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On H-infinity Control and Large-Scale Systems

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To Petter and Silas

Abstract

In this thesis, a class of linear time-invariant systems is identified for which a particular type of H-infinity optimal control problem can be solved explicitly. It follows that the synthesized controller can be given on a simple explicit form. More specifically, the controller can be written in terms of the matrices of the system's state-space representation. The result has applications in the control of large-scale systems, as well as for the control of infinite-dimensional systems, with certain properties.

For the large-scale applications considered, the controller is both globally optimal as well as possesses a structure compatible with the information-structure of the system. This decentralized property of the controller is obtained without any structural constraints or regularization techniques being part of the synthesis procedure. Instead, it is a result of its particular form. Examples of applications are electrical networks, temperature dynamics in buildings and water irrigation systems.

In the infinite-dimensional case, the explicitly stated controller solves the infinite-dimensional H-infinity synthesis problem directly without the need of approximation techniques. An important application is diffusion equations. Moreover, the presented results can be used for evaluation and benchmarking of general purpose algorithms for H-infinity control.

The systems considered in this thesis are shown to belong to a larger class of systems for which the H-infinity optimal control problem can be translated into a static problem at a single frequency. In certain cases, the static problem can be solved through a simple least-squares argument. This procedure is what renders the simple and explicit expression of the controller previously described. Moreover, the given approach is in contrast to conventional methods to the problem of H-infinity control, as they are in general performed numerically.

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I would like to thank my supervisors Professor Anders Rantzer, Professor Bo Bernhardsson and Doctor Richard Pates for their support, encouragement and advice throughout this work. Anders, you have taught me so much, made me see possibilities when I have struggled to, and been the greatest ambassador for my work. Richard, thank you for all the discussions, it has been invaluable to have you on board on this journey. Bosse, your ability to see plausibility in theoretical results has been oh so useful. And also, thank you for introducing me to a new exiting project! My thanks extend to Professor Kirsten Morris for great collaborations and discussions, and for making me take the leap into infinite-dimensional spaces.

I am very grateful for being a part of the Department of Automatic Control at Lund University. I would like to thank all of my colleagues, former and current, for making it such a great place to work at. Thank you former and current office mates for letting cats play a central role in our everyday life. Thank you Leif for making this thesis look pretty. Thank you Eva and other fellow JäLM-are and JäLM-husare for your devotion and positive spirit at our meetings. Thank you everyone in the technical and administrative staff for making the department run smoothly. Thank you Gustav, Martin Heyden, Christian Rosdahl, Eva and Anders Robertsson for proofreading. Thank you Karl Johan for being a marvellous role-model to us all, and for always offering your help. Thank you Pontus G for letting me optimize. Thank you Olof for the annual Påsklunch and making me do Toughest. Thank you Andreas for being a great mentor. Thank you Mika for my very own cookie jar, among many many other things. I also want to thank my colleagues abroad at IMA for looking after me during my stay.

Most importantly, I would like to thank my family and my friends. Petter, Silas, Snäckan och Loui. Mamma, pappa, Bella och Mimmi. Nonno och Nonna i himlen. Ie, Lucci och kussarna med familjer. Maria och trion. Ni är mitt allt. But most of all, in regards to this work, thank you Pappsen for always inspiring me, challenging me and believing in me. With you by my side, I have always followed my dreams.

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Lidström, C., A. Rantzer, and K. A. Morris (2016). "H-infinity optimal control for infinite-dimensional systems with strictly negative generator". In: *2016 IEEE 55th Conference on Decision and Control (CDC),* IEEE, pp. 5275–5280.

is owing to the Thematic year 2015-2016 organized on Control Theory and its Applications at the Institute for Mathematics and its Applications, University of Minnesota, Minneapolis, USA.

Contents

1

Systems of Large Scale or Infinite Dimension

In our everyday lives, we are dependent upon numerous systems of large scale. Transportation networks, the power grid, water supply systems, cellular networks and the internet are only a few examples. The goal of these large-scale systems is to support multiple entities simultaneously with their main functionality, often through some kind of network structure.

In this thesis, a large-scale system is defined as a system composed of a large number of components. Hence, it is not necessarily a system of large spatial size. Also, note that the given definition does not only include man-made systems, such as those previously mentioned, but also systems naturally occurring, such as river networks or ecosystems. However, the systems considered in this thesis are further restricted to be linear and time-invariant, which confines the type of dynamics that can be analysed. This might seem limiting but it is a well-known fact that linear models can be used to describe the behaviour of systems around an operating point. Moreover, such a representation is often sufficient for the purpose of control design, which is the main focus of this thesis.

Although confined to linear dynamics, models of large-scale systems can still become very complex. For instance, they could have high dimension, reflective of the many components apparent in these systems. Sometimes they are even represented as infinite-dimensional systems, i.e modelled by partial differential equations that can describe the dynamics of physical quantities that evolve both in time and in space. In synthesis, such systems often have to be treated through high-order approximations.

The design and synthesis of controllers for systems of large scale or infinite dimension are often obstructed by the complexity in their models. However, in this thesis, an approach to the so called problem of H_∞ control is presented that circumvents the complexity for certain such systems while still providing controllers that achieve optimal performance.

1.1 Outline of the Thesis

The format of this thesis is a compilation of publications. It consists of four introductory chapters, including the current one, followed by five papers. In the following chapter, a review of selected works within the fields of control of large-scale systems and H_∞ control of infinite-dimensional systems is given. Thereafter, in Chapter 3, the main contributions are presented as well as related to the works reviewed in Chapter 2. Finally, the thesis is concluded in Chapter 4, which also includes directions for further research.

The remaining part of this chapter is divided into four sections. In the first two sections, the reader is offered further background to the problems of control of large-scale and infinite-dimensional systems. Then, the method of H_∞ control is briefly described. Finally, in the last section, the included publications are summarized and the contributions made by the author of this thesis are specified.

1.2 Large-Scale Systems in Our Society

The power grid is truly a system of large scale. Moreover, controllers are a fundamental part of the system. For instance, they regulate the power balance on the grid so that electricity supply can be guaranteed [Kundur et al., 1994]. However, the increase in use of renewable energy, such as energy from the wind turbines seen in Figure 1.1, raises new demands on the power grid. In comparison to traditional generators, often driven by coal or nuclear power, the renewable energy sources are much less predictable. Researchers investigate, for instance, how management of the demand side [Taneja et al., 2010; Blarke and Jenkins, 2013], electricity market regulation [Klessmann et al., 2008] and energy storage solutions [Blarke and Jenkins, 2013; Castillo and Gayme, 2014] can compensate for the volatile behaviour of renewables.

Another large-scale application is that of resource efficient temperature regulation in buildings. In [Statens Energimyndighet, 2017] it was reported that 53% of the total energy use during 2017 within the housing and service sector in Sweden, was due to heating. Minimizing the energy used for heating is of course important from an environmental aspect. In fact, resource efficiency is one of the main goals of the United Nations 2030 Agenda. It is addressed world-wide through the development of so called smart buildings and societies, see e.g. [Snoonian, 2003; Hazyuk et al., 2012]. If heated through electricity, an apartment building is a major user on the power grid. In relation to the previous paragraph, regulating the activation of heating or cooling devices can act as a buffer on the grid and be used in times of high demand. This has, for instance, been trialed in a pilot project with an

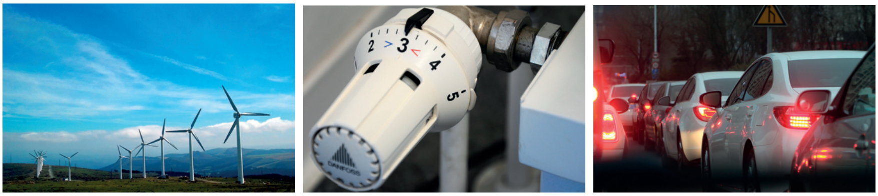

Figure 1.1 The energy harvested by wind turbines (left) is an example of a renewable energy source. Within buildings, the temperature is often regulated on a room-by-room basis through thermostats (middle). Traffic jams (right) are becoming an increasingly occurring problem as more and more people are choosing to live in urban areas. (Free images from Pixabay.com)

office building in Malmö, Sweden, with promising results [e.on, 2019].

Concerning transportation networks, congestion on our roads is a prominent issue and something many of us experience daily. Traffic control is a field of research just on its own, which is actually the case for many large-scale applications. Efforts are made within this field to make our travels on the roads both safe and swift, see e.g. [Papageorgiou et al., 2003; Urmson et al., 2008; Bojarski et al., 2016; Ferrara et al., 2018; Nilsson, 2019].

1.3 Implications of Scale and Dimension for Control

In the design of controllers, one has traditionally considered the scenario of a single process element to be controlled by a single control element. This setup is not the one found within large-scale systems. Instead, large-scale systems can be seen as an interconnection of numerous process and control elements. Moreover, they often lack centralized information and computing capability, which is assumed to be available in the traditional setup. In fact, in a large-scale system, computations often have to be made locally at the control elements, and information exchange between components is limited.

The differences described to the traditional setup impose complexity in the control design for large-scale systems. For instance, complexity is introduced in the form of constraints on the structure of the controller. Moreover, the dimensionality and uncertainty of the models, much due to that simplifications are needed for analysis, are other challenges. In fact, classical control methods often have to be re-invented for the purpose of control of large-scale systems, see [Bakule, 2008] for an overview of some of the approaches to control of large-scale systems.

In controller design for infinite-dimensional systems, the synthesis problem is generally approached by first approximating the partial differential equations by a system of ordinary differential equations. The design is then performed on this finite-dimensional approximation of the original system. However, to ensure high accuracy, approximations of high order often need to be used. This renders both complexity and high computational demand in synthesis. It is also difficult to ensure that the designed controller works, as predicted, on the original infinite-dimensional system. In contrast to approximative methods, it is often easier to determine the performance of controllers derived by approaches that work directly in the infinite-dimensional realm. However, such approaches are in general difficult use. For an introduction to the control of infinite-dimensional systems and the issues that can arise in design, see [Morris, 2010].

1.4 The Method of H_∞ Control

In this thesis, a classical controller synthesis method is analysed for the purpose of control of systems of large scale, as well as for the control of infinite-dimensional systems. The considered method is that of H_∞ control. In the H_∞ control framework, a system's performance is given by its behaviour when subject to worst-case disturbances, see [Zhou et al., 1996] and [Van Keulen, 1993] for a comprehensive presentation of the problem for finite and infinite-dimensional systems, respectively. Moreover, H_∞ control is a method within the theory of robust control. This theory describes methods for how to design controllers that guarantee some pre-specified behaviour of the system in spite of model uncertainties or the impact of disturbances.

There are many additional design specifications to that of H_∞ control that can be considered for the synthesis of controllers. Examples are how well the system can attenuate disturbances assumed to be Gaussian white noise and how well the controller can adapt to changes in the dynamics of the system. The many design specifications offered within the field of control theory all have areas of application for which they are particularly suitable. However, in this thesis, it is primarily the theoretical aspects of controller design that have been investigated and the focus has been devoted to the method of H_∞ control.

1.5 Included Publications and Statement of Contribution

This thesis was prepared by Carolina Bergeling at the Department of Automatic Control, Lund University, during the time period from June 2013 to April 2019 (excluding May 2018 to January 2019 due to parental leave) as a partial fulfilment of the requirements for obtaining the PhD degree. The results presented in this thesis were conducted by Carolina Bergeling under the supervision of Professor Anders Rantzer, Professor Bo Bernhardsson and Doctor Richard Pates.

In this section, the publications included in this thesis are summarized. Moreover, a statement is given to specify what is contributed by whom in each paper. Notice that Carolina Bergeling has changed her surname from Lidström to Bergeling during the course of her PhD-studies. Both surnames are used in the papers below.

Paper I

Lidström, C. and A. Rantzer (2016). "Optimal H-infinity state feedback for systems with symmetric and Hurwitz state matrix". In: *American Control Conference (ACC), 2016*. IEEE, pp. 3366–3371.

In Paper I, an H_∞ optimal state feedback law is stated explicitly on a simple form. It is applicable to finite-dimensional linear and time-invariant systems with symmetric and Hurwitz state matrix. Moreover, the control law as well as the optimal performance value are expressed in terms of the matrices of the system's state-space representation.

The control law is shown to scale well for certain large-scale applications. Examples include temperature dynamics in buildings and networks of buffers. The control law is also shown to comply with the structure of the system. Moreover, for a subclass of the systems, the property of internal positivity is preserved in closed-loop.

The paper also includes an extension of the control law that incorporates coordination among a heterogeneous group of linear and time-invariant systems, with the aforementioned properties necessary for applicability. The extended control law is composed of a decentralized and a centralized term, where the centralized term is identical for all subsystems. Hence, it can be implemented in a distributed manner.

Authors' contribution: C. Lidström contributed with a conjecture giving the structure of the optimal control law as well as an initial statement and proof of the main theorem. The initial proof was, as is the final version given in the paper, based on the so called KYP-lemma. C. Lidström derived the crucial step of the proof, giving the optimal choice of matrices that fulfil the inequality of this lemma. Furthermore, C. Lidström prepared the manuscript. A. Rantzer revised the results and reviewed the manuscript. Some of the applications to distributed control were formed in discussions between the two authors.

Paper II

Lidström, C., R. Pates, and A. Rantzer (2017). "H-infinity optimal distributed control in discrete time". In: *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*. IEEE, pp. 3525–3530.

Paper II includes the discrete-time analogue of the continuous-time state feedback result stated in Paper I. The translation is non-trivial and a comparison to the continuous time result is included. Furthermore, an explicit expression for an optimal proportional integral controller is given, which is based on the continuous-time result presented in [Rantzer et al., 2017]. Examples illustrate how the explicitly stated control laws can be used in a distributed manner for the control of large-scale systems.

Authors' contribution: C. Lidström derived the results and prepared the manuscript, however, the proof of the main theorem relies upon an idea suggested by A. Rantzer. Also, the local condition in Section 4.2 is a result of discussions between C. Bergeling and R. Pates. The results as well as the manuscript were reviewed by R. Pates and A. Rantzer.

Paper III

Bergeling, C., R. Pates, and A. Rantzer (2019). "H-infinity optimal control for systems with a bottleneck frequency". *Submitted to IEEE Transactions on Automatic Control*.

The first theorem of Paper III characterizes a class of systems for which the H_∞ optimal control problem can be translated into a static problem at a single frequency. Moreover, it is shown that for a subclass of the considered systems, an optimal controller can be given explicitly on a simple form. The systems considered in Paper I and II are examples in this class. However, the class of systems presented in Paper III goes beyond systems with symmetric and Hurwitz state matrix. Further, several examples of large-scale applications are included, such as control of electrical networks and water irrigation systems.

Authors' contribution: The first theorem of the paper was jointly derived by C. Bergeling and A. Rantzer. The remaining results were derived by C. Bergeling, however, based on discussions with R. Pates and A. Rantzer. The example on droop control was suggested by R. Pates. Moreover, the manuscript was prepared by C. Bergeling and reviewed by R. Pates and A. Rantzer.

Paper IV

Lidström, C., A. Rantzer, and K. A. Morris (2016). "H-infinity optimal control for infinite-dimensional systems with strictly negative generator". In: *2016 IEEE 55th Conference on Decision and Control (CDC),* IEEE, pp. 5275–5280.

In Paper IV, the infinite-dimensional analogue to the finite-dimensional result presented in Paper I is given. The infinite-dimensional systems considered are linear and time-invariant with self-adjoint and strictly negative state operator as well as bounded input and output operators. Diffusion equations are an important example in this class. Similar to Paper I, an H_∞ optimal control law can be stated on a very simple and explicit form for a certain case of state feedback.

Authors' contribution: A. Rantzer and K. A. Morris suggested the idea of an extension of the main result given in Paper I to infinite-dimensional systems. C. Lidström formalized the theorem and its proof as well as prepared the manuscript. A. Rantzer and K. A. Morris revised the proof and reviewed the manuscript.

Paper V

Bergeling, C., K. A. Morris, and A. Rantzer (2019). "Closed-form H-infinity optimal control for parabolic systems". *Submitted to Automatica*.

In Paper V, the problem of H_∞ optimal state estimation, or filtering, is studied for a certain class of infinite-dimensional systems. Similarly to Paper IV, an optimal observer can be stated explicitly. The filtering problem is highly related to the state feedback problem considered in Paper IV, which is also included in this paper, however, with a new proof. The results are illustrated through several examples. Furthermore, the computational time of numerically determining an approximation of the explicitly stated controller is compared to the computational time of a general purpose algorithm for H_∞ controller synthesis. Also, an application to optimal actuator and sensor placement is described.

Authors' contribution: C. Bergeling formalized the theorems and their proofs, prepared the manuscript and performed the numerical comparison. K. A. Morris provided some initial code upon which the numerical comparison is based. A. Rantzer and K. A. Morris revised the proofs and reviewed the manuscript.

Additional publications

In addition to the publications included in this thesis, the author has been part of the following works during her PhD studies:

Bergeling, C., R. Pates, and A. Rantzer (2019). "On closed-form H-infinity output feedback control". *Submitted to the 2019 IEEE Conference on Decision and Control*.

Pates, R., C. Lidström, and A. Rantzer (2017). "Control using local distance measurements cannot prevent incoherence in platoons". In: *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*. IEEE, pp. 3461–3466.

Rantzer, A., C. Lidström, and R. Pates (2017). "Structure preserving H-infinity optimal PI control". *IFAC-PapersOnLine* **50**:1, pp. 2573–2576.

Ryu, E. K., A. B. Taylor, C. Bergeling, and P. Giselsson (2018). "Operator splitting performance estimation: tight contraction factors and optimal parameter selection". *arXiv preprint arXiv:1812.00146*.

2

Challenges in Control

In the 1960s and 1970s, research on the topic of large-scale systems and control was initiated at several institutions around the world [Bakule, 2008]. Today, the interest for this field of research has been renewed by the advances made in wireless communication. The field of control of infinite-dimensional systems emerged around the same time as that of control of large-scale systems [Padhi and Ali, 2009]. By now, it is a well-established area within systems and controls research.

In this chapter, existing literature on control of large-scale and infinite-dimensional systems are reviewed. It is divided into two sections, of which the first one considers large-scale systems while the second one is devoted to infinite-dimensional systems. The main focus of the review is to provide further details on the research challenges within these fields. However, it will also specifically focus on existing literature that is highly related to the contributions presented in this thesis.

2.1 Control of Large-Scale Systems

In the preface of [Siljak, 2011], the author writes "Complexity is a central problem in modern system theory and practice. Because of our intensive and limitless desire to build and to control ever larger and more sophisticated systems, the orthodox concept of a high performance system driven by a central computer has become obsolete. [...] It is becoming apparent that a "well-organized complexity" is the way of the future." This quote captures the need to move past the setup traditionally considered in control, as was also explained in Section 1.3. However, although moving away from the concept of centrality is to prefer for the purpose of control of large-scale systems, it is not straightforward how to compute or even how to construct controllers suitable for these systems.

The complexity in design of controllers for large-scale systems stems from the dimensionality of the problem, the requirements on the structure of the controller, by the so called information structure of the system, as well as the unavoidable uncertainty in the models used. The research devoted to address the challenges the complexity imposes can be divided into design of decentralized control laws, distributed or efficient computation in synthesis and control systems architecture. In the following three sections, selected literature within these areas are described. However, a single work within the field of control of large-scale systems often treats elements of all three problem areas. Hence, there is no harsh divide between the literature reviewed in the different sections.

From Centralized to Decentralized

Control of large-scale systems is often referred to as decentralized control. This is due to the nonclassical information structure apparent in these systems which demands its controllers to be based on local, or more generally non-central, information rather than the full set of information available. The latter is often referred to as global information.

The following quote from the introduction of [Bakule, 2008] summarizes the concept of a decentralized controller: "A system is considered large-scale if it is necessary to partition the given analysis or synthesis into manageable sub-problems. As a result, the overall plant is no longer controlled by a single controller but by several independent controllers which all together represent a decentralized controller. This is the fundamental difference between feedback control of small and large systems usually described by the idea of information structure." Distributed control is similar to decentralized control but it often involves a central entity that supervise the full set of control actions. However, in this thesis, the two notions will be used interchangeably and refer to the control of systems with nonclassical information structure.

Information structure To illustrate the difference between a classical and a nonclassical information structure, consider the feedback interconnections, or closed-loop systems, depicted in Figure 2.1. The diagram to the right depicts a system with classical information structure in which the global information of the system is available to the controller. Furthermore, the controller is in charge of the full set of control input signals.

In contrast to the classical setup, the diagram to the left in Figure 2.1 depicts a system with nonclassical information structure. Notice that the information available to a controller K_i in Figure 2.1 (left) is not the globally available information of the overall system. Furthermore, a specific controller can only impact some of the subsystems G_j. However, even if the block-diagram in Figure 2.1 (left) looks vastly different from the traditional setup shown in Figure 2.1 (right), it can be condensed into the form shown

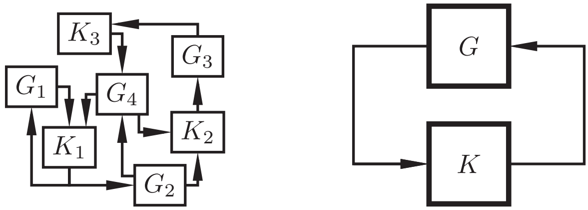

Figure 2.1 (Left) Feedback interconnection of subsystems G_j and controllers K_i. (Right) Traditional feedback interconnection of system G and controller K.

in Figure 2.1 (right). It is important to stress that such a representation implicitly demands a certain structure of the controller K.

Traditionally, the information structure of the system has been prespecified and the design problem been solely to determine the decentralized control laws. Moreover, the design is to be based on some notion of the desired closed-loop behaviour, as is the case in any synthesis procedure. The desired behaviour could be, simply, that of closed-loop stability or to additionally achieve performance requirements such as the ability to attenuate certain disturbances. The conventional methods of control offer a wide range of performance measures and many of them, if not all, have been studied in the setting of decentralized control.

Global design Often, the design of decentralized controllers is based on the full model of the system. Constraints on the structure of the controller, as imposed by the information structure of the system, are then added to a general synthesis procedure. However, enforcing the controller to have a certain structure could greatly complicate the analysis or even make it intractable [Papadimitriou and Tsitsiklis, 1986; Lessard and Lall, 2011; Wang and Chen, 2002].

Procedures for a range of information structures have been reported since the problem of decentralized control design was first addressed, see e.g. [Sandell et al., 1978; Vidyasagar, 1981; Bakule, 2008; Siljak, 2011]. For instance, the system property called diagonal dominance, see [Grosdidier and Morari, 1986], has been intensely studied. It is also common to perform procedures in order to simplify the model of a large-scale system and through this lower the complexity in synthesis. For an introduction to model order reduction of large-scale systems' models see for example the first couple of chapters in [Mohammadpour and Grigoriadis, 2010].

In [Rotkowitz and Lall, 2005], a number of important cases have been derived for which the decentralized control problem is in fact equivalent to a convex optimization problem. Similarly, [Jovanović and Dhingra, 2016] summarizes several distributed controller synthesis problems that are also convex. It covers for example problems of symmetric systems, consensus type, optimal selection of sensors and actuators and decentralized control of positive systems. Furthermore, on the topic of positive systems, [Tanaka and Langbort, 2011; Briat, 2013; Rantzer, 2015] present methods for the design of distributed controllers such systems, in the framework of several different types of performance measures.

Most of the previously reviewed works consider stabilization, disturbance attenuation or robustness to model uncertainty as performance specifications. In addition, design of distributed predictive control for large-scale systems is considered in [Katebi and Johnson, 1997] and the framework of model predictive control is further utilized in [Venkat et al., 2007]. Moreover, control methods of adaptive nature have also been considered in the large-scale setting, see e.g. [Jain and Khorrami, 1997].

Local design In comparison to global design, a local design prcedure is based on partial knowledge of the dynamics or structure of the system. For instance, the information structure can be assumed to belong to a class of information structures rather than being defined specifically. In [Lestas and Vinnicombe, 2006], local stability conditions are given that are independent of the interconnection topology of the system's network, or in other words its information structure, as well as the size of the system. Similarly, in [Pates and Vinnicombe, 2017], the authors present a local certificate that can be used to guarantee stability of the overall system. Methods similar to those in [Lestas and Vinnicombe, 2006] and [Pates and Vinnicombe, 2017] are passivity-based approaches for control, see e.g [Ortega et al., 2008], as well methods based on so called integral quadratic constraints, see e.g. [Kao et al., 2009] and [Khong and Rantzer, 2014].

The local design methods are often more efficient, computationally, than global design methods. However, the simplicity in synthesis of local design approaches comes at a price, as the limited information available in the control design most probably imposes conservatism. In other words, it could be the case that the closed-loop performance given a locally designed control law is far from globally optimal.

Besides complying with the structural requirements imposed by the information structure of the system, it is of interest to design control laws that are less rigid to changes in the dynamics or structure of the system. For instance, if an additional process or control element is added to the system in Figure 2.1 (left), it would be preferable if only a subset of the controllers were in need of updating their policies. This scenario is common among large-scale systems, as they could be expanded to provide their functionality to an increasing number of users. The described requirement is to keep updates from becoming far too computationally complex and time-consuming as well as to achieve robustness towards model uncertainty.

The local design methods can often handle changes in the structure or dynamics of the system. This is because they can be accounted for by the design criteria, which often only specifies the dynamics or structure to be of a certain type. On the contrary, in a global design setup, the synthesis often has to be redone whenever the model is updated.

Optimality versus simplicity in design In comparison to centralized control, the property of decentralization often results in losses in performance. This was investigated in [Delvenne and Langbort, 2006] where the performance of a decentralized controller was shown to be only half as good as a centralized controller. It is inevitably so that one has to consider the interplay between the achieved performance and simplicity of the design procedure as a part of the design process. However, it is not always the case that optimal performance is unachievable with a decentralized control law. In fact, examples of this are given in this thesis. Moreover, several works on fundamental limitations in distributed control of large-scale systems have recently been published, and showcase the inherent limitations of decentralized control in certain applications, see, e.g. [Tegling et al., 2017; Tegling, 2018; Pates et al., 2017; Bamieh et al., 2012].

Efficient computation

It is one thing to be able to synthesize decentralized controllers and another to be able to do it in an efficient way. The computational complexity of the actual implemented controller is an additional concern, however, often related to the efficiency of the synthesis procedure. In the previous section, the synthesis problem of decentralized control was studied and solvability was related to convexity of the problem. However, although a problem is convex, it is not necessarily computationally fast to solve, although this is the case for many of the methods reviewed in the previous section.

In [Wang et al., 2018], the issue with computational scalability of traditional distributed optimal control methods is addressed. The work is based on [Wang et al., 2019; Anderson and Matni, 2017] and shows that given certain separability of the control objective functions and system constraints, the global optimization problem can be decomposed into parallel subproblems. Given further sparsity constraints, the subproblems can be solved efficiently. Moreover, the method in [Wang et al., 2018] clearly incorporates both scalable synthesis as well as efficient computation when the controller is in use.

In [Lestas and Vinnicombe, 2006], previously mentioned, the stability certificates are shown to scale well with the network size. Similar to this, [Jönsson and Kao, 2010] presents a scalable stability criterion for interconnected systems with heterogeneous linear time-invariant components. The criterion is based only on the individual components and the spectrum of the interconnection matrix, which is what maintains scalability of the analysis. Moreover, [D'Andrea and Dullerud, 2003] presents a state-space based synthesis approach for systems with certain types of information structures. It generalizes standard results in control to tractable computational tools suitable for interconnected systems.

The so called alternating direction method of multipliers, see [Boyd et al., 2011], is used for computing sparse controllers in for example [Fardad et al., 2011; Dörfler et al., 2014] and [Lin et al., 2013]. More specifically, it is the mathematical technique of regularization that guarantees sparsity in the controller. Moreover, the computational approach alternates between promoting the sparsity of the controller and optimizing the closed-loop performance. In [Dhingra and Jovanović, 2016], another algorithm is used for computing sparse controllers that has both a theoretical guarantee of convergence and fast computation speed in practice. Further, symmetries in the model of the system are taken advantage of in [Wu and Jovanović, 2017] and other methods for efficient computation are found in [Waki et al., 2006; Andersen et al., 2014; Benner, 2004].

Control Systems Architecture

In the design of large-scale systems, it is just as important to design the architecture of the control system as it is to design the control laws. In other words, the problem of control systems architecture is to design the placement of controllers as well as the communication network they depend on. In [Matni and Chandrasekaran, 2016], this design problem is interpreted as the solution of a particular linear inverse problem. Furthermore, the design problem can be formulated as a convex optimization problem that can be solved efficiently. In [Rantzer, 2018], the author applies the same idea to prove that network realizability of controllers can be enforced using convex constraints on the closed-loop.

In some cases, the synthesis procedure renders a suitable architecture without that being the primary function of the design method. For instance, in [Bamieh et al., 2002; Curtain, 2011], control problems are investigated for so called spatially invariant systems. Given their solution of the design problem, the resulting controller has a degree of spatial localization similar to the plant, due to which it possibly could be implemented in a distributed manner. Similarly, the approach in [D'Andrea and Dullerud, 2003] renders controllers that adopt and preserve the distributed spatial structure of the system. Further, passivity is used as the primary design tool in [Arcak, 2007]. The controllers designed can be implemented with local information and ensure stability of the overall closed-system. In certain cases, the closed-loop system exhibits an interconnection structure that inherits the passivity properties of its components.

2.2 Control of Infinite-Dimensional Systems

Many of the results in control, firstly derived for finite-dimensional systems, have been translated into the infinite-dimensional realm, see e.g. [Curtain and Zwart, 2012]. Hence, there is an extensive literature on the of control of infinite-dimensional systems. The following review will therefore focus on H_∞ control, which is the method considered in this thesis. Moreover, note that some of the results covered in the previous section on control of large-scale systems, actually considers infinite-dimensional systems, i.e. the works on spatially invariant and spatially distributed systems.

As was already mentioned in the previous chapter, controller synthesis for infinite-dimensional systems is often approached by first approximating the partial differential equations by a system of ordinary differential equations. However, a major drawback of this approach is that the controller designed for the finite-dimensional approximation may not stabilize the original system. Hence, it needs to be verified that the controller synthesized for the finite-dimensional approximation performs well on the original infinite-dimensional systems. Sufficient conditions for certain problems have been derived, see e.g. [Özbay et al., 2018; Ito and Morris, 1998; Morris, 2001], which covers H_∞ control and the class of systems considered in this thesis.

In order to achieve accuracy, the finite-dimensional approximations often need to be of high order, which could complicate computations. In finite dimensions, H_∞ synthesis is generally performed though iteratively solving a series of so called algebraic Riccati equations, see [Doyle et al., 1989]. The method in [Arnold and Laub, 1984] is one example of an algorithm for numerically solving such equations. However, it works poorly when the order of the system is large, i.e. when the system of equations is of large dimension. Other methods for solving algebraic Riccati equations are the matrix sign function method [Byers, 1987] and the method based on game-theory presented in [Lanzon et al., 2008]. However, even though synthesis techniques with algebraic Riccati equations have existed for decades, there is no generally accepted algorithm for systems of large order, such as high order approximations of infinite-dimensional systems [Kasinathan et al., 2014]. Moreover, it is not uncommon for numerical issues to arise, particularly when attempting to compute a controller near optimal attenuation [Lanzon et al., 2008]. More specifically, it is the so called sign-indefiniteness of the quadratic term in the H_∞ type algebraic Riccati equation, and the need for an iterative procedure to find the optimal attenuation, that complicate the computations. Several approaches to the problem of H_∞ control for infinite-dimensional systems, and the computational difficulties that arise, are described in [Özbay et al., 2018].

Methods to control that are not based on approximations of the infinite-dimensional system are often called direct methods. Again, the advantage of a direct approach is that since the controller is designed for the original model, actual performance is easier to determine. There are both state-space based and frequency domain based solutions to the H_∞ control problem in infinite dimensions, see e.g. [Bensoussan and Bernhard, 1993; Van Keulen, 1993; Foias et al., 1996; Özbay et al., 2018]. The frequency domain approach often requires one to determine the transfer function of the system, which in general can be hard. For a tutorial on transfer functions of infinite-dimensional systems, see [Curtain and Morris, 2009]. In the state-space based approach to the H_∞ control problem, the synthesis involves solving an infinite-dimensional operator-valued Riccati equation or inequality, see, e.g. [Bensoussan and Bernhard, 1993] and [Van Keulen, 1993]. On the latter approach, the author in [Van Keulen, 1993, p. 184] writes "In general, it is impossible to find explicit solutions to (infinite-dimensional) Riccati equations. Therefore, one usually considers (numerical) approximations".

3

Contributions

In this chapter, the main contributions of the included publications are highlighted as well as related to the existing literature presented in the previous chapter. The chapter is divided into three sections where the first section covers the theoretical contributions. The remaining sections describe the application of the results to control of large-scale systems and infinite-dimensional systems.

Notation and preliminaries The following mathematical notation and basic concepts of control theory can be found in standard text books on the subject, see, e.g. [Zhou et al., 1996]. Moreover, [Zhou et al., 1996] offers a comprehensive representation of the H_∞ control problem treated in the following section.

The real and complex numbers are denoted $\mathbb{R}$ and $\mathbb{C}$, respectively. Moreover, $\mathbb{R}^{n\times m}$ and $\mathbb{C}^{n\times m}$ are the spaces of n-by-m real-valued and complex-valued matrices. For vectors, only the length is specified, e.g. $\mathbb{R}^n$ is the space of real-valued vectors of length n. The identity matrix is written as I.

If a scalar, vector or matrix x belongs to a set X, we write $x \in X$. The transpose of a matrix $M \in \mathbb{R}^{m\times n}$ is written M^T while the conjugate transpose of a matrix $M \in \mathbb{C}^{m\times n}$ is written M^*. $M \in \mathbb{R}^{n\times n}$ is said to be Hurwitz if all its eigenvalues have negative real part. Further, for $M \in \mathbb{C}^{n\times n}$, positive and negative definiteness are denoted $M \succ 0$ and $M \prec 0$, respectively.

The l_2-norm of a vector $v \in \mathbb{C}^n$ is denoted $|v|$. The l_2-induced matrix norm is denoted $\|M\|$, for $M \in \mathbb{C}^{n\times m}$. It holds that

$$\|M\| = \sup_{|v|=1} |Mv|.$$

The space of square-integrable functions over $[0, \infty)$ is denoted $L_2[0, \infty)$ and its norm, denoted $\|\cdot\|_2$, is given by, for $f \in L_2[0, \infty)$,

$$\|f\|_2 = \left(\int_0^\infty |f(t)|^2 \, dt\right)^{\frac{1}{2}}.$$

Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{k\times n}$ and $D \in \mathbb{R}^{k\times n}$ define a linear time-invariant continuous-time system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t) + Du(t),\end{aligned} \quad x(0) = x_0, \quad t \geq 0, \tag{3.1}$$

where x is the state, u is the input, y is the output and n the order of the system. Moreover, x_0 is called the initial state. The transfer function, or transfer matrix, of the system is given by

$$G(s) = C(sI - A)^{-1}B + D,$$

where s is the Laplace variable. The dual to the system G is denoted G^T and defined as

$$G^T(s) = B^T(sI - A^T)^{-1}C^T + D^T.$$

The so called poles of the system (3.1) are the eigenvalues of the matrix A in (3.1). The system is said to be input-output stable, or simply stable, if its poles have strictly negative real-part. If A in (3.1) is Hurwitz, the system is stable. If (3.1) is a stable system with transfer function G, then the so called H_∞ norm of G is defined as

$$\|G\|_\infty := \sup_{\omega\in\mathbb{R}} \|G(j\omega)\|.$$

Assuming that (3.1) has zero initial state, i.e. $x_0 = 0$, the H_∞ norm can also be expressed as

$$\|G\|_\infty = \sup_{\|u\|_2=1} \|y\|_2.$$

Consider (3.1) with $C = I$ and $D = 0$. The static state feedback law $u = Kx$, where $K \in \mathbb{R}^{m\times n}$, or simply the controller K is said to stabilize the system, or to be a stabilizing controller, if $A + BK$ is Hurwitz.

3.1 An H_∞ Optimal Controller on a Simple Explicit Form

In this section, an explicit solution to a particular H_∞ optimal control problem will be presented. Also, a more general result on the class of systems studied in this thesis is given.

The Problem Considered

Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t) + Hw(t), \qquad x(0) = 0, \quad t \geq 0, \tag{3.2a}$$

$$z(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \tag{3.2b}$$

$$y(t) = x(t), \tag{3.2c}$$

where $x(t) \in \mathbb{R}^n$ is again the state of the system, $w(t) \in \mathbb{R}^l$ is an unknown disturbance and the system can be controlled through the signal $u(t) \in \mathbb{R}^m$. The signal z is the so called regulated output of the system. Moreover, y is the measurement of the system. A, B and H are real-valued matrices of appropriate dimensions, and such that there exist a $K \in \mathbb{R}^{m\times n}$ for which $A + BK$ is Hurwitz.

The control input $u(t)$ is to be constructed as $u(t) = Ky(t)$, where K is a real-valued matrix of appropriate dimension. In fact, as $y(t) = x(t)$ in (3.2), i.e. the entire state vector can be measured, the control law can be written as $u(t) = Kx(t)$. In other words, it is a static state feedback law.

More specifically, K should be chosen such that the controller stabilizes the system (3.2) and that the following objective function is minimized

$$\sup_{\|w\|_2=1} \|z\|_2.$$

The objective function defines the performance of the closed-loop system, as measured in H_∞ control. It is implicitly assumed that the disturbance w belongs to the space $L_2[0,\infty)$, in other words, the disturbance signal is assumed to have finite energy. The described problem can be written compactly as

$$\gamma_o := \inf_{K\in\mathbb{R}^{m\times n} \text{ stab.}} \sup_{\|w\|_2=1} \|z\|_2, \tag{3.3}$$

where "stab." is short for stabilizing. Note that

$$\|z\|_2 = \sqrt{\|x\|^2 + \|u\|^2}.$$

In words, the objective (3.3) is to find a stabilizing static state feedback controller K such that ratio of the energy of the state and control input signals to the energy of any disturbance $w \in L_2[0,\infty)$ is minimized. Hence, the controller should be designed so that the impact of a disturbance on the closed-loop system's dynamics is optimally attenuated. The value γ_o is called the optimal performance value.

The problem (3.3) can be written in the frequency domain as through the following procedure. Denote the transfer matrix of (3.2) by G, i.e given

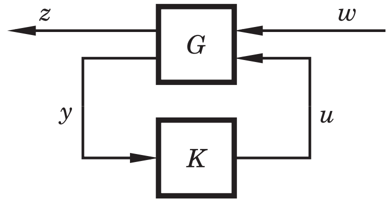

Figure 3.1 Feedback interconnection of system G and controller K. The signals z, y, u and w are the regulated output, measurement, control input and disturbance, respectively.

inputs w and u and outputs z and y. It follows that G can be divided into four blocks as

$$G(s) = \begin{bmatrix} G_{zw}(s) & G_{zu}(s) \\ G_{yw}(s) & G_{yu}(s) \end{bmatrix} = \left[\begin{array}{c|c} (sI-A)^{-1}H & (sI-A)^{-1}B \\ 0 & I \\ \hline (sI-A)^{-1}H & (sI-A)^{-1}B \end{array}\right]. \quad (3.4)$$

Moreover, as previously described, the controller K maps the measurement signal y to the control input u, see Figure 3.1 for a depiction of the closed-loop system.

The closed-loop system's transfer matrix, i.e. the transfer matrix of the system from w to z in Figure 3.1, can be written in terms of the so called lower linear fractional transformation, denoted F_l, as

$$F_l(G, K) := G_{zw} + G_{zu}K(I - G_{yu}K)^{-1}G_{yw}.$$

In this description it is assumed that the controller K is such that the inverse of $I - G_{yu}K$ exists. The problem (3.3) can now be given in the frequency domain as

$$\inf_{K \in \mathbb{R}^{m\times n},\ K \text{ stab.}} \|F_l(G, K)\|_\infty, \quad (3.5)$$

from which it becomes clear that it is an H_∞ control problem. It is well known that optimality can be achieved by a static controller in the case of H_∞ state feedback, see [Khargonekar et al., 1988]. Hence, it is nonrestrictive to specify the set of controllers K as $\mathbb{R}^{m\times n}$.

Further comments on the objective The objective in (3.5), or more specifically the choice of the signal z in (3.2), will now be discussed. For simplicity, in (3.2), assume that $H = B$ and that there are as many control inputs as there are states in (3.2), i.e., B is a square matrix. Furthermore, denote $P(s) = (sI - A)^{-1}B$. Then, (3.4) is given by

$$G = \left[\begin{array}{c|c} P & P \\ 0 & I \\ \hline P & P \end{array}\right].$$

Moreover,

$$F_l(G, K) = \begin{bmatrix} P(I + KP)^{-1} \\ KP(I + KP)^{-1} \end{bmatrix}.$$

The H_∞ optimal control problem (3.5) can thus be written as

$$\inf_{K \in \mathbb{R}^{m \times n} \text{ stab.}} \left\| \begin{bmatrix} P(I + KP)^{-1} \\ KP(I + KP)^{-1} \end{bmatrix} \right\|_\infty . \tag{3.6}$$

In general, closed-loop performance is concerned with the behaviour of the systems corresponding to the four transfer functions $(I + PK)^{-1}$, $(I + PK)^{-1}P$, $K(I + PK)^{-1}$ and $(I + PK)^{-1}PK$, see [Zhou et al., 1996] for more details on this statement. The considered performance objective (3.6) implies properties on two of these transfer functions. However, in the applications considered in this thesis, P often has the characteristics of a low-pass filter. Thus, small

$$\|P(I + KP)^{-1}\|_\infty$$

implies that $\|(I+P(j\omega)K)^{-1}\|$ is small at low frequencies. This is generally the performance requirement aimed for. Also, in this example, as P and K are square with the same dimensions, we have that

$$\|K(I + P(j\omega)K)^{-1}\| \le \|K\| \|(I + P(j\omega)K)^{-1}\|.$$

Thus, $\|(I + P(j\omega)K)^{-1}\|$ small for low frequencies implies that $\|K(I + P(j\omega)K)^{-1}\|$ is small for low frequencies, as long as $\|K\|$ is kept small.

The explicit solution

In this section, an explicit solution to (3.5), or equivalently (3.3), given a certain class of systems (3.2), is presented. It follows from the explicit solution that the synthesized static state feedback law also can be stated explicitly, and on a simple form. The first part of this section considers systems of the form (3.2) with A symmetric and Hurwitz while the second part considers more general systems (3.2).

The KYP approach A problem closely related to (3.5) is the following: given $\gamma > 0$, find a stabilizing $K \in \mathbb{R}^{m\times n}$, if any exists, such that

$$\|F_l(G, K)\|_\infty < \gamma. \tag{3.7}$$

Note that γ can not be chosen as the value of (3.5), i.e. γ_o. In fact, it must hold that $\gamma > \gamma_o$ for there to possibly exist a solution. Hence, a controller K that solves (3.7) is suboptimal.

The problem (3.7) can equivalently be written in terms of a linear matrix inequality constraint by the KYP-lemma, see [Rantzer, 1996] for the version used in this thesis. For simplicity, $H = I$ in this section. The equivalent statements are

i) There exists a stabilizing $K \in \mathbb{R}^{m\times n}$ such that

$$\|F_l(G, K)\|_\infty < \gamma.$$

ii) There exist matrices $X \in \mathbb{R}^{n\times n}$, $X = X^T \succ 0$, and $Y \in \mathbb{R}^{m\times n}$ such that

$$\begin{bmatrix} XA^T + AX + Y^TB^T + BY & I & X & Y^T \\ I & -\gamma^2 I & 0 & 0 \\ X & 0 & -I & 0 \\ Y & 0 & 0 & -I \end{bmatrix} \prec 0.$$

The two statements are related by $K = YX^{-1}$. Moreover, the matrix inequality in ii) can be rewritten as

$$\underbrace{(X + A)(X + A)^2 + (Y^T + B)(Y^T + B)^T}_{=:\, F(X,Y)} - AA^T - BB^T + \gamma^{-2} I \prec 0, \tag{3.8}$$

through the use of the Schur complement lemma and completion of squares. The equivalent statements i) and ii) will now be used to find an explicit solution to (3.5) when A in (3.2) is symmetric and Hurwitz.

In (3.8), it is clear that if the term $F(X, Y)$ is made as small as possible, it allows for the performance value γ to be chosen as small as possible. In fact, if the matrix A is symmetric and Hurwitz, then the term $F(X, Y)$ can be made equal to zero by a certain choice of matrices X and Y.

From symmetry and Hurwitz stability of A it follows that A is negative definite, i.e. $A \prec 0$. Hence, it is possible to pick $X = -A$. Moreover, the matrix Y can be chosen as $-B^T$. This particular choice of matrices X and Y makes the term $F(X, Y)$ equal to zero, i.e. $F(-A, -B^T) = 0$. Now, given $X = -A$ and $Y = -B^T$, it follows that

$$K = B^T A^{-1}.$$

Moreover, the suboptimal performance level γ is bounded through (3.8) as

$$-A^2 - BB^T + \gamma^{-2} I \prec 0,$$

which is equivalent to $\gamma > \|(A^2 + BB^T)^{-1}\|^{\frac{1}{2}}$. However, it can be shown that the controller $K = B^T A^{-1}$ actually achieves the performance level

$$\|(A^2 + BB^T)^{-1}\|^{\frac{1}{2}}$$

and that this is in fact the optimal performance level γ_o. This result is stated as Theorem 1 in Paper I.

The static problem approach The optimal controller $K = B^T A^{-1}$ can also be obtained through a procedure very different to that previously described. Consider the following optimization problem

$$\text{minimize} \quad |\bar{x}|^2 + |\bar{u}|^2 \tag{3.9a}$$

$$\text{subject to} \quad 0 = A\bar{x} + B\bar{u} + H\bar{w}, \tag{3.9b}$$

where $\bar{x}$, $\bar{u}$ and $\bar{w}$ are vectors of appropriate dimensions. Moreover, $\bar{w}$ is given and $\bar{x}$ and $\bar{u}$ are to be chosen so as to minimize the objective function, given the constraint. This is a standard least-squares type problem with solution

$$\begin{bmatrix} \bar{x}_* \\ \bar{u}_* \end{bmatrix} = \begin{bmatrix} -A^T \\ -B^T \end{bmatrix} (AA^T + BB^T)^{-1} H\bar{w}, \tag{3.10}$$

where the matrix $AA^T + BB^T$ is invertible by assumption. The solution suggests that $\bar{u}_* = B^T A^{-T} \bar{x}_*$. Notice that this is exactly the optimal controller previously derived. However, in the case with A symmetric, it can be written as $\bar{u}_* = B^T A^{-1} \bar{x}_*$.

It can be shown that the problem (3.5) is lower-bounded by the supremum of the squareroot of (3.9) over $|\bar{w}| = 1$, i.e.

$$\gamma_o \geq \|H^T (AA^T + BB^T)^{-1} H\|^{\frac{1}{2}},$$

see the proof of Theorem 2 in Paper III for details. For systems (3.2) with A symmetric and Hurwitz, the controller $K = B^T A^{-1}$ is always stabilizing and achieves this lower bound. Hence, it is a solution to (3.5). The discrete time counterpart to this result is stated in Paper II.

Notice that (3.10) suggests that the controller $K = B^T A^{-T}$ could be a good guess of an optimal controller for systems (3.2) where A is not symmetric as well, however, invertible. Systems for which this is in fact the case are presented in Paper III. Moreover, for any system (3.2), it is sufficient to check if $K = B^T A^{-T}$ is stabilizing and if it achieves the performance value

$$\|H^T (AA^T + BB^T)^{-1} H\|^{\frac{1}{2}},$$

for $K = B^T A^{-T}$ to be optimal.

Systems with a bottleneck frequency

For certain systems (3.2), such as those with A symmetric and Hurwitz, the following claim can be made: there exist an $\omega_0 \in \mathbb{R}$ and a stabilizing controller $K_0 \in \mathbb{R}^{m\times n}$, of appropriate dimension, such that

$$K_0 \quad \text{belongs to the set} \quad \arg\min_{C\in\mathbb{C}^{m\times n}} \|F_l(G(j\omega_0), C)\| \tag{3.11}$$

and

$$\omega_0 \quad \text{belongs to the set} \quad \arg\max_{\omega\in\mathbb{R}} \|F_l(G(j\omega), K_0)\|. \tag{3.12}$$

It follows that K_0 minimizes $\|F_l(G, K)\|_\infty$ over $K \in \mathbb{R}^{m\times n}$, which is proven in Paper III. In fact, these properties characterize a class of systems G, not necessarily on the form (3.2), for which the H_∞ optimal control problem is translated into a static problem at a single frequency. More specifically, the frequency ω_0 and the static problem given by the optimization problem in (3.11), i.e.

$$\min_{C\in\mathbb{C}^{m\times n}} \|F_l(G(j\omega_0), C)\|. \tag{3.13}$$

The frequency ω_0 can be interpreted as a bottleneck frequency at which disturbance attenuation is the most crucial.

For G as defined in (3.4), the problem (3.13) has the solution $C_* = -B^T(-j\omega_0 - A^T)^{-1}$. This is given by Theorem 2 in Paper III. Notice that C_* is highly related to G_{yu}^T. Moreover, for this to be a solution to (3.5) it suffices to prove that there exists an $\omega_0 \in \mathbb{R}$ for which C_* is real-valued and stabilizing and such that (3.12) is fulfilled when $K_0 = -B^T(-j\omega_0 - A^T)^{-1}$. Clearly, for (3.2) with Hurwitz and Symmetric A, this holds for $\omega_0 = 0$.

Further comments Unstable open-loop systems (3.2) are not included in the examples given in this thesis. However, they are covered by the included results. Moreover, a slight variation in the problem setup is needed for systems (3.2) with imaginary axis poles, see, e.g. the techniques described in [Stoorvogel, 1992, Sec. 4.7]. However, they are not studied in any more detail in this thesis. It is also possible to consider (3.2) where the signals z and y are more general, see Paper I and Paper III for this.

3.2 Applications in Decentralized and Distributed Control

The H_∞ optimal state feedback controller derived in the previous section, i.e., $K = B^T A^{-T}$, has several features that makes it suitable for the control of large-scale systems. Firstly, since it is given explicitly, no computations are needed for it to be synthesized, only when implemented. Moreover, if the matrices A and B are sparse, it is often the case that the controller $K = B^T A^{-T}$ is sparse as well. Consider for example (3.2) with A diagonal and Hurwitz and B sparse. Then $K = B^T A^{-1}$ is an example of a decentralized control law that is also globally optimal.

The decentralized properties of the controller $K = B^T A^{-T}$ are illustrated by the following example, extracted from Paper III.

Example 1
Consider (3.2a) to be given by the N subsystems $i = 1, \ldots, N$ with dynamics

$$\dot{x}_i(t) = -a_i x_i(t) + \sum_{(i,j)\in\mathcal{E}} (u_{ij}(t) - u_{ji}(t)) + w_i(t), \quad x_i(0) = 0, \quad t \geq 0.$$

Here, x_i is the state of subsystem i, w_i is a disturbance and the control inputs u_{ij} are to be designed. Furthermore, (i, j) is in the set $\mathcal{E}$ if and only if subsystems i and j are connected and $a_i > 0$ for all $i \in \{1, \ldots, N\}$. The overall system can be written on the form (3.2a) in which A is diagonal and Hurwitz. Hence, it follows that $K = B^T A^{-1}$ solves (3.5) for the given system. Moreover, $K = B^T A^{-T}$ is equivalent to

$$u_{ij}(t) = -x_i(t)/a_i + x_j(t)/a_j.$$

Notice that a control signal u_{ij} is decentralized as it is only composed of the states that it directly affects. Furthermore, the control law scales well with the order of the system as each control input can be computed locally and with simple computations. It is also the case that an extra subsystem can be added without the need to change any of the already existing control laws, see Paper I for more details on this claim. □

The sparsity patterns of the matrices A and B considered in Example 1 are not the only types of sparsity patterns for which the controller $K = B^T A^{-T}$ is decentralized. In fact, in Papers I, II and III, several systems are presented for which the explicitly stated controller can be implemented in a distributed manner. The following model for the temperature dynamics in a building is considered in both Paper I and Paper III.

Example 2
Consider a building with N rooms. The average temperature in room i is denoted T_i. The temperature dynamics is governed by Fourier's law of thermal conduction and given, around an operating point, as follows

$$m_i c \dot{T}_i = p_i(T_{out} - T_i) + \sum_{j\in\mathcal{E}_i} p_{ij}\,(T_j - T_i) + u_i + d_i, \tag{3.14}$$

where m_i is the air mass of room i and c is the specific heat capacity of air. Furthermore, $\mathcal{E}_i$ is the set of rooms that share a wall/floor/ceiling with room i. The heat conduction coefficients p_i and $p_{ij} = p_{ji}$ are constant, real-valued and strictly positive. Moreover, T_{out} is the outdoor temperature and inputs d_i and u_i are disturbance and control inputs, respectively.

Disturbances occur for example when a window is opened or when there is a change in outdoor temperature. Furthermore, it is assumed that the average temperature of each room can be measured as well as controlled, the latter through heating and cooling devices modelled by the control inputs u_i.

The overall system can be written as

$$E\dot{T} = PT + u + w$$

where

$$T = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_N \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix},$$

E is a diagonal matrix with positive elements $E_{ii} = m_i c$, $P \prec 0$ and the i:th entry of w is equal to $d_i + p_i T_{out}$. Consider the variable transformation $x = E^{\frac{1}{2}} T$. Then, the system can be written as

$$\dot{x} = Ax + Bu + Hw,$$

where

$$A = E^{-1/2} P E^{-1/2} \prec 0, \quad B = E^{-1/2} \quad \text{and} \quad H = E^{-1/2}.$$

The system is now on the form (3.2a) with A is symmetric and Hurwitz. Hence, $K = B^T A^{-1} = P^{-1} E$ solves the problem (3.5) for (3.2) where (3.2a) is defined as above. Moreover, in words, the regulated output

$$z = \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} E^{\frac{1}{2}} T \\ u \end{bmatrix},$$

means that the temperature deviation in each room, as weighted by the air mass, should be made as small as possible with minimum control effort.

Of course, it is not computationally efficient to compute the inverse of the matrix P when the number of rooms is large, as the size of the matrix is $N \times N$. However, as the matrix P is sparse, computation can be done in a distributed manner throughout the building. This will now be illustrated.

For simplicity, consider the case of 3 rooms in a line, i.e. room 1 and 2 share a wall and room 2 and room 3 share a wall as depicted in Figure 3.2. Moreover, assume that $p_i = 1$ for $i = 1, \dots, 3$ and that $p_{12} = p_{23} = 1$. Then,

$$P = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 1 & -2 \end{bmatrix}.$$

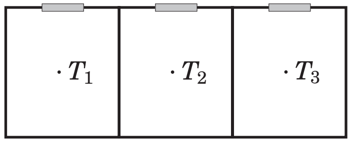

Figure 3.2 Three rooms in a line, where each room has a window. The temperature measurement in each room is indicated by a dot.

The inverse of P is dense. However, consider the system of equations that is equivalent to $Pu = ET$,

$$\begin{bmatrix} 1 & 0 & 0 \\ -\alpha & 1 & 0 \\ 0 & -\beta & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 \\ 0 & \alpha-3 & 1 \\ 0 & 0 & \beta-2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} cm_1T_1 \\ cm_2T_2 \\ cm_3T_3 \end{bmatrix}$$

where $\alpha = 0.5$ and $\beta = 0.4$. The control law can ideally be implemented as follows, where delays due to communication are neglected. The arrows show the propagation of information.

Controller 1 computes $x_1 = cm_1T_1$

$\rightarrow$ Controller 2 computes $x_2 = \alpha x_1 + cm_2T_2$

$\rightarrow$ Controller 3 computes $u_3 = (\beta x_2 + cm_3T_3)/(\beta - 2)$

$\rightarrow$ Controller 2 computes $u_2 = (x_2 - u_3)/(\alpha - 3)$

$\rightarrow$ Controller 1 computes $u_1 = (u_2 - x_1)/2$.

The flow of information is through the building and back again. Notice that each controller makes computations based only on local information and information from its neighbours. □

Limitations and comparison to existing works Since the problem of H_∞ control was first formulated in [Zames, 1981], several solution techniques for the general H_∞ control problem, i.e. not necessarily the case of state feedback, have been proposed. Examples are Youla-Kucera parametrization, see e.g. [Zhou et al., 1996], Riccati-based approaches [Doyle et al., 1989] and the optimization-based approach that uses linear matrix inequalities [Gahinet and Apkarian, 1994]. However, sparsity in the controller's structure is generally not a trait of controllers derived through these so called conventional approaches. For instance, the solution of an algebraic Riccati equation is often dense. Also, in practice, the conventional methods need to be performed numerically. Furthermore, to achieve optimality by these general purpose methods, the computational procedure needs to be iterated which is time consuming. This is not the case with the explicit control law we have obtained. However, it is only applicable to a certain class of systems and objectives.

Decentralized or distributed controllers are often obtained through imposing structural constraints on the controller as a part the synthesis procedure. This is the tool used in many of the works reviewed in Section 2.1. However, as was also discussed there, imposing such constraints might render losses in performance or intractability of the problem. The control design problem we consider does not include any structural constraints on the controller. Furthermore, the controller is not sparse due to regularization. Instead, it is simply the structural sparsity of the system that decides if the explicitly stated controller will be sparse or not. In some sense, the controller naturally inherits a structure compatible with the information structure of the system. This property of the controller is similar to that exhibited by the spatially invariant and spatially distributed systems commented on in the previous chapter. However, the systems we can consider are not restricted to be spatially invariant. Moreover, the presented results show that structural constraints, or regularization techniques, are not necessary for synthesis of distributed controllers.

If an H_∞ optimal synthesis problem is solvable, it generally has several solutions. In other words, there exist several optimal control laws. Naturally, they will have different properties. In the design of controllers for large-scale systems, properties of sparsity in the structure of the control law is of importance. Hence, the choice of an optimal controller is crucial. The explicitly stated controller indicates an optimal control systems architecture that, for certain applications of large-scale systems, can be implemented in a decentralized or distributed manner, as was previously illustrated through examples. However, it is important to stress, again, that our results are only applicable to a certain class of systems and objective, and only for the problem of H_∞ control. However, due to its strong properties, the explicitly stated control law could be used for benchmarking of general purpose methods for synthesis of decentralized controllers.

In comparison to the H_∞ control problem, the regular H_2 optimal control problem, the type of which the state feedback case is, has one unique solution. Hence, the structure given by the unconstrained synthesis is the only globally optimal structure possible. On the contrary, when performance is measured through the L_1 or L_∞ induced norms, there are yet again several solutions. Existing works that consider the latter types of control was reported in Chapter 1. However, they offer the controller as the unspecified solution to linear programs, i.e. the controllers are not given explicitly but can be efficiently computed.

3.3 Applications for Infinite-Dimensional Systems

The infinite-dimensional linear and time-invariant systems considered in this thesis can be described in the time-domain similarly to (3.2). However, the state now evolves on an infinite-dimensional Hilbert space. Furthermore, what was before the state matrix A is instead an operator acting on this infinite-dimensional space. More specifically, for this type of representation to be valid, the operator A with domain $D(A)$ needs to generate a strongly continuous semigroup on the space. Consider the sections with mathematical preliminaries in Paper IV and V for further details as well as [Curtain and Zwart, 2012].

The state feedback result for (3.2) with symmetric and Hurwitz state matrix A is translated into the infinite-dimensional realm in Paper IV. This covers systems modelled by certain parabolic equations such as the heat equation. In the following example, extracted from Paper IV, the problem considered is to control the temperature of the rod in Figure 3.3 when under the impact of a disturbance. The example showcases the use of the explicitly expressed controller, or closed-form controller as we often refer to it in the infinite-dimensional setting.

Example 3

The temperature at time t and position x is denoted $z(x,t)$ and governed by

$$\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t) + u(t) + w(t) \quad 0 < x < l,\ z(x,0) = 0,\ t \geq 0, \tag{3.15}$$

where l is the length of the rod and $w(t) \in \mathbb{R}$ is the disturbance at time t. The disturbance is uniformly distributed along the length of the rod. Similarly, the control input u can impact the temperature along the full length of the rod and the temperature is assumed to be zero at the end points. Notice that the state of the system, i.e. the temperature, is denoted z in the infinite-dimensional setup as opposed to x in the finite-dimensional setting.

The H_∞ state feedback problem (3.5) is as follows: Given $w \in L_2[0,\infty)$, minimize the H_∞ norm of the closed-loop system's transfer function from w to

$$\zeta = \begin{bmatrix} z \\ u \end{bmatrix}$$

over the set of stabilizing controllers K, represented by bounded linear operators, that map $z(x,t)$ to $u(t)$. This is exactly the problem considered in the finite-dimensional case and similarly, an optimal controller can be given on closed form.

Figure 3.3 Rod of length l with one-dimensional spatial coordinate x.

In this example, the explicitly stated, or closed-form, controller is equivalent to

$$u(t) = \int_0^l \underbrace{\frac{s(s-l)}{2}}_{:= f(s)} z(s,t)\, ds.$$

Hence, the control signal is a weighted integral of the deviation in temperature along the spatial coordinate. Moreover, the weight $f(s)$ determines the scalar signal for controlling the temperature profile, as a compromise between the deviation in temperature from zero and the cost for changing the temperature. □

Limitations and comparison to existing works It is rare to obtain the solution to an infinite-dimensional H_∞ problem explicitly. Hence, the results presented in this thesis are a rarity in the theory of H_∞ control. Moreover, the explicitly stated solution is a powerful tool both for its primary purpose of control and as a means for benchmarking of general purpose algorithms, especially those intended for systems of large order. Of course, it is again important to stress that the controller is only applicable to a certain class of infinite-dimensional systems, which is in contrast to the general purpose methods found in the literature reviewed in the previous chapter.

The state feedback law we derive will depend on the infinite-dimensional state vector for computation. However, as our results follows through in finite dimensions as well, an approximation of the controller can be obtained explicitly. Hence, the difference between the implemented and the exact controller can be calculated and used to analyse the closed-loop behaviour.

4

Conclusions

In this thesis, the problems of control of linear time-invariant large-scale systems as well as infinite-dimensional systems are addressed. More specifically, the method of H_∞ control, and in particular the case of state feedback, is considered. One of the main contributions is the identification of a class of systems for which an H_∞ optimal control law can be given on a simple explicit form. The control law is shown to have applications both to the problem of control of systems of large scale as well as to the control of infinite-dimensional systems.

More generally, a class of systems is characterized for which the H_∞ optimal control problem can be translated into a static problem at a single frequency. Hence, systems of this class has a bottleneck frequency at which the disturbance attenuation is the most crucial. For some of the systems, and objectives, the static problem can be solved explicitly. This is what renders the simple and explicit expression of the control law. Examples of systems for which this can be done are models for temperature dynamics in buildings, water irrigation and electrical networks.

For systems with sparse structure, such as the large-scale applications mentioned, the explicity stated control law is an example of a decentralized controller that is also globally optimal. In fact, its decentralized structure is obtained without any structural constraints or regularization techniques being part of the synthesis. Instead, it is a result of its explicit form, through which it inherits a structure compatible with the information structure of the system. For infinite-dimensional systems, the explicitly stated control law is important both for its primary purpose of control and as a means in evaluation and benchmarking of general purpose algorithms for H_∞ control. Moreover, an important application is diffusion equations.

Directions for further research include to investigate time-delayed systems as well as systems with nonlinearities in the presented framework. For instance, it is a natural extension to consider control input saturation as this is a common feature among the physical systems considered in this thesis. Initial analysis for a certain class of non-linear systems has presented promising results. Another direction for further research is to study how to efficiently solve the static synthesis problem for a general system in the class, such as for the case of output feedback. It is also of interest to investigate the possibility of explicit solutions to control problems in which the performance is measured through norms other than the H_∞ norm.

In the continuation of the work for infinite-dimensional systems, there are several areas of application to explore, e.g. systems in financial mathematics and other problems in heat transfer. Furthermore, the possibility of an extension to unbounded input and output maps, such as in the case of boundary control, is worth to investigate. Moreover, it could be studied how the results can be used to improve the performance of general purpose algorithms for H_∞ synthesis. This is also the case for the application of the results to the problems of optimal actuator and sensor placement. Initial discussion on these topics are included in Paper V.

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Paper I

Optimal H-infinity State Feedback for Systems with Symmetric and Hurwitz State Matrix

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Abstract

We address H_∞ state feedback and give a simple form for an optimal control law applicable to linear time-invariant systems with symmetric and Hurwitz state matrix. More specifically, the control law as well as the minimal value of the norm can be expressed in the matrices of the system's state space representation, given separate cost on state and control input. Thus, the control law is transparent, easy to synthesize and scalable. If the plant possesses a compatible sparsity pattern, it is also distributed. Examples of such sparsity patterns are included. Furthermore, if the state matrix is diagonal and the control input matrix is a node-link incidence matrix, the open-loop system's property of internal positivity is preserved by the control law. Finally, we give an extension of the optimal control law that incorporate coordination among subsystems. Examples demonstrate the simplicity in synthesis and performance of the optimal control law.

1. Introduction

Systems with a high density of sensors and actuators often lack centralized information and computing capability. Thus, structural constraints, e.g, on information exchange among subsystems, have to be incorporated into the design procedure of the control system. However, imposing such constraints may greatly complicate controller synthesis.

We address H_∞ structured static state feedback, a problem that is recognized as genuinely hard given arbitrary plant and controller structures. However, we give a simple form for an optimal control law applicable to linear time-invariant (LTI) systems with symmetric and Hurwitz state matrix that is distributed if the system possesses a compatible sparsity pattern. Consider the following LTI system

$$\dot{x} = \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}}_{=:\,A} x + \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{=:\,B} u + w \tag{4.1}$$

where the state x, the control input u and the disturbance w are real valued. The static state feedback controllers

$$K_1 = \begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & \frac{1}{3} & -\frac{1}{2} \end{bmatrix} \quad \text{and} \quad K_2 = \begin{bmatrix} 0.93 & -0.11 & 0.00 \\ -0.05 & -0.17 & -0.01 \\ 0.04 & 0.16 & -0.26 \end{bmatrix}$$

both minimize the H_∞ norm of the closed-loop system from disturbance w to penalized variables x and u, i.e., when $u = K_1x$ and $u = K_2x$, respectively. However, they have different structural properties, e.g., K_1 is sparser than K_2. Furthermore, the feedback law $u = K_1x$ is distributed as the matrix K_1 has the same structure as the sparse matrix B^T. This is not the case for controller K_2. Controller K_1 can be given on the simple form we propose. More specifically K_1 can be written as $K_1 = B^TA^{-1}$. Controller K_2 is derived by the algebraic Riccati equation (ARE) approach. That is, iteration over an ARE-constraint until the minimal value of the norm is obtained, see [Zhou et al., 1996] for details. Controllers synthesized by the ARE method are often dense, as is the case for controller K_2. Moreover, as the control law we give, i.e., $u = B^TA^{-1}x$, is optimal, it is equal in performance to any centrally derived optimal controller. Additionally, it is transparent in its structure, easy to synthesize and scalable.

In the 1980's, synthesis of controllers that achieve H_∞ norm specifications became a major research area and was formulated in [Zames, 1981]. The state-space based solution approach to the synthesis problem paved the way for optimization tools to be used, e.g., see [Doyle et al., 1989]. The H_∞ norm condition can be turned into a linear matrix inequality (LMI) by the Kalman-Yakubovich-Popov lemma [Gahinet and Apkarian, 1994], see Lemma 1 in Appendix for the version used in this paper. As the theory on H_∞ control emerged, a decentralized version took form, e.g., see [Zhai et al., 2001]. Imposing general sparsity constraints on the controller might complicate the design procedure. However, design is simplified if the constrained set of controllers is quadratically invariant with respect to the given system [Rotkowitz and Lall, 2006]. It is also simplified if the closed-loop system is constrained to be internally positive [Tanaka and Langbort, 2011]. Our method results in a control law that is equal in performance to the central non-structured controller of the system. This is not the case in the methods previously mentioned. However, they treat more general classes of systems.

The optimal control law $u = B^T A^{-1} x$ only requires some relatively inexpensive matrix calculations for its synthesis, especially for sparse systems. This is in contrast to general H_∞ controller synthesis where more expensive computational methods are required. Additionally, its structure is transparent, which is not often the case in H_∞ controller synthesis. The H_∞ framework treats worst-case disturbance as opposed to stochastic disturbance in the H_2 framework. However, the transparent structure and simple synthesis of the derived optimal feedback law might motivate its use even when some characteristics of the disturbance are known. Moreover, we show that it can be extended to incorporate coordination in a system of heterogeneous subsystems, given a linear coordination constraint. The coordinated control law is a superposition of a decentralized and a centralized part, where the latter is equal for all agents. This structure might be well suited for distributed control purposes as well. See [Madjidian and Mirkin, 2014] for a similar problem treated in the H_2 framework. Furthermore, if A is diagonal and $-BB^T$ is Metzler, the closed-loop system with the optimal control law, from disturbance to state, is internally positive. Thus, for such systems the property of internal positivity is preserved in the closed-loop system.

The outline of this paper is as follows. This section is ended with some notation. In Section 2, the main result is stated and proved. Section 3 treats system sparsity patterns that result in a distributed control law. Section 4 treats the result on internal positivity while Section 5 gives an extension of the control law that incorporates coordination. In Section 6, the performance of our optimal control law is compared, by a numerical example, to an optimal controller synthesized by the ARE approach. Concluding remarks are given in Section 7.

The set of real numbers is denoted $\mathbb{R}$ and the space n-by-m real-valued matrices is denoted $\mathbb{R}^{n\times m}$. The identity matrix is written as I when its size is clear from context, otherwise I_n to denote it is of size n-by-n. Similarly, a column vector of all ones is written **1** if its length is clear form context, otherwise $\mathbf{1}_n$ to denote it is of length n.

For a matrix M, the inequality $M \geq 0$ means that M is entry-wise nonnegative and $M \in \mathbb{R}^{n\times n}$ is said to be Hurwitz if all eigenvalues have negative real part. The matrix M is said to be Metzler if its off-diagonal entries are nonnegative and the spectral norm of M is denoted $\|M\|$. Furthermore, for a square symmetric matrix M, $M \prec 0$ ($M \preceq 0$) means that M is negative (semi)definite while $M \succ 0$ ($M \succeq 0$) means M is positive (semi)definite.

The H_∞ norm of a transfer function $F(s)$ is written as $\|F(s)\|_\infty$. It is well known that this operator norm equals the induced 2-norm, that is

$$\|F\|_\infty = \sup_{v\neq 0} \frac{\|Fv\|_2}{\|v\|_2}.$$

2. An Optimal H_∞ State Feedback Law

Consider a LTI system

$$\dot{x} = Ax + Bu + w \tag{4.2}$$

where the state matrix $A \in \mathbb{R}^{n\times n}$ is symmetric and Hurwitz and the state $x \in \mathbb{R}^n$ can be measured. Moreover, the control input $u \in \mathbb{R}^m$, disturbance $w \in \mathbb{R}^n$ and matrix $B \in \mathbb{R}^{n\times m}$. Given (4.2), consider a stabilizing static state feedback law $u := Kx$, where $K \in \mathbb{R}^{m\times n}$. Then, the transfer function of the closed-loop system from disturbance w to penalized variables x and u, is given by

$$G_K(s) = \begin{bmatrix} I \\ K \end{bmatrix} (sI - A - BK)^{-1}. \tag{4.3}$$

For (4.2) with A symmetric and Hurwitz, an optimal H_∞ static state feedback controller K, i.e., a matrix K such that $\|G_K\|_\infty$ is minimized, can be given explicitly in the matrices A and B. This is the main result of this paper and it is stated in the following theorem.

Theorem 1
Consider the system (4.2) with A symmetric and Hurwitz. Then, the norm $\|G_K\|_\infty$ is minimized by the static state feedback controller $K_* = B^T A^{-1}$. The minimal value of the norm is $\|(A^2 + BB^T)^{-1}\|^{\frac{1}{2}}$. □

Proof. Given $\gamma > 0$, the following statements are equivalent.

(i) There exists a stabilizing controller K such that

$$\|G_K\|_\infty = \sup_{\omega\in\mathbb{R}} \left\| \begin{bmatrix} I \\ K \end{bmatrix} (j\omega I - A - BK)^{-1} \right\| < \gamma.$$

(ii) There exist matrices K and $P = P^T$, $P \succ 0$, such that

$$\begin{bmatrix} (A+BK)^T P + P(A+BK) & P & [I \;\; K^T] \\ P & -\gamma^2 I & 0 \\ [I \;\; K^T]^T & 0 & -I \end{bmatrix} \prec 0.$$

(iii) There exist matrices $X = X^T$, $X \succ 0$, and Y such that

$$\begin{bmatrix} AX + XA + BY + Y^T B^T & I & [X \;\; Y^T] \\ I & -\gamma^2 I & 0 \\ [X \;\; Y^T]^T & 0 & -I \end{bmatrix} \prec 0.$$

(iv) There exist matrices $X = X^T$, $X \succ 0$, and Y such that

$$(X + A)^2 + \left(Y + B^T\right)^T \left(Y + B^T\right) - A^2 - BB^T + \gamma^{-2} I \prec 0.$$

(v)

$$-A^2 - BB^T + \gamma^{-2} I \prec 0.$$

(vi)

$$\gamma > \| \left(A^2 + BB^T\right)^{-1} \|^{\frac{1}{2}}.$$

The equivalence between (i) and (ii) is given by the KYP-lemma, see Lemma 1 given in the Appendix. Statement (ii) can be equivalently written as (iii) after right- and left-multiplication with $\mathrm{diag}(P^{-1}, I, I)$ and change of variables $(P^{-1}, KP^{-1}) \to (X, Y)$. The equivalence between (iii) and (iv) is obtained by applying the Schur complement lemma and completion of squares to the inequality in (iii). Choosing $X = -A$ and $Y = -B^T$ shows equivalence between (iv) and (v). It is possible to choose $X = -A$ as A is symmetric and Hurwitz, i.e., $A \prec 0$. Finally, notice that $A^2 + BB^T \succ 0$. Thus, $\left(A^2 + BB^T\right)^{-1} \succ 0$ and

$$\text{(v)} \iff \gamma^2 I \succ \left(A^2 + BB^T\right)^{-1} \iff \text{(vi)}.$$

Given $X = -A$ and $Y = -B^T$, γ is minimized and $K_* = YX^{-1} = B^T A^{-1}$ minimizes the norm in (i). Now, define $\gamma_* := \| \left(A^2 + BB^T\right)^{-1} \|^{\frac{1}{2}}$ and assume that $\|G_{K_*}\|_\infty \neq \gamma_*$. Then $\|G_{K_*}\|_\infty$ has to be strictly larger than or strictly smaller than γ_*. Consider $\|G_{K_*}\|_\infty > \gamma_*$. This statement contradicts statement (i) and (vi) and is therefore false. Now, consider instead $\|G_{K_*}\|_\infty < \gamma_*$. This statement contradicts that γ is minimized and is therefore also false. Hence, the statement $\|G_{K_*}\|_\infty \neq \gamma_*$ is false and

$$\|G_{K_*}\|_\infty = \| \left(A^2 + BB^T\right)^{-1} \|^{\frac{1}{2}}.$$

□

Remark 1
The result stated in Theorem 1 can be made more general. However, we only give some comments on this here, the details are left to the reader. One can consider Hw instead of w in (4.2), where H is a real matrix of appropriate size. Then, the optimal control law is still given by $K_* = B^T A^{-1}$, i.e., its form is not altered by H. However, the value of the norm becomes $\|H^T(A^2 + BB^T)^{-1}H\|^{\frac{1}{2}}$. Notice that if H is a column vector, the expression inside the norm is a scalar.

If the considered system is stable and diagonalizable but not symmetric, a variable transformation can be used in order to be able to apply the result in Theorem 1. If Du, with $D \in \mathbb{R}^{q\times m}$, is penalized instead of u, and $R := D^T D$ is invertible, the control law becomes $K_* = R^{-1}B^T A^{-1}$ and the norm is given by $\|(A^2 + BR^{-1}B^T)^{-1}\|^{\frac{1}{2}}$. If the penalized variables x and u are scaled by scalar nonzero coefficients, the optimal control law will only be scaled by a scalar nonzero coefficient. □

Synthesis of optimal state feedback controllers generally requires additional computation beyond what is needed to compute K_* from Theorem 1, i.e., some relatively simple matrix calculations. Moreover, controllers generated by other methods are rarely as transparent as K_*. The transparency simplifies analysis of the controller's structure and enables scalability, which will be exploited in the following section.

In order for Theorem 1 to be applicable, the system of interest has to have a state space representation with symmetric and Hurwitz state matrix A. The symmetry property of A demands that states that affect each other do so with equal rate coefficient. Such representations appear, for instance, in buffer networks and models of temperature dynamics in buildings. We will now give an example of the latter.

Example 1
Consider a building with three rooms as depicted in Figure 1. The average temperature T_i in each room $i = 1$, 2 and 3, around some steady state, is given by the following model

$$\begin{aligned} \dot{T}_1 &= -r_1 T_1 + r_{12}(T_2 - T_1) + u_1 + w_1 \\ \dot{T}_2 &= -r_2 T_2 + r_{12}(T_1 - T_2) + r_{23}(T_3 - T_2) + u_2 + w_2 \\ \dot{T}_3 &= -r_3 T_3 + r_{23}(T_2 - T_3) + u_3 + w_3 \end{aligned} \tag{4.4}$$

governed by heat balance. The parameters $r_\bullet$ are constant, real-valued and positive. They are the rate coefficients of the system. For instance, r_{12} is the rate coefficient of the heat transfer through the wall between room 1 and 2. Changes in outdoor temperature and disturbances specific for each room, such as a window is opened, are modeled by disturbances w_i. The average

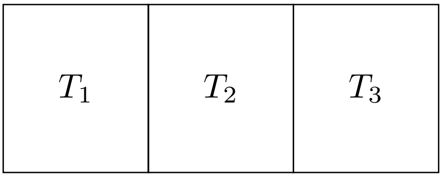

Figure 1. Schematic of a building with three rooms. The average temperature in each room $i = 1, 2$ and 3 is denoted T_i and given by (4.4).

temperatures can be measured as well as controlled, the latter through heating and cooling devices given by control inputs u_i.

If (4.4) is written on form (4.2), it is easy to see that the corresponding matrix A is symmetric. Thus, Theorem 1 is applicable to (4.4), assuming that parameters $r_\bullet$ are such that A is also Hurwitz. Given a disturbance, the feedback law with K_* from Theorem 1 tries to keep the average temperature as close to the steady state as possible while minimizing the cost that comes with heating and cooling. □

3. Distributed and Scalable

The structure of the optimal controller K_* given in Theorem 1 is clearly dependent on the structure of matrices A and B in (4.2). For instance, if A is diagonal and B is sparse, K_* has the same sparsity pattern as B^T. Moreover, controller K_* is distributed if (4.2) possesses a compatible sparsity pattern. This is demonstrated in Example 2 below. It is worthwhile to point out that for some sparsity patterns of (4.2) the representation $K_*^{-1}u = x$ instead of $u = K_* x$ might be beneficial for computation of u. That is, if B^T is invertible.

Example 2
Consider the following LTI system, containing three subsystems denoted S_1, S_2 and S_3,

$$\begin{aligned} S_1 : \quad & \dot{x}_1 = A_1 x_1 + B_1 u_1 + w_1 \\ S_2 : \quad & \dot{x}_2 = A_2 x_2 + B_2 u_1 + B_3 u_2 + w_2 \\ S_3 : \quad & \dot{x}_3 = A_3 x_3 + B_4 u_2 + w_3 \end{aligned} \tag{4.5}$$

where each subsystem S_i, i =1, 2 and 3, has finite state dimension $n_i \geq 1$, each control input u_i, i =1, 2 and 3, is a vector of finite length $m_i \geq 1$ and the matrices are of suitable dimension. Furthermore, matrices A_1, A_2

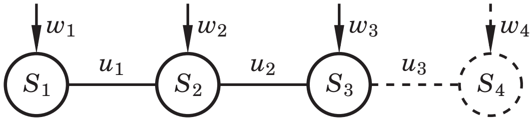

Figure 2. Graphical representation of (4.5) in solid lines. Additional subsystem S_4 and control input u_3 in dashed lines.

and A_3 are assumed to be symmetric and Hurwitz. Then, Theorem 1 is applicable to (4.5) and results in the optimal controller

$$K_* = \begin{bmatrix} B_1^T A_1^{-1} & B_2^T A_2^{-1} & 0 \\ 0 & B_3^T A_2^{-1} & B_4^T A_3^{-1} \end{bmatrix}. \tag{4.6}$$

Notice that, if (4.5) is written on form (4.2) the optimal controller K_* has the same sparsity pattern as B^T. Thus, each control input vector u_i is only constructed from the states it affects in (4.5). If we consider each subsystem S_i in (4.5) to represent an area of the physical system it models, the optimal controller (4.6) is distributed according to these areas. See Figure 2 for a graphical representation of the system, drawn in solid lines. Each subsystem S_i is depicted by a circular node while each control input u_i is given by a link connecting the subsystems it affects in (4.5). Each disturbance w_i is drawn as an arrow that points toward the subsystem it affects in (4.5). □

Now, we will demonstrate the scalability of the optimal control law. Consider that a fourth subsystem denoted S_4, of finite dimension $n_4 \geq 1$, is connected to (4.5) via a third control input denoted u_3, of finite length $m_3 \geq 1$, as depicted by the dashed lines in Figure 2. The dynamics of subsystem S_4 and the altered dynamics of subsystem S_3 are then given by

$$\begin{aligned} S_3: \quad \dot{x}_3 &= A_3 x_3 + B_4 u_2 + B_5 u_3 + w_3 \\ S_4: \quad \dot{x}_4 &= A_4 x_4 + B_6 u_3 + w_4 \end{aligned}$$

where matrix A_4 is also assumed to be symmetric and Hurwitz. Then, Theorem 1 is still applicable and the extended optimal controller becomes

$$K_* = \begin{bmatrix} B_1^T A_1^{-1} & B_2^T A_2^{-1} & 0 & 0 \\ 0 & B_3^T A_2^{-1} & B_4^T A_3^{-1} & 0 \\ 0 & 0 & B_5^T A_3^{-1} & B_6^T A_4^{-1} \end{bmatrix}.$$

The expansion of the system does not alter the initial control inputs u_1 and u_2. Thus, for systems with this type of sparsity pattern, the control law $u = K_* x$ is easily scalable. Moreover, the control law is still distributed as the additional control input u_3 is only constructed from states x_3 and x_4.

4. Preserves Internal Positivity

We will now consider (4.2) with diagonal and Hurwitz matrix A and where $-BB^T$ is Metzler. Then, the closed-loop system from disturbance w to state x with the optimal control law $u = K_* x$, from Theorem 1, is internally positive by Lemma 2, given in Appendix. This result is stated in Corollary 1 below and demonstrated in Example 3.

Corollary 1
Consider (4.2) with A diagonal and Hurwitz. Then, the closed-loop system from w to x with $K_* = B^T A^{-1}$ is internally positive if and only if $-BB^T$ is Metzler. □

Proof. Theorem 1 is applicable as A is Hurwitz and clearly symmetric. The closed-loop system from w to output $y := x$, with $K_* = B^T A^{-1}$, is

$$\dot{x} = (A + BK_*)\,x + w, \quad y = x,$$

where $A + BK_* = A + BB^T A^{-1}$. This system is internally positive by Lemma 2 in the Appendix, if and only if $A + BK_*$ is Metzler, as the other matrices are entry-wise nonnegative. As A is diagonal and Hurwitz, i.e, all diagonal elements are negative, it is easy to see that it is necessary and sufficient that $-BB^T$ is Metzler for $A + BK_*$ to be Metzler. □

Remark 2
If B is a node-link incidence matrix, see [Newman, 2010] for a formal definition of this notion, the matrix product $-BB^T$ is Metzler. The B-matrix given in Example 3 below is an example of a node-link incidence matrix. □

Example 3
Consider three buffers of some quantity connected via links with flow u_1 and u_2 as depicted in Figure 3. The dynamics of the levels in the buffers, around some steady state depicted by the dashed lines in Figure 3, is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix}}_{=:\,A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}}_{=:\,B} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + w. \tag{4.7}$$

State x_i corresponds to the level in buffer i =1, 2 and 3, respectively. Each buffer has some internal dynamics dependent on its own state, as given by matrix A. However, with different rate coefficients for the different buffers.

We want to construct a control law that minimizes the impact from disturbance w to the penalized variables x and u in the H_∞ norm sense.

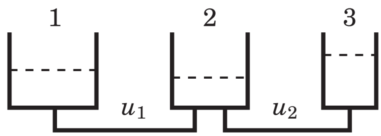

Figure 3. Three buffers denoted 1, 2 and 3 connected via links with flow u_1 and u_2, respectively. The dashed lines represent some steady state of the system.

That is, we want to keep the system at its steady state, i.e, $x_i = 0$ for all i, while also keeping the cost down, i.e, the magnitude of the control input.

Given the matrix B in (4.7), $-BB^T$ is Metzler. Thus, by Corollary 1, the closed-loop system from w to x with the optimal control law given by Theorem 1, i.e.,

$$K_* = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{4} \end{bmatrix},$$

is internally positive. This implies that, in closed-loop with controller K_*, the states x_i of (4.7) will always be nonnegative, i.e., the buffer levels will never go below their steady state values, given nonnegative disturbance. To get some further intuition of what controller K_* does, consider control input u_1. It is given by $u_1 = x_1 - x_2/2$. Thus, u_1 is strictly positive if $x_1 > x_2/2$ and the controller K_* redistributes the quantity of buffer 1 and buffer 2 relative to their internal rate coefficients. As in the previous example, K_* has the same sparsity pattern as B^T and thus each control input only considers local information, i.e., from the buffers it connects. □

5. Coordination in the H_∞ Framework

In this section we will extend the optimal control law given by Theorem 1 in order to include coordination. The problem formulation is as follows. Consider a LTI system of ν subsystems

$$\dot{x}_i = A_i x_i + B_i u_i + w_i, \quad i = 1, \ldots, \nu \tag{4.8}$$

where A_i, for $i = 1, \ldots, \nu$, is symmetric and Hurwitz. Furthermore, the control inputs u_i have to coordinate in order to fulfill the following constraint

$$u_1 + u_2 + \cdots + u_\nu = 0. \tag{4.9}$$

Given penalized variables x and u and the coordination constraint in (4.9), we want to construct an optimal H_∞ static state feedback controller for (4.8). The resulting control law is given by Corollary 2.

Corollary 2
Consider ν subsystems as in (4.8) with symmetric and Hurwitz state matrices and coordination constraint (4.9). Then,

$$u_i = B_i^T A_i^{-1} x_i - \frac{1}{\nu} \sum_{k=1}^{\nu} B_k^T A_k^{-1} x_k \quad \text{for } i = 1, \ldots, \nu \tag{4.10}$$

minimizes the norm of the closed-loop system from w to the penalized variables x and u. □

Proof. Rewrite control input u_1 in terms of the other control inputs given (4.9), i.e.,

$$u_1 = -u_2 - u_3 \ldots - u_\nu, \tag{4.11}$$

and define $\tilde{u} = [u_2, u_3, \ldots, u_\nu]^T$. Then,

$$u = \underbrace{\begin{bmatrix} -\mathbf{1}_{\nu-1}^T \\ I_{\nu-1} \end{bmatrix}}_{D} \tilde{u}$$

and the overall system of (4.8) can be written as

$$\dot{x} = \underbrace{\text{diag}(A_1, \ldots, A_\nu)}_{=:\,A} x + \underbrace{\text{diag}(B_1, \ldots, B_\nu)}_{=:\,B} D\tilde{u} + w$$

with penalized variables x and $u = D\tilde{u}$. Define $R = D^T D = I + \mathbf{1}\mathbf{1}^T$ and notice that $R^{-1} = I - \frac{1}{\nu}\mathbf{1}\mathbf{1}^T$. The optimal control law by Theorem 1, see also Remark 1, is then

$$\begin{aligned}
\tilde{u} &= R^{-1} D^T B^T A^{-1} x \\
&= \left(I_{\nu-1} - \frac{1}{\nu}\mathbf{1}_{\nu-1}\mathbf{1}_{\nu-1}^T\right) \begin{bmatrix} -\mathbf{1}_{\nu-1}^T \\ I_{\nu-1} \end{bmatrix}^T B^T A^{-1} x \\
&= \left(\begin{bmatrix} 0 & I_{\nu-1} \end{bmatrix} - \frac{1}{\nu}\mathbf{1}_{\nu-1}\mathbf{1}_{\nu}^T\right) B^T A^{-1} x.
\end{aligned}$$

Thus, u_i for $i = 2, \ldots, \nu$, i.e., the elements in $\tilde{u}$, is

$$u_i = B_i^T A_i^{-1} x_i - \frac{1}{\nu} \sum_{k=1}^{\nu} B_k^T A_k^{-1} x_k. \tag{4.12}$$

Now, consider u_1 again,

$$
\begin{aligned}
u_1 &\overset{(4.11)}{=} -\sum_{i=2}^{\nu} u_i \overset{(4.12)}{=} -\sum_{i=2}^{\nu}\left(B_i^T A_i^{-1} x_i - \frac{1}{\nu}\sum_{k=1}^{\nu} B_k^T A_k^{-1} x_k\right) \\
&= -\left(\sum_{k=1}^{\nu} B_k^T A_k^{-1} x_k - B_1^T A_1^{-1} x_1 - \frac{\nu-1}{\nu}\sum_{k=1}^{\nu} B_k^T A_k^{-1} x_k\right) \\
&= B_1^T A_1^{-1} x_1 - \frac{1}{\nu}\sum_{k=1}^{\nu} B_k^T A_k^{-1} x_k,
\end{aligned}
$$

i.e., it has the same structure as (4.12). Thus, the optimal control law is given by

$$
u_i = B_i^T A_i^{-1} x_i - \frac{1}{\nu}\sum_{k=1}^{\nu} B_k^T A_k^{-1} x_k
$$

for each subsystem $i = 1, \ldots, \nu$ in (4.8). □

Remark 3

The first term of u_i in (4.10) is a local term, only dependent upon the subsystem i, while the second term is dependent on global information of the overall system. However, as this term is equal for all control inputs u_i, (4.10) might still be appropriate for distributed control use. □

In [Madjidian and Mirkin, 2014], a similar type of problem is considered, however in the H_2 framework with stochastic disturbances and the necessity of homogeneous subsystems. The optimal control law derived in [Madjidian and Mirkin, 2014] and the one we suggest in (4.10) are similar in structure. However, our approach can treat heterogeneous systems in addition to homogeneous ones. On the contrary, it is only applicable to systems with symmetric and Hurwitz state matrix, properties that are not necessary in [Madjidian and Mirkin, 2014].

6. Numerical Example

Consider a system of the same structure as (4.1) given in Section 1, i.e., a system

$$
\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \underbrace{\begin{bmatrix}-a_1 & 0 & 0\\ 0 & -a_2 & 0\\ 0 & 0 & -a_3\end{bmatrix}}_{=:\,A}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \underbrace{\begin{bmatrix}-b_1 & 0 & 0\\ b_2 & b_3 & -b_4\\ 0 & 0 & b_5\end{bmatrix}}_{=:\,B}\begin{bmatrix}u_{12}\\ u_2\\ u_{23}\end{bmatrix} + \begin{bmatrix}w_1\\ w_2\\ w_3\end{bmatrix} \tag{4.13}
$$

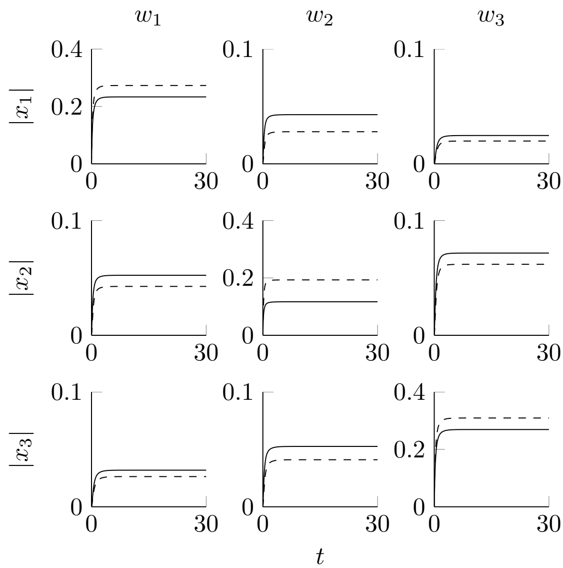

Figure 4. Average step response for states x_1, x_2 and x_3 for closed-loop systems with controller K_* (solid lines) and K_G (dashed lines).

where $a_i > 0$, for i =1, 2 and 3, and $b_j > 0$, for $j = 1, \ldots, 5$, and penalized variables x and u. We will now compare the optimal controller given by Theorem 1, i.e., K_*, and an optimal controller derived by the ARE-approach, see [Zhou et al., 1996], denoted K_G for global. In the latter approach, we consider the minimal value of the H_∞ norm of (4.3) given by Theorem 1 and iterate over the ARE-constraint until this minimal value is reached. See [Mathworks, 2015] for the software used. Controllers K_1 and K_2 given in Section 1 are examples of controllers K_* and K_G treated here, respectively.

Controllers K_* and K_G are optimal and thus they both obtain the minimal value of the H_∞ norm of (4.3). Now we want to compare how they affect the closed-loop dynamics more in detail. We randomly generate values of the parameters a_i and b_j in $(0.1, 5]$ and compare the step response of the states of (4.13) in closed-loop with K_* and K_G. In other words, given constant disturbance of value 1. The average dynamics over 50 such randomly generated systems is shown in Figure 4. To clarify, we average over the absolute value of the step response in each time instance.

The system (4.13) can be depicted by the graph given in Figure 5, as described in Section 3. If we compare the step responses shown in Figure 4, it seems as if controller K_* is better at attenuating local disturbances than K_G is. With local disturbances we mean the disturbance that points towards

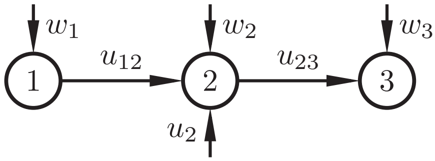

Figure 5. Associated graph of (4.13).

the state in Figure 5. This is at the expense of larger impact on distance. However, overall they are comparable in performance.

We will end this numerical example by commenting on controller K_2 given in Section 1, that is an example of controller K_G treated in this numerical example. Some entries of K_2 are small in magnitude compared to the other entries, i.e., entries (2,1), (2,3) and (3,1), where the first number in each parenthesis is the row and the second is the column. However, only entry (3,1) can be replaced with a zero for the controller to still achieve the optimal bound. Furthermore, for systems of much larger dimension than (4.1), this type of reduction analysis might be difficult.

7. Conclusions

We give a simple form for an optimal H_∞ static state feedback law applicable to LTI systems with symmetric and Hurwitz state matrix. More specifically, this simple form is given in the matrices of the system's state space representation which makes the structure of the controller transparent. It also simplifies synthesis and enables scalability of the control law, especially given sparse systems. Furthermore, given compatible system sparsity patterns, the control law is distributed. The examples we give consider diagonal or block diagonal state matrices and somewhat more general sparsity patterns of the remaining system matrices. Given some further constraints on the system's matrices, the closed-loop system from disturbance to state becomes internally positive. Furthermore, we extend the optimal control law in order to incorporate coordination among subsystems. The resulting coordinated control law is similar for all subsystems. More specifically, for each subsystem, it is a superposition of a local term and an averaged centralized term where the latter is equal for all subsystems involved in the coordination. In conclusion, our control law is well suited for distributed control purposes.

Future research directions include to consider saturation constraints on the optimal control law as such are common in the systems intended for its application. Furthermore, to investigate the existence of an analogous optimal control law given output feedback instead of state feedback. For an extension of the result to infinite-dimensional systems, see [Lidström et al., 2016].

Appendix

LEMMA 1—THE KALMAN-YAKUBOVICH-POPOV LEMMA
Given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $M \in \mathbb{R}^{(n+m)\times(n+m)}$, $M = M^T$, with $\det(j\omega I - A) \neq 0$ and (A, B) controllable, the following two statements are equivalent:

(i)

$$\begin{bmatrix}(j\omega I - A)^{-1}B \\ I\end{bmatrix}^* M \begin{bmatrix}(j\omega I - A)^{-1}B \\ I\end{bmatrix} \preceq 0$$

$\forall \omega \in \mathbb{R} \cup \{\infty\}$.

(ii) There exists a matrix $P \in \mathbb{R}^{n\times n}$ such that $P = P^T$ and

$$M + \begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} \preceq 0$$

The corresponding equivalence for strict inequalities holds even if (A, B) is not controllable. □

Proof. See [Rantzer, 1996]. □

REMARK 4
If the upper left corner of M is positive semidefinite, it follows from (1) and Hurwitz stability of A that $P \succeq 0$ [Rantzer, 1996]. □

LEMMA 2
The LTI system

$$\dot{x} = Ax + Bv, \quad y = Cx + Dv$$

is internally positive if and only if

(i) A is Metzler, and

(ii) $B \geq 0$, $C \geq 0$ and $D \geq 0$. □

Proof. See [Kaczorek, 2001]. □

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Paper II

H-infinity Optimal Distributed Control in Discrete Time

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Abstract

We give closed-form expressions for H-infinity optimal state feedback laws applicable to linear time-invariant discrete time systems with symmetric and Schur state matrix. This class includes networked systems with local dynamics in each node and control action along each edge. Furthermore, the structure of the controllers mimics that of the system, which makes them suitable for distributed control purposes.

1. Introduction

We study structured H_∞ control and give a class of linear time-invariant (LTI) discrete time systems for which distributed controllers are optimal. To give a flavour of our results, consider the subsystems

$$x_i(t+1) = a_i x(t) + b \sum_{(i,j)\in\mathcal{E}} u_{ij}(t) + d_i(t).$$

Here $i \in (1,\ldots,N)$, $0 < a_i < 1$, $b > 0$, $u_{ij} = -u_{ji}$ and $\mathcal{E}$ is the edge set of a network with N nodes. This system is naturally associated with a graph, such as that in Figure 1. Each subsystem is depicted by a node, and the edges describe the couplings between subsystems through the control signals u_{ij}. We show that the static state feedback law

$$u_{ij}(t) = \frac{b}{a_i - 1} x_i(t) - \frac{b}{a_j - 1} x_j(t)$$

minimizes the H_∞ norm of the closed-loop system from the disturbance d to the state x and control input u when $a_i^2 + 2b^2 k_i < a_i$, where k_i is the degree of node i. This constraint is related to the speed of information propagation through the network as well as its connectivity. Note that each control input u_{ij} is only comprised of the states it directly affects, with a proportional term related to these subsystems. Therefore not only is this control optimal, it is also easy to apply, even when the number of subsystems N is large.

Control of large-scale and complex systems is most often performed in a distributed manner. This is due to the practical impossibility of having access to information about the overall system when deciding the control actions. However, it is not straightforward to translate the conventional control synthesis methods to synthesis of controllers suitable in a large-scale setting. In fact, the optimal distributed control problem can often be intractable.

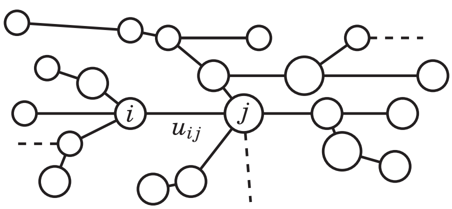

Figure 1. Part of a network of N nodes. The dashed lines illustrates where the connections are to the rest of the network. Node i depicts subsystem i while edge (i, j) is related to the control input u_{ij} for which $u_{ij} = -u_{ji}$.

Imagine the system described previously to be written compactly as $x(t+1) = Ax + Bu + d$. The optimal control law can then be written as $u = B^T(I - A)^{-1}x$. In fact, this control law is optimal as long as A is symmetric and Schur and $A^2 + BB^T \prec A$. Also, it is explicitly given which is a rarity in the case of H_∞ control. Furthermore, our theory naturally suggests a controller with a structure related to the structure of the considered system, which makes it a candidate for distributed control. This link between the structure of the system and that of the controller is similar to what is described for spatially invariant systems [Bamieh et al., 2002; Bamieh and Voulgaris, 2005]. However, the systems we consider are not restricted to be spatially invariant, though our results are only valid for H_∞ norm performance requirements. Further, synthesis of structured controllers is simplified when the closed-loop system is required to be positive [Tanaka and Langbort, 2011; Rantzer, 2015]. Although, as we do not include this requirement, it is hard to compare it with our approach. In [Delvenne and Langbort, 2006; Wang et al., 2014; Pates and Vinnicombe, 2017], they instead consider a localized approach to the design of distributed controllers. Of course, this can be conservative. In our case, a local information pattern is optimal whenever the dynamics can be divided into subsystems that only share control inputs.

H_∞ control, let alone distributed H_∞ control, has mainly been treated in the continuous time setting since these questions were first brought up half a century ago. See [Doyle et al., 1989] for the first state space based solution to the centralized non-structured H_∞ control problem. In fact, in order to solve the centralized H_∞ control problem in discrete time an additional criterion on the system needs to be fulfilled [Limebeer et al., 1989; Yaesh and Shaked, 1989; Gu et al., 1989]. However, it is of great importance to study the problem of distributed H_∞ control in discrete time, as controllers are almost always implemented digitally.

The work presented covers the non-trivial translation of the continuous time problems studied by the authors in [Lidström and Rantzer, 2016] and [Rantzer et al., 2017]. Besides the static state feedback law described previously, we give a closed-form expression for a state feedback controller with integral action which requires minimum control effort and guarantees a specified level of disturbance attenuation. It has similar structure preserving properties to the static state feedback controller and can track references.

The outline is as follows. In Section 2 we present general results on the optimal state feedback controllers, and state their closed-forms as well as the system requirements needed for them to be applicable. In Section 3, we discuss their relation to distributed control. Section 4 displays how the results are related to their continuous time counterparts. The introduction is ended with the notation used.

The set of real numbers is denoted $\mathbb{R}$ and the space of n-by-m real-valued matrices is denoted $\mathbb{R}^{n\times m}$. The identity matrix is written as I. Given a matrix M, the spectral norm of M is denoted $\|M\|$ and the Moore-Penrose pseudo-inverse of M is denoted $M^\dagger$. A square matrix $M \in \mathbb{R}^{n\times n}$ is said to be Hurwitz if all eigenvalues have negative real part. It is said to be Schur if all eigenvalues are strictly inside the unit circle. Furthermore, for a square symmetric matrix M, $M \prec 0$ ($M \preceq 0$) means that M is negative (semi)definite while $M \succ 0$ ($M \succeq 0$) means M is positive (semi)definite. The H_∞ norm of a proper and real rational stable transfer function $T(z)$ is written as $\|T\|_\infty$ and given by $\|T\|_\infty = \sup_{\omega\in\mathbb{R}} \|T(e^{j\omega})\|$.

2. Closed-Form H_∞ Optimal State Feedback in Discrete Time

The first part of this section treats H_∞ optimal static state feedback when the performance requirement is to minimize the impact from process disturbance on the state and control input. In the second part, we require the controller to track a reference in addition to a disturbance rejection requirement, with minimum control effort. Naturally, the latter controller has integral action. Further, we specify a class of systems for which the optimal controllers can be stated on closed-form. This class includes a broad range of linear networked systems. We will show how these results can be used for distributed control in Section 3.

2.1 Optimal Static State Feedback

Consider the discrete time LTI system

$$x(t+1) = Ax + Bu + Hd \tag{4.1}$$

where the state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m$, disturbance $d \in \mathbb{R}^l$ and the matrices A, B and H are of appropriate dimensions. Furthermore, consider the regulated output

$$\zeta = \begin{bmatrix} x \\ u \end{bmatrix}. \tag{4.2}$$

The objective is to find a stabilizing static state feedback law $u = Kx$, $K \in \mathbb{R}^{m\times n}$, that minimizes the H_∞ norm of the closed-loop system from d to ζ. We denote the transfer function of the closed-loop system with (4.1)-(4.2) and K by $T_{d\to\zeta}[K]$. It is given by

$$T_{d\to\zeta}[K](z) = \begin{bmatrix} I \\ K \end{bmatrix} (zI - A - BK)^{-1} H. \tag{4.3}$$

The following theorem gives a closed-form optimal control law, with respect to the objective described above, that is applicable to (4.1) with symmetric and Schur stable matrix A and for which $A^2 + BB^T \prec A$. The latter constraint is related to the sample time used in the time discretization of the equivalent continuous time system. This is discussed in more detail in Section 4.

THEOREM 1
Consider (4.3) with A symmetric and Schur and $A^2 + BB^T \prec A$. Then, $\|T_{d\to\zeta}[K]\|_\infty$ is minimized by $K_{\text{opt}} = B^T(A-I)^{-1}$ and the minimal value of the norm is $\|H^T\left((A-I)^2 + BB^T\right)^{-1} H\|^{\frac{1}{2}}$. □

The proof of Theorem 1 is given in the Appendix.

REMARK 1
Note that the control law is independent of how the disturbance enters the system, i.e, H in (4.1). □

2.2 Optimal PI control

Consider the discrete time LTI system

$$\begin{aligned} x(t+1) &= Ax + Bu + Bd \\ q(t+1) &= q + x \end{aligned} \tag{4.4}$$

where the state $x \in \mathbb{R}^n$, integral of the state x, i.e., $q \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m$, disturbance $d \in \mathbb{R}^m$ and the matrices A and B are of appropriate dimensions. The objective is to find a stabilizing state feedback controller K, which maps $r - x$ to u, where $r \in \mathbb{R}^n$ is a reference signal. Furthermore, it should track r with minimum control effort and also guarantees a certain level of disturbance attenuation. The closed-loop transfer functions from r to u and d to q are denoted $T_{r\to u}[K]$ and $T_{d\to q}[K]$, respectively. The following theorem states a closed-form optimal control law, with respect to the objective described above, that is applicable to (4.4) with A symmetric and $0 \prec A \prec I$.

THEOREM 2
Consider (4.4) with A symmetric and $0 \prec A \prec I$. Define $\gamma = \|\left((I-A)^{-1}B\right)^\dagger\|$ and assume that $\tau > 0$ fulfills

$$\tau\left(\tau I - \gamma B^T(I-A)^{-2}B\right) \succeq B^T(I-A)^{-4}AB.$$

Then, the problem

$$\begin{aligned} \text{minimize} \quad & \|T_{r\to u}[K]\|_\infty \\ \text{subject to} \quad & \|T_{d\to q}[K]\|_\infty \le \tau \end{aligned}$$

over stabilizing K, is solved by

$$\hat{K}_{\text{opt}}(z) = k\left(B^T(I-A)^{-2} + \frac{1}{z-1}B^T(I-A)^{-1}\right),$$

where $k = \gamma/\tau$. The optimal value is γ. □

The proof of Theorem 2 is given in the Appendix.

Remark 2
The parameter τ determines the bandwidth of the control loop where a smaller τ corresponds to disturbance rejection over a wider frequency range. □

Remark 3
Note that K enters as an ordinary negative feedback as compared to the previous subsection, where the negative feedback sign was incorporated in the control law. □

3. Distributed Implementation

This section concerns structured control and describes a particular type of systems for which Theorem 1 and 2 result in distributed controllers. The considered systems are comprised of subsystems with local dynamics, that only share control inputs. Furthermore, each control input only affects two subsystems. Depicting the subsystems as nodes and the control inputs as edges between the nodes they affect, the overall system can be illustrated by a network graph.

It is evident from the results given in the previous section that the open-loop system need to be structured itself in order for the optimal control laws to be structured. This is natural from a large-scale system point of view as the dynamics of such systems most definitely would be highly localized. For the networked systems considered in this section, the control signals drive the interaction among the subsystems. In a transportation network this is related to routing the flow of commodities. Furthermore, the dynamics in each subsystem is diffusive, so they act as infinite buffers. The dynamics could also describe a linear approximation of the behavior of more involved dynamics around an operating point.

3.1 Static State Feedback Case

Consider a network with N nodes or subsystems,

$$x_i(t+1) = a_i x_i + b \sum_{(i,j)\in\mathcal{E}} u_{ij} + d_i \tag{4.5}$$

where $i \in (1, ..., N)$, $b \in \mathbb{R}$, $b > 0$ and $\mathcal{E}$ is edge set. The overall system can be written on the form (4.1) with A diagonal and B with columns of one element equal to 1 and one equal to -1, scaled by b, while the remaining elements are zero. See Figure 1 for an illustration of the system. Each node in the system's graph represents a subsystem i while an edge (i, j) illustrates how control signal u_{ij} enters the system. Furthermore, $u_{ij} = -u_{ji}$, i.e., what is drawn from system j is added to system i. This class of systems includes linear models of transportation and buffer networks. The Corollary below follows from Theorem 1 and gives a closed-form expression for an optimal distributed static state feedback law for (4.5).

Corollary 1
Consider a graph with a set of nodes $\mathcal{V}$ and edges $\mathcal{E}$. Let the dynamics in each node $i \in \mathcal{V}$ be given by (4.5) with $0 < a_i < 1$, $b > 0$ and $u_{ij} = -u_{ji}$. Furthermore, consider the N subsystems to be written on the form (4.1)-(4.2), where $x = \{x_i\}_{i\in\mathcal{V}}$, $u = \{u_{ij}\}_{(i,j)\in\mathcal{E}}$ and $d = \{d_i\}_{i\in\mathcal{V}}$ and assume that $A^2 + BB^T \prec A$. Then, the control law

$$u_{ij} = \frac{b}{a_i - 1}x_i - \frac{b}{a_j - 1}x_j$$

minimizes the H_∞ norm of the transfer function from the disturbance d to the regulated output ζ. □

Proof. The overall system is given by $x(t+1) = Ax + Bu + d$, where A is diagonal and $0 \prec A \prec I$ with $A^2 + BB^T \prec A$. The controller structure then follows from Theorem 1. □

Remark 4
In the distributed case, the constraint $A^2 + BB^T \prec A$ can be approximated by a local constraint. See Section 4 for more details. □

3.2 Distributed optimal PI control

Consider a slight variation to the subsystems in (4.5) with an extra state for each subsystem i as the integral of x_i, denoted q_i,

$$\begin{aligned} x_i(t+1) &= a_i x_i + b_i(u_i + d_i) + \sum_{(i,j)\in\mathcal{E}} (u_{ij} + d_{ij}), \\ q_i(t+1) &= q_i + x_i. \end{aligned} \tag{4.6}$$

In this system, the disturbances enter in the same way as the control inputs and again $u_{ij} = -u_{ji}$ as well as $d_{ij} = -d_{ji}$. The corollary given next follows from Theorem 2.

Corollary 2
Consider a graph with a set of nodes $\mathcal{V}$ and edges $\mathcal{E}$. Let the dynamics in each node $i \in \mathcal{V}$ be given by (4.6) with $0 < a_i < 1$, $b_i \neq 0$ for at least one i, $u_{ij} = -u_{ji}$, $d_{ij} = -d_{ji}$ and $q_i(0) = 0$. Denote $e_i = r_i - x_i$, where r_i is the reference signal for subsystem i. Furthermore, consider the overall system written on the form (4.4). Define $\gamma = \|((I-A)^{-1}B)^{+}\|/\tau$ and assume that

$$\tau(\tau - \gamma B^T(I-A)^{-2}B) \succeq B^T(I-A)^{-4}AB.$$

Then, the controller

$$\begin{aligned} p_i(t+1) &= p_i(t) + e_i(t), \\ u_{ij}(t) &= k\left(p_i/(1-a_i) - p_j/(1-a_j)\right) + k\left(e_i/(1-a_i)^2 - e_j/(1-a_j)^2\right), \\ u_i(t) &= k\left(p_i b_i/(1-a_i) + e_i b_i/(1-a_i)^2\right), \end{aligned}$$

with $k = \gamma/\tau$, minimizes the H_∞ norm of the transfer function from r to u while keeping the L_2-gain from d to q bounded by τ. □

Proof. The overall system is given by $x(t+1) = Ax + Bu + Bd$, where A is diagonal and $0 \prec A \prec I$. Furthermore, the assumptions in Theorem 2 hold, and the controller structure thus follows from Theorem 2. □

3.3 Numerical Example

Consider the buffer network depicted in Figure 2, where N is the total number of buffers. The network has a fork structure where the leftmost part, the root, has n buffers while the upper and lower branch has n_1 and n_2 buffers, respectively. The dynamics of the content in the buffers, around some operating point, is

$$\begin{aligned} x_1(t+1) &= a_1 x_1 + b_1(u_1 + d_1) + u_{12} + d_{12}, \\ x_i(t+1) &= a_i x_i + \sum_{(i,j)\in\mathcal{E}} u_{ij} + d_{ij}, \quad \forall\, i \in 2, \ldots, N. \end{aligned}$$

The disturbance d_{ij} enters on edge (i, j). The control inputs and disturbances satisfy $u_{ij} = -u_{ji}$ and $d_{ij} = -d_{ji}$, respectively. Each $0 < a_i < 1$, so given a non-zero initial state the buffers will eventually be empty if no control is used and after any disturbance has abated. The content in the buffers dissipates faster through the lower branch than through the upper.

The system described is part of the class of systems treated in the previous subsections. We will now show the closed-loop behavior of the system with the distributed static state feedback and distributed PI controller, respectively. As we consider the number of buffers N to be large, a localized

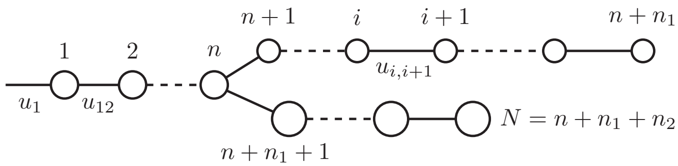

Figure 2. Buffer network with N buffers. Buffer 1 has inputs u_1 and u_{12}. A general buffer i has inputs $u_{i,k}$, where k is the number of its neighbors. The network has a fork structure where, from the left, the first part has n buffers. The upper branch has n_1 buffers while the lower branch has n_2 buffers.

control approach is the only practical design for implementation. Our results suggest control inputs u_{ij} that only require local information regardless of the size of the system N.

Figure 3, on the following page, shows the levels in some of the buffers in the network, over time. At time $t = 20$ a constant disturbance enters in node 1. The disturbance is then processed through the network via the edges. The two upper plots and the lower right plot show the time trajectories of the three first buffers in the head, the upper branch and the lower branch, respectively. The dotted lines show the references, while the dashed and solid lines show the trajectories given the static and PI controller, respectively. The bottom left plot shows the time trajectory of the disturbance. You can see how the process is evolved through the network, starting at the head and following the branches by comparing the peaks of the time trajectories.

4. Discussion

4.1 Comparison to Continuous Time Results

We will now compare Theorem 1 with its continuous time counterpart stated by the authors in [Lidström and Rantzer, 2016]. For clarity, we include that result next.

Theorem 3—[Lidström and Rantzer, 2016]
Consider

$$G_{d\to\zeta}[K](s) = \begin{bmatrix} I \\ K \end{bmatrix} (sI - A_c - BK)^{-1}H,$$

with A_c symmetric and Hurwitz. Then $\|G_{d\to z}[K]\|_\infty$ is minimized by $K_{\text{opt}} = B^T A_c^{-1}$ and the minimal value of the norm is given by $\|H^T(A_c^2 + BB^T)^{-1}H\|^{\frac{1}{2}}$. □

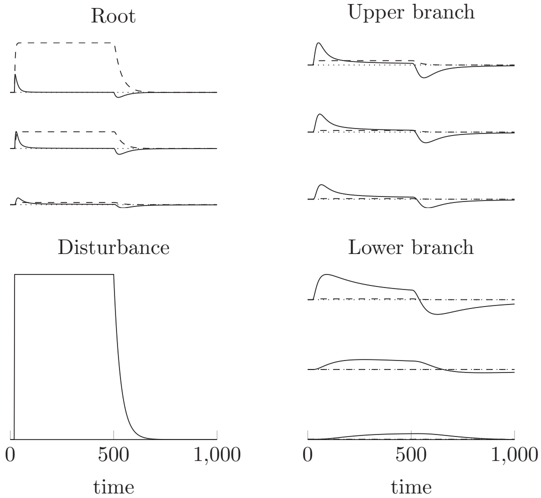

Figure 3. Numerical example with buffer network of N buffers. The two upper plots and the lower right plot show the time trajectories of the three first buffers in the root, upper branch and lower branch, respectively. For instance the upper left plot show the time trajectories of node 1 in the top, node 2 in the middle and node 3 at the bottom. The plots for the branches are constructed in the same manner. The dotted lines show the references, while the dashed and solid lines show the trajectories given the static and PI controllers, respectively. The bottom left plot shows the disturbance that enters in node 1. It is evident that the PI controller is able to track the reference while the static controller leaves a stationary error.

The statements in Theorem 1 and 3 are very similar, however, with one main difference. That is, the extra requirement on the matrices of the system's state space representation that is required in the discrete time case, i.e., $A^2 + BB^T \prec A$. If we consider the discretization of the open-loop continuous time system with time period h, we get

$$x(t+1) = e^{A_c h} x(t) + \int_0^h e^{A_c \tau} Bu(\tau) d\tau.$$

For small h and given the assumption that $u(t)$ is constant during each time period, we can use the approximation

$$x(t+1) \approx (I + A_c h)x(t) + hBu(t).$$

The constraint then becomes

$$(I + A_c h)^2 + h^2 BB^T \prec I + A_c h,$$

which is equivalent to $h < \|A_c^{\frac{1}{2}}(A_c^2 + BB^T)^{-1}A_c^{\frac{1}{2}}\|$. Thus, for small enough h, it is always fulfilled. Similarly to the discussion above, one can compare Theorem 2 to its continuous time equivalent stated in [Rantzer et al., 2017].

The constraint $A^2 + BB^T \prec A$ reveals that $A \succ 0$ in the discrete time case. Thus, the class of discrete time systems that can be considered for both Theorem 1 and Theorem 2 are non-oscillatory. This also maps to the class of continuous time systems considered in Theorem 3, as symmetric matrices do not have imaginary eigenvalues.

The optimal controller given by Theorem 3 is clearly related to the controller resulting from bisecting over the continuous time algebraic Riccati equation (CARE). That is, if we denote the solution to the CARE by P, the controller is given by $K = -B^T P$ [Stoorvogel, 1992]. In the discrete time setting, they are not as clearly related. The controller given by bisection over the discrete time ARE (DARE) is $K = -(I + B^T PB)^{-1}B^T PA$ where P is the solution to the DARE [Stoorvogel, 1992]. The expression for the DARE controller is more involved than the expression we give for the controller proposed in Theorem 1.

4.2 Local Condition for $A^2 + BB^T \prec A$

Consider the systems described in Section 3. They are of the form (4.1) with A diagonal and B sparse. In fact, given the network description of these systems, the matrix B relates the nodes and edges. B/b is generally called the node-link incidence matrix of the network graph. From this, it is possible to approximate the constraint $A^2 + BB^T \prec A$ by a local constraint. Denote the i:th diagonal element of A by a_i. Furthermore, write $BB^T = b^2L$. The inequality can then be written as

$$A^2 - A + b^2L \prec 0.$$

Further, define the diagonal matrix D by $D_{ii} := L_{ii}$. Then the inequality becomes

$$\begin{aligned}
0 &\succ D^{\frac{1}{2}}\left(D^{-1}(A^2 - A) + b^2D^{-\frac{1}{2}}LD^{-\frac{1}{2}}\right)D^{\frac{1}{2}},\\
0 &\succ D^{-1}(A^2 - A) + b^2L_{\text{sym}},
\end{aligned}$$

where L_{sym} is the symmetric normalized Laplacian of the network's graph. It is well-known that $\lambda_{\max}(L_{\text{sym}}) \leq 2$. Therefore, satisfying the local condition

$$a_i^2 - a_i + 2b^2D_{ii} < 0.$$

is sufficient to guarantee that $A^2 + BB^T \prec A$. The entry D_{ii} is the degree of node i, i.e., the number of nodes it is directly connected to, and often denoted k_i. Thus, the constraint $A^2 + BB^T \prec A$ is related to the speed of information propagation through the network and its connectivity, via a_i, b, the bound on the maximum eigenvalue of the symmetric normalized Laplacian and the node degree. Note that a similar analysis can be made for the inequality constraint on τ in Corollary 2, where it is necessary that $\tau I - \gamma B^T(I-A)^{-2}B \succ 0$.

5. Conclusions and Future Works

We define a class of systems and performance objectives for which the optimal H_∞ controller is structured. It includes networked systems with local dynamics in each node and control action along each edge. Some assumptions on this class of systems are only sufficient. It is left as future work to characterize the both sufficient and necessary systems properties. Furthermore, combining the discrete time results presented with their continuous time counterparts could bring some further intuition into the case of sampled data control for networked systems.

Appendix

To prove Theorem 1, we need the following lemma.

Lemma 1
Assume $A \in \mathbb{R}^{n\times n}$ symmetric and Schur, and $B \in \mathbb{R}^{n\times m}$. Then, the following statements are equivalent

(i) $A^2 + BB^T \prec A$,

(ii) $(A-I)\left((A-I)^2 + BB^T\right)^{-1}(A-I) + A - I \succ 0$.

Proof. Note that $A - I \prec 0$ as A is symmetric and Schur. Then,

$$\begin{aligned}\text{(ii)} &\iff \left((A-I)^2 + BB^T\right)^{-1} + (A-I)^{-1} \succ 0 \\ &\iff -A + I \succ (A-I)^2 + BB^T \iff \text{(i)},\end{aligned}$$

where in the first step we have multiplied (ii) with $(A-I)^{-1}$ from both left and right. □

Next, we give the proof of Theorem 1.

Proof of Theorem 1. The proof is divided into two parts. The first part considers a lower bound on the minimal norm-value. In the second part, we show stabilizability of K_{opt} and that the lower bound is achieved for K_{opt}.

The minimal norm-value can be lower bounded as follows

$$\inf_K \|T_{d\to\zeta}[K]\|_\infty \;=\; \inf_K \sup_\omega \|T_{d\to\zeta}[K](e^{j\omega})\| \;\geq\; \inf_K \|T_{d\to\zeta}[K](1)\|. \quad (4.7)$$

Moreover, given $\gamma > 0$,

$$\|T_{d\to\zeta}[K](1)\| \leq \gamma$$

is equivalent to

$$\|T_{d\to\zeta}[K](1)d\|^2 \leq \gamma^2\|d\|^2$$

for all $d \in \mathbb{C}^l$. Further, it can be written as

$$\|x\|^2 + \|Kx\|^2 \leq \gamma^2\|d\|^2$$

for all x, d such that $x = (I - A - BK)^{-1}Hd$. Denote $u = Kx$, and rewrite the latter equality as

$$(I - A)x - Bu = Hd.$$

Now, for fixed d, consider the problem

$$\begin{aligned} &\text{minimize } \|x\|^2 + \|u\|^2 \\ &\text{subject to } (I - A)x - Bu = Hd. \end{aligned}$$

This problem lower bounds γ as the constraint $u = Kx$ is removed. The solution to the problem is

$$\begin{bmatrix} x_* \\ u_* \end{bmatrix} = \begin{bmatrix} I - A^T \\ -B^T \end{bmatrix} \left((A - I)(A - I)^T + BB^T\right)^{-1} Hd.$$

Thus, given (4.7) and symmetry of A we have that

$$\inf_K \|T_{d\to\zeta}[K]\|_\infty \geq \|H^T((A - I)^2 + BB^T)^{-1}H\|^{\frac{1}{2}}.$$

We will now prove that $K_{\text{opt}} = B^T(A - I)^{-1}$ is optimal by showing that it is stabilizing and achieves the lower bound given above. For K_{opt} to be stabilizing, $A_{cl} := A + BB^T(A - I)^{-1}$ has to be Schur. It is equivalent to existence of a matrix $P \succ 0$ such that $A_{cl}PA_{cl}^T - P \prec 0$. One such P is $P = (A-I)^2$, which is valid as $A \prec I$. Note that, given the assumptions on A and B, $A_{cl}PA_{cl}^T - P \prec 0$ with $P = (A - I)^2$ is equivalent to $(A - I)^2 + BB^T \succ 0$, which is true as $A \prec I$.

To show that K_{opt} achieves the lower bound, rewrite

$$T_{d\to\zeta}[K_{\text{opt}}](e^{j\omega})^* T_{d\to\zeta}[K_{\text{opt}}](e^{j\omega}) = H^T G^{-1}(j\omega)H$$

where

$$\begin{aligned}G(j\omega) &= (e^{j\omega} - I)(A - I)M^{-1}(A - I)(e^{-j\omega} - I) \\ &\qquad - (e^{j\omega} + e^{-j\omega} - 2I)(A - I) + M \\ &= (2 - 2\cos(\omega))\underbrace{\left((A - I)M^{-1}(A - I) + A - I\right)}_{=:\,N} + M,\end{aligned}$$

and $M := (A - I)^2 + BB^T$. It follows from Lemma 1 that $N \succ 0$ as $A^2 + BB^T \prec A$ by assumption. Therefore, it holds that $G(j\omega) \succeq M$ and moreover that $G(j\omega)^{-1} \preceq M^{-1}$ for $\omega \in [0, 2\pi)$. Hence,

$$\begin{aligned}T_{d\to\zeta}[K_{\text{opt}}](e^{j\omega})^* T_{d\to\zeta}[K_{\text{opt}}](e^{j\omega}) &= H^T G^{-1}(j\omega) H \\ &\preceq H^T M^{-1} H \\ &= H^T((A - I)^2 + BB^T)^{-1} H \\ &= T_{d\to\zeta}[K_{\text{opt}}](1)^* T_{d\to\zeta}[K_{\text{opt}}](1)\end{aligned}$$

from which it follows that

$$\|T_{d\to\zeta}[K_{\text{opt}}]\|_\infty = \|H^T((A - I)^2 + BB^T)^{-1} H\|^{\frac{1}{2}}$$

and the proof is complete. □

To prove Theorem 2 we need the following lemma.

Lemma 2—[Rantzer et al., 2017]
Let $A \in \mathbb{C}^{n\times m}$. Then,

$$\min_{X\in\mathbb{C}^{q\times n}} \|X\| \text{ s.t. } AXA = A$$

has the minimal value $\|A^\dagger\|$, attained by $\hat{X} = A^\dagger$. □

Proof of Theorem 2. Define $P(z) := (zI - A)^{-1}B$. Then, $T_{r\to u}[K] = (I + KP)^{-1}K$. Now, define

$$M := I - kB^T(I - A)^{-2}B$$

and note that the given assumption on τ together with $A \succ 0$ yields $M \succ 0$. Firstly, we will show that $\hat{K}_{\text{opt}}$ is stabilizing. Factorize B as $B = GF^T$, where G and F have full column rank. Then,

$$\begin{aligned}T_{r\to u}[\hat{K}_{\text{opt}}](z) &= k\left((z - 1)I + kB^T(I - A)^{-2}B\right)^{-1} B^T(I - A)^{-2}(zI - A) \\ &= k\left((z - 1)I + kG^T(I - A)^{-2}GF^TF\right)^{-1} G^T(I - A)^{-2}(zI - A),\end{aligned}$$

so the poles of $T_{r\to u}[\hat{K}_{\text{opt}}]$ are the eigenvalues of

$$I - kG^T(I-A)^{-2}GF^TF,$$

i.e., the eigenvalues of M that are not equal to 1. Clearly, as $0 \prec M \preceq I$, $T_{r\to u}[\hat{K}_{\text{opt}}]$ is stable. Thus, $\hat{K}_{\text{opt}}$ is stabilizing.

Now, we will show that $\|T_{d\to q}[\hat{K}_{\text{opt}}]\|_\infty \leq \tau$. From the definition of k we have that $I \preceq k^2\tau^2B^T(I-A)^{-2}B$ which is equivalent to

$$\begin{aligned} &B^T(2(1-\cos(\omega))A + (I-A)^2)^{-1}B \\ &\qquad\qquad \preceq \tau^2\left[2(1-\cos(\omega))M + k^2(B^T(I-A)^{-2}B)^2\right] \end{aligned}$$

and further to

$$\left\| \tfrac{1}{e^{j\omega}-1}P(e^{j\omega})\left(I + \hat{K}_{\text{opt}}(e^{j\omega})P(e^{j\omega})\right)^{-1} \right\| \leq \tau.$$

The latter inequality is precisely $\|T_{d\to q}[\hat{K}_{\text{opt}}](e^{j\omega})\| \leq \tau$ and thus $\hat{K}_{\text{opt}}$ fulfills the constraint.

Finally, we will show that $\hat{K}_{\text{opt}}$ is in fact optimal. Again, consider the constraint with τ. In general $PT_{r\to u}[K]P = P - P(I+KP)^{-1}$ so the constraint demands

$$P(1)T_{r\to u}[K](1)P(1) = P(1).$$

Now, consider the minimization of $\|T_{r\to u}[K](1)\|$ subject to the equality above. Then, by Lemma 2, $T_{r\to u}[K](1) = P(1)^\dagger$ with optimal value $\|P(1)^\dagger\|$. Now, as $T_{r\to u}[\hat{K}_{\text{opt}}](1) = P(1)^\dagger$, it follows that $\hat{K}_{\text{opt}}$ is a feasible solution to the static problem at $\omega = 0$. Furthermore, to show that $\hat{K}_{\text{opt}}$ is optimal for the non static problem we need to show that $\|T_{r\to u}[\hat{K}_{\text{opt}}]\|_\infty$ is achieved at $\omega = 0$. Consider $T_{r\to u}[\hat{K}_{\text{opt}}]T_{r\to u}[\hat{K}_{\text{opt}}]^* \preceq \gamma^2 I$ which is equivalent to

$$\begin{aligned} &k^2B^T(I-A)^{-2}(2(1-\cos(\omega))A + (I-A)^2)(I-A)^{-2}B \\ &\qquad\qquad \preceq \gamma^2\left[2(1-\cos(\omega))M + k^2(B^T(I-A)^{-2}B)^2\right]. \end{aligned}$$

This inequality holds trivially for $\omega = 0$. It holds for all other other $\omega \in \mathbb{R}$ provided that

$$k^2B^T(I-A)^{-4}AB \preceq \gamma^2\left[I - kB^T(I-A)^{-2}B\right]$$

which is equivalent to the assumption on τ. Thus, $\|T_{r\to u}[\hat{K}_{\text{opt}}]\|_\infty$ takes the minimal value γ and the proof is complete. □

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Paper IV

H-infinity Optimal Control for Infinite-Dimensional Systems with Strictly Negative Generator

Carolina Lidström, Anders Rantzer and Kirsten Morris

Abstract

A simple form for the optimal H-infinity state feedback of linear time-invariant infinite-dimensional systems is derived. It is applicable to systems with bounded input and output operators and a closed, densely defined, self-adjoint and strictly negative state operator. However, unlike other state-space algorithms, the optimal control is calculated in one step. Furthermore, a closed-form expression for the L2-gain of the closed-loop system is obtained. The result is an extension of the finite-dimensional case, derived by the first two authors. Examples demonstrate the simplicity of synthesis as well as the performance of the control law.

1. Introduction

Infinite-dimensional models are often needed when the physical system of interest is both temporally and spatially distributed. For instance, heat conduction systems can be modelled by a parabolic partial differential equation known as the heat equation, see [Renardy and Rogers, 2006] for details on this equation. We consider H_∞ state feedback control of linear and time-invariant infinite-dimensional systems. The H_∞ control problem was first formulated for finite-dimensional systems, see [Zhou et al., 1996] and the references therein. There are both state-space based and frequency domain based solutions to the H_∞ control problem for infinite-dimensional systems, as in the finite-dimensional case. In the frequency domain approach, see [Foias et al., 1996], one needs to determine the transfer function of the system, which in general can be hard. In the state-space based approach to this problem, the synthesis involves solving an infinite-dimensional operator-valued Riccati equation or inequality, see [Bensoussan and Bernhard, 1993] and [Van Keulen, 1993]. Closed-form solutions are generally hard or not possible to obtain. However, we show that for certain infinite-dimensional systems, it is not only possible to give an analytic solution to the infinite-dimensional operator-valued Riccati inequality, but also the resulting controller has a very simple form.

We consider infinite-dimensional systems with bounded input and output operators and where the state evolves on a separable Hilbert space. Moreover, the state operator is closed, densely defined, self-adjoint and strictly negative. Thus, it generates an exponentially stable strongly continuous semigroup. See [Curtain and Zwart, 1995] for further details. We give a simple form for an optimal H_∞ state feedback law applicable to these systems, given that the state and control input are penalized separately. More specifically, the control law is given by the product of the adjoint of the control input operator and the inverse of the state operator. Furthermore, we provide a closed-form expression for the L_2-gain of the closed-loop system's transfer function. The result is the analog to the result for finite-dimensional systems derived by the first two authors in [Lidström and Rantzer, 2016]. The heat equation is an example of a system to which the derived control law is applicable. Examples are given in Section 4 that show the simplicity of synthesis and the performance of the control law.

As mentioned earlier, closed-form solutions of the operator-valued Riccati equation are generally hard or impossible to obtain. Therefore, one common approach is to consider the state-space based synthesis problem for a finite-dimensional approximation of the original system. In this procedure one has to ensure that the controller synthesized for the approximated system stabilizes the original system and also provides performance that approaches optimal as the approximation order increases; see [Morris, 2010].

This can be problematic but there are conditions under which this approach works, see [Ito and Morris, 1998] for H_∞ state feedback. However, the approximation order can be large and the multiple solutions of the Riccati equation required mean that computation can be intensive. Furthermore, it is difficult to determine the performance degradation resulting from the use of an approximated controller.

The result in this paper is important in several respects. First, for systems with self-adjoint generator to which the result directly applies, it provides an explicit characterization of the optimal controller. No iteration is required. This controller will be approximated in implementation, however the difference between the implemented and exact controller can be calculated. Furthermore, the result may be used in evaluation and benchmarking of algorithms for general systems.

The outline of this paper is as follows. Section 2 gives some mathematical preliminaries and the notation used. The main theorem is stated in Section 3 together with its proof. In Section 4, we illustrate the simplicity of synthesis and the performance of the derived control law by means of an example. Section 4 also includes some further discussion. Concluding remarks are given in Section 5.

2. Mathematical Preliminaries

The notations $\mathbb{R}$ and $\mathbb{C}$ stand for the set of real and complex numbers, respectively, while the set of nonnegative real numbers is denoted $\mathbb{R}_+$. The notation $\mathrm{Re}(x)$ where $x \in \mathbb{C}$ denotes the real part of x. We will only consider linear operators on separable Hilbert spaces, where we denote the inner product and norm by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively.

The domain of an operator T is denoted by $D(T)$, the adjoint of T is denoted by T^* and the inverse of T, if it exists, is denoted by T^{-1}. An operator T is called self-adjoint if $T^* = T$ and $D(T^*) = D(T)$. The set of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$ is denoted $\mathcal{L}(\mathcal{X}, \mathcal{Y})$, and $\mathcal{L}(\mathcal{X}) = \mathcal{L}(\mathcal{X}, \mathcal{X})$. The norm of an operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is defined as follows

$$\|T\| = \sup_{\substack{x \in D(T) \\ x \neq 0}} \frac{\|Tx\|_{\mathcal{Y}}}{\|x\|_{\mathcal{X}}}.$$

Definition 1—[Curtain and Zwart, 1995, p. 606, Def. A.3.71]
A self-adjoint operator A on the Hilbert space $\mathcal{Z}$ is nonnegative if $\langle Az, z \rangle \geq 0$ for all $z \in D(A)$, A is positive if $\langle Az, z \rangle > 0$ for all nonzero $z \in D(A)$ and A is strictly positive (coercive) if there exists an $m > 0$ such that

$$\langle Az, z \rangle \geq m\|z\|^2 \text{ for all } z \in D(A).$$

□

We will use the notation $A \succ 0$ for strict positivity of the self-adjoint operator A. We will use the terminology strictly negative denoted $A \prec 0$ when $-A \succ 0$.

REMARK 1
Let $\mathcal{Z}$ be a Hilbert space and consider a self-adjoint strictly negative operator A. It is clear from the definition of strict negativity that A is injective, thus A^{-1} exists. Furthermore, it can be shown that it is bounded, positive and $A^{-1} \in \mathcal{L}(\mathcal{Z})$. See [Curtain and Zwart, 1995, Ex. A.4.2] for details on this. □

DEFINITION 2— [CURTAIN AND ZWART, 1995, P. 15, DEF. 2.1.2]
A strongly continuous semigroup is an operator-valued function $S(t)$ from $\mathbb{R}_+$ to $\mathcal{L}(\mathcal{Z})$ that satisfies the following properties

1. $S(0) = I$,
2. $S(t+\tau) = S(t)S(\tau)$ for $t, \tau \geq 0$,
3. $\lim_{t\to 0,\, t>0} S(t)z = z$ for all $z \in \mathcal{Z}$.

□

DEFINITION 3— [CURTAIN AND ZWART, 1995, P. 215, DEF. 5.1.1]
A strongly continuous semigroup, $S(t)$, on a Hilbert space Z is *exponentially stable* if there exist constants $M, \alpha > 0$ such that $\|T(t)\| \leq Me^{-\alpha t}$ for all $t \geq 0$.

DEFINITION 4— [CURTAIN AND ZWART, 1995, P. 20, DEF. 2.1.8]
The generator $A : D(A) \to \mathcal{Z}$ of a strongly continuous semigroup $S(t)$ on a Hilbert space Z is defined by

$$D(A) = \left\{ z \in \mathcal{X} \;\middle|\; \lim_{\substack{t\to 0\\ t>0}} \frac{S(t)z - z}{t} \text{ exists} \right\}$$

$$Az = \lim_{\substack{t\to 0\\ t>0}} \frac{S(t)z - z}{t} \quad \text{for all } z \in D(A).$$

□

REMARK 2
If A is the generator of a strongly continuous semigroup as in Definition 4, then the domain of A, i.e., $D(A)$, is dense in $\mathcal{Z}$ and A is a closed operator, see [Curtain and Zwart, 1995, p. 21, Th. 2.1.10]. □

LEMMA 1—[CURTAIN AND ZWART, 1995, P. 33, COR. 2.2.3]
Sufficient conditions for a closed, densely defined operator on a Hilbert space to be the infinitesimal generator of a strongly continuous semigroup satisfying $\|S(t)\| \leq e^{wt}$ are:

$$\mathrm{Re}(\langle Az, z\rangle) \leq w\|z\|^2 \quad \text{for } z \in D(A),$$

$$\mathrm{Re}(\langle A^* z, z\rangle) \leq w\|z\|^2 \quad \text{for } z \in D(A^*).$$

□

REMARK 3
If A is self-adjoint, then the sufficient condition becomes $\langle Az, z\rangle \leq w\|z\|^2$ for $z \in D(A)$. Furthermore, if A is strictly negative by Definition 1, the condition clearly holds for some $w < 0$. Thus, by Definition 3, $S(t)$ is exponentially stable. Hence, A is the generator of an exponentially stable strongly continuous semigroup. □

If A is the generator of a strongly continuous semigroup $S(t)$ on the Hilbert space $\mathcal{Z}$, then for all $z_0 \in D(A)$, the differential equation on $\mathcal{Z}$

$$\frac{dz(t)}{dt} = Az(t), \quad z(0) = z_0,$$

has the unique solution $z(t) = S(t)z_0$. Consider an input $u \in L_2(0, t; \mathcal{U})$, where $\mathcal{U}$ is a Hilbert space and $L_p(\Omega; \mathcal{X})$ is the class of Lebesque measurable $\mathcal{X}$-valued functions f with

$$\int_{\Omega} |f(t)|^p dt < \infty, \quad p \in [0, \infty].$$

Given u and an operator $B \in \mathcal{L}(\mathcal{U}, \mathcal{Z})$, the differential equation

$$\frac{dz(t)}{dt} = Az(t) + Bu(t), \quad z(0) = z_0,$$

has the following solution at any time t

$$z(t) = S(t)z_0 + \int_0^t S(t-s)Bu(s)ds.$$

If we consider an output signal

$$y(t) = Cz(t) + Du(t)$$

where $C \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$ and $D \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$, the output at any time t given an input u is

$$y(t) = CS(t)z_0 + C\int_0^t S(t-\tau)Bu(\tau)d\tau + Du(t).$$

The Laplace transform of $y(t)$ given $z_0 = 0$ yields the transfer function of the system, denoted G, as follows

$$\hat{y}(s) = G(s)\hat{u}(s).$$

In what follows, the considered systems are assumed to be causal.

Definition 5— [Morris, 2010, p. 10, Def. 2.5]
A system is *externally stable* or *L_2-stable* if for every input $u \in L_2(0, \infty; \mathcal{U})$, the output $y \in L_2(0, \infty; \mathcal{Y})$. If a system is externally stable, the maximum ratio between the norm of the input and the norm of the output is called the *L_2-gain*. □

Define

$$H_\infty = \left\{ G : \mathbb{C}_0^+ \to \mathbb{C} \;\middle|\; G \text{ analytic and } \sup_{\operatorname{Re} s > 0} \|G(s)\| < \infty \right\},$$

where $\mathbb{C}_0^+$ are all complex numbers with real part larger than zero, with norm

$$\|G\|_\infty = \sup_{\operatorname{Re} s > 0} \|G(s)\|.$$

The lemma below is stated for systems with finite-dimensional input and output spaces, e.g., $\mathcal{U}$ and $\mathcal{Y}$ are $\mathbb{R}$, but it generalises to infinite-dimensional ones. The notation $M(H_\infty)$ stands for matrices with entries in H_∞.

Lemma 2— [Morris, 2010, p. 10, Def. 2.6]
A linear system is externally stable if and only if its transfer function matrix $G \in M(H_\infty)$. In this case, $\|G\|_\infty$ is the *L_2-gain* of the system and we say that G is a *stable transfer function*. □

Definition 6— [Morris, 2010, p. 10, Def. 2.9]
The pair (A, B) is exponentially stabilizable if there exists a $K \in \mathcal{L}(\mathcal{Z}, \mathcal{U})$ such that $A + BK$ generates an exponentially stable strongly continuous semigroup. □

3. Main Theorem

Consider a linear time-invariant infinite-dimensional system

$$\frac{dz(t)}{dt} = Az(t) + Bu(t) + Hd(t) \tag{4.1}$$

where the state $z(t) \in \mathcal{Z}$ and $\mathcal{Z}$ is a separable Hilbert space. The operator A is closed, densely defined, self-adjoint and strictly negative. Then by

Lemma 1, a version of the Lumer-Philips Theorem, A is the generator of an exponentially stable strongly continuous semigroup on $\mathcal{Z}$. See Remark 3 for further comments on this statement. The state $z(t)$ is assumed to be measurable with initial condition $z(0) = 0$. Furthermore, the control signal $u(t) \in \mathcal{U}$ and the disturbance $d \in L_2(0, \infty; \mathcal{V})$, where $\mathcal{U}$ and $\mathcal{V}$ are Hilbert spaces, and $B \in \mathcal{L}(\mathcal{U}, \mathcal{Z})$ and $H \in \mathcal{L}(\mathcal{V}, \mathcal{Z})$.

Consider H_∞ state feedback of (4.1) given unit cost on the state $z(t)$ and control input $u(t)$, separately, i.e., the cost function is given by

$$\zeta(t) = \begin{bmatrix} z(t) \\ u(t) \end{bmatrix}.$$

Given a stabilizing static state feedback controller $K \in \mathcal{L}(\mathcal{Z}, \mathcal{U})$, i.e., $u(t) = Kz(t)$, the closed-loop system from the disturbance $d(t)$ to the controlled output $\zeta(t)$ is given by

$$\begin{aligned} \frac{dz(t)}{dt} &= (A + BK)z(t) + Hd(t) \\ \zeta(t) &= \begin{bmatrix} I \\ K \end{bmatrix} z(t) \end{aligned} \tag{4.2}$$

where $A + BK$ generates an exponentially stable strongly continuous semigroup. We denote the Laplace transform of the closed-loop system given a controller K by G_K, i.e.,

$$\hat{\zeta}(s) = G_K(s)\hat{d}(s).$$

In the following theorem, we give a closed-form expression for a state feedback controller K that minimizes the L_2-gain of G_K. The optimal control law can be considered to be constant without restriction, see [Morris, 2010] for further details to this statement. The notation B^* indicates the adjoint of the operator B.

Theorem 1
Consider the system (4.1) where A is closed, densely defined, self-adjoint and strictly negative, $B \in \mathcal{L}(\mathcal{U}, \mathcal{Z})$ and $H \in \mathcal{L}(\mathcal{V}, \mathcal{Z})$, where $\mathcal{Z}$, $\mathcal{U}$ and $\mathcal{V}$ are Hilbert spaces. Then, $\|G_K\|_\infty$ is minimized by the state feedback controller $K_{\text{opt}} = B^* A^{-1}$ and the minimal value of the norm is given by $\|H^*(A^2 + BB^*)^{-1}H\|^{\frac{1}{2}}$. □

Proof. The proof is divided into two parts. In the first part we show that

$$\|G_{K_{\text{opt}}}\| \leq \|H^*(A^2 + BB^*)^{-1}H\|^{\frac{1}{2}}.$$

In the second part of the proof, we show that no controller can achieve strict inequality. Hence, equality holds. In both parts of the proof, we use the following equivalence given by the strict bounded real lemma in infinite dimensions, see [Curtain, 1993, Theorem 1.1], applied to (4.2): Given $\gamma > 0$ and a controller $K \in \mathcal{L}(\mathcal{Z}, \mathcal{U})$, the following two statements are equivalent

(i) $A + BK$ generates an exponentially stable strongly continuous semigroup $T(t)$ on the Hilbert space $\mathcal{Z}$ and

$$\|G_K\|_\infty < \gamma.$$

(ii) There exists a self-adjoint, *nonnegative* operator $\tilde{P} \in \mathcal{L}(\mathcal{Z})$ such that

$$(A + BK)^* \tilde{P} + \tilde{P}(A + BK) + I + K^* K + \gamma^{-2} \tilde{P} H H^* \tilde{P} \prec 0. \quad (4.3)$$

First, as A is closed, densely defined, self-adjoint and strictly negative then by Lemma 1, A is the generator of an exponentially stable strongly continuous semigroup on $\mathcal{Z}$, denoted $S(t)$. Furthermore, we know that (A, B) is exponentially stabilizable as $S(t)$ is exponentially stable. The domain of $A + BK$, i.e., $D(A + BK)$, is equal to the domain of A as $BK \in \mathcal{L}(\mathcal{Z})$.

For the first part of the proof consider (ii) and set $\tilde{P} = -A^{-1}$, $K = K_{\text{opt}} = B^* A^{-1}$ and take any γ with

$$\|H^*(A^2 + BB^*)^{-1} H\|^{\frac{1}{2}} < \gamma.$$

It is possible to set $\tilde{P} = -A^{-1}$ as A is self-adjoint and strictly negative, thus $-A^{-1}$ is self-adjoint, nonnegative and $A^{-1} \in \mathcal{L}(\mathcal{Z})$, see Remark 1. Now, we will prove that $\|G_{K_{\text{opt}}}\|_\infty < \gamma$ by the equivalence between (ii) and (i). First, notice that

$$\tilde{P}(A + BK) = -A^{-1}(A + BB^* A^{-1}) = -I - K^* K.$$

Thus, (4.3) can be equivalently written as

$$-I - K^* K + \gamma^{-2} A^{-1} H H^* A^{-1} \prec 0. \quad (4.4)$$

Inequality (4.4) holds if and only if

$$\begin{bmatrix} I + K^* K & -A^{-1} H \\ -H^* A^{-1} & \gamma^2 I \end{bmatrix} \succ 0 \quad (4.5)$$

by the Schur Complement Lemma for bounded linear operators, see [Dritschel and Rovnyak, 2010, Def. 3.1 and Lem. A.1]. Again, by the same Lemma, inequality (4.5) is equivalent to

$$\gamma^2 I - H^*(A^2 + BB^*)H \succ 0. \quad (4.6)$$

where we have used that

$$\gamma^2 I - H^* A^{-1}(I + K^* K)^{-1} A^{-1} H = \gamma^2 I - H^*(A^2 + BB^*)H.$$

Inequality (4.6) is true by the definition of γ. Hence, $\|G_{K_{\text{opt}}}\| < \gamma$ by the equivalence between (ii) and (i).

For the second part of the proof, consider again (4.3). Given a self-adjoint, nonnegative operator $\tilde{P}$ that solves (4.3), we can construct a self-adjoint, strictly positive operator $P_\epsilon \succ 0$ by $P_\epsilon = \tilde{P} + \epsilon I$, where $\epsilon > 0$ is some small real number. Then, we can define

$$M_\epsilon = (A + BK)^* P_\epsilon + P_\epsilon(A + BK) + I + K^* K + \gamma^{-2} P_\epsilon H H^* P_\epsilon$$

and we know that $M_0 \prec 0$. Furthermore,

$$M_\epsilon = M_0 + \epsilon 2A + \epsilon(K^* B^* + BK) + I + K^* K + \gamma^{-2}(P_\epsilon H H^* P_\epsilon - P_0 H H^* P_0).$$

The right-hand side of this equality is negative for small ϵ as $2A \prec 0$ and K, B, P and H are bounded. Thus, $M_\epsilon \prec 0$, i.e., the following holds

$$(A + BK)^* P + P(A + BK) + I + K^* K + \gamma^{-2} P H H^* P \prec 0$$

for some $P \succ 0$. This P is invertible and we can rewrite the inequality further as

$$P^{-1}(A + BK)^* + (A + BK)P^{-1} + P^{-2} + P^{-1} K^* K P^{-1} + \gamma^{-2} H H^* \prec 0.$$

We perform the change of variables

$$(P^{-1}, KP^{-1}) \to (X, Y),$$

thus $X \in \mathcal{L}(\mathcal{Z})$ and $Y \in \mathcal{L}(\mathcal{U}, \mathcal{Z})$, and sum of squares to write the inequality as follows

$$(X + A)^2 + (Y^* + B)(Y^* + B)^* - A^2 - BB^* + \gamma^{-2} H H^* \prec 0.$$

The first two terms of the operator expression are always nonnegative and thus no controller can satisfy a bound γ smaller than $\|H^*(A^2 + BB^*)^{-1} H\|^{\frac{1}{2}}$. Hence the controller constructed in the first part is optimal and the proof is complete. □

4. Control of the Heat Equation

In this section, we illustrate the simplicity in synthesis of the control law given by Theorem 1. The example concerns control of the heat equation,

Figure 1. Rod of length l with one-dimensional spatial coordinate x.

see (4.7) below, which describes the distribution of heat, or variation in temperature, in a region over time. The equation also describes other types of diffusion, such as chemical diffusion.

Consider the following partial differential equation that models heat propagation in a rod of length l

$$\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t) \quad 0 < x < l,\ t \geq 0. \tag{4.7}$$

The temperature at time t at position x is $z(x,t) \in \mathcal{Z} = L_2(0,l)$. See Figure 1 for a depiction of the rod.

To fully determine the temperature of the rod, the initial temperature profile as well as the boundary conditions have to be specified. As we consider H_∞ control, the initial temperature is set to zero. We will consider Dirichlet boundary conditions, i.e.,

$$z(0,t) = 0, \quad z(l,t) = 0.$$

Define the operator A as

$$A = \frac{d^2 z}{dx^2}$$

with domain

$$D(A) = \left\{ z \in L_2(0,l) \;\middle|\; z,\ \frac{dz}{dt} \text{ locally absolutely continuous,} \right.$$
$$\left. \frac{d^2 z}{dx^2} \in L_2(0,l) \text{ with } z(0) = 0,\ z(l) = 0 \right\}.$$

This operator fulfills the requirements for Theorem 1, i.e., it is closed, densely defined, self-adjoint and strictly negative. For a proof of this see [Tucsnak and Weiss, 2009, pp. 92-94]. Thus, by Lemma 1, A generates an exponentially stable strongly continuous semigroup $S(t)$ on $L_2(0,l)$, the state z evolves on the space $L_2(0,l)$ and we can write (4.7) as

$$\dot{z}(t) = Az(t), \quad z(x,0) = 0.$$

Now, suppose the temperature is controlled by an input $u(t)$ and affected by a disturbance $d(t)$ as follows

$$\dot{z}(t) = Az(t) + Bu(t) + Hd(t), \quad z(x, 0) = 0,$$

where $B, H \in \mathcal{L}(\mathbb{R}, L_2(0, l))$, $u \in L_2(0, \infty; \mathbb{R})$ and the disturbance $d \in L_2(0, \infty; \mathbb{R})$. Given the properties stated for the system, Theorem 1 is applicable. We will now, given some explicit examples of operators B and H, write down the closed-form expression for the control law given by Theorem 1.

The structure of the optimal control law, i.e., $K_{\text{opt}} = B^* A^{-1}$ is not dependent upon the operator H, as can be seen in Theorem 1. We will only consider

$$(Hd)(x) = d(t) \text{ for all } 0 < x < l.$$

In other words, the disturbance is uniformly distributed along the entire rod. We will treat operators B defined by

$$Bu = \chi_{[0,\alpha]}(x)u(t) \tag{4.8}$$

where $0 < \alpha \leq l$ and

$$\chi_{[0,\alpha]}(x) = \begin{cases} 1 & \text{if } 0 < x < \alpha \\ 0 & \text{otherwise.} \end{cases}$$

Thus, for $\alpha = l$ the control input is uniformly distributed along the entire rod while for instance for $\alpha = l/2$ it is only distributed in $0 < x < l/2$ while it is zero for the remaining part of the rod. The adjoint of operator B defined in (4.8) is

$$B^* y(x, t) = \int_0^\alpha y(x, t)dx \text{ for } y \in L_2(0, l).$$

Consider the following equality, as a step towards explicitly stating the optimal control law $u(t) = K_{\text{opt}} z(x, t) = B^* A^{-1} z(x, t)$,

$$z(x, t) = Ay(x, t), \quad y \in D(A).$$

The function $y(x, t)$ can be written as

$$y(x, t) = \int_0^l G(x, s)z(s, t)ds$$

where

$$G(x, s) = \begin{cases} \dfrac{(s-l)}{l}x & \text{if } 0 < x < s \\ \dfrac{s}{l}(x - l) & \text{if } s < x < l \end{cases}$$

is the Green's function of A. Note that $G(x, s)$ is piece-wise linear in x with $G(0, s) = G(l, s) = 0$. Now, if $\alpha = 1$ in (4.8), then

$$u(t) = B^* A^{-1} z(x, t) = \int_0^l \int_0^l G(x, s) z(s, t)\, ds\, dx$$
$$= \int_0^l \underbrace{\left[\int_0^l G(x, s)\, dx\right]}_{:= f(s)} z(s, t)\, ds \tag{4.9}$$

where

$$f(s) = \frac{s(s-l)}{2}.$$

The control input is thus a weighted integral of the deviation in temperature along the spatial coordinate. The quadratic weight $f(s)$ determines the scalar signal for controlling the temperature profile, as a compromise between the deviation in temperature from zero and the cost for changing the temperature. The general form of the control signal, i.e., without any specific value on α, is similarly given by

$$u(t) = \int_0^l \left(\int_0^\alpha G(x, s)\, dx\right) z(s, t)\, ds$$
$$= \int_0^\alpha f_1(s) z(s, t)\, ds + \int_\alpha^l f_2(s) z(s, t)\, ds$$

where

$$f_1(s) = \frac{s(s-l)}{2} + \frac{s(l-\alpha)^2}{2l} \quad \text{and} \quad f_2(s) = \frac{\alpha^2}{2l}(s-l).$$

The weighting function is altered dependent on if the spatial coordinate is less than or larger than α, to account for the asymmetry in B. Notice that $f_1(s) = f(s)$ and the second integral is zero when $\alpha = l$ as expected from the firstly derived expression for $u(t)$ in (4.9).

Given a constant disturbance $d(t) = 1$ for $t \geq 0$, the state $z(x, t)$ is determined numerically in MATLAB, see [Mathworks, 2015], by the finite element method for 200 time steps with interval length 0.01 and spatial segments of length 0.1, with $l = 3$. The integrals in the expression of the control law are approximated numerically by the trapezoidal rule.

In Figure 2a, the time trajectory of the temperature at the midpoint, i.e., $x = l/2$, is shown for the control input operators B defined by (4.8) given $\alpha = l$ and $\alpha = l/2$ as well as $B := 0$. Clearly, when $\alpha = l$ we get the best disturbance attenuation as shown by the solid line. When $\alpha = l/2$,

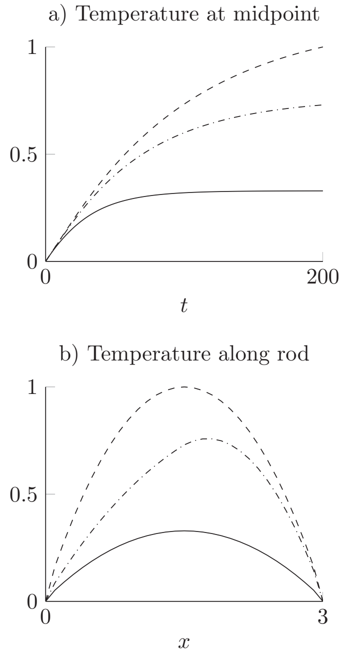

Figure 2. Response from unit disturbance for system with $B = 0$ is given by the dashed lines, with B defined in (4.8) with $\alpha = l$ is given by the solid lines and with $\alpha = l/2$ is given by the dashed dotted lines. a) Temperature at $x = l/2$ over time, b) temperature along the rod at $t = 200$.

the controller is not able to attenuate the disturbance as effectively and of course with $B = 0$ the system evolves only according to the heat equation with a disturbance. In Figure 2b we show the temperature distribution of the rod at the final time $t = 200$. Here one can see that the temperature distribution given with $\alpha = l/2$ is not symmetric along x. This is due to that the control input operator B in this case is asymmetric in x. The temperature distributions are normalized such that $z(200, l/2)$ given $\alpha = 0$ is equal to 1.

5. Conclusions

We give a closed-form expression for an optimal H_∞ state feedback controller applicable to systems with bounded input and output operators and closed, densely defined, self-adjoint and strictly negative state operator. We demonstrate, by means of an example, the simplicity of synthesis of the control law as well as its performance. The control law may be used in evaluation and benchmarking of general purpose algorithms for H_∞ controller synthesis. Future work includes comparison of a finite-dimensional approximation of the optimal controller to a controller derived by a general purpose algorithm. Further, to investigate possible benefits of having a closed-form expression for an optimal controller in the synthesis of controllers for large scale systems.

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Enkla regulatorer för storskaliga system

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Vi är alla beroende av ett flertal storskaliga system i vår vardag. Exempelvis så är de allra flesta hushåll kopplade till elnätet och vi nyttjar vägnätet för att pendla till jobbet. Reglerteknik kan användas för att försäkra att dessa system fungerar på ett bra sätt. I själva verket utnyttjas reglertekniken redan inom många användningsområden, bland annat för att garantera stabil elförsörjning via vårt elnät.

Reglerteknik baseras på återkoppling av information från till exempel mätningar. Föreställ dig ett rum där temperaturen regleras med hjälp av ett värme/kyla-system. Detta kan vara ett element eller en luftkonditioneringsenhet. I rummet finns även en termometer för att avläsa temperaturen. Vi önskar nu att temperaturen hålls så nära 21 grader som möjligt trots störningar. Exempel på störningar kan vara att ett fönster öppnas eller att utomhustemperaturen sjunker. Istället för att behöva styra värme/kyla-enheten på egen hand, så att temperaturen hålls på önskad nivå, kan en reglerteknisk styrlag användas och göra det automatiskt.

Effektiv temperaturreglering i bostäder är en del av forskningen för att skapa smarta samhällen.

Att skapa reglertekniska styrlagar för just storskaliga system kräver tyvärr ofta många och svåra beräkningar. Tänk dig ett flerbostadshus där temperaturen i varje rum ska regleras. Likt tidigare är varje rum utrustat med en termometer och en värme/kyla-enhet. Däremot är temperaturnivån i ett rum endast tillgänglig för värme/kyla-enheten som är installerad i det specifika rummet.

Förutom att hålla temperaturen i rummen på önskad nivå, trots störningar, är målet nu att koordinera värme/kyla-enheterna på ett sådant sätt att den totala energiförbrukningen blir den minsta möjliga. För att inte behöva installera en central beräkningsenhet, som även måste inhämta information från alla värme/kyla-enheter, ska målet kunna uppnås genom beräkningar utförda lokalt i varje rum. För att kunna koordinera sin verkan måste därför enheterna kommunicera med varandra och utbyta information om temperaturen i rummen. Det är även önskvärt att de lokala beräkningarna kan utföras med så lite informationsutbyte mellan enheterna som möjligt. Denna begänsning på informationsdelning måste tas i beaktning vid designen av styrlagen.

Det beskrivna scenariot kan formuleras som ett matematiskt problem. Om rummen är många blir dock beräkningarna som krävs för att lösa problemet, alltså för att bestämma beteendet hos varje enskild värme/kyla-enheten, mycket svåra att genomföra. Denna avhandling ger en lösning på detta problem, nämligen en styrlag som rentav kan anges på en enkel matematisk form och som endast kräver att en värme/kyla-enhet kommunicerar med enheterna i angränsande rum. Det behövs alltså inte ett beräkningsprogram för att beräkna lösningen till problemet. Samtidigt krävs det endast begränsat med kommunikation för att styrlagen ska kunna användas.

Designen av en reglerteknisk styrlag är beroende av hur man definierar systemets prestanda, vilket kan göras på många olika sätt. I detta arbete har styrlagen designats så att systemet kan hantera värsta möjliga störningar. Styrlagen kan alltså reglera systemet så att det beter sig på önskat sätt även när det utsätts för dessa störningar.

Avhandlingen undersöker egentligen en klass av matematiska modeller till vilken modellen för temperaturreglering tillhör. Andra möjliga tillämpningar är bevattningssystem och elektriska nätverk. Det visar sig att det matematiska designproblemet för system i denna klass kan förenklas. I själva verket visar det sig att man endast behöver ta hänsyn till systemets mest dominanta beteende, vilket förenklar problemet.

Styrlagen visar sig ha ett flertal fördelaktiga egenskaper för just storskaliga tillämpningar. Exempelvis tar den hänsyn till de restriktioner som finns på informationsdelning mellan olika delar av systemet, likt i fallet med temperaturreglering. Den visar sig även vara ett värdefullt verktyg för reglering av fysikaliska system där storheten beror både på tid och rum. Ett exempel är temperaturdynamiken hos ett objekt, vilken beskrivs med hjälp av värmeledningsekvationen. Dessutom kan styrlagen användas som ett riktmärke vid utvecklingen av designmetoder för generella system.

För att vidareutveckla styrlagen som presenteras i denna avhandling är det viktigt att införa olinjär dynamik i analysen. Detta kan användas för att beskriva exempelvis maxvärden på styrsignaler, vilket i praktiken alltid existerar. Det är även av intresse att undersöka om styrlagar med liknande egenskaper kan bestämmas när prestandan hos systemet beskrivs på ett annorlunda sätt.