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On uniqueness and continuity for the quasi-linear, bianisotropic Maxwell equations, using an entropy condition

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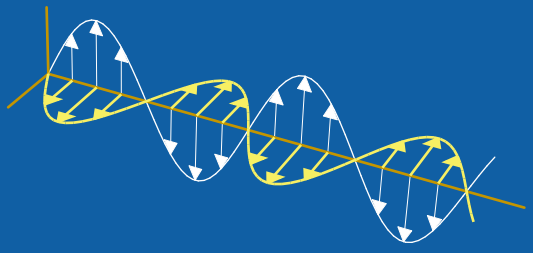

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Abstract

The quasi-linear Maxwell equations describing electromagnetic wave propagation in nonlinear media permit several weak solutions, which may be discontinuous (shock waves). It is often conjectured that the solutions are unique if they satisfy an additional entropy condition. The entropy condition states that the energy contained in the electromagnetic fields is irreversibly dissipated to other energy forms, which are not described by the Maxwell equations. We use the method employed by Kružkov to scalar conservation laws to analyze the implications of this additional condition in the electromagnetic case, *i.e.*, systems of equations in three dimensions. It is shown that if a certain term can be ignored, the solutions are unique.

1 Introduction

There are three classical questions regarding a given mathematical problem: does it have a solution, is the solution unique, and does the solution change only little when we perturb the data? When all these questions are answered in the positive, we say the problem is well posed, in the sense of Hadamard [9]. One reason for this statement is that these properties guarantee reproducible results from simulations. For instance, if the model permits multiple solutions, how can we be sure which one we are calculating and that it is physically relevant?

In this paper, we treat the questions of uniqueness and continuity for solutions of the Maxwell equations, when modeling nonlinear media. Using a technique developed by Kružkov [17] for scalar conservation laws, we study the consequences of postulating an additional condition to the Maxwell equations, known as the entropy condition. We show that if a certain term can be ignored, the solution is indeed unique and depends continuously on given data.

The Maxwell equations alone are not sufficient to describe wave propagation through a material. They must be supplemented by constitutive relations, modeling the interaction between the electromagnetic fields and the material. These relations are often linear, but for large field strengths it is necessary to include some nonlinear interactions as well. When the material reacts much faster than the typical time scale of the wave, we may assume an instantaneous model. In this case, the Maxwell equations takes the mathematical structure of a symmetric system of hyperbolic conservation laws. In our case, the key word in this classification is "hyperbolic" [6, p. 401], in the sense that we can diagonalize the system of equations into a system of weakly coupled, scalar transport equations, allowing wave solutions.

Nonlinear hyperbolic conservation laws have been extensively studied, mostly from the perspective of continuum mechanics and thermodynamics. Much of the early engineering work up to 1948 is reported in [3], and a recent survey of mainly the mathematical aspects of this field is given in [5]. A nice introduction to the numerical treatment as well as a summary of theoretical results is found in [7], and the subject is treated in text books on partial differential equations [6, 12, 30]. One of the key results is that these equations permit solutions which become discontinuous in finite time, even if the initial data is infinitely differentiable. This means we

cannot guarantee the existence of classical derivatives, and the solution must be interpreted in a weak sense, *e.g.*, as a distribution or a measure.

It is well known that weak solutions to nonlinear hyperbolic conservation laws are not necessarily unique, see *e.g.*, [6, p. 142]. One remedy to this problem is to define the hyperbolic conservation law as the limit of a parabolic equation, which has well-defined solutions. This is the technique of vanishing viscosity, and was first introduced in [10]. This programme has been quite successful, but some difficulties remain, especially for systems of equations in several space variables. However, it has been shown that if the limit can be suitably defined, the solution satisfies an *entropy condition*, which can be defined independently of the limit process. This entropy condition is well motivated from a modeling point of view, and can often be shown to be a means of selecting the unique, physically relevant solution. When uniqueness proofs fail, it is often conjectured that the entropy condition provides unique solutions [7, p. 32].

There are many kinds of entropy conditions. Probably the first was considered by Jouguet [14], followed by Oleĭnik's condition E for a scalar equation [25], which was later extended by Liu [22], and a similar condition for strictly hyperbolic systems was formulated by Lax [20]. These conditions essentially require that when the equations allow a discontinuous solution, the characteristics should cross each other, which can be interpreted as "loss of information" or increase of entropy. There are also conditions for systems of conservation laws which are directly linked to the physical entropy, especially in gas dynamics. From an energy conservation point of view, this can also be considered as the dissipation of the energy defined by the conservation law. Dafermos has proposed an entropy condition requiring this dissipation to be maximal [4].

This paper is organized as follows. In Section 2 we present the notation used in the paper and the constitutive relations leading to the formulation of the Maxwell equations as a symmetric system of hyperbolic, quasi-linear conservation laws. We postulate the entropy condition in Section 3, and discuss the relevant interpretation of this condition. In Section 4 we treat the questions of uniqueness and continuous dependence on data for our solution using the technique of "doubling the variables" introduced by Kružkov for a scalar conservation law in [17]. We conclude by giving an explicit example in Section 5 of a situation where the Maxwell equations alone permit two solutions, and use the entropy condition to choose the relevant one. Some final remarks are made in Section 6.

2 The quasi-linear Maxwell equations

In this paper we use a slight modification of the Heaviside-Lorentz units for our fields [13, p. 781], *i.e.*, all electromagnetic fields are scaled to units of $\sqrt{\text{energy/volume}}$,

$$\begin{cases} \boldsymbol{E} = \sqrt{\epsilon_0}\boldsymbol{E}_{\text{SI}} \\ \boldsymbol{H} = \sqrt{\mu_0}\boldsymbol{H}_{\text{SI}}, \end{cases} \qquad \begin{cases} \boldsymbol{D} = 1/\sqrt{\epsilon_0}\boldsymbol{D}_{\text{SI}} \\ \boldsymbol{B} = 1/\sqrt{\mu_0}\boldsymbol{B}_{\text{SI}}, \end{cases} \qquad \boldsymbol{J} = \sqrt{\mu_0}\boldsymbol{J}_{\text{SI}}, \tag{2.1}$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ is the electric and magnetic field strength, respectively, and $\boldsymbol{D}$ and $\boldsymbol{B}$ is the electric and magnetic flux density, respectively, and $\boldsymbol{J}$ is the electric current density. The index SI is used to indicate the field in SI units. We use the scaled time $t = c_0 t_{\mathrm{SI}}$, where $c_0 = 1/\sqrt{\epsilon_0 \mu_0}$ is the speed of light in vacuum, and the constants ϵ_0 and μ_0 are the permittivity and permeability of free space, respectively. The six-vector notation from [8, 27], *i.e.*,

$$\boldsymbol{e} = \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}, \quad \boldsymbol{d} = \begin{pmatrix} \boldsymbol{D} \\ \boldsymbol{B} \end{pmatrix}, \quad \boldsymbol{j} = \begin{pmatrix} \boldsymbol{J} \\ \boldsymbol{0} \end{pmatrix} \quad \nabla \times \mathbf{J} = \begin{pmatrix} \mathbf{0} & -\nabla \times \mathbf{I} \\ \nabla \times \mathbf{I} & \mathbf{0} \end{pmatrix}, \tag{2.2}$$

enables us to write the Maxwell equations in the compact form

$$\nabla \times \mathbf{J}\boldsymbol{e} + \partial_t \boldsymbol{d} = -\boldsymbol{j}. \tag{2.3}$$

In this paper we treat the six-vectors as column vectors, *i.e.*, we write the scalar product as $\boldsymbol{e}^{\mathrm{T}}\boldsymbol{d} = \sum_{i=1}^{6} e_i d_i$. This is merely for notational convenience and does not capture the full mathematical structure, which is not needed here. For more ambitious attempts to construct a six-vector notation, we refer to [8, 21].

The Maxwell equations must be supplemented by a constitutive relation, whose purpose is to model the interaction of the electromagnetic field with the material. When the material reacts very fast to stimulance, we can model it with an instantaneous constitutive model, where the values of the electric flux density $\boldsymbol{D}$ and the magnetic flux density $\boldsymbol{B}$ are completely determined by the values of the electric field strength $\boldsymbol{E}$ and magnetic field strength $\boldsymbol{H}$ at the same point in spacetime. We write this as

$$\boldsymbol{d}(\boldsymbol{r}, t) = \boldsymbol{d}(\boldsymbol{e}(\boldsymbol{r}, t)), \tag{2.4}$$

where $\boldsymbol{d}(\boldsymbol{e})$ is the gradient of a scalar function $\phi(\boldsymbol{e})$ with respect to $\boldsymbol{e}$, *i.e.*, in terms of thermodynamics, the field gradient of the thermodynamic potential (or the free energy density or the free enthalpy density) [2, 18]. We use the notation $\boldsymbol{d}(\boldsymbol{e}) = \phi'(\boldsymbol{e})$ to denote this derivative, *i.e.*, $d_i(\boldsymbol{e}) = \partial\phi/\partial e_i$, $i = 1, \ldots, 6$. The model is passive if we require that the symmetric 6×6 matrix $\boldsymbol{d}'(\boldsymbol{e}) = \phi''(\boldsymbol{e})$, where $[\boldsymbol{d}'(\boldsymbol{e})]_{ij} = \partial^2\phi/\partial e_i \partial e_j$, is a positive definite matrix, which is the case if the scalar function $\phi(\boldsymbol{e})$ is a convex function.

The Maxwell equations with an instantaneously reacting constitutive model is

$$\nabla \times \mathbf{J}\boldsymbol{e} + \boldsymbol{d}'(\boldsymbol{e})\partial_t \boldsymbol{e} = -\boldsymbol{j}, \tag{2.5}$$

and since $\boldsymbol{d}'(\boldsymbol{e})$ is positive definite and symmetric, this is by definition a quasi-linear, symmetric, hyperbolic system of partial differential equations [30, p. 360]. The source free version of this system has been extensively studied in [27], where it is shown that the equations in general support two waves, the ordinary and the extraordinary wave, each with its own refractive index.

Due to the quasi-linearity, the system (2.5) may exhibit shock solutions, *i.e.*, even if we give smooth data, the solution becomes discontinuous in finite time. This

means the derivatives cannot be classically defined everywhere, but we can make a weak formulation of the problem by requiring the equality

$$\int_{\mathbb{R}} \int_{\mathbb{R}^3} [-\boldsymbol{e}^{\mathrm{T}} \nabla \times \mathbf{J} \boldsymbol{\varphi} - \boldsymbol{d}(\boldsymbol{e})^{\mathrm{T}} \partial_t \boldsymbol{\varphi} + \boldsymbol{j}^{\mathrm{T}} \boldsymbol{\varphi}] \, \mathrm{d}V \, \mathrm{d}t = 0 \tag{2.6}$$

to hold for all six-vector test functions $\boldsymbol{\varphi}$ defined on $\mathbb{R}^3 \times \mathbb{R}$, *i.e.*, vector-valued functions which are infinitely differentiable with compact support. We do not consider static fields in this paper, *i.e.*, if $\boldsymbol{j} = \mathbf{0}$ for $t < 0$, then $\boldsymbol{e} = \mathbf{0}$ for $t < 0$.

One problem with the weak formulation is that we lose uniqueness, *i.e.*, there are several weak solutions $\boldsymbol{e}$ which satisfy the above criteria. In the following section we present an entropy condition which guarantees uniqueness of the weak solutions.

3 The entropy condition

When the solutions to (2.5) are smooth, we can derive an equation representing the conservation of energy. First, we note the identities

$$\begin{cases} \boldsymbol{e}^{\mathrm{T}} \nabla \times \mathbf{J} \boldsymbol{e} = \nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) = \nabla \cdot \boldsymbol{S}(\boldsymbol{e}) \\ \boldsymbol{e}^{\mathrm{T}} \partial_t \boldsymbol{d}(\boldsymbol{e}) = \partial_t (\boldsymbol{e}^{\mathrm{T}} \boldsymbol{d}(\boldsymbol{e}) - \phi(\boldsymbol{e})) = \partial_t \eta(\boldsymbol{e}), \end{cases} \tag{3.1}$$

where the last identity follows from $\boldsymbol{d}(\boldsymbol{e}) = \phi'(\boldsymbol{e})$. The vector $\boldsymbol{S}(\boldsymbol{e})$ is the Poynting vector, and the scalar, convex function $\eta(\boldsymbol{e})$ is the electromagnetic energy density. Multiplying (2.5) with $\boldsymbol{e}^{\mathrm{T}}$ now implies the Poynting theorem (conservation of energy)

$$\nabla \cdot \boldsymbol{S}(\boldsymbol{e}) + \partial_t \eta(\boldsymbol{e}) = -\boldsymbol{e}^{\mathrm{T}} \boldsymbol{j}. \tag{3.2}$$

When the solutions to (2.5) are not smooth, this equation is no longer valid since the derivatives are not defined. We propose to replace it with the inequality

$$\boxed{\nabla \cdot \boldsymbol{S}(\boldsymbol{e}) + \partial_t \eta(\boldsymbol{e}) \leq -\boldsymbol{e}^{\mathrm{T}} \boldsymbol{j},} \tag{3.3}$$

which is interpreted in a weak sense, *i.e.*, for all nonnegative test functions φ, the inequality

$$\int_{\mathbb{R}} \int_{\mathbb{R}^3} [-\boldsymbol{S}(\boldsymbol{e}) \cdot \nabla \varphi - \eta(\boldsymbol{e}) \partial_t \varphi + \boldsymbol{e}^{\mathrm{T}} \boldsymbol{j} \varphi] \, \mathrm{d}V \, \mathrm{d}t \leq 0 \tag{3.4}$$

holds. The inequality (3.3) is called the *entropy condition*, and is here postulated in addition to the Maxwell equations. Observe that since (3.3) is postulated and interpreted in the weak sense, it is valid for non-smooth solutions. The pair of functions $\eta(\boldsymbol{e})$ and $\boldsymbol{S}(\boldsymbol{e})$ are known in the mathematical literature as an entropy/entropy-flux pair, see *e.g.*, [6, pp. 604–611], [12, pp. 70–71], and [30, pp. 436–445]. The existence of such a pair is nontrivial in the general case, and is a special property of the system. Similar conditions are often present for systems of nonlinear conservation laws, such as the equations governing gas dynamics, see *e.g.*, [7, pp. 21–35] and [4, 19, 22].

3.1 Why the term "entropy"?

It is quite obvious that no constitutive relation can capture all of the physical processes which occur when electromagnetic waves interact with matter. There is always some interaction that is left out, and if we choose not to model it, we must assume that the electromagnetic energy used in the interaction is lost in an irreversible process. If the process were not irreversible, we would have to include it in our equations if the equations are supposed to be realistic. Since the electromagnetic energy is lost, it must be a nonincreasing function of time (except for the energy fed to the system), which is the essence of the entropy inequality (3.3). Namely, we can choose a suitable sequence of test functions $\{\varphi\}$ converging to a function constant on $\mathbb{R}^3 \times [t_1, t_2]$ to find[1]

$$\int_{\mathbb{R}^3} \eta(\boldsymbol{e})\,\mathrm{d}V\bigg|_{t=t_2} \leq \int_{\mathbb{R}^3} \eta(\boldsymbol{e})\,\mathrm{d}V\bigg|_{t=t_1} - \int_{t_1}^{t_2}\int_{\mathbb{R}^3} \boldsymbol{e}^{\mathrm{T}}\boldsymbol{j}\,\mathrm{d}V\,\mathrm{d}t, \tag{3.5}$$

when all the integrals are defined. For active sources $\boldsymbol{j}$, *i.e.*, sources which radiate electromagnetic energy, the term $-\int_{t_1}^{t_2}\int_{\mathbb{R}^3} \boldsymbol{e}^{\mathrm{T}}\boldsymbol{j}\,\mathrm{d}V\,\mathrm{d}t$ is positive when $t_1 < t_2$ and represents the energy fed to the system.

The inequality sign in (3.5) represents the fact that there is a loss of electromagnetic energy with increasing time. The irreversible processes that are not modeled by the Maxwell equations can be represented with an energy density term TS, where T is the temperature and S is the entropy density. The first law of thermodynamics states that the total internal energy

$$\mathcal{U} = \int_{\mathbb{R}^3} (\eta(\boldsymbol{e}) + TS)\,\mathrm{d}V \tag{3.6}$$

is constant for an isolated system (no exchange of heat or work, *i.e.*, for time intervals when $\boldsymbol{j} = \boldsymbol{0}$). If the integral of the electromagnetic energy is nonincreasing, it then follows that the integral of TS must be nondecreasing. This means the entropy must be nondecreasing under isothermal conditions, which is consistent with the second law of thermodynamics. This shows that the term "entropy condition" is justified for electromagnetic waves, if we interpret entropy as a representation of the dissipative processes not modeled by the Maxwell equations. Thus, we think of entropy as "missing information" about the system. For a further discussion on the interpretation of entropy, we refer to [29].

3.2 The entropy condition for vanishing viscosity solutions

We have previously postulated the entropy condition in addition to the Maxwell equations. The question may be raised if there exists solutions that satisfy both these criteria. There is at present no definite answer to this question, but we can show that if we make a parabolic regularization of the Maxwell equations,

$$\nabla \times \mathbf{J}\boldsymbol{e}_\delta + \partial_t \boldsymbol{d}(\boldsymbol{e}_\delta) = -\boldsymbol{j} + \delta\nabla^2\boldsymbol{e}_\delta, \tag{3.7}$$

[1]This procedure is performed in detail in Section 4.2.

and the solution $\boldsymbol{e}_\delta$ is uniformly bounded in the supremum norm and converges almost everywhere to $\boldsymbol{e}$ as $\delta \to 0$, this limit solution satisfies the entropy condition (3.3), see [7, p. 27] and [30, p. 438]. This method of constructing solutions to quasi-linear hyperbolic equations is called the vanishing viscosity method, and is a standard method in partial differential equation theory. It can be shown that for each $\delta > 0$, the initial value problem for (3.7) is well posed with solutions infinitely differentiable in the interior domain and continuous on the boundary [30, p. 338]. For similar systems of equations in one spatial dimension and scalar equations in several dimensions, there exists a limiting function as $\delta \to 0$. However, there are still some questions regarding the convergence for systems of equations in several space dimensions, that have not been resolved, and we can only conjecture the existence of a limit $\boldsymbol{e} = \lim_{\delta \to 0} \boldsymbol{e}_\delta$, see [26] and [7, p. 32].

The parabolic equation (3.7) is actually the equivalent[2] (or modified) equation corresponding to certain numerical schemes used to solve hyperbolic equations, *e.g.*, the Lax-Friedrichs scheme in one spatial dimension [7, p. 181]. The viscosity parameter δ is then typically of order $(\Delta x)^2/\Delta t$, where Δx and Δt is the discretization in space and time, respectively, implying $\delta \to 0$ as the discretization is refined. In this context, the entropy is a measure of what goes on on a finer scale than we are observing, *i.e.*, on a scale of order δ.

4 Kružkov's method for entropy solutions

In this section we use a method due to Kružkov (see [17] and [6, pp. 608–611]) to study uniqueness and continuous dependence on data for solutions which satisfy the Maxwell equations as well as an entropy condition,

$$\begin{cases} \nabla \times \mathbf{J}\boldsymbol{e} + \partial_t \boldsymbol{d}(\boldsymbol{e}) = -\boldsymbol{j} \\ \nabla \cdot \boldsymbol{S}(\boldsymbol{e}) + \partial_t \eta(\boldsymbol{e}) \leq -\boldsymbol{e}^{\mathrm{T}} \boldsymbol{j}, \end{cases} \tag{4.1}$$

where $\boldsymbol{S}(\boldsymbol{e}) = \boldsymbol{E} \times \boldsymbol{H}$ and $\eta(\boldsymbol{e}) = \boldsymbol{e}^{\mathrm{T}} \boldsymbol{d}(\boldsymbol{e}) - \phi(\boldsymbol{e})$. The idea is to study the difference between the energies for two potentially different solutions, slightly perturbed in space and time. This enables us to obtain an inequality similar to (3.5), but with a new energy which is zero only when the two solutions are equal almost everywhere. However, the inequality also comprises a term which eludes further analysis. This is further commented at the end of Section 4.1.

Suppose we have two solutions, $\boldsymbol{e}$ and $\tilde{\boldsymbol{e}}$, satisfying

$$\begin{cases} \nabla_{\boldsymbol{x}} \times \mathbf{J}\boldsymbol{e} + \partial_t \boldsymbol{d}(\boldsymbol{e}) = -\boldsymbol{j} \\ \nabla_{\boldsymbol{x}} \cdot \boldsymbol{S}(\boldsymbol{e}) + \partial_t \eta(\boldsymbol{e}) \leq -\boldsymbol{e}^{\mathrm{T}} \boldsymbol{j}, \end{cases} \quad \text{and} \quad \begin{cases} \nabla_{\boldsymbol{y}} \times \mathbf{J}\tilde{\boldsymbol{e}} + \partial_s \boldsymbol{d}(\tilde{\boldsymbol{e}}) = -\tilde{\boldsymbol{\jmath}} \\ \nabla_{\boldsymbol{y}} \cdot \boldsymbol{S}(\tilde{\boldsymbol{e}}) + \partial_s \eta(\tilde{\boldsymbol{e}}) \leq -\tilde{\boldsymbol{e}}^{\mathrm{T}} \tilde{\boldsymbol{\jmath}}, \end{cases} \tag{4.2}$$

where $\boldsymbol{j}$ and $\tilde{\boldsymbol{\jmath}}$ may be different. Note that we have labeled the independent variables differently for the two solutions, *i.e.*, $\boldsymbol{e} = \boldsymbol{e}(\boldsymbol{x}, t)$, $\boldsymbol{j} = \boldsymbol{j}(\boldsymbol{x}, t)$, $\tilde{\boldsymbol{e}} = \tilde{\boldsymbol{e}}(\boldsymbol{y}, s)$ and

[2] For a numerical scheme of order n approximating a given equation, the equivalent (or modified) equation is defined as the equation which is approximated to order $n + 1$ by the scheme.

$\tilde{\boldsymbol{j}} = \tilde{\boldsymbol{j}}(\boldsymbol{y}, s)$. This is helpful when handling the differential operators in the following. We add[3] $-\tilde{\boldsymbol{e}}^{\mathrm{T}}(\nabla_{\boldsymbol{x}} \times \mathbf{J}\boldsymbol{e} + \partial_t \boldsymbol{d}(\boldsymbol{e}) + \boldsymbol{j}) + \tilde{\boldsymbol{e}}^{\mathrm{T}} \nabla_{\boldsymbol{x}} \times \mathbf{J}\tilde{\boldsymbol{e}} + \partial_t \phi(\tilde{\boldsymbol{e}}) = 0$ to the entropy condition for $\boldsymbol{e}$, which implies

$$\begin{aligned} 0 &\geq \nabla_{\boldsymbol{x}} \cdot \boldsymbol{S}(\boldsymbol{e}) + \partial_t \eta(\boldsymbol{e}) + \boldsymbol{e}^{\mathrm{T}} \boldsymbol{j} - \tilde{\boldsymbol{e}}^{\mathrm{T}}(\nabla_{\boldsymbol{x}} \times \mathbf{J}\boldsymbol{e} + \partial_t \boldsymbol{d}(\boldsymbol{e}) + \boldsymbol{j}) + \tilde{\boldsymbol{e}}^{\mathrm{T}} \nabla_{\boldsymbol{x}} \times \mathbf{J}\tilde{\boldsymbol{e}} + \partial_t \phi(\tilde{\boldsymbol{e}}) \\ &= \nabla_{\boldsymbol{x}} \cdot \boldsymbol{S}(\boldsymbol{e} - \tilde{\boldsymbol{e}}) + \partial_t \eta(\boldsymbol{e}, \tilde{\boldsymbol{e}}) + (\boldsymbol{e} - \tilde{\boldsymbol{e}})^{\mathrm{T}} \boldsymbol{j}, \end{aligned} \tag{4.3}$$

where

$$\eta(\boldsymbol{e}, \tilde{\boldsymbol{e}}) = (\boldsymbol{e} - \tilde{\boldsymbol{e}})^{\mathrm{T}} \boldsymbol{d}(\boldsymbol{e}) - \phi(\boldsymbol{e}) + \phi(\tilde{\boldsymbol{e}}). \tag{4.4}$$

Note that the terms $\tilde{\boldsymbol{e}}^{\mathrm{T}} \nabla_{\boldsymbol{x}} \times \mathbf{J}\tilde{\boldsymbol{e}}$ and $\partial_t \phi(\tilde{\boldsymbol{e}})$ are identically zero since $\tilde{\boldsymbol{e}}$ does not depend on $\boldsymbol{x}$ or t, and are included in order to obtain better symmetry in the inequality (4.3). Repeating the procedure for the set of equations with independent variables $(\boldsymbol{y}, s)$, results in the entropy inequalities

$$\begin{cases} \nabla_{\boldsymbol{x}} \cdot \boldsymbol{S}(\boldsymbol{e} - \tilde{\boldsymbol{e}}) + \partial_t \eta(\boldsymbol{e}, \tilde{\boldsymbol{e}}) + (\boldsymbol{e} - \tilde{\boldsymbol{e}})^{\mathrm{T}} \boldsymbol{j} \leq 0 \\ \nabla_{\boldsymbol{y}} \cdot \boldsymbol{S}(\tilde{\boldsymbol{e}} - \boldsymbol{e}) + \partial_s \eta(\tilde{\boldsymbol{e}}, \boldsymbol{e}) + (\tilde{\boldsymbol{e}} - \boldsymbol{e})^{\mathrm{T}} \tilde{\boldsymbol{j}} \leq 0. \end{cases} \tag{4.5}$$

These inequalities are interpreted in a weak sense, *i.e.*, they are defined through their effect on nonnegative test functions. Thus, the inequalities

$$\begin{cases} \iiiint \Big\{ -\boldsymbol{S}(\boldsymbol{e} - \tilde{\boldsymbol{e}}) \cdot \nabla_{\boldsymbol{x}} \varphi - \eta(\boldsymbol{e}, \tilde{\boldsymbol{e}}) \partial_t \varphi + (\boldsymbol{e} - \tilde{\boldsymbol{e}})^{\mathrm{T}} \boldsymbol{j} \varphi \Big\} \, \mathrm{d}V(\boldsymbol{x}) \, \mathrm{d}V(\boldsymbol{y}) \, \mathrm{d}t \, \mathrm{d}s \leq 0 \\ \iiiint \Big\{ -\boldsymbol{S}(\tilde{\boldsymbol{e}} - \boldsymbol{e}) \cdot \nabla_{\boldsymbol{y}} \varphi - \eta(\tilde{\boldsymbol{e}}, \boldsymbol{e}) \partial_s \varphi + (\tilde{\boldsymbol{e}} - \boldsymbol{e})^{\mathrm{T}} \tilde{\boldsymbol{j}} \varphi \Big\} \, \mathrm{d}V(\boldsymbol{x}) \, \mathrm{d}V(\boldsymbol{y}) \, \mathrm{d}t \, \mathrm{d}s \leq 0, \end{cases} \tag{4.6}$$

must hold for all test functions $\varphi(\boldsymbol{x}, \boldsymbol{y}, t, s) \geq 0$. The integrations are performed over $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}$, but we suppress the integration limits in order to simplify the notation. We now observe the symmetry $\boldsymbol{S}(\boldsymbol{e} - \tilde{\boldsymbol{e}}) = \boldsymbol{S}(\tilde{\boldsymbol{e}} - \boldsymbol{e})$, and expand $\partial_t \varphi$ and $\partial_s \varphi$ as $\partial_t \varphi = \frac{1}{2}\partial_t \varphi + \frac{1}{2}\partial_s \varphi + \frac{1}{2}\partial_t \varphi - \frac{1}{2}\partial_s \varphi$ and $\partial_s \varphi = \frac{1}{2}\partial_s \varphi + \frac{1}{2}\partial_t \varphi + \frac{1}{2}\partial_s \varphi - \frac{1}{2}\partial_t \varphi$. After adding the inequalities we obtain

$$\begin{aligned} 0 \geq \iiiint \Big\{ &-\boldsymbol{S}(\boldsymbol{e} - \tilde{\boldsymbol{e}}) \cdot (\nabla_{\boldsymbol{x}} \varphi + \nabla_{\boldsymbol{y}} \varphi) - \frac{1}{2}[\eta(\boldsymbol{e}, \tilde{\boldsymbol{e}}) + \eta(\tilde{\boldsymbol{e}}, \boldsymbol{e})](\partial_t \varphi + \partial_s \varphi) \\ &- \frac{1}{2}[\eta(\boldsymbol{e}, \tilde{\boldsymbol{e}}) - \eta(\tilde{\boldsymbol{e}}, \boldsymbol{e})](\partial_t \varphi - \partial_s \varphi) + (\boldsymbol{e} - \tilde{\boldsymbol{e}})^{\mathrm{T}}(\boldsymbol{j} - \tilde{\boldsymbol{j}}) \varphi \Big\} \, \mathrm{d}V(\boldsymbol{x}) \, \mathrm{d}V(\boldsymbol{y}) \, \mathrm{d}t \, \mathrm{d}s, \end{aligned} \tag{4.7}$$

which is the general expression with an arbitrary test function φ. To proceed with the analysis, we now choose a special test function which somewhat simplifies this inequality.

[3]This distribution is well-defined, since $\tilde{\boldsymbol{e}}(\boldsymbol{y}, s)$ does not depend on $\boldsymbol{x}$ or t. This implies that the derivatives operate only on the test function.

4.1 Choosing the proper test function

Following Kružkov's classical uniqueness proof, we employ the special test function

$$\varphi(\boldsymbol{x},\boldsymbol{y},t,s) = J_\delta^{(3)}\left(\frac{\boldsymbol{x}-\boldsymbol{y}}{2}\right) J_\delta\left(\frac{t-s}{2}\right)\psi\left(\frac{\boldsymbol{x}+\boldsymbol{y}}{2},\frac{t+s}{2}\right), \tag{4.8}$$

where J_δ is a nonnegative mollifier, having unit integral and converging to the Dirac measure as $\delta \to 0$. The mollifier in space $J_\delta^{(3)}$ can be written as the product $J_\delta^{(3)}(\boldsymbol{x}) = J_\delta(x_1)J_\delta(x_2)J_\delta(x_3)$. Since the support of the mollifiers shrinks to zero when $\delta \to 0$, this choice of test function brings the variables $\boldsymbol{x}$ and $\boldsymbol{y}$, and t and s, respectively, close to each other as the parameter $\delta \to 0$. This is similar to restricting the tensor product between two distributions to the diagonal, see [11].

Introducing the new variables

$$\bar{\boldsymbol{x}} = \frac{\boldsymbol{x}+\boldsymbol{y}}{2}, \quad \bar{\boldsymbol{y}} = \frac{\boldsymbol{x}-\boldsymbol{y}}{2}, \quad \bar{t} = \frac{t+s}{2}, \quad \bar{s} = \frac{t-s}{2}, \tag{4.9}$$

the inequality (4.7) is written

$$\iiiint \left\{-\boldsymbol{S}(\boldsymbol{e}-\tilde{\boldsymbol{e}})\cdot\nabla_{\bar{\boldsymbol{x}}}\psi - \frac{1}{2}[\eta(\boldsymbol{e},\tilde{\boldsymbol{e}})+\eta(\tilde{\boldsymbol{e}},\boldsymbol{e})]\partial_{\bar{t}}\psi + (\boldsymbol{e}-\tilde{\boldsymbol{e}})^{\mathrm{T}}(\boldsymbol{j}-\tilde{\boldsymbol{j}})\psi\right\}$$
$$J_\delta^{(3)} J_\delta \,\mathrm{d}V(\bar{\boldsymbol{x}})\,\mathrm{d}V(\bar{\boldsymbol{y}})\,\mathrm{d}\bar{t}\,\mathrm{d}\bar{s}$$
$$\leq \iiiint \frac{1}{2}[\eta(\boldsymbol{e},\tilde{\boldsymbol{e}})-\eta(\tilde{\boldsymbol{e}},\boldsymbol{e})] J_\delta^{(3)} J_\delta'\psi \,\mathrm{d}V(\bar{\boldsymbol{x}})\,\mathrm{d}V(\bar{\boldsymbol{y}})\,\mathrm{d}\bar{t}\,\mathrm{d}\bar{s}. \tag{4.10}$$

The explicit expression for the energy term $\frac{1}{2}[\eta(\boldsymbol{e},\tilde{\boldsymbol{e}})+\eta(\tilde{\boldsymbol{e}},\boldsymbol{e})]$ is found from the definition of $\eta(\boldsymbol{e},\tilde{\boldsymbol{e}})$,

$$\frac{1}{2}[\eta(\boldsymbol{e},\tilde{\boldsymbol{e}})+\eta(\tilde{\boldsymbol{e}},\boldsymbol{e})] = \frac{1}{2}(\boldsymbol{e}-\tilde{\boldsymbol{e}})^{\mathrm{T}}(\boldsymbol{d}(\boldsymbol{e})-\boldsymbol{d}(\tilde{\boldsymbol{e}})), \tag{4.11}$$

and in Appendix A it is shown that we can introduce a third rank tensor $K_{ijk}(\boldsymbol{e},\tilde{\boldsymbol{e}})$, defined in (A.8), to write

$$\frac{1}{2}[\eta(\boldsymbol{e},\tilde{\boldsymbol{e}})-\eta(\tilde{\boldsymbol{e}},\boldsymbol{e})] = K_{ijk}(\boldsymbol{e},\tilde{\boldsymbol{e}})(e_i-\tilde{e}_i)(e_j-\tilde{e}_j)(e_k-\tilde{e}_k), \tag{4.12}$$

where summation over repeated indices is assumed. As seen in Appendix A, the tensor K_{ijk} is related to the third derivative of the thermodynamic potential, ϕ'''. Since ϕ is a quadratic function for linear materials, we see that this term must be due to the nonlinearity of our constitutive relation.

The explicit form of the entropy inequality (4.7) is thus

$$\iiiint \left\{-\boldsymbol{S}(\boldsymbol{e}-\tilde{\boldsymbol{e}})\cdot\nabla_{\bar{\boldsymbol{x}}}\psi - \frac{1}{2}(\boldsymbol{e}-\tilde{\boldsymbol{e}})^{\mathrm{T}}(\boldsymbol{d}(\boldsymbol{e})-\boldsymbol{d}(\tilde{\boldsymbol{e}}))\partial_{\bar{t}}\psi + (\boldsymbol{e}-\tilde{\boldsymbol{e}})^{\mathrm{T}}(\boldsymbol{j}-\tilde{\boldsymbol{j}})\psi\right\}$$
$$J_\delta^{(3)} J_\delta \,\mathrm{d}V(\bar{\boldsymbol{x}})\,\mathrm{d}V(\bar{\boldsymbol{y}})\,\mathrm{d}\bar{t}\,\mathrm{d}\bar{s}$$
$$\leq \iiiint K_{ijk}(\boldsymbol{e},\tilde{\boldsymbol{e}})(e_i-\tilde{e}_i)(e_j-\tilde{e}_j)(e_k-\tilde{e}_k) J_\delta^{(3)} J_\delta'\psi \,\mathrm{d}V(\bar{\boldsymbol{x}})\,\mathrm{d}V(\bar{\boldsymbol{y}})\,\mathrm{d}\bar{t}\,\mathrm{d}\bar{s}, \tag{4.13}$$

where no approximations are made so far. It is conjectured that the term on the right hand side of the inequality is negligible, since it is cubic in the difference $\boldsymbol{e}-\tilde{\boldsymbol{e}}$ and should therefore be small compared to the other terms when $|\boldsymbol{e}-\tilde{\boldsymbol{e}}|$ is small. However, the differentiated mollifier J_δ' could change this assumption. It should be noted that in the case of a scalar conservation law, which Kružkov studied, it is possible to choose the functions corresponding to $\boldsymbol{S}$ and η such that this term does not appear. To see why it is desirable to obtain control over this term, we spend the following subsections showing that this implies that our solutions are unique and depend continuously on data.

4.2 Uniqueness and continuous dependence on data

If we assume the term on the right hand side of (4.13) can be replaced with zero, we are free to take the limit $\delta \to 0$ in the mollifiers since the terms inside the curly brackets are summable over $\bar{\boldsymbol{x}}$ and $\bar{t}$. This implies that the integrals over $\bar{\boldsymbol{y}}$ and $\bar{s}$ only contribute when $(\bar{\boldsymbol{y}}, \bar{s})$ is close to $(\boldsymbol{0}, 0)$, which implies $\bar{\boldsymbol{x}} \approx \boldsymbol{x} \approx \boldsymbol{y}$ and $\bar{t} \approx t \approx s$. Hence the limit $\delta \to 0$ provides the inequality

$$\iint \Big\{ -\boldsymbol{S}(\boldsymbol{e}-\tilde{\boldsymbol{e}}) \cdot \nabla_{\bar{\boldsymbol{x}}}\psi - \frac{1}{2}(\boldsymbol{e}-\tilde{\boldsymbol{e}})^{\mathrm{T}}(\boldsymbol{d}(\boldsymbol{e})-\boldsymbol{d}(\tilde{\boldsymbol{e}}))\partial_{\bar{t}}\psi + (\boldsymbol{e}-\tilde{\boldsymbol{e}})^{\mathrm{T}}(\boldsymbol{j}-\tilde{\boldsymbol{j}})\psi \Big\} \,\mathrm{d}V(\bar{\boldsymbol{x}})\,\mathrm{d}\bar{t} \leq 0, \quad (4.14)$$

and from this point on we use the variables $\bar{\boldsymbol{x}}$ and $\bar{t}$ to emphasize that they are the mean values of the variables $\boldsymbol{x}$ and $\boldsymbol{y}$, and t and s, respectively. Following [6, pp. 608–611] we choose the test function $\psi(\bar{\boldsymbol{x}}, \bar{t}) = \alpha(\bar{\boldsymbol{x}})\beta(\bar{t})$ according to

$$\begin{cases} \alpha : \mathbb{R}^3 \to \mathbb{R} \quad \text{is smooth,} \\ \alpha(\bar{\boldsymbol{x}}) = 1 \quad \text{if} \quad |\bar{\boldsymbol{x}}| \leq r, \\ \alpha(\bar{\boldsymbol{x}}) = 0 \quad \text{if} \quad |\bar{\boldsymbol{x}}| \geq r + r_0, \\ |\nabla_{\bar{\boldsymbol{x}}}\alpha(\bar{\boldsymbol{x}})| \leq 2/r_0, \end{cases} \quad (4.15)$$

and

$$\begin{cases} \beta : \mathbb{R} \to \mathbb{R} \text{ is Lipschitz continuous,} \\ \beta(\bar{t}) = 0 \quad \text{if} \quad \bar{t} \leq t_1 \text{ or } \bar{t} \geq t_2 + \Delta t, \\ \beta(\bar{t}) = 1 \quad \text{if} \quad t_1 + \Delta t \leq \bar{t} \leq t_2, \\ \beta \text{ is linear on } [t_1, t_1 + \Delta t] \text{ and } [t_2, t_2 + \Delta t], \end{cases} \quad (4.16)$$

where Δt satisfies $0 < \Delta t < t_2 - t_1$. Strictly speaking, β is not a test function, but we can use a suitable sequence of proper test functions to construct this limit. Our

inequality is now written

$$\begin{aligned}\frac{1}{\Delta t}\int_{t_2}^{t_2+\Delta t}\frac{1}{2}\int_{\mathbb{R}^3}(\boldsymbol{e}-\tilde{\boldsymbol{e}})^{\mathrm{T}}(\boldsymbol{d}(\boldsymbol{e})-\boldsymbol{d}(\tilde{\boldsymbol{e}}))\alpha(\bar{\boldsymbol{x}})\,\mathrm{d}V(\bar{\boldsymbol{x}})\,\mathrm{d}\bar{t}\\ +\int_{t_1}^{t_2+\Delta t}\int_{r<|\bar{\boldsymbol{x}}|<r+r_0}\boldsymbol{S}(\boldsymbol{e}-\tilde{\boldsymbol{e}})\cdot\nabla_{\bar{\boldsymbol{x}}}\alpha(\bar{\boldsymbol{x}})\beta(\bar{t})\,\mathrm{d}V(\bar{\boldsymbol{x}})\,\mathrm{d}\bar{t}\\ \leq\frac{1}{\Delta t}\int_{t_1}^{t_1+\Delta t}\frac{1}{2}\int_{\mathbb{R}^3}(\boldsymbol{e}-\tilde{\boldsymbol{e}})^{\mathrm{T}}(\boldsymbol{d}(\boldsymbol{e})-\boldsymbol{d}(\tilde{\boldsymbol{e}}))\alpha(\bar{\boldsymbol{x}})\,\mathrm{d}V(\bar{\boldsymbol{x}})\,\mathrm{d}\bar{t}\\ -\int_{t_1}^{t_2+\Delta t}\int_{\mathbb{R}^3}(\boldsymbol{e}-\tilde{\boldsymbol{e}})^{\mathrm{T}}(\boldsymbol{j}-\tilde{\boldsymbol{j}})\alpha(\bar{\boldsymbol{x}})\beta(\bar{t})\,\mathrm{d}V(\bar{\boldsymbol{x}})\,\mathrm{d}\bar{t}\end{aligned}\tag{4.17}$$

and the integral containing $\boldsymbol{S}(\boldsymbol{e}-\tilde{\boldsymbol{e}})$ vanishes as $r\to\infty$ since $\boldsymbol{S}(\boldsymbol{e}-\tilde{\boldsymbol{e}})$ is a quadratic function of $\boldsymbol{e}-\tilde{\boldsymbol{e}}$ and $|\boldsymbol{e}|^2$ and $|\tilde{\boldsymbol{e}}|^2$ are integrable, which means the integral must disappear in this limit. We next let $\Delta t\to 0$ to deduce the fundamental energy estimate

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^3}(\boldsymbol{e}-\tilde{\boldsymbol{e}})^{\mathrm{T}}(\boldsymbol{d}(\boldsymbol{e})-\boldsymbol{d}(\tilde{\boldsymbol{e}}))\,\mathrm{d}V(\bar{\boldsymbol{x}})\bigg|_{\bar{t}=t_2}\leq\frac{1}{2}\int_{\mathbb{R}^3}(\boldsymbol{e}-\tilde{\boldsymbol{e}})^{\mathrm{T}}(\boldsymbol{d}(\boldsymbol{e})-\boldsymbol{d}(\tilde{\boldsymbol{e}}))\,\mathrm{d}V(\bar{\boldsymbol{x}})\bigg|_{\bar{t}=t_1}\\ -\int_{t_1}^{t_2}\int_{\mathbb{R}^3}(\boldsymbol{e}-\tilde{\boldsymbol{e}})^{\mathrm{T}}(\boldsymbol{j}-\tilde{\boldsymbol{j}})\,\mathrm{d}V(\bar{\boldsymbol{x}})\,\mathrm{d}\bar{t}\end{aligned}\tag{4.18}$$

for every pair $t_1<t_2$. We emphasize that this estimate was obtained by assuming that a term cubic in the difference $\boldsymbol{e}-\tilde{\boldsymbol{e}}$ could be ignored.

Our first question concerns the uniqueness of entropy solutions, *i.e.*, can different solutions be generated by the same data? From (4.18) we see the answer is negative. The currents may be assumed to start at a specific time, *i.e.*, $\boldsymbol{j}=\tilde{\boldsymbol{j}}=\boldsymbol{0}$ for $t\leq 0$, and causality implies $\boldsymbol{e}=\tilde{\boldsymbol{e}}=\boldsymbol{0}$ for $\bar{t}=0$. By choosing $t_2=T$ and $t_1=0$ and using the same currents for the two solutions, $\boldsymbol{j}=\tilde{\boldsymbol{j}}$ everywhere, we obtain

$$\frac{1}{2}\int_{\mathbb{R}^3}(\boldsymbol{e}-\tilde{\boldsymbol{e}})^{\mathrm{T}}(\boldsymbol{d}(\boldsymbol{e})-\boldsymbol{d}(\tilde{\boldsymbol{e}}))\,\mathrm{d}V(\bar{\boldsymbol{x}})\bigg|_{\bar{t}=T}\leq 0\tag{4.19}$$

for every finite time $T>0$. Assuming the model saturates for high field strengths, *i.e.*, $\boldsymbol{d}(\boldsymbol{e})$ can be bounded by a linear function of $\boldsymbol{e}$, there exists positive constants $C_<$ and $C_>$ such that

$$C_<|\boldsymbol{e}-\tilde{\boldsymbol{e}}|^2\leq(\boldsymbol{e}-\tilde{\boldsymbol{e}})^{\mathrm{T}}(\boldsymbol{d}(\boldsymbol{e})-\boldsymbol{d}(\tilde{\boldsymbol{e}}))\leq C_>|\boldsymbol{e}-\tilde{\boldsymbol{e}}|^2,\tag{4.20}$$

we see that (4.19) implies $\boldsymbol{e}=\tilde{\boldsymbol{e}}$ almost everywhere. Thus we have uniqueness.

Our second question concerns the continuous dependence of the solution on given data. We use Hölder's inequality to estimate the source term

$$\begin{aligned}-\int_0^T\int_{\mathbb{R}^3}(\boldsymbol{e}-\tilde{\boldsymbol{e}})^{\mathrm{T}}(\boldsymbol{j}-\tilde{\boldsymbol{j}})\,\mathrm{d}V(\bar{\boldsymbol{x}})\,\mathrm{d}\bar{t}\leq\int_0^T\int_{\mathbb{R}^3}|(\boldsymbol{e}-\tilde{\boldsymbol{e}})^{\mathrm{T}}(\boldsymbol{j}-\tilde{\boldsymbol{j}})|\,\mathrm{d}V(\bar{\boldsymbol{x}})\,\mathrm{d}\bar{t}\\ \leq\int_0^T\left(\int_{\mathbb{R}^3}|\boldsymbol{e}-\tilde{\boldsymbol{e}}|^2\,\mathrm{d}V(\bar{\boldsymbol{x}})\right)^{1/2}\left(\int_{\mathbb{R}^2}|\boldsymbol{j}-\tilde{\boldsymbol{j}})|^2\,\mathrm{d}V(\bar{\boldsymbol{x}})\right)^{1/2}\mathrm{d}\bar{t}.\end{aligned}\tag{4.21}$$

Using the notation $\|\boldsymbol{e}-\tilde{\boldsymbol{e}}\| = \left(\int_{\mathbb{R}^3} |\boldsymbol{e}-\tilde{\boldsymbol{e}}|^2 \,\mathrm{d}V\right)^{1/2}$ and (4.20), the estimate (4.18) implies

$$\frac{C_<}{2}\,\|\boldsymbol{e}-\tilde{\boldsymbol{e}}\|_{\bar{t}=T}^2 \le \int_0^T \|\boldsymbol{e}-\tilde{\boldsymbol{e}}\|\cdot\|\boldsymbol{j}-\tilde{\boldsymbol{j}}\|\,\mathrm{d}\bar{t}, \tag{4.22}$$

where we used $\|\boldsymbol{e}-\tilde{\boldsymbol{e}}\|_{\bar{t}=0}^2 = 0$. Since this inequality is valid for all $T>0$, the term on the left hand side can be replaced by its supremum. After dividing by $\sup_{\bar{t}\in[0,T]}\|\boldsymbol{e}-\tilde{\boldsymbol{e}}\|$, we find

$$\sup_{\bar{t}\in[0,T]}\|\boldsymbol{e}-\tilde{\boldsymbol{e}}\| \le \frac{2}{C_<}\int_0^T \|\boldsymbol{j}-\tilde{\boldsymbol{j}}\|\,\mathrm{d}\bar{t}, \tag{4.23}$$

for every $T>0$. This shows that the norm of the difference between two solutions is bounded by the norms of the difference between the difference between the sources. Thus we have continuous dependence of the solution on input data for each finite time T.

4.3 Initial/boundary-value problem

To simplify and streamline the presentation, the analysis so far has been for all of space and time, which means there are no initial or boundary values involved. In this subsection, we give a brief review on how to treat a finite region $\Omega\subset\mathbb{R}^3$ instead of all space. We also allow for initial values by making the following weak formulation of the Maxwell equations instead of (2.6),

$$\begin{aligned}\int_0^\infty\int_\Omega [-\boldsymbol{e}^{\mathrm{T}}\nabla\times\mathbf{J}\boldsymbol{\varphi} - \boldsymbol{d}(\boldsymbol{e})^{\mathrm{T}}\partial_t\boldsymbol{\varphi} + \boldsymbol{j}^{\mathrm{T}}\boldsymbol{\varphi}]\,\mathrm{d}V\,\mathrm{d}t \\ + \int_0^\infty\int_{\partial\Omega}\boldsymbol{S}(\boldsymbol{\varphi},\boldsymbol{e})\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S\,\mathrm{d}t - \int_\Omega \boldsymbol{d}(\boldsymbol{e}_0)^{\mathrm{T}}\boldsymbol{\varphi}\,\mathrm{d}V\Big|_{t=0} = 0,\end{aligned} \tag{4.24}$$

where $\hat{\boldsymbol{n}}$ denotes the unit normal pointing out of the region Ω, and $\boldsymbol{S}(\boldsymbol{\varphi},\boldsymbol{e}) = \boldsymbol{\varphi}_E\times\boldsymbol{H} - \boldsymbol{\varphi}_H\times\boldsymbol{E}$, with $\boldsymbol{\varphi}_E$ and $\boldsymbol{\varphi}_H$ denoting the parts of the six-vector test function $\boldsymbol{\varphi}$ corresponding to the electric and magnetic field, respectively. We denote the initial values by $\boldsymbol{e}(\boldsymbol{x},0)=\boldsymbol{e}_0(\boldsymbol{x})$. Instead of the estimate (4.18) we now obtain

$$\begin{aligned}\int_0^T\int_{\partial\Omega}\boldsymbol{S}(\boldsymbol{e}-\tilde{\boldsymbol{e}})\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S(\bar{\boldsymbol{x}})\,\mathrm{d}\bar{t} + \frac{1}{2}\int_\Omega(\boldsymbol{e}-\tilde{\boldsymbol{e}})^{\mathrm{T}}(\boldsymbol{d}(\boldsymbol{e})-\boldsymbol{d}(\tilde{\boldsymbol{e}}))\,\mathrm{d}V(\bar{\boldsymbol{x}})\Big|_{\bar{t}=T} \\ \le \frac{1}{2}\int_\Omega(\boldsymbol{e}_0-\tilde{\boldsymbol{e}}_0)^{\mathrm{T}}(\boldsymbol{d}(\boldsymbol{e}_0)-\boldsymbol{d}(\tilde{\boldsymbol{e}}_0))\,\mathrm{d}V(\bar{\boldsymbol{x}}) - \int_0^T\int_\Omega(\boldsymbol{e}-\tilde{\boldsymbol{e}})^{\mathrm{T}}(\boldsymbol{j}-\tilde{\boldsymbol{j}})\,\mathrm{d}V(\bar{\boldsymbol{x}})\,\mathrm{d}\bar{t},\end{aligned} \tag{4.25}$$

once again under the assumption that we can ignore the cubic term in (4.13). The initial values are given from the problem formulation, but it remains to divide the integral of Poynting's vector over the boundary, representing the net flow of energy

across the boundary, into parts representing energy flow in and out of the region. We use the energy splitting (change of variables) [8, 15]

$$\boldsymbol{E}_{\pm} = \frac{-\hat{\boldsymbol{n}} \times (\hat{\boldsymbol{n}} \times \boldsymbol{E}) \pm \hat{\boldsymbol{n}} \times \boldsymbol{H}}{2}, \tag{4.26}$$

to decompose the energy flux into

$$\boldsymbol{S}(\boldsymbol{e} - \tilde{\boldsymbol{e}}) \cdot \hat{\boldsymbol{n}} = |\boldsymbol{E}_+ - \tilde{\boldsymbol{E}}_+|^2 - |\boldsymbol{E}_- - \tilde{\boldsymbol{E}}_-|^2. \tag{4.27}$$

Assuming we can choose boundary data such that the incoming energy flux $|\boldsymbol{E}_- - \tilde{\boldsymbol{E}}_-|^2$ is given, we obtain

$$\begin{aligned} \int_0^T \int_{\partial\Omega} |\boldsymbol{E}_+ - \tilde{\boldsymbol{E}}_+|^2 \,\mathrm{d}S(\bar{\boldsymbol{x}}) \,\mathrm{d}\bar{t} + \frac{1}{2} \int_\Omega (\boldsymbol{e} - \tilde{\boldsymbol{e}})^{\mathrm{T}} (\boldsymbol{d}(\boldsymbol{e}) - \boldsymbol{d}(\tilde{\boldsymbol{e}})) \,\mathrm{d}V(\bar{\boldsymbol{x}}) \Big|_{\bar{t}=T} \\ \leq \int_0^T \int_{\partial\Omega} |\boldsymbol{E}_- - \tilde{\boldsymbol{E}}_-|^2 \,\mathrm{d}S(\bar{\boldsymbol{x}}) \,\mathrm{d}\bar{t} + \frac{1}{2} \int_\Omega (\boldsymbol{e}_0 - \tilde{\boldsymbol{e}}_0)^{\mathrm{T}} (\boldsymbol{d}(\boldsymbol{e}_0) - \boldsymbol{d}(\tilde{\boldsymbol{e}}_0)) \,\mathrm{d}V(\bar{\boldsymbol{x}}) \\ - \int_0^T \int_\Omega (\boldsymbol{e} - \tilde{\boldsymbol{e}})^{\mathrm{T}} (\boldsymbol{j} - \tilde{\boldsymbol{j}}) \,\mathrm{d}V(\bar{\boldsymbol{x}}) \,\mathrm{d}\bar{t}, \end{aligned} \tag{4.28}$$

with everything on the right hand side given by initial/boundary data or the sources $\boldsymbol{j} - \tilde{\boldsymbol{j}}$. It is easy to see that this estimate provides us with the same conclusions regarding uniqueness and continuous dependence on data as in the previous subsection.

5 One-dimensional example

We give an example of a situation where we have several solutions to the Maxwell equations (2.5), but the entropy condition (3.3) helps us finding the relevant solution. Assuming no sources and the initial values

$$\boldsymbol{e}(\boldsymbol{x}, 0) = \begin{cases} \boldsymbol{e}^{\mathrm{l}} & z < 0 \\ \boldsymbol{e}^{\mathrm{r}} & z > 0, \end{cases} \tag{5.1}$$

where the constant six-vectors $\boldsymbol{e}^{\mathrm{l}}$ and $\boldsymbol{e}^{\mathrm{r}}$ denote the left and right state, respectively, the Maxwell equations reduce to the one-dimensional equations,

$$\hat{\boldsymbol{z}} \times \mathbf{J} \partial_z \boldsymbol{e} + \partial_t \boldsymbol{d}(\boldsymbol{e}) = \mathbf{0}, \tag{5.2}$$

where $\hat{\boldsymbol{z}}$ is the unit vector in the z direction. This is a Riemann problem, *i.e.*, the propagation of a step function, which is the archetype problem when studying discontinuous solutions, or shock waves. For an isotropic, nonmagnetic material, we can further reduce the Maxwell equations to the well investigated system [1, 16, 28]

$$\begin{cases} \partial_z H + \partial_t D(E) = 0 \\ \partial_z E + \partial_t H = 0. \end{cases} \tag{5.3}$$

Note that this can be converted to the p-system in gas dynamics by making D the dependent variable instead of E.

The entropy condition reduces to

$$\partial_z(EH) + \partial_t \eta(E,H) \le 0, \tag{5.4}$$

where

$$\eta(E,H) = ED(E) - \int_0^E D(E')\,\mathrm{d}E' + \frac{H^2}{2}. \tag{5.5}$$

We study the constitutive relation for an instantaneously reacting Kerr medium,

$$D(E) = E + E^3 \quad \Rightarrow \quad \eta(E,H) = \frac{E^2}{2} + \frac{3E^4}{4} + \frac{H^2}{2}, \tag{5.6}$$

and choose the initial values, corresponding to (5.1), as

$$\begin{pmatrix} E^{\mathrm{l}} \\ H^{\mathrm{l}} \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} E^{\mathrm{r}} \\ H^{\mathrm{r}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.7}$$

It can be verified that the one-dimensional Maxwell equations (5.3) allow two solutions for these initial values (see Figure 1): the shock wave solution

$$\begin{pmatrix} E \\ H \end{pmatrix} = \begin{cases} (1,\sqrt{2})^{\mathrm{T}} & z < \frac{1}{\sqrt{2}}t \\ (0,0)^{\mathrm{T}} & z > \frac{1}{\sqrt{2}}t \end{cases} \tag{5.8}$$

and the rarefaction wave solution[4]

$$\begin{pmatrix} E \\ H \end{pmatrix} = \begin{cases} (1,\sqrt{2})^{\mathrm{T}} & z < \frac{1}{2}t \\ (f(z/t), g(z/t))^{\mathrm{T}} & \frac{1}{2}t < z < t \\ (0,0)^{\mathrm{T}} & z > t, \end{cases} \tag{5.9}$$

where f and g are differentiable functions satisfying

$$f(1/2) = 1, \quad g(1/2) = \sqrt{2}, \quad f(1) = g(1) = 0, \tag{5.10}$$

thus providing a smooth transition from the left state $(1,\sqrt{2})$ to the right state $(0,0)$. For a discontinuous solution which is equal to $(E^{\mathrm{l}}, H^{\mathrm{l}})$ when $z < vt$ and equal to $(E^{\mathrm{r}}, H^{\mathrm{r}})$ when $z > vt$, the entropy condition becomes

$$(E^{\mathrm{r}}H^{\mathrm{r}} - E^{\mathrm{l}}H^{\mathrm{l}}) - v(\eta(E^{\mathrm{r}},H^{\mathrm{r}}) - \eta(E^{\mathrm{l}},H^{\mathrm{l}})) \le 0. \tag{5.11}$$

[4]Actually, this is not the true solution; it should also contain an additional, small shock wave of amplitude $[\![E]\!] \sim 0.01$ propagating to the left with speed ~ -0.5, where $[\![E]\!]$ denotes the discontinuity in E over the shock. We have chosen to exclude it to keep the example simple. The qualitative behavior of the solution is dominated by the continuous rarefaction wave (5.9), which is drastically different from the nonphysical shock solution (5.8).

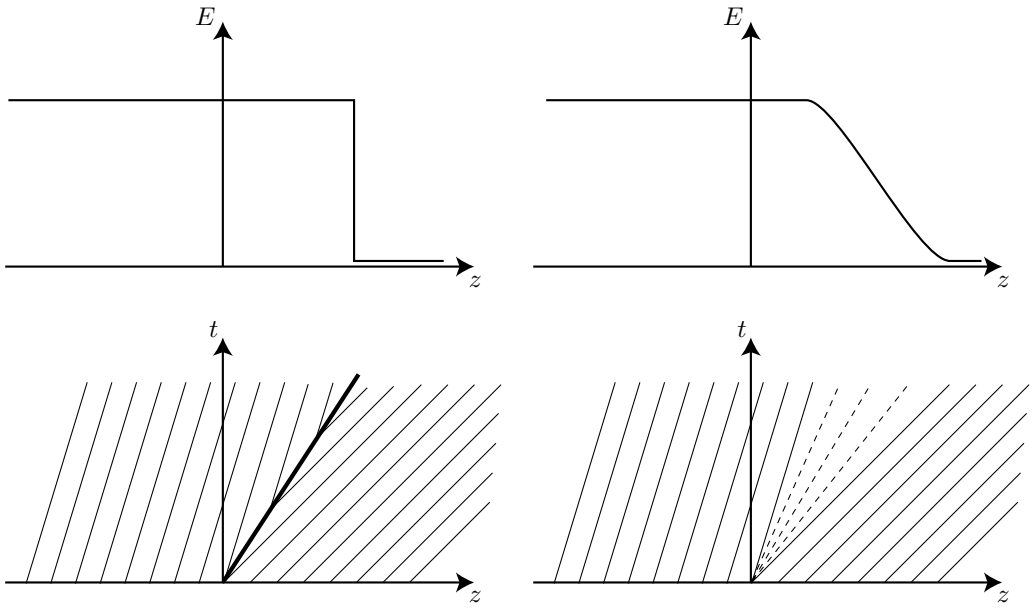

Figure 1: Top row: the two solutions with initial values (5.7) for a given time t. The solution to the left is the (nonphysical) shock wave (5.8), and the one to the right is the rarefaction (5.9). Bottom row: the '+'-characteristics for the two solutions. The '+'-characteristics are the curves in spacetime along which the waves propagating in the positive z-direction are constant, *i.e.*, in order to find the field at a certain point in space and time, we follow the characteristic curve back in time to the initial values. There are also '−'-characteristics, corresponding to waves propagating in the negative z-direction, but we have chosen initial values such that these waves can be ignored. Note that for the discontinuous solution, the characteristics originate from the shock front, indicated by the bold line.

Calculating the expression on the left hand side for the discontinuous solution (5.8), we find it is equal to $1/4\sqrt{2} \not\leq 0$. The entropy condition is violated, and the true solution must be (5.9), which can be shown to satisfy the entropy condition. Thus, the entropy condition has helped us in choosing the correct solution, where the Maxwell equations alone are not sufficient.

It should be noted that if we exchange the left and the right states in the initial value problem, *i.e.*,

$$\begin{pmatrix} E^{\mathrm{l}} \\ H^{\mathrm{l}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} E^{\mathrm{r}} \\ H^{\mathrm{r}} \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}, \tag{5.12}$$

the entropy condition is satisifed for the shock solution

$$\begin{pmatrix} E \\ H \end{pmatrix} = \begin{cases} (0,0)^{\mathrm{T}} & z < \frac{1}{\sqrt{2}}t \\ (1,\sqrt{2})^{\mathrm{T}} & z > \frac{1}{\sqrt{2}}t, \end{cases} \tag{5.13}$$

since in this case $(E^{\mathrm{r}}H^{\mathrm{r}} - E^{\mathrm{l}}H^{\mathrm{l}}) - v(\eta(E^{\mathrm{r}}, H^{\mathrm{r}}) - \eta(E^{\mathrm{l}}, H^{\mathrm{l}})) = -1/4\sqrt{2} \leq 0$. Thus, the initial values (5.12) generates a unique, physical, shock solution, where electromagnetic energy is dissipated. This solution is depicted in Figure 2.

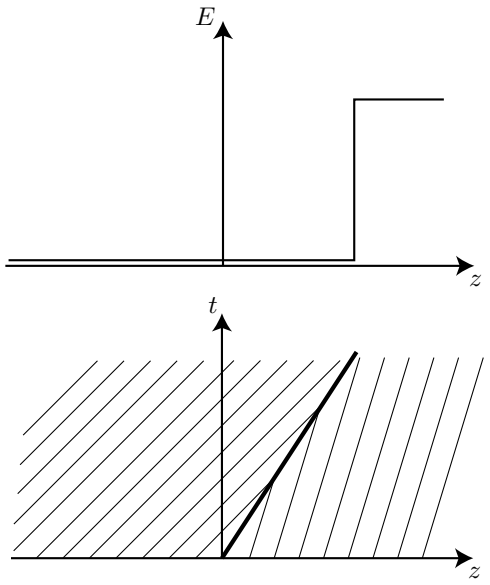

Figure 2: Top: the (physical) shock wave (5.13) for a given time t. Bottom: the '+'-characteristics corresponding to this solution. Note that the characteristics cross each other on the shock front, indicated by the bold line.

6 Conclusions

In the previous sections, the constitutive relation $\boldsymbol{d}(\boldsymbol{e})$ may depend on additional parameters without any change in the analysis. In particular, we allow for a dependence on the spatial variable, *i.e.*, $\boldsymbol{d}(\boldsymbol{x}, \boldsymbol{e})$. It is also easily seen that this form of constitutive relation allows for a coupling between the electric and magnetic field and any anisotropic effects, as long as the 6×6 matrix $\boldsymbol{d}'(\boldsymbol{e})$ is positive definite for all $\boldsymbol{e}$. Thus, our presentation comprises inhomogeneous, bianisotropic, instantaneously reacting nonlinear models.

It must be stressed that the results obtained in this paper are subject to the assumption that the cubic term in (4.13) is negligible. We have not been able to prove this conjecture, but one way might be to study the conservation of pseudo-momentum $\boldsymbol{D} \times \boldsymbol{B}$, as is done for one-dimensional shock profiles in continuum mechanics [23, 24]. This results in three additional conservation laws (one for each component of the pseudo-momentum), which might bring additional information to the problem. For instance, the problematic term can be related to the balance of forces across shock fronts, but the usefulness of this approach in three-dimensional electromagnetics is unclear.

Since it seems reasonable that entropy solutions to the Maxwell equations are unique and depend continuously on data, numerical methods for treating these equations should incorporate the entropy condition. One way to do this is by choosing a numerical scheme based on vanishing viscosity, where the viscosity parameter is of the same order as the discretization as explained at the end of Section 3.2.

In the Introduction, we listed the three questions of existence, uniqueness and continuity. The latter two have been treated in this paper using Kružkov's method,

but the questions remain open. There is also only empirical evidence regarding the existence of solutions satisfying the entropy condition. If it is possible to answer these questions by the vanishing viscosity technique, that answer will most probably also shed additional light on the problems with uniqueness and continuity treated in this paper.

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Appendix A Analysis of an energy term

In this appendix, we analyze the term $\frac{1}{2}[\eta(\boldsymbol{e},\tilde{\boldsymbol{e}}) - \eta(\tilde{\boldsymbol{e}},\boldsymbol{e})]$. The definition of the energy $\eta(\boldsymbol{e},\tilde{\boldsymbol{e}})$ is

$$\eta(\boldsymbol{e},\tilde{\boldsymbol{e}}) = (\boldsymbol{e}-\tilde{\boldsymbol{e}})^{\mathrm{T}}\boldsymbol{d}(\boldsymbol{e}) - \phi(\boldsymbol{e}) + \phi(\tilde{\boldsymbol{e}}), \tag{A.1}$$

which enables us to write

$$\frac{1}{2}[\eta(\boldsymbol{e},\tilde{\boldsymbol{e}}) - \eta(\tilde{\boldsymbol{e}},\boldsymbol{e})] = (\boldsymbol{e}-\tilde{\boldsymbol{e}})^{\mathrm{T}}\frac{\boldsymbol{d}(\boldsymbol{e})+\boldsymbol{d}(\tilde{\boldsymbol{e}})}{2} - \phi(\boldsymbol{e}) + \phi(\tilde{\boldsymbol{e}}). \tag{A.2}$$

This is in fact a cubic function of $\boldsymbol{e}-\tilde{\boldsymbol{e}}$. To see this, first note that we can write

$$\begin{aligned}\phi(\boldsymbol{e}) - \phi(\tilde{\boldsymbol{e}}) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r}\phi(r\boldsymbol{e}+(1-r)\tilde{\boldsymbol{e}})\,\mathrm{d}r \\ &= (\boldsymbol{e}-\tilde{\boldsymbol{e}})^{\mathrm{T}}\int_0^1 \phi'(r\boldsymbol{e}+(1-r)\tilde{\boldsymbol{e}})\,\mathrm{d}r \\ &= (\boldsymbol{e}-\tilde{\boldsymbol{e}})^{\mathrm{T}}\int_0^1 \boldsymbol{d}(r\boldsymbol{e}+(1-r)\tilde{\boldsymbol{e}})\,\mathrm{d}r.\end{aligned} \tag{A.3}$$

We then have

$$\begin{aligned}(\boldsymbol{e}-\tilde{\boldsymbol{e}})^{\mathrm{T}}\frac{\boldsymbol{d}(\boldsymbol{e})+\boldsymbol{d}(\tilde{\boldsymbol{e}})}{2} - \phi(\boldsymbol{e}) + \phi(\tilde{\boldsymbol{e}}) = \frac{1}{2}(\boldsymbol{e}-\tilde{\boldsymbol{e}})^{\mathrm{T}} \\ \left\{\boldsymbol{d}(\boldsymbol{e}) - \int_0^1 \boldsymbol{d}(r\boldsymbol{e}+(1-r)\tilde{\boldsymbol{e}})\,\mathrm{d}r + \boldsymbol{d}(\tilde{\boldsymbol{e}}) - \int_0^1 \boldsymbol{d}(r\boldsymbol{e}+(1-r)\tilde{\boldsymbol{e}})\,\mathrm{d}r\right\},\end{aligned} \tag{A.4}$$

and we repeat the trick in (A.3) to find

$$\begin{aligned}\boldsymbol{d}(\boldsymbol{e}) - \int_0^1 \boldsymbol{d}(r\boldsymbol{e}+(1-r)\tilde{\boldsymbol{e}})^{\mathrm{T}}\,\mathrm{d}r &= \int_0^1 \{\boldsymbol{d}(\boldsymbol{e}) - \boldsymbol{d}(r\boldsymbol{e}+(1-r)\tilde{\boldsymbol{e}})\}\,\mathrm{d}r \\ &= \int_0^1 (\boldsymbol{e}-(r\boldsymbol{e}+(1-r)\tilde{\boldsymbol{e}}))^{\mathrm{T}} \int_0^1 \boldsymbol{d}'(q\boldsymbol{e}+(1-q)(r\boldsymbol{e}+(1-r)\tilde{\boldsymbol{e}}))\,\mathrm{d}q\,\mathrm{d}r \\ &= (\boldsymbol{e}-\tilde{\boldsymbol{e}})^{\mathrm{T}} \int_0^1 (1-r) \int_0^1 \boldsymbol{d}'(q\boldsymbol{e}+(1-q)(r\boldsymbol{e}+(1-r)\tilde{\boldsymbol{e}}))\,\mathrm{d}q\,\mathrm{d}r, \quad \text{(A.5)}\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{d}(\tilde{\boldsymbol{e}}) - \int_0^1 \boldsymbol{d}(r\boldsymbol{e}+(1-r)\tilde{\boldsymbol{e}})^{\mathrm{T}}\,\mathrm{d}r &= \int_0^1 \{\boldsymbol{d}(\tilde{\boldsymbol{e}}) - \boldsymbol{d}(r\boldsymbol{e}+(1-r)\tilde{\boldsymbol{e}})\}\,\mathrm{d}r \\ &= \int_0^1 (\tilde{\boldsymbol{e}}-(r\boldsymbol{e}+(1-r)\tilde{\boldsymbol{e}}))^{\mathrm{T}} \int_0^1 \boldsymbol{d}'(q\tilde{\boldsymbol{e}}+(1-q)(r\boldsymbol{e}+(1-r)\tilde{\boldsymbol{e}}))\,\mathrm{d}q\,\mathrm{d}r \\ &= (\tilde{\boldsymbol{e}}-\boldsymbol{e})^{\mathrm{T}} \int_0^1 r \int_0^1 \boldsymbol{d}'(q\tilde{\boldsymbol{e}}+(1-q)(r\boldsymbol{e}+(1-r)\tilde{\boldsymbol{e}}))\,\mathrm{d}q\,\mathrm{d}r \\ &= (\tilde{\boldsymbol{e}}-\boldsymbol{e})^{\mathrm{T}} \int_0^1 (1-r) \int_0^1 \boldsymbol{d}'(q\tilde{\boldsymbol{e}}+(1-q)(r\tilde{\boldsymbol{e}}+(1-r)\boldsymbol{e}))\,\mathrm{d}q\,\mathrm{d}r, \quad \text{(A.6)}\end{aligned}$$

where the last line follows from a change of variables $r \to 1-r$. The sum of these terms involve the expression

$$\begin{aligned}&\int_0^1 (1-r) \int_0^1 \{\boldsymbol{d}'(q\boldsymbol{e}+(1-q)(r\boldsymbol{e}+(1-r)\tilde{\boldsymbol{e}})) - \boldsymbol{d}'(q\tilde{\boldsymbol{e}}+(1-q)(r\tilde{\boldsymbol{e}}+(1-r)\boldsymbol{e}))\}\,\mathrm{d}q\,\mathrm{d}r \\ &\qquad = \int_0^1 (1-r) \int_0^1 (\boldsymbol{e}-\tilde{\boldsymbol{e}})^{\mathrm{T}}(q+(1-q)(r-(1-r))) \\ &\int_0^1 \boldsymbol{d}''(p(q\boldsymbol{e}+(1-q)(r\boldsymbol{e}+(1-r)\tilde{\boldsymbol{e}}))+(1-p)(q\boldsymbol{e}+(1-q)(r\boldsymbol{e}+(1-r)\tilde{\boldsymbol{e}})))\,\mathrm{d}p\,\mathrm{d}q\,\mathrm{d}r \\ &\qquad = (\boldsymbol{e}-\tilde{\boldsymbol{e}})^{\mathrm{T}} \int_0^1 \int_0^1 \int_0^1 (1-r)(2q+2r-2qr-1) \\ &\boldsymbol{d}''(p(q\boldsymbol{e}+(1-q)(r\boldsymbol{e}+(1-r)\tilde{\boldsymbol{e}}))+(1-p)(q\boldsymbol{e}+(1-q)(r\boldsymbol{e}+(1-r)\tilde{\boldsymbol{e}})))\,\mathrm{d}p\,\mathrm{d}q\,\mathrm{d}r. \quad \text{(A.7)}\end{aligned}$$

Since $\boldsymbol{d}'' = \phi'''$ we can introduce the third rank tensor

$$\begin{aligned}K_{ijk}(\boldsymbol{e},\tilde{\boldsymbol{e}}) &= \int_0^1 \int_0^1 \int_0^1 (1-r)(2q+2r-2qr-1) \\ &\frac{\partial^3\phi}{\partial e_i \partial e_j \partial e_k}(p(q\boldsymbol{e}+(1-q)(r\boldsymbol{e}+(1-r)\tilde{\boldsymbol{e}}))+(1-p)(q\boldsymbol{e}+(1-q)(r\boldsymbol{e}+(1-r)\tilde{\boldsymbol{e}})))\,\mathrm{d}p\,\mathrm{d}q\,\mathrm{d}r, \quad \text{(A.8)}\end{aligned}$$

to write

$$(\boldsymbol{e}-\tilde{\boldsymbol{e}})^{\mathrm{T}}\frac{\boldsymbol{d}(\boldsymbol{e})+\boldsymbol{d}(\tilde{\boldsymbol{e}})}{2} - \phi(\boldsymbol{e}) + \phi(\tilde{\boldsymbol{e}}) = K_{ijk}(\boldsymbol{e},\tilde{\boldsymbol{e}})(e_i-\tilde{e}_i)(e_j-\tilde{e}_j)(e_k-\tilde{e}_k), \quad \text{(A.9)}$$

where summation over repeated indices is assumed. With ϕ a quadratic function for linear materials, we see that this term must be due to the nonlinearity of the constitutive relation.

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