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SMOOTH BLENDING OF NONLINEAR CONTROLLERS USING DENSITY FUNCTIONS

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Abstract

A new convergence criterion for nonlinear systems was recently derived by the first author. The criterion is similar to Lyapunov's second theorem but differs in several respects. In particular, it has a remarkable convexity property in the context of control synthesis. While the set of control Lyapunov functions for a given system may not even be connected, the corresponding set of "density" functions is always convex. In this paper, we will demonstrate how this opens new possibilities in synthesis of nonlinear controllers. We also give an alternative version of the criterion, that proves attractivity of a set rather than a point in the state space.

1 Introduction

A fundamental problem in nonlinear systems is the search for control laws that achieve stability of a certain point or set in the state space. Many methods have been proposed, for example backstepping, feedback linearization and passivity based control [3, 2, 5].

In this paper, we address the stabilization problem using the new criterion of [4]. Convexity is exploited to make smooth transitions between different control laws. In particular, the transition from a globally stabilizing nonlinear controller to a local linear controller is addressed.

Consider a nonlinear control system of the form

$$\dot{x} = f(x) + g(x)u(x) \qquad x(t) \in \mathbf{R}^n$$

For a fixed control law $u(x)$, the stability citerion of [4] can be applied provided that there exists a non-negative scalar function $\rho(x)$ which is integrable outside a neighborhood of zero and satisfies the divergence inequality

$$\nabla \cdot ((f + gu)\rho) > 0$$

almost everywhere in the state space. This criterion is *convex in the pair* $(\rho, u\rho)$. In particular, if the two control laws $u_1(x)$

and $u_2(x)$ satisfy the criterion together with $\rho_1(x)$ and $\rho_2(x)$ respectively, then the control law

$$u(x) = \frac{\rho_1(x)}{\rho_1(x) + \rho_2(x)}u_1(x) + \frac{\rho_2(x)}{\rho_1(x) + \rho_2(x)}u_2(x)$$

satisfies the criterion with $\rho(x) = \rho_1(x) + \rho_2(x)$. Note that $u \approx u_1$ in regions of the state space where $\rho_1 >> \rho_2$ and vice versa. This will be used for design of $u(x)$ by "blending" two control laws $u_1(x)$ and $u_2(x)$.

The outline of the paper is as follows. Each of the following two sections is devoted to an example applying the idea described above. The second example concerns swing-up of an inverted pendulum and the original criterion of [4] is not directly applicable. This motivates the introduction of a modified (but still not fully satisfactory) criterion in section4.

2 Local modification of a global controller

For the system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = f_u(x, y) := \begin{bmatrix} x^2 + y \\ u \end{bmatrix}$$

a globally stabilizing controller can be found, e.g. using backstepping:

$$u_N(x, y) = -2(x + y + x^2 + xy + x^3)$$

This design gives an oscillatory behavior near the origin. Suppose that a different local controller has been designed based on linearization. It is then natural to ask how the transition between the local and the global controller can be done without loss of global stability. This problem will now be addressed using the method outlined before.

The controller $u_N(x, y)$ satisfies the inequality $\nabla \cdot (f_{u_N}\rho_N) > 0$ (see Theorem 1 below) together with

$$\rho_N = \left(x^2 + (y + x^2 + x)^2\right)^{-2}$$

For the system $(\dot{x}, \dot{y}) = f_{u_L}(x, y)$ with the linear controller
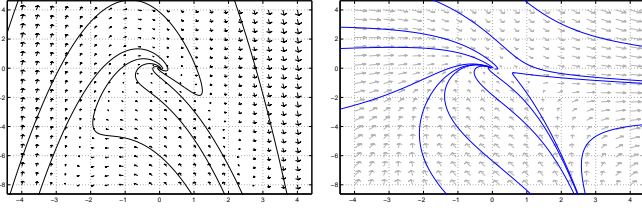
$$u_L(x, y) = -x - 2y$$

Figure 1: Left: Phase plot for globally stabilizing nonlinear controller. Right: Phase plot for linear controller with good local performance.

Figure 2: Left: Phase plot for pendulum with energy control. Right: Graph of the density function $1/E^2$.

let the matrix $P > 0$ define a Lyapunov function that is decreasing inside the ellipsoid $[x\ y]P[x\ y]' = 1$. Then

$$\rho_L(x,y) = \max\{(([x\ y]P[x\ y]'))^{-3} - 1, 0\}$$

satisfies

$$\nabla \cdot (f_{u_L}\rho_L) > 0 \text{ inside the ellipsoid}$$
$$\nabla \cdot (f_{u_L}\rho_L) = 0 \text{ outside the ellipsoid}$$

In fact ρ_L can be replaced by a smooth approximation without violating these conditions.

The controller

$$u(x,y) = \frac{\rho_L}{\rho_N + \rho_L} u_L(x,y) + \frac{\rho_N}{\rho_N + \rho_L} u_N(x,y)$$

gives

$$\nabla \cdot (f_u(\rho_N + \rho_L))$$
$$= \nabla \cdot (f_{u_N}\rho_N) + \nabla \cdot (f_{u_L}\rho_L) > 0 \qquad x \neq 0$$

It is identical to u_N outside the ellipsoid $[x\ y]P[x\ y]' = 1$ and it is close to u_L for small (x,y). Hence our problem has been solved.

3 Swing-up of an inverted pendulum

Let us consider the problem to find a control law for swing-up of an inverted pendulum. The dynamics can be written as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = f_u(x,y) = \begin{bmatrix} y \\ \sin x + u \cos x \end{bmatrix}$$

The energy and its time-derivative can be described as

$$E = y^2/2 + \cos x - 1$$
$$\dot{E} = u y \cos x$$

For almost all initial conditions, the feedback $u_E = -y \cos x E$ steers towards the right energy [1]. To prove this, introduce

$$\rho_0(x,y) = \frac{1}{E^2}$$
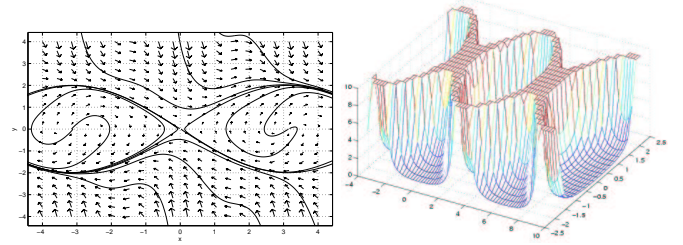$$\nabla \cdot (f_{u_E}\rho_0) = \frac{\cos^2 x}{E^2}\left(\frac{y^2}{2} + 1 - \cos x\right) \geq 0$$
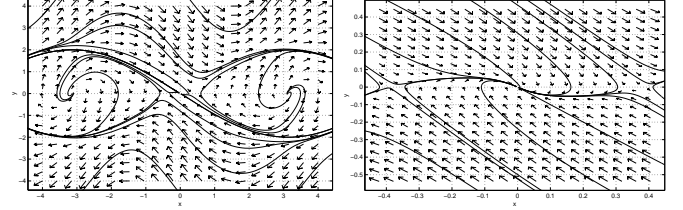

Figure 3: Left: Phase plot for pendulum swing-up control. Right: Magnified phase plot near upright equilibrium.

Note however, that ρ_0 is not integrable outside a neighborhood of the equilibria, so even the generalized Theorem 1 in section 4 does not apply as it stands. Instead Theorem 2 needs to be applied with X as a manifold.

The controller $u_L(x,y) = -2\sin(x) - 2y$ is locally stabilizing with Lyapunov function $V(x,y) = 8\sin(x/2)^2 + (2\sin(x/2) + y)^2$. With

$$\rho_E = \frac{1}{E^2 + \max(0, 1 - x^2 - y^2)}$$
$$\rho_L = \max(0, V(x,y)^{-1} - 100)$$
$$u(x,y) = \frac{\rho_L}{\rho_E + \rho_L} u_L(x,y) + \frac{\rho_E}{\rho_E + \rho_L} u_E(x,y)$$

the pendulum swings up for almost all initial conditions. Phase plots with smoothened max-operator are shown in Figure 3

4 Convergence to a set

Consider the equation

$$\dot{x} = f(x) \tag{1}$$

where $f \in C^1(\mathbf{R}^n, \mathbf{R}^n)$ and assume existence of $A, B > 0$ such that $|f(x)| \leq A|x| + B$ (This is the classical condition in order to guarantee existence of solutions on $[0, +\infty)$).

In the following we denote by $\phi(t, z)$ the solution of (1) with initial condition z and by $\phi(t, Z)$ the set $\{\phi(t, z),\ z \in Z\}$. Moreover $|\cdot|$ is the Lebesgue measure in $\mathbf{R}^n$, d is the usual distance in $\mathbf{R}^n$, $\overline{B(0, r)}$ is the closed ball centered at 0 with radius r and $S_r = S + \overline{B(0, r)}$ for any given set S.

The following theorem modifies Theorem 1 in [4] to the case in which the attraction set is a general closed invariant set.

Theorem 1 *Assume that $S \subset \mathbf{R}^n$ is a closed set, invariant for (1) and $\rho \in C^1(\mathbf{R}^n \backslash S, \mathbf{R}) \cap L^1(\mathbf{R}^n \backslash S_\varepsilon)$ for all $\varepsilon > 0$. If $\rho(x) > 0$ and $\nabla \cdot (\rho f)(x) > 0$ for almost all $x \in \mathbf{R}^n \backslash S$ and f is bounded in S_r for some $r > 0$, then $\lim_{t \to +\infty} d(\phi(t, x), S) = 0$ for almost all $x \in \mathbf{R}^n$.*

Before giving the proof of the theorem, we recall some preliminary results from [4] which we state in the form we will need in the following.

Theorem 2 *Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $P \subset X$ be a finite measure set and $T : X \to X$ be such that $\mu(T^{-1}Y) \leq \mu(Y)$ for all $Y \in \mathcal{A}$ and $\mu(T^{-1}Y) < \mu(Y)$ if $\mu(Y) > 0$. The set $Z = \{x \in P : T^n x \in P \text{ for infinitely many } n\}$ then has zero measure.*

Lemma 1 *If S, f and ρ are as in the statement of Theorem 1 and $Y \subseteq (\mathbf{R}^n \backslash S)$ is such that $\int_Y \rho(x) dx < +\infty$, then for $t > 0$ one has*

$$\int_{\phi(t,Y)} \rho(x) dx - \int_Y \rho(x) dx$$
$$= \int_0^t \int_{\phi(s,Y)} \nabla \cdot (\rho f)(x) dx ds$$

Proof of Theorem 1 For $k, m \in \mathbf{N}$, we define the following sets:

$$Z_k^m = \left\{ x \in \mathbf{R}^n : \limsup_{l \to \infty} d(\phi(\tfrac{l}{k}, x), S) > \tfrac{1}{m} \right\}$$

$$Z^m = \left\{ x \in \mathbf{R}^n : \limsup_{t \to \infty} d(\phi(t, x), S) > \tfrac{1}{m} \right\}$$

$$Z = \left\{ x \in \mathbf{R}^n : \limsup_{t \to \infty} d(\phi(t, x), S) > 0 \right\}$$

where $l \in \mathbf{N}$ and $t \in \mathbf{R}$. Note that if $x \notin Z$ then $\lim_{t \to +\infty} d(\phi(t, x), S) = 0$.

We have to prove that $|Z| = 0$. The first step is to prove that $|Z_k^m| = 0$ for all $k, m \in \mathbf{N}$. This fact is a consequence of Lemma 1 and Theorem 2 where $X = \mathbf{R}^n \backslash S$, $\mathcal{A}$ is the usual σ-algebra in $\mathbf{R}^n$, $P = \{x \in \mathbf{R}^n : d(x, S) > \tfrac{1}{m}\}$, $\mu(Y) = \int_Y \rho(x) dx$ and $Tx = \phi(\tfrac{1}{k}, x)$.

The second step is to prove that $Z^m \subseteq \cup_{k \in \mathbf{N}} Z_k^m$ for all fixed m (sufficiently large: $m > \tfrac{1}{r}$). Actually we prove that $x \notin \cup_{k \in \mathbf{N}} Z_k^m$ implies $x \notin Z^m$. $x \notin \cup_{k \in \mathbf{N}} Z_k^m$ means that $x \notin Z_k^m$ for all k, i.e. for all k there exists n_k such that $d(\phi(\tfrac{n}{k}, x), S) \leq \tfrac{1}{m}$ for all $n > n_k$. We prove that for all $\varepsilon > 0$ there exists t_ε such that $d(\phi(t, x), S) < \tfrac{1}{m} + \varepsilon$ for all $t > t_\varepsilon$.

Let $\varepsilon > 0$ be fixed, $K \in \mathbf{R}$ be such that $|f(x)| \leq K$ for all $x \in S_r$, $k \in \mathbf{N}$ such that $k > \max\{\tfrac{K}{\varepsilon}, \tfrac{K}{r - \tfrac{1}{m}}\}$, n_k such that $d(\phi(\tfrac{n}{k}, x), S) \leq \tfrac{1}{m}$ for all $n > n_k$, $t_\varepsilon = \tfrac{n_k}{k}$, $t > t_\varepsilon$ and $n_t \in \mathbf{N}$ be such that $|t - \tfrac{n_t}{k}| < \tfrac{1}{k}$. Let us remark that $\phi(\tau, x) \in S_r$ for all $\tau \in [\tfrac{n_t}{k}, t]$. Infact if $z \in S_{\frac{1}{m}}$, the minimum time t_r for a

solution starting at z to reach the boundary of S_r satisfies the inequality $t_r \geq \tfrac{1}{K}(r - \tfrac{1}{m})$:

$$r - \frac{1}{m} \leq |\phi(t_r, z) - z| \leq \int_0^{t_r} |f(\phi(s, z))| ds \leq t_r K.$$

Due to this fact we also have that

$$d(\phi(t, x), \phi(\tfrac{n_t}{k}, x))$$
$$= |\int_{\frac{n_t}{k}}^t f(\phi(s, x)) dx|' \leq (t - \frac{n_t}{k}) K < \frac{K}{k} < \varepsilon$$

and

$$d(\phi(t, x), S) \leq d(\phi(\tfrac{n_t}{k}, x), \phi(t, x)) + d(\phi(\tfrac{n}{k}, x), S)$$
$$< \varepsilon + \frac{1}{m}$$

Since $|Z_k^m| = 0$ for each $m, k \in \mathbf{R}^n$, we get that also $|Z^m| = 0$. Finally we just need to note that $Z \subseteq \cup_{m > \frac{1}{r}} Z^m$ in order to get that $|Z| = 0$.

Acknowledgements

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