u} \in H^{-\frac{1}{2}}(\partial\Omega; \mathbb{C}^3), \hat{\nu} \cdot \mathbf{u} = 0, \text{div}_{\partial\Omega} \mathbf{u} \in H^{-\frac{1}{2}}(\partial\Omega) \} \\ H^{-\frac{1}{2}}(\text{rot}, \partial\Omega) \stackrel{\text{def}}{=} \{ \mathbf{u} \in H^{-\frac{1}{2}}(\partial\Omega; \mathbb{C}^3), \hat{\nu} \cdot \mathbf{u} = 0, \text{rot}_{\partial\Omega} \mathbf{u} \in H^{-\frac{1}{2}}(\partial\Omega; \mathbb{C}^3) \} \end{cases}$$

where the surface divergence,  $\text{div}_{\partial\Omega}$ , and the surface rotation,  $\text{rot}_{\partial\Omega}$ , are defined by duality and restriction

$$\begin{cases} (\text{div}_{\partial\Omega} \mathbf{u}, \phi) = -(\mathbf{u}, \text{grad}_{\partial\Omega} \phi), & \forall \phi \in D(\partial\Omega) \\ \text{rot}_{\partial\Omega} \mathbf{u} = \hat{\nu} \cdot (\nabla \times \mathbf{u})|_{\partial\Omega} \end{cases}$$

and the surface gradient,  $\text{grad}_{\partial\Omega}$ , is defined by the orthogonal projection of  $\nabla$  on the surface  $\partial\Omega$ .

We also define the function spaces

$$\begin{cases} H_{\#}(\text{div}, Y) \stackrel{\text{def}}{=} \{ \mathbf{u} \in H(\text{div}, Y), \mathbf{u} \text{ } Y\text{-periodic} \} \\ H_{\#}(\text{rot}, Y) \stackrel{\text{def}}{=} \{ \mathbf{u} \in H(\text{rot}, Y), \mathbf{u} \text{ } Y\text{-periodic} \} \end{cases}$$

and

$$\begin{cases} H_{\#}^1(Y) \stackrel{\text{def}}{=} \{ \text{the completion of } C_{\#}^{\infty}(Y) \text{ in the } H^1(Y)\text{-norm} \} \\ H_{\#}^1(Y)/\mathbb{C} \stackrel{\text{def}}{=} \{ \phi \in H_{\#}^1(Y), \text{equivalent up to a complex constant} \} \end{cases}$$

If  $\gamma$  denotes the unit sphere in  $\mathbb{R}^3$ , the following norms are used in the paper

$$\begin{cases} \|\mathbf{u}\|_{\gamma} = \left\{ \iint_{\gamma} |\mathbf{u}(\hat{\mathbf{x}})|^2 d\gamma \right\}^{1/2} \\ \|\mathbf{u}\|_{\infty} = \sup_{|\hat{\mathbf{x}}|=1} |\mathbf{u}(\hat{\mathbf{x}})| \end{cases}$$

and  $d\gamma$  denotes the surface measure on the unit sphere in  $\mathbb{R}^3$ .

We conclude this appendix by stating the Lax-Milgram theorem [13].

**Theorem B.1 (Lax-Milgram).** *Assume that  $H$  is a Hilbert space, with norm  $\|\cdot\|$ . Moreover, assume*

$$B : H \times H \rightarrow \mathbb{C}$$



is a sesquilinear functional on  $H$ , for which there exists constants  $a, b > 0$ , such that

$$|B[u, v]| \leq a\|u\|\|v\|, \quad \forall u, v \in H$$

and

$$b\|u\|^2 \leq |B[u, u]|, \quad \forall u \in H$$

Finally, let  $f : H \rightarrow \mathbb{C}$  be a bounded linear functional on  $H$ .

Then there exists a unique  $u \in H$  such that

$$B[u, v] = f(v), \quad \forall v \in H.$$

## Appendix C Two-scale convergence

**Definition C.1.** A sequence  $\{\mathbf{u}^\varepsilon\}$  in  $L^2(\Omega; \mathbb{C}^3)$  two-scale converges to  $\mathbf{u}_0 \in L^2(\Omega \times Y; \mathbb{C}^3)$  if

$$\lim_{\varepsilon \searrow 0} \iiint_{\Omega} \mathbf{u}^\varepsilon(\mathbf{x}) \cdot \phi(\mathbf{x}, \mathbf{x}/\varepsilon) dv_{\mathbf{x}} = \iiint_{\Omega} \iiint_Y \mathbf{u}_0(\mathbf{x}, \mathbf{y}) \cdot \phi(\mathbf{x}, \mathbf{y}) dv_{\mathbf{y}} dv_{\mathbf{x}}$$

for every  $\phi \in D(\Omega; C_{\#}^\infty(Y; \mathbb{C}^3))$ . We denote this by  $\mathbf{u}^\varepsilon \xrightarrow{2-s} \mathbf{u}_0$ .

The class of test functions can be enlarged to all admissible test functions defined below [2].

**Definition C.2.** We say that  $\phi \in L^2(\Omega; L_{\#}^2(Y; \mathbb{C}^3))$  is an admissible test function if  $\phi(\mathbf{x}, \mathbf{x}/\varepsilon)$  is measurable and

$$\lim_{\varepsilon \searrow 0} \|\phi(\mathbf{x}, \mathbf{x}/\varepsilon)\|_{L^2(\Omega; \mathbb{C}^3)} = \|\phi(\mathbf{x}, \mathbf{y})\|_{L^2(\Omega \times Y; \mathbb{C}^3)}.$$

**Remark C.1.** Some examples of admissible test functions are  $L^2(\Omega; C_{\#}(Y; \mathbb{C}^3))$  and for  $\Omega$  bounded  $L_{\#}^2(Y; C(\overline{\Omega}; \mathbb{C}^3))$ .

We cite two important theorems by Nguetseng [22].

**Theorem C.1 (Nguetseng).** Let  $u^\varepsilon \in L^2(\Omega)$ . Suppose that there exists a constant  $C > 0$  such that

$$\|u^\varepsilon\|_{L^2(\Omega)} \leq C \text{ for all } \varepsilon$$

Then a subsequence (still denoted by  $\varepsilon$ ) can be extracted from  $\varepsilon$  such that, letting  $\varepsilon \searrow 0$

$$\iiint_{\Omega} u^\varepsilon(\mathbf{x}) \Psi(\mathbf{x}, \mathbf{x}/\varepsilon) dv_{\mathbf{x}} \rightarrow \iiint_{\Omega} \iiint_Y u_0(\mathbf{x}, \mathbf{y}) \Psi(\mathbf{x}, \mathbf{y}) dv_{\mathbf{y}} dv_{\mathbf{x}}$$

for all  $\Psi \in C_0(\overline{\Omega}; C_{\#}(Y))$ , where  $u_0 \in L^2(\Omega; L_{\#}^2(Y))$ . Moreover,

$$\iiint_{\Omega} u^\varepsilon(\mathbf{x}) v(\mathbf{x}) w(\mathbf{x}/\varepsilon) dv_{\mathbf{x}} \rightarrow \iiint_{\Omega} \iiint_Y u_0(\mathbf{x}, \mathbf{y}) v(\mathbf{x}) w(\mathbf{y}) dv_{\mathbf{y}} dv_{\mathbf{x}}$$

for all  $v \in C_0(\overline{\Omega})$ , and all  $w \in L_{\#}^2(Y)$ .

We note that if  $u^\varepsilon$  is a sequence in  $L^2(\Omega)$ , which two-scale converges to the limit  $u_0 \in L^2(\Omega \times Y)$ , then  $u^\varepsilon$  also converges to  $u(\mathbf{x}) = \iint_Y u_0(\mathbf{x}, \mathbf{y}) dv_{\mathbf{y}}$  in  $L^2(\Omega)$  weakly [2]. Moreover, if  $u^\varepsilon$  converges strongly to  $u(\mathbf{x})$  in  $L^2(\Omega)$ , then  $u^\varepsilon$  two-scale converges to the same limit  $u(\mathbf{x})$ . The second theorem is,

**Theorem C.2 (Nguetseng).** *Let  $u^\varepsilon \in H^1(\Omega)$ . Suppose that there exists a constant  $C > 0$  such that*

$$\|u^\varepsilon\|_{H^1(\Omega)} \leq C \text{ for all } \varepsilon$$

*Then a subsequence (still denoted by  $\varepsilon$ ) can be extracted from  $\varepsilon$  such that, letting  $\varepsilon \searrow 0$*

$$u^\varepsilon \rightarrow u \text{ in } H^1(\Omega)\text{-weak}$$

*and*

$$\begin{aligned} & \iiint_{\Omega} \frac{\partial u^\varepsilon(\mathbf{x})}{\partial x_j} v(\mathbf{x}) w(\mathbf{x}/\varepsilon) dv_{\mathbf{x}} \\ & \rightarrow \iiint_{\Omega} \iiint_Y \left\{ \frac{\partial u(\mathbf{x})}{\partial x_j} + \frac{\partial u_1(\mathbf{x}, \mathbf{y})}{\partial y_j} \right\} v(\mathbf{x}) w(\mathbf{y}) dv_{\mathbf{y}} dv_{\mathbf{x}} \end{aligned}$$

*$j = 1, 2, 3$ , for all  $v \in C_0(\overline{\Omega})$ , and all  $w \in L^2_{\#}(Y)$ , where  $u_1 \in L^2(\Omega; H^1_{\#}(Y)/\mathbb{C})$ .*

In addition to these two theorems we observe that taking  $w = 1$  we get from Theorem C.1

$$\iiint_{\Omega} u^\varepsilon(\mathbf{x}) v(\mathbf{x}) dv_{\mathbf{x}} \rightarrow \iiint_{\Omega} u(\mathbf{x}) v(\mathbf{x}) dv_{\mathbf{x}}$$

for all  $v \in C_0(\overline{\Omega})$ , where

$$u(\mathbf{x}) = \iiint_Y u_0(\mathbf{x}, \mathbf{y}) dv_{\mathbf{y}}$$

is the usual weak  $L^2(\Omega)$ -limit of  $u^\varepsilon(\mathbf{x})$ . It follows that  $u_0$  is uniquely expressed in the form

$$u_0(\mathbf{x}, \mathbf{y}) = u(\mathbf{x}) + \tilde{u}_0(\mathbf{x}, \mathbf{y})$$

where

$$\iiint_Y \tilde{u}_0(\mathbf{x}, \mathbf{y}) dv_{\mathbf{y}} = 0$$

**Lemma C.1.** *Let  $\mathbf{f} \in H^1_{\#}(Y; \mathbb{C}^3)$  and assume that  $\nabla_{\mathbf{y}} \times \mathbf{f}(\mathbf{y}) = \mathbf{0}$ . Moreover, assume  $\langle \mathbf{f} \rangle = 0$ . Then there exists a unique function  $q \in H^1_{\#}(Y)/\mathbb{C}$  such that*

$$\mathbf{f}(\mathbf{y}) = \nabla_{\mathbf{y}} q(\mathbf{y})$$

**Proof of Lemma C.1:** The periodicity of the function  $\mathbf{f} \in H_{\#}^1(Y; \mathbb{C}^3)$  implies that  $\mathbf{f}$  has a Fourier expansion

$$\mathbf{f}(\mathbf{y}) = \sum_n \mathbf{f}_n e^{i\mathbf{k}_n \cdot \mathbf{y}}$$

where the vector  $\mathbf{k}_n$  is defined as

$$\mathbf{k}_n = 2\pi n_1 \hat{\mathbf{e}}_1 + 2\pi n_2 \hat{\mathbf{e}}_2 + 2\pi n_3 \hat{\mathbf{e}}_3$$

and where  $n_1, n_2, n_3$  are integers and  $n = (n_1, n_2, n_3)$ . The sequence  $\mathbf{f}_n$  belongs to  $(\ell_1^2)^3$ . The assumption that  $\langle \mathbf{f} \rangle = 0$  implies that  $\mathbf{f}_{(0,0,0)} = 0$ . Moreover, the coefficients  $\mathbf{f}_n$  satisfy

$$\mathbf{k}_n \times \mathbf{f}_n = \mathbf{0} \quad \text{for all } n$$

Therefore  $\mathbf{f}_n$  has the form

$$\mathbf{f}_n = \hat{\mathbf{k}}_n (\hat{\mathbf{k}}_n \cdot \mathbf{f}_n)$$

Define  $q_n$  as

$$\begin{cases} q_n = -i(\hat{\mathbf{k}}_n \cdot \mathbf{f}_n)/k_n & \text{for } n \neq (0, 0, 0) \\ q_{(0,0,0)} & \text{arbitrary} \end{cases}$$

where  $k_n = |\mathbf{k}_n|$ . The coefficients  $q_n \in (\ell_1^2)^3$  and

$$\mathbf{f}_n = i\mathbf{k}_n q_n \text{ for all } n$$

and

$$\mathbf{f}(\mathbf{y}) = \sum_n i\mathbf{k}_n q_n e^{i\mathbf{k}_n \cdot \mathbf{y}} = \nabla_{\mathbf{y}} q(\mathbf{y})$$

where

$$q(\mathbf{y}) = \sum_n q_n e^{i\mathbf{k}_n \cdot \mathbf{y}} \in H_{\#}^1(Y)/\mathbb{C}$$

since  $q_{(0,0,0)}$  is arbitrary and the lemma is proved. ■

The obvious vector analogous theorems are:

**Theorem C.3.** Let  $\mathbf{u}^\varepsilon \in L^2(\Omega; \mathbb{C}^3)$ . Suppose that there exists a constant  $C > 0$  such that

$$\|\mathbf{u}^\varepsilon\|_{L^2(\Omega; \mathbb{C}^3)} \leq C \text{ for all } \varepsilon$$

Then a subsequence (still denoted by  $\varepsilon$ ) can be extracted from  $\varepsilon$  such that, letting  $\varepsilon \searrow 0$

$$\iiint_{\Omega} \mathbf{u}^\varepsilon(\mathbf{x}) \cdot \Psi(\mathbf{x}, \mathbf{x}/\varepsilon) dv_{\mathbf{x}} \rightarrow \iiint_{\Omega} \iiint_Y \mathbf{u}_0(\mathbf{x}, \mathbf{y}) \cdot \Psi(\mathbf{x}, \mathbf{y}) dv_{\mathbf{y}} dv_{\mathbf{x}}$$

for all  $\Psi \in C_0(\overline{\Omega}; C_{\#}(Y; \mathbb{C}^3))$ , where  $\mathbf{u}_0 \in L^2(\Omega; L_{\#}^2(Y; \mathbb{C}^3))$ . Moreover,

$$\iiint_{\Omega} \mathbf{u}^\varepsilon(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) w(\mathbf{x}/\varepsilon) dv_{\mathbf{x}} \rightarrow \iiint_{\Omega} \iiint_Y \mathbf{u}_0(\mathbf{x}, \mathbf{y}) \cdot \mathbf{v}(\mathbf{x}) w(\mathbf{y}) dv_{\mathbf{y}} dv_{\mathbf{x}}$$

for all  $\mathbf{v} \in C_0(\overline{\Omega}; \mathbb{C}^3)$ , and all  $w \in L_{\#}^2(Y)$ .

The field  $\mathbf{u}_0$  is uniquely expressed in the form

$$\mathbf{u}_0(\mathbf{x}, \mathbf{y}) = \mathbf{u}(\mathbf{x}) + \tilde{\mathbf{u}}_0(\mathbf{x}, \mathbf{y})$$

where

$$\iiint_Y \tilde{\mathbf{u}}_0(\mathbf{x}, \mathbf{y}) dv_{\mathbf{y}} = \mathbf{0}$$

We have the following results proved in [26]:

**Theorem C.4.** *Let  $\mathbf{u}^\varepsilon \in H(\operatorname{div}, \Omega)$ . Suppose that there exists a constant  $C > 0$  such that*

$$\|\mathbf{u}^\varepsilon\|_{H(\operatorname{div}, \Omega)} \leq C \text{ for all } \varepsilon$$

*Then a subsequence (still denoted by  $\varepsilon$ ) can be extracted from  $\varepsilon$  such that, letting  $\varepsilon \searrow 0$*

$$\mathbf{u}^\varepsilon \rightarrow \mathbf{u} \text{ in } L^2(\Omega; \mathbb{C}^3)\text{-weak}$$

*and*

$$\begin{aligned} & \iiint_{\Omega} \nabla_{\mathbf{x}} \cdot \mathbf{u}^\varepsilon(\mathbf{x}) v(\mathbf{x}) w(\mathbf{x}/\varepsilon) dv_{\mathbf{x}} \\ & \rightarrow \iiint_{\Omega} \iiint_Y \{\nabla_{\mathbf{x}} \cdot \mathbf{u}(\mathbf{x}) + \nabla_{\mathbf{y}} \cdot \mathbf{u}_1(\mathbf{x}, \mathbf{y})\} v(\mathbf{x}) w(\mathbf{y}) dv_{\mathbf{y}} dv_{\mathbf{x}} \end{aligned}$$

*for all  $v \in C_0(\overline{\Omega})$ , and all  $w \in L^2_{\#}(Y)$ , where  $\mathbf{u}(\mathbf{x}) = \iiint_Y \mathbf{u}_0(\mathbf{x}, \mathbf{y}) dv_{\mathbf{y}}$ ,  $\mathbf{u}_0$  is the two-scale limit of  $\mathbf{u}^\varepsilon$ , and  $\mathbf{u}_1 \in L^2(\Omega; H_{\#}(\operatorname{div}, Y))$ .*

**Theorem C.5.** *Let  $\mathbf{u}^\varepsilon \in H(\operatorname{rot}, \Omega)$ . Suppose that there exists a constant  $C > 0$  such that*

$$\|\mathbf{u}^\varepsilon\|_{H(\operatorname{rot}, \Omega)} \leq C \text{ for all } \varepsilon$$

*Then a subsequence (still denoted by  $\varepsilon$ ) can be extracted from  $\varepsilon$  such that, letting  $\varepsilon \searrow 0$*

$$\mathbf{u}^\varepsilon \rightarrow \mathbf{u}_0 \text{ in } L^2(\Omega; \mathbb{C}^3)\text{-weak}$$

*and*

$$\begin{aligned} & \iiint_{\Omega} \nabla \times \mathbf{u}^\varepsilon(\mathbf{x}) \cdot v(\mathbf{x}) \mathbf{w}(\mathbf{x}/\varepsilon) dv_{\mathbf{x}} \\ & \rightarrow \iiint_{\Omega} \iiint_Y \{\nabla_{\mathbf{x}} \times \mathbf{u}_0(\mathbf{x}, \mathbf{y}) + \nabla_{\mathbf{y}} \times \mathbf{u}_1(\mathbf{x}, \mathbf{y})\} \cdot v(\mathbf{x}) \mathbf{w}(\mathbf{y}) dv_{\mathbf{y}} dv_{\mathbf{x}} \end{aligned}$$

*for all  $v \in C_0(\overline{\Omega})$ , and all  $\mathbf{w} \in L^2_{\#}(Y; \mathbb{C}^3)$ , where  $\mathbf{u}_1 \in L^2(\Omega; H_{\#}(\operatorname{rot}, Y))$ .*

**Proof of Theorem C.5:** From Theorem C.3 we get

$$\iint_{\Omega} \mathbf{u}^\varepsilon(\mathbf{x}) \cdot \Psi(\mathbf{x}, \mathbf{x}/\varepsilon) dv_{\mathbf{x}} \rightarrow \iiint_{\Omega} \iint_Y \mathbf{u}_0(\mathbf{x}, \mathbf{y}) \cdot \Psi(\mathbf{x}, \mathbf{y}) dv_{\mathbf{y}} dv_{\mathbf{x}}$$

and

$$\iiint_{\Omega} \nabla \times \mathbf{u}^\varepsilon(\mathbf{x}) \cdot \Psi(\mathbf{x}, \mathbf{x}/\varepsilon) dv_{\mathbf{x}} \rightarrow \iiint_{\Omega} \iint_Y \chi_0(\mathbf{x}, \mathbf{y}) \cdot \Psi(\mathbf{x}, \mathbf{y}) dv_{\mathbf{y}} dv_{\mathbf{x}}$$

for all  $\Psi \in C_0(\overline{\Omega}; C_{\#}(Y; \mathbb{C}^3))$ , where  $\mathbf{u}_0, \chi_0 \in L^2(\Omega; L^2_{\#}(Y; \mathbb{C}^3))$ . Choose test functions  $\Psi \in C_0(\overline{\Omega}; C_{\#}(Y; \mathbb{C}^3))$  such that  $\nabla_{\mathbf{y}} \times \Psi = 0$ . We get by integration by parts

$$\begin{aligned} \iiint_{\Omega} \nabla \times \mathbf{u}^\varepsilon(\mathbf{x}) \cdot \Psi(\mathbf{x}, \mathbf{x}/\varepsilon) dv_{\mathbf{x}} &= \iiint_{\Omega} \mathbf{u}^\varepsilon(\mathbf{x}) \cdot \nabla \times \Psi(\mathbf{x}, \mathbf{x}/\varepsilon) dv_{\mathbf{x}} \\ &= \iiint_{\Omega} \mathbf{u}^\varepsilon(\mathbf{x}) \cdot \nabla_{\mathbf{x}} \times \Psi(\mathbf{x}, \mathbf{x}/\varepsilon) dv_{\mathbf{x}} \\ &\rightarrow \iiint_{\Omega} \iint_Y \mathbf{u}_0(\mathbf{x}, \mathbf{y}) \cdot \nabla_{\mathbf{x}} \times \Psi(\mathbf{x}, \mathbf{y}) dv_{\mathbf{y}} dv_{\mathbf{x}} \\ &= \iiint_{\Omega} \iint_Y \nabla_{\mathbf{x}} \times \mathbf{u}_0(\mathbf{x}, \mathbf{y}) \cdot \Psi(\mathbf{x}, \mathbf{y}) dv_{\mathbf{y}} dv_{\mathbf{x}}. \end{aligned}$$

This means that

$$\iiint_{\Omega} \iint_Y (\chi_0(\mathbf{x}, \mathbf{y}) - \nabla_{\mathbf{x}} \times \mathbf{u}_0(\mathbf{x}, \mathbf{y})) \cdot \Psi(\mathbf{x}, \mathbf{y}) dv_{\mathbf{y}} dv_{\mathbf{x}} = 0$$

for all  $\Psi \in C_0(\overline{\Omega}; C_{\#}(Y; \mathbb{C}^3))$  such that  $\nabla_{\mathbf{y}} \times \Psi = 0$ . By the decomposition of  $L^2(\Omega; \mathbb{C}^3)$  (e.g., see [9]) there exists a function  $\mathbf{u}_1 \in L^2(\Omega; H_{\#}(\text{rot}, Y))$  such that

$$\nabla_{\mathbf{y}} \times \mathbf{u}_1 = \chi_0(\mathbf{x}, \mathbf{y}) - \nabla_{\mathbf{x}} \times \mathbf{u}_0(\mathbf{x}, \mathbf{y}).$$

■

**Theorem C.6 (Wellander [29] or [30]).** *Let  $\mathbf{u}^\varepsilon \in H(\text{rot}, \Omega)$ . Suppose that there exists a constant  $C > 0$  such that*

$$\|\mathbf{u}^\varepsilon\|_{H(\text{rot}, \Omega)} \leq C \text{ for all } \varepsilon$$

*Then a subsequence (still denoted by  $\varepsilon$ ) can be extracted from  $\varepsilon$  such that, letting  $\varepsilon \searrow 0$*

$$\mathbf{u}^\varepsilon \xrightarrow{2-s} \mathbf{u}(\mathbf{x}) + \nabla_{\mathbf{y}} \phi(\mathbf{x}, \mathbf{y})$$

where  $\phi \in L^2(\Omega; H_{\#}^1(Y))$  is a scalar-valued function satisfying

$$\iiint_Y \nabla_{\mathbf{y}} \phi(\mathbf{x}, \mathbf{y}) dv_{\mathbf{y}} = \mathbf{0}$$

Moreover,

$$\nabla \times \mathbf{u}^\varepsilon \rightharpoonup \nabla \times \mathbf{u}(\mathbf{x}) \text{ in } L^2(\Omega; \mathbb{C}^3)$$

**Theorem C.7 (Wellander [29] or [30]).** Let  $\mathbf{u}^\varepsilon \in H(\operatorname{div}, \Omega)$ . Suppose that there exists a constant  $C > 0$  such that

$$\|\mathbf{u}^\varepsilon\|_{H(\operatorname{div}, \Omega)} \leq C \text{ for all } \varepsilon$$

Then a subsequence (still denoted by  $\varepsilon$ ) can be extracted from  $\varepsilon$  such that, letting  $\varepsilon \searrow 0$

$$\mathbf{u}^\varepsilon \xrightarrow{2-s} \mathbf{u}_0(\mathbf{x}, \mathbf{y})$$

and

$$\varepsilon \nabla \cdot \mathbf{u}^\varepsilon \xrightarrow{2-s} \nabla_{\mathbf{y}} \cdot \mathbf{u}_0(\mathbf{x}, \mathbf{y})$$

**Proof of Theorem C.7:** From Theorem C.3 we get

$$\iiint_{\Omega} \mathbf{u}^\varepsilon(\mathbf{x}) \cdot \Psi(\mathbf{x}, \mathbf{x}/\varepsilon) dv_{\mathbf{x}} \rightarrow \iiint_{\Omega} \iiint_Y \mathbf{u}_0(\mathbf{x}, \mathbf{y}) \cdot \Psi(\mathbf{x}, \mathbf{y}) dv_{\mathbf{y}} dv_{\mathbf{x}}$$

and

$$\iiint_{\Omega} \nabla \cdot \mathbf{u}^\varepsilon(\mathbf{x}) \Psi(\mathbf{x}, \mathbf{x}/\varepsilon) dv_{\mathbf{x}} \rightarrow \iiint_{\Omega} \iiint_Y \chi_0(\mathbf{x}, \mathbf{y}) \Psi(\mathbf{x}, \mathbf{y}) dv_{\mathbf{y}} dv_{\mathbf{x}}$$

for all  $\Psi \in C_0(\overline{\Omega}; C_{\#}(Y; \mathbb{C}^3))$  and  $\Psi \in C_0(\overline{\Omega}; C_{\#}(Y))$ , where  $\mathbf{u}_0 \in L^2(\Omega; L_{\#}^2(Y; \mathbb{C}^3))$  and  $\chi_0 \in L^2(\Omega; L_{\#}^2(Y))$ .

We get by integration by parts

$$\begin{aligned} \iiint_{\Omega} \varepsilon \nabla \cdot \mathbf{u}^\varepsilon(\mathbf{x}) \Psi(\mathbf{x}, \mathbf{x}/\varepsilon) dv_{\mathbf{x}} &= - \iiint_{\Omega} \varepsilon \mathbf{u}^\varepsilon(\mathbf{x}) \cdot \nabla \Psi(\mathbf{x}, \mathbf{x}/\varepsilon) dv_{\mathbf{x}} \\ &= - \iiint_{\Omega} \varepsilon \mathbf{u}^\varepsilon(\mathbf{x}) \cdot \nabla_{\mathbf{x}} \Psi(\mathbf{x}, \mathbf{x}/\varepsilon) dv_{\mathbf{x}} - \iiint_{\Omega} \mathbf{u}^\varepsilon(\mathbf{x}) \cdot \nabla_{\mathbf{y}} \Psi(\mathbf{x}, \mathbf{x}/\varepsilon) dv_{\mathbf{x}} \\ &\rightarrow - \iiint_{\Omega} \iiint_Y \mathbf{u}_0(\mathbf{x}, \mathbf{y}) \cdot \nabla_{\mathbf{y}} \Psi(\mathbf{x}, \mathbf{y}) dv_{\mathbf{y}} dv_{\mathbf{x}} \\ &= \iiint_{\Omega} \iiint_Y \nabla_{\mathbf{y}} \cdot \mathbf{u}_0(\mathbf{x}, \mathbf{y}) \Psi(\mathbf{x}, \mathbf{y}) dv_{\mathbf{y}} dv_{\mathbf{x}}. \end{aligned}$$

■

## Appendix D Vector spherical harmonics

The vector spherical harmonics are defined as [8]

$$\begin{cases} \mathbf{A}_{1n}(\hat{\mathbf{x}}) = \frac{1}{\sqrt{l(l+1)}} \nabla \times (\mathbf{x} Y_n(\hat{\mathbf{x}})) = \frac{1}{\sqrt{l(l+1)}} \nabla Y_n(\hat{\mathbf{x}}) \times \mathbf{x} \\ \mathbf{A}_{2n}(\hat{\mathbf{x}}) = \frac{1}{\sqrt{l(l+1)}} \mathbf{x} \nabla Y_n(\hat{\mathbf{x}}) \\ \mathbf{A}_{3n}(\hat{\mathbf{x}}) = \hat{\mathbf{x}} Y_n(\hat{\mathbf{x}}) \end{cases}$$

where the spherical harmonics are denoted  $Y_n(\hat{\mathbf{x}})$ . The index  $n$  is a multi-index for the integer indices  $l = 0, 1, 2, 3, \dots$ ,  $m = 0, 1, \dots, l$ , and  $\sigma = \text{e, o}$  (even and odd in the azimuthal angle). From these definitions we see that the first two vector spherical harmonics,  $\mathbf{A}_{1n}(\hat{\mathbf{x}})$  and  $\mathbf{A}_{2n}(\hat{\mathbf{x}})$ , are tangential to the unit sphere  $\gamma$  in  $\mathbb{R}^3$  and they are related as

$$\begin{cases} \hat{\mathbf{x}} \times \mathbf{A}_{1n}(\hat{\mathbf{x}}) = \mathbf{A}_{2n}(\hat{\mathbf{x}}) \\ \hat{\mathbf{x}} \times \mathbf{A}_{2n}(\hat{\mathbf{x}}) = -\mathbf{A}_{1n}(\hat{\mathbf{x}}) \end{cases}$$

The vector spherical harmonics form an orthonormal set over the unit sphere  $\gamma$  in  $\mathbb{R}^3$ , *i.e.*,

$$\iint_{\gamma} \mathbf{A}_{\tau n}(\hat{\mathbf{x}}) \cdot \mathbf{A}_{\tau' n'}(\hat{\mathbf{x}}) d\gamma = \delta_{nn'} \delta_{\tau\tau'} \quad (\text{D.1})$$

The radiating solutions to the Maxwell equations in vacuum are defined as

$$\begin{cases} \mathbf{u}_{1n}(k_0 \mathbf{x}) = h_l^{(1)}(k_0 x) \mathbf{A}_{1n}(\hat{\mathbf{x}}) \\ \mathbf{u}_{2n}(k_0 \mathbf{x}) = \frac{1}{k_0} \nabla \times \left( h_l^{(1)}(k_0 x) \mathbf{A}_{1n}(\hat{\mathbf{x}}) \right) \end{cases}$$

where  $h_l^{(1)}(k_0 x)$  is the spherical Hankel function of the first kind [1]. These vector waves satisfy

$$\nabla \times (\nabla \times \mathbf{u}_{\tau n}(k_0 \mathbf{x})) - k_0^2 \mathbf{u}_{\tau n}(k_0 \mathbf{x}) = \mathbf{0}, \quad \tau = 1, 2 \quad (\text{D.2})$$

and they also satisfy the radiation condition in (2.6). Another representation of the definition of the vector waves is

$$\begin{cases} \mathbf{u}_{1n}(k_0 \mathbf{x}) = h_l^{(1)}(k_0 x) \mathbf{A}_{1n}(\hat{\mathbf{x}}) \\ \mathbf{u}_{2n}(k_0 \mathbf{x}) = \frac{(k_0 x h_l^{(1)}(k_0 x))'}{k_0 x} \mathbf{A}_{2n}(\hat{\mathbf{x}}) + \sqrt{l(l+1)} \frac{h_l^{(1)}(k_0 x)}{k_0 x} \mathbf{A}_{3n}(\hat{\mathbf{x}}) \end{cases}$$

where  $'$  denotes differentiation w.r.t. the argument of the spherical Hankel function. A simple consequence of these definitions is

$$\begin{cases} \mathbf{u}_{1n}(k_0 \mathbf{x}) = \frac{1}{k_0} \nabla \times \mathbf{u}_{2n}(k_0 \mathbf{x}) \\ \mathbf{u}_{2n}(k_0 \mathbf{x}) = \frac{1}{k_0} \nabla \times \mathbf{u}_{1n}(k_0 \mathbf{x}). \end{cases} \quad (\text{D.3})$$

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