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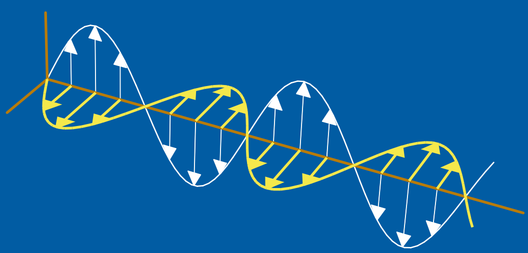

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Abstract

Using exact dispersion relations for electromagnetic wave propagation in layered, periodic media, consisting of two phases, we derive explicit asymptotic solutions for small wavenumbers. These solutions are compared to the numerical solutions of the exact dispersion relations, and applications to homogenization problems are discussed. The results can be used as test cases for homogenization techniques intended for finite scale homogenization, that is, where the wavelength is not assumed infinitely large compared to the microscale.

1 Introduction

Many interesting problems in wave propagation depend on the dispersion relation, *i.e.*, a relation between the frequency ω and the wavenumber k. The problem that has motivated the research presented in this paper is homogenization, where the goal is to model a material with a complicated microstructure with a homogeneous medium, making it easy to treat the wave propagation. The problem is to compute the properties of the fictitious homogeneous medium, which is usually done in the limit of infinitely long wavelength. Many ways have been designed to compute the homogenization, but the one that most directly concerns the dispersion relation is the effective mass approximation, where the effective permittivity is computed by differentiating the dispersion relation $\omega(k)$ at $k = 0$. This technique has recently gotten considerable attention from mathematicians, see for instance [2, 3, 8, 10]. For a more thorough review of homogenization techniques, we refer to the books [4, 7, 14].

Some recent contributions [3, 10, 11, 15, 16] have shown that it is possible to do some homogenization even when the wavelength is not infinitely long. This corresponds to studying the dispersion relation for $k \neq 0$. In this paper, we first show that the homogenized permittivity is linked to the dispersion relation through a very simple formula in the case of nonmagnetic, stratified media. We then use the exact expressions for the dispersion relation for piecewise constant media given in [18] (maybe more easily accessible in [12, Appendix A]), to find the asymptotic solution when $k \to 0$. This provides explicit expressions for the homogenized permittivity up to a given order in k, although we only give a few terms. The results can be used as test cases for more general homogenization approaches.

Wave propagation in onedimensional periodic structures is described by Hill's equation [6, p. 178], with Mathieu's equation as a special case. More information on Mathieu functions can be found in, for instance, [1, Ch. 20] but we do not need much of the deep properties of these functions.

2 Basic equations

We study layered media, periodic in the z-direction with period a. The Floquet-Bloch theorem [5, 9] then states that the typical wave can be written (ignoring the

x and y dependence)

$$\boldsymbol{E}(z) = \mathrm{e}^{\mathrm{i}k_z z}\widetilde{\boldsymbol{E}}(z) \tag{2.1}$$

where $\widetilde{\boldsymbol{E}}$ is periodic in z. This implies pseudoperiodic boundary conditions, $\boldsymbol{E}(z+a) = \mathrm{e}^{\mathrm{i}k_z a}\boldsymbol{E}(z)$. The wavenumber k_z is called the Bloch wave number, and is a free parameter in the range $k_z \in [-\pi/a, \pi/a]$.

Inserting (2.1) into Maxwell's equations, the dispersion relation is determined from the eigenproblem

$$\nabla \times \boldsymbol{E} = \mathrm{i}\omega\mu_0\boldsymbol{H} \tag{2.2}$$

$$\nabla \times \boldsymbol{H} = -\mathrm{i}\omega\epsilon_0\epsilon(z)\boldsymbol{E} \tag{2.3}$$

where we assumed an isotropic, nonmagnetic medium depending only on z. The fields are given in SI units, μ_0 is the permeability of vacuum, and ϵ_0 is the permittivity of vacuum. The eigenvalue ω depends on the wavenumber k_z through the pseudoperiodic boundary conditions.

We apply a Fourier transform in x and y to write

$$(\partial_z\hat{\boldsymbol{z}} + \mathrm{i}\boldsymbol{k}_{xy}) \times \boldsymbol{E} = \mathrm{i}\omega\mu_0\boldsymbol{H} \tag{2.4}$$

$$(\partial_z\hat{\boldsymbol{z}} + \mathrm{i}\boldsymbol{k}_{xy}) \times \boldsymbol{H} = -\mathrm{i}\omega\epsilon_0\epsilon(z)\boldsymbol{E} \tag{2.5}$$

where the Fourier variable $\boldsymbol{k}_{xy}$ can take any value in $\mathbb{R}^2$, although we are particularly interested in small values in homogenization problems. Eliminating $\boldsymbol{H}$ implies

$$(\partial_z\hat{\boldsymbol{z}} + \mathrm{i}\boldsymbol{k}_{xy}) \times [(\partial_z\hat{\boldsymbol{z}} + \mathrm{i}\boldsymbol{k}_{xy}) \times \boldsymbol{E}] = c^{-2}\omega^2\epsilon(z)\boldsymbol{E} \tag{2.6}$$

where $c = 1/\sqrt{\epsilon_0\mu_0}$ is the speed of light in vacuum. The left hand side is

$$\begin{aligned}
&(\partial_z\hat{\boldsymbol{z}} + \mathrm{i}\boldsymbol{k}_{xy}) \times [(\partial_z\hat{\boldsymbol{z}} + \mathrm{i}\boldsymbol{k}_{xy}) \times \boldsymbol{E}] = (\partial_z\hat{\boldsymbol{z}} + \mathrm{i}\boldsymbol{k}_{xy}) \times (\partial_z\hat{\boldsymbol{z}} \times \boldsymbol{E} + \mathrm{i}\boldsymbol{k}_{xy} \times \boldsymbol{E}) \\
&= \partial_z^2\hat{\boldsymbol{z}} \times (\hat{\boldsymbol{z}} \times \boldsymbol{E}) + \partial_z\hat{\boldsymbol{z}} \times (\mathrm{i}\boldsymbol{k}_{xy} \times \boldsymbol{E}) + \mathrm{i}\boldsymbol{k}_{xy} \times (\partial_z\hat{\boldsymbol{z}} \times \boldsymbol{E}) + \mathrm{i}\boldsymbol{k}_{xy} \times (\mathrm{i}\boldsymbol{k}_{xy} \times \boldsymbol{E}) \\
&= -\partial_z^2[\mathbf{I}_3 - \hat{\boldsymbol{z}}\hat{\boldsymbol{z}}] \cdot \boldsymbol{E} + \mathrm{i}k_{xy}\partial_z[\hat{\boldsymbol{z}} \times (\hat{\boldsymbol{k}}_{xy} \times \boldsymbol{E}) + \hat{\boldsymbol{k}}_{xy} \times (\hat{\boldsymbol{z}} \times \boldsymbol{E})] + k_{xy}^2[\mathbf{I}_3 - \hat{\boldsymbol{k}}_{xy}\hat{\boldsymbol{k}}_{xy}] \cdot \boldsymbol{E}
\end{aligned} \tag{2.7}$$

where we use a hat to indicate unit vectors. Writing $\boldsymbol{E} = E_z\hat{\boldsymbol{z}} + E_k\hat{\boldsymbol{k}}_{xy} + E_\perp\hat{\boldsymbol{z}} \times \hat{\boldsymbol{k}}_{xy}$ implies $\hat{\boldsymbol{z}} \times (\hat{\boldsymbol{k}}_{xy} \times \boldsymbol{E}) + \hat{\boldsymbol{k}}_{xy} \times (\hat{\boldsymbol{z}} \times \boldsymbol{E}) = \hat{\boldsymbol{k}}_{xy}E_z + \hat{\boldsymbol{z}}E_k$, and we have

$$-\mathrm{i}k_{xy}\partial_z E_k + (c^{-2}\omega^2\epsilon(z) - k_{xy}^2)E_z = 0 \tag{2.8}$$

$$\partial_z^2 E_k - \mathrm{i}k_{xy}\partial_z E_z + c^{-2}\omega^2\epsilon(z)E_k = 0 \tag{2.9}$$

$$\partial_z^2 E_\perp + (c^{-2}\omega^2\epsilon(z) - k_{xy}^2)E_\perp = 0 \tag{2.10}$$

where (2.10) is Hill's equation, see for instance [6, p. 178]. A special case is Mathieu's equation, when $\epsilon(z) = \epsilon_1 + \epsilon_2\cos(2\pi z/a)$. Combining the first and second equation results in

$$\partial_z\left[\frac{c^{-2}\omega^2\epsilon(z)}{c^{-2}\omega^2\epsilon(z) - k_{xy}^2}\partial_z E_k\right] + c^{-2}\omega^2\epsilon(z)E_k = 0 \tag{2.11}$$

and we remind of the pseudoperiodic boundary conditions $E_k(z+a) = \mathrm{e}^{\mathrm{i}k_z a}E_k(z)$ and $E_\perp(z+a) = \mathrm{e}^{\mathrm{i}k_z a}E_\perp(z)$.

3 Implications for homogenization

This investigation is motivated by the problem of homogenization, where we extract effective material parameters from a given geometry by considering propagation of waves with long wavelength, *i.e.*, small wavenumbers. In a physical sense, the effective permittivity is defined by the relation

$$\left\langle \boldsymbol{\epsilon}\cdot(\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\boldsymbol{E})\right\rangle = \boldsymbol{\epsilon}_{\mathrm{eff}}\cdot\left\langle \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\boldsymbol{E}\right\rangle \tag{3.1}$$

where the mean value is taken over the unit cell and the field $\boldsymbol{E}$ is pseudoperiodic, *i.e.*, $\boldsymbol{E}(\boldsymbol{x}+\boldsymbol{a}) = \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{a}}\boldsymbol{E}(\boldsymbol{x})$ where $\boldsymbol{a}$ is a lattice vector. In our case, the field $\boldsymbol{E}$ is the solution to the equations at the end of Section 2. Note that this definition is not a strict mathematical one, but is rather based on physical intuition. However, as is shown in [15, 16], it is mathematically motivated in our case.

Starting with the $E_\perp$-field, we can use equation (2.10) to find

$$\begin{aligned}\left\langle \epsilon(z)\mathrm{e}^{-\mathrm{i}k_z z}E_\perp\right\rangle = \frac{1}{c^{-2}\omega^2}\left\langle \mathrm{e}^{-\mathrm{i}k_z z}(k_{xy}^2-\partial_z^2)E_\perp\right\rangle &= \frac{1}{c^{-2}\omega^2}\left\langle (k_{xy}^2+k_z^2-\partial_z^2)\mathrm{e}^{-\mathrm{i}k_z z}E_\perp\right\rangle \\ &= \frac{k_{xy}^2+k_z^2}{c^{-2}\omega^2}\left\langle \mathrm{e}^{-\mathrm{i}k_z z}E_\perp\right\rangle \end{aligned} \tag{3.2}$$

where we used that $\mathrm{e}^{-\mathrm{i}k_z z}\partial_z E_\perp = (\partial_z + \mathrm{i}k_z)\mathrm{e}^{-\mathrm{i}k_z z}E_\perp$ and that the mean value of the derivative of a periodic function, $\partial_z \mathrm{e}^{-\mathrm{i}k_z z}E_\perp$, is zero.

For E_k and E_z, matters are a bit more messy. These fields are described by equations (2.8) and (2.9), and using the same technique as above we can prove

$$\left\langle \epsilon(z)\mathrm{e}^{-\mathrm{i}k_z z}E_k\right\rangle = \frac{k_z^2}{c^{-2}\omega^2}\left\langle \mathrm{e}^{-\mathrm{i}k_z z}E_k\right\rangle - \frac{k_{xy}k_z}{c^{-2}\omega^2}\left\langle \mathrm{e}^{-\mathrm{i}k_z z}E_z\right\rangle \tag{3.3}$$

$$\left\langle \epsilon(z)\mathrm{e}^{-\mathrm{i}k_z z}E_z\right\rangle = \frac{k_{xy}^2}{c^{-2}\omega^2}\left\langle \mathrm{e}^{-\mathrm{i}k_z z}E_z\right\rangle - \frac{k_{xy}k_z}{c^{-2}\omega^2}\left\langle \mathrm{e}^{-\mathrm{i}k_z z}E_k\right\rangle \tag{3.4}$$

As is explained in [16], it is sufficient to consider only field components orthogonal to the propagation direction, since these are the only ones concerned with the wave propagation. Since the wave vector is $k_{xy}\hat{\boldsymbol{k}}_{xy} + k_z\hat{\boldsymbol{z}}$, the orthogonal polarization is proportional to $k_z\hat{\boldsymbol{k}}_{xy} - k_{xy}\hat{\boldsymbol{z}}$, which corresponds to a combination of E_k and E_z proportional to $k_zE_k - k_{xy}E_z$. This combination satisfies

$$\left\langle \epsilon(z)\mathrm{e}^{-\mathrm{i}k_z z}(k_zE_k - k_{xy}E_z)\right\rangle = \frac{k_{xy}^2+k_z^2}{c^{-2}\omega^2}\left\langle \mathrm{e}^{-\mathrm{i}k_z z}(k_zE_k - k_{xy}E_z)\right\rangle \tag{3.5}$$

Thus, for both polarizations we have the conclusion

$$\epsilon_{\mathrm{eff}} = \frac{k_{xy}^2+k_z^2}{c^{-2}\omega^2} \tag{3.6}$$

which is also used in [17] and [14, pp. 227–228]. With the asymptotic dispersion relations (5.2) and (5.3) derived in Section 5, this can be used to obtain explicit results for the effective permittivity. We remark that this result is different from the effective mass approximation, where the effective permittivity is obtained by differentiating the dispersion relation as

$$(\epsilon_{\text{eff}}^{-1})_{ij} = \frac{1}{2}\frac{\partial^2[c^{-2}\omega(\boldsymbol{k})^2]}{\partial k_i \partial k_j} \tag{3.7}$$

This formula is based on the idea that the group velocity should be the same for waves propagating in the heterogeneous medium and in the fictitious homogeneous medium. The results are identical at the origin ($k_{xy} = k_z = 0$), which is the classical homogenization regime.

4 Piecewise constant media

With $\epsilon(z)$ being piecewise constant, say, ϵ_1 for $0 < z < a_1$ and ϵ_2 for $a_1 < z < a$, we have the following equations for $E_\perp$ and E_k:

$$\begin{cases} E'' + (c^{-2}\omega^2\epsilon_1 - k_{xy}^2)E = 0 & 0 < z < a_1 \\ E'' + (c^{-2}\omega^2\epsilon_2 - k_{xy}^2)E = 0 & a_1 < z < a \end{cases} \tag{4.1}$$

In the following, we use the notation $a_2 = a - a_1$ to indicate the thickness of material 2 when appropriate. Introducing $k_1^2 = c^{-2}\omega^2\epsilon_1 - k_{xy}^2$ and $k_2^2 = c^{-2}\omega^2\epsilon_2 - k_{xy}^2$, the general solution is

$$E(z) = \begin{cases} A\mathrm{e}^{\mathrm{i}k_1 z} + B\mathrm{e}^{-\mathrm{i}k_1 z} & 0 < z < a_1 \\ C\mathrm{e}^{\mathrm{i}k_2 z} + D\mathrm{e}^{-\mathrm{i}k_2 z} & a_1 < z < a \end{cases} \tag{4.2}$$

Looking at the original equations, it is clear that we must require $E_\perp$, $\partial_z E_\perp$, E_k, and $\frac{c^{-2}\omega^2\epsilon(z)}{c^{-2}\omega^2\epsilon(z)-k^2}\partial_z E_k$ to be continuous across the boundaries. Taking into account the pseudoperiodicity $E_\perp(z+a) = \mathrm{e}^{\mathrm{i}k_z a}E_\perp(z)$, the conditions for $E_\perp$ implies

$$(A+B)\mathrm{e}^{\mathrm{i}k_z a} = C\mathrm{e}^{\mathrm{i}k_2 a} + D\mathrm{e}^{-\mathrm{i}k_2 a} \tag{4.3}$$

$$A\mathrm{e}^{\mathrm{i}k_1 a_1} + B\mathrm{e}^{-\mathrm{i}k_1 a_1} = C\mathrm{e}^{\mathrm{i}k_2 a_1} + D\mathrm{e}^{-\mathrm{i}k_2 a_1} \tag{4.4}$$

$$(\mathrm{i}k_1 A - \mathrm{i}k_1 B)\mathrm{e}^{\mathrm{i}k_z a} = \mathrm{i}k_2 C\mathrm{e}^{\mathrm{i}k_2 a} - \mathrm{i}k_2 D\mathrm{e}^{-\mathrm{i}k_2 a} \tag{4.5}$$

$$\mathrm{i}k_1 A\mathrm{e}^{\mathrm{i}k_1 a_1} - \mathrm{i}k_1 B\mathrm{e}^{-\mathrm{i}k_1 a_1} = \mathrm{i}k_2 C\mathrm{e}^{\mathrm{i}k_2 a_1} - \mathrm{i}k_2 D\mathrm{e}^{-\mathrm{i}k_2 a_1} \tag{4.6}$$

This system of equations has nontrivial solutions only if the following determinant is zero:

$$0 = \begin{vmatrix} \mathrm{e}^{\mathrm{i}k_z a} & \mathrm{e}^{\mathrm{i}k_z a} & -\mathrm{e}^{\mathrm{i}k_2 a} & -\mathrm{e}^{-\mathrm{i}k_2 a} \\ \mathrm{e}^{\mathrm{i}k_1 a_1} & \mathrm{e}^{-\mathrm{i}k_1 a_1} & -\mathrm{e}^{\mathrm{i}k_2 a_1} & -\mathrm{e}^{-\mathrm{i}k_2 a_1} \\ k_1\mathrm{e}^{\mathrm{i}k_z a} & -k_1\mathrm{e}^{\mathrm{i}k_z a} & -k_2\mathrm{e}^{\mathrm{i}k_2 a} & k_2\mathrm{e}^{-\mathrm{i}k_2 a} \\ k_1\mathrm{e}^{\mathrm{i}k_1 a_1} & -k_1\mathrm{e}^{-\mathrm{i}k_1 a_1} & -k_2\mathrm{e}^{\mathrm{i}k_2 a_1} & k_2\mathrm{e}^{-\mathrm{i}k_2 a_1} \end{vmatrix} \tag{4.7}$$

It is shown in [18] that this is equivalent to the condition (see also [12] and [6, pp. 181–186])

$$0 = \cos(k_z a) - \cos(k_1 a_1)\cos(k_2 a_2) + \frac{1}{2}\left(\frac{k_1}{k_2} + \frac{k_2}{k_1}\right)\sin(k_1 a_1)\sin(k_2 a_2) \tag{4.8}$$

where $a = a_1 + a_2$. Requiring E_k and $\frac{c^{-2}\omega^2\epsilon(z)}{c^{-2}\omega^2\epsilon(z)-k_{xy}^2}\partial_z E_k$ to be continuous implies

$$(A+B)\mathrm{e}^{\mathrm{i}k_z a} = C\mathrm{e}^{\mathrm{i}k_2 a} + D\mathrm{e}^{-\mathrm{i}k_2 a} \tag{4.9}$$

$$A\mathrm{e}^{\mathrm{i}k_1 a_1} + B\mathrm{e}^{-\mathrm{i}k_1 a_1} = C\mathrm{e}^{\mathrm{i}k_2 a_1} + D\mathrm{e}^{-\mathrm{i}k_2 a_1} \tag{4.10}$$

$$(\frac{\mathrm{i}\epsilon_1}{k_1}A - \frac{\mathrm{i}\epsilon_1}{k_1}B)\mathrm{e}^{\mathrm{i}k_z a} = \frac{\mathrm{i}\epsilon_2}{k_2}C\mathrm{e}^{\mathrm{i}k_2 a} - \frac{\mathrm{i}\epsilon_2}{k_2}D\mathrm{e}^{-\mathrm{i}k_2 a} \tag{4.11}$$

$$\frac{\mathrm{i}\epsilon_1}{k_1}A\mathrm{e}^{\mathrm{i}k_1 a_1} - \frac{\mathrm{i}\epsilon_1}{k_1}B\mathrm{e}^{-\mathrm{i}k_1 a_1} = \frac{\mathrm{i}\epsilon_2}{k_2}C\mathrm{e}^{\mathrm{i}k_2 a_1} - \frac{\mathrm{i}\epsilon_2}{k_2}D\mathrm{e}^{-\mathrm{i}k_2 a_1} \tag{4.12}$$

with the corresponding determinant condition

$$0 = \begin{vmatrix} \mathrm{e}^{\mathrm{i}k_z a} & \mathrm{e}^{\mathrm{i}k_z a} & -\mathrm{e}^{\mathrm{i}k_2 a} & -\mathrm{e}^{-\mathrm{i}k_2 a} \\ \mathrm{e}^{\mathrm{i}k_1 a_1} & \mathrm{e}^{-\mathrm{i}k_1 a_1} & -\mathrm{e}^{\mathrm{i}k_2 a_1} & -\mathrm{e}^{-\mathrm{i}k_2 a_1} \\ \frac{\epsilon_1}{k_1}\mathrm{e}^{\mathrm{i}k_z a} & -\frac{\epsilon_1}{k_1}\mathrm{e}^{\mathrm{i}k_z a} & -\frac{\epsilon_2}{k_2}\mathrm{e}^{\mathrm{i}k_2 a} & \frac{\epsilon_2}{k_2}\mathrm{e}^{-\mathrm{i}k_2 a} \\ \frac{\epsilon_1}{k_1}\mathrm{e}^{\mathrm{i}k_1 a_1} & -\frac{\epsilon_1}{k_1}\mathrm{e}^{-\mathrm{i}k_1 a_1} & -\frac{\epsilon_2}{k_2}\mathrm{e}^{\mathrm{i}k_2 a_1} & \frac{\epsilon_2}{k_2}\mathrm{e}^{-\mathrm{i}k_2 a_1} \end{vmatrix} \tag{4.13}$$

which can be simplified to [12, 18]

$$0 = \cos(k_z a) - \cos(k_1 a_1)\cos(k_2 a_2) + \frac{1}{2}\left(\frac{\epsilon_1 k_2}{\epsilon_2 k_1} + \frac{\epsilon_2 k_1}{\epsilon_1 k_2}\right)\sin(k_1 a_1)\sin(k_2 a_2) \tag{4.14}$$

5 Asymptotic solutions

When looking for small eigenvalues ω, it is reasonable to assume the dimensionless quantities $k_{xy}a$, $k_z a$, and $c^{-1}\omega a$ to be small. This allows for a Taylor expansion of the determinant conditions, and we use the program Maple to handle the lengthy calculations necessary in this section. Observe that we explicitly include the scale a in this section, in order to obtain dimensionless quantities in the series involved.

The procedure is as follows: we assume that $k_{xy}a$ and $k_z a$ both scale as a generic parameter ka, and that the solution ω can be written as

$$(c^{-1}\omega a)^2 = \alpha_0(ka)^2 + \alpha_1(ka)^4 + \mathrm{O}((ka)^6) \tag{5.1}$$

We then substitute this together with $k_{1,2}^2 = c^{-2}\omega^2\epsilon_{1,2} - k_{xy}^2$ in the exact dispersion relations (4.8) and (4.14), and let Maple compute the series expansion with respect to ka. Solving the dispersion relation for each level of ka, we find the following asympotic solution, where we write $f_1 = a_1/a$ and $f_2 = a_2/a$ to indicate the volume fractions of the materials,

$$(c^{-1}\omega a)^2 = \frac{(k_{xy}a)^2 + (k_z a)^2}{f_1\epsilon_1 + f_2\epsilon_2} - \frac{1}{12}\frac{(\epsilon_1 - \epsilon_2)^2(f_1 f_2)^2}{(f_1\epsilon_1 + f_2\epsilon_2)^3}\left[(k_{xy}^2 a)^2 + (k_z a)^2\right]^2 + \mathrm{O}((ka)^6) \tag{5.2}$$

for (4.8), and

$$\begin{aligned}(c^{-1}\omega a)^2 &= (f_1\epsilon_1^{-1} + f_2\epsilon_2^{-1})(k_{xy}a)^2 + (f_1\epsilon_1 + f_2\epsilon_2)^{-1}(k_z a)^2 \\ &- \frac{1}{24}\frac{(\epsilon_1-\epsilon_2)^2(f_1 f_2)^2}{(\epsilon_1\epsilon_2)^2(f_1\epsilon_1 + f_2\epsilon_2)^3}\left[(f_1\epsilon_1 + f_2\epsilon_2)^2(k_{xy}a)^2 - \epsilon_1\epsilon_2(k_z a)^2\right]^2 + \mathrm{O}((ka)^6)\end{aligned} \tag{5.3}$$

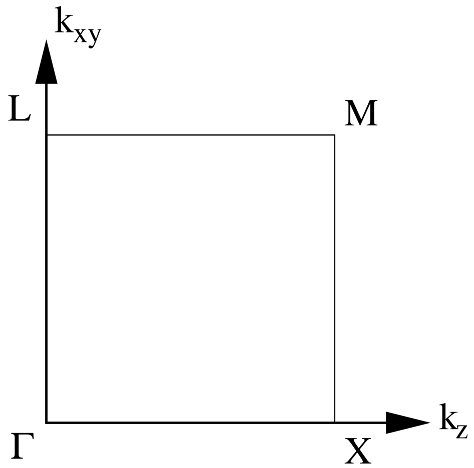

Figure 1: Definition of the wave vector points used in figures 2, 3, and 4. Γ is the origin, X corresponds to $(k_z a, k_{xy} a) = (\pi, 0)$, M corresponds to $(k_z a, k_{xy} a) = (\pi, \pi)$, and L corresponds to $(k_z a, k_{xy} a) = (0, \pi)$. Since there is no welldefined period in x and y, the limits for k_{xy} are arbitrary and have been chosen for symmetry.

for (4.14). The second term, proportional to $(ka)^4$, in these expressions is the dominating contribution to the spatial dispersion for large wavelengths. We see that this term is zero in both cases if $\epsilon_1 = \epsilon_2$, corresponding to a homogeneous material.

In (5.2), there is no formal difference between the wavenumber k_{xy}, concerned with propagation in the xy-plane, and the wavenumber k_z, concerned with propagation in the z-direction. Thus, (5.2) is an isotropic dispersion relation.

But in (5.3), there is clearly a difference between k_{xy} and k_z, and the dispersion relation is anisotropic. Furthermore, it is seen that the second term in (5.3) can be forced to zero by choosing $k_z = k_{xy}(f_1\epsilon_1 + f_2\epsilon_2)/\sqrt{\epsilon_1\epsilon_2}$. This means there exists directions of propagation where the spatial dispersion is minimal.

5.1 Numerical illustrations

In this subsection, we compare numerical solutions to the dispersion relations (4.8) and (4.14) with the asymptotic solutions (5.2) and (5.3). The plots in figures 2, 3, and 4, show $c^{-1}\omega a$ as a function of $k_z a$ and $k_{xy} a$ in the meaning that these parameters are varied linearly between the points Γ, X, M, and L, depicted in Figure 1.

The results are given for three different contrasts. In each case, we choose $f_1 = f_2 = 1/2$ and $\epsilon_1 = 1$, and choose $\epsilon_2 = 2$, 10, and 50. It can be seen that the asymptotic solution is very good for small wave numbers (large wavelengths), but may fail miserably for large contrasts and high wavenumbers. Higher order terms are necessary in these cases.

The isotropy of the asymptotic dispersion relation (5.2) causes the dashed blue curves in the top part of the figures to be symmetric. It is interesting to note, that the exact solution (solid red curves) is *not* symmetric.

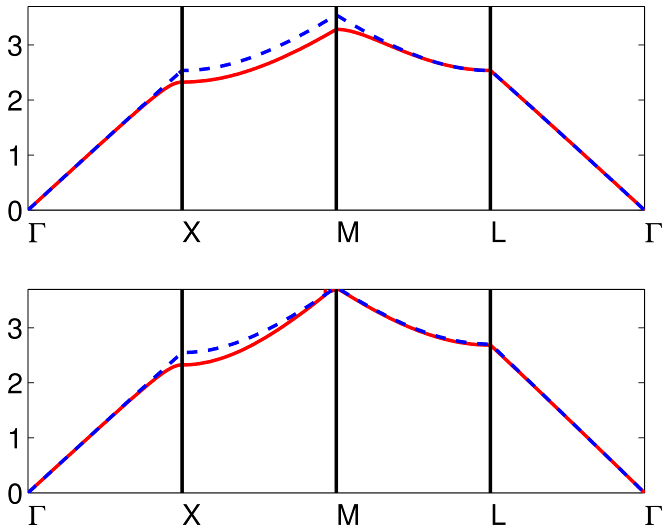

Figure 2: Dispersion relations for $f_1 = f_2 = 1/2$, $\epsilon_1 = 1$, and $\epsilon_2 = 2$. The solid red curve in the upper diagram is the exact solution (4.8), and the dashed blue curve is the asymptotic solution (5.2). The solid red curve in the lower diagram is the exact solution (4.14), and the dashed blue curve is the asymptotic solution (5.3).

5.2 Homogenized permittivity

As described in Section 3, the effective permittivity is given by

$$\epsilon_{\text{eff}} = \frac{k_{xy}^2 + k_z^2}{c^{-2}\omega^2} \tag{5.4}$$

for both polarizations. We insert the asymptotic dispersion relations (5.2) and (5.3) in this formula to obtain

$$\begin{aligned} \epsilon_{\text{eff}} = \left[(k_{xy}a)^2 + (k_z a)^2\right] \Bigg\{ & \frac{(k_{xy}a)^2 + (k_z a)^2}{f_1\epsilon_1 + f_2\epsilon_2} \\ & - \frac{1}{12}\frac{(\epsilon_1 - \epsilon_2)^2 (f_1 f_2)^2}{(f_1\epsilon_1 + f_2\epsilon_2)^3} \left[(k_{xy}^2 a)^2 + (k_z a)^2\right]^2 + \mathrm{O}((ka)^6) \Bigg\}^{-1} \\ = f_1\epsilon_1 + f_2\epsilon_2 + & \frac{1}{12}\frac{(\epsilon_1 - \epsilon_2)^2 (f_1 f_2)^2}{f_1\epsilon_1 + f_2\epsilon_2} \left[(k_{xy}^2 a)^2 + (k_z a)^2\right] + \mathrm{O}((ka)^4) \end{aligned} \tag{5.5}$$

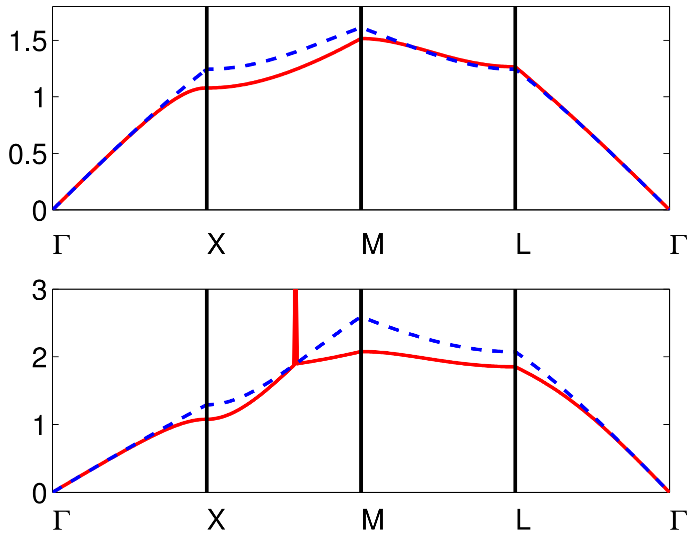

Figure 3: Dispersion relations for $f_1 = f_2 = 1/2$, $\epsilon_1 = 1$, and $\epsilon_2 = 10$. The spike in the solid red curve in the lower diagram is due to numerical difficulties in solving (4.14) for ω.

for the $E_\perp$ field, and

$$\epsilon_{\text{eff}} = \left[(k_{xy}a)^2 + (k_z a)^2\right] \Bigg\{ (f_1\epsilon_1^{-1} + f_2\epsilon_2^{-1})(k_{xy}a)^2 + (f_1\epsilon_1 + f_2\epsilon_2)^{-1}(k_z a)^2$$
$$- \frac{1}{24}\frac{(\epsilon_1 - \epsilon_2)^2(f_1 f_2)^2}{(\epsilon_1\epsilon_2)^2(f_1\epsilon_1 + f_2\epsilon_2)^3}\left[(f_1\epsilon_1 + f_2\epsilon_2)^2(k_{xy}a)^2 - \epsilon_1\epsilon_2(k_z a)^2\right]^2 + + \mathrm{O}((ka)^6)\Bigg\}^{-1}$$
$$= \frac{(k_{xy}a)^2 + (k_z a)^2}{(f_1\epsilon_1^{-1} + f_2\epsilon_2^{-1})(k_{xy}a)^2 + (f_1\epsilon_1 + f_2\epsilon_2)^{-1}(k_z a)^2}$$
$$+ \frac{1}{24}\frac{(\epsilon_1 - \epsilon_2)^2(f_1 f_2)^2}{(\epsilon_1\epsilon_2)^2(f_1\epsilon_1 + f_2\epsilon_2)^3}\frac{[(f_1\epsilon_1 + f_2\epsilon_2)^2(k_{xy}a)^2 - \epsilon_1\epsilon_2(k_z a)^2]^2\,[(k_{xy}a)^2 + (k_z a)^2]}{\left[(f_1\epsilon_1^{-1} + f_2\epsilon_2^{-1})(k_{xy}a)^2 + (f_1\epsilon_1 + f_2\epsilon_2)^{-1}(k_z a)^2\right]^2}$$
$$+ \mathrm{O}((ka)^4) \quad (5.6)$$

for the E_k and E_z fields. The dominating terms correspond to classical homogenization results. For polarizations parallel to the material interfaces, the effective material is simply the arithmetic average of the permittivity. For polarizations orthogonal to the interfaces, we obtain the harmonic average, which is easiest seen by setting $k_z = 0$ in (5.6), corresponding to a dominating E_z component. The first term for the effective permittivity in (5.6) is the classical formula for the effective permittivity for propagation of extraordinary rays in uniaxial media [13, p. 340], with the arithmetic average and the harmonic average as the principal values of the permittivity matrix.

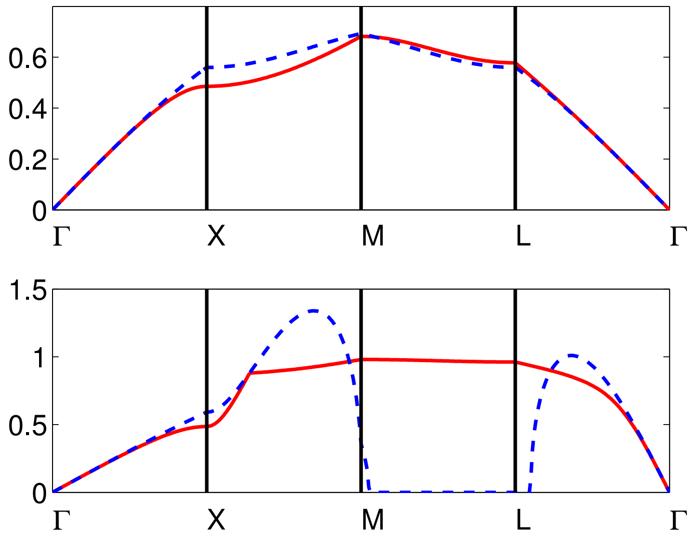

Figure 4: Dispersion relations for $f_1 = f_2 = 1/2$, $\epsilon_1 = 1$, and $\epsilon_2 = 50$. Note that the asymptotic solution (dashed blue curve) only gives imaginary solutions in a large interval.

The terms proportional to $(ka)^2$ in the equations above do not lend themselves easily to interpretation. However, they both share the property of being proportional to $(\epsilon_1 - \epsilon_2)^2(f_1 f_2)^2$. This means they are small whenever the contrast is small, or the volume fraction of one material is small. Thus, spatial dispersion can be expected to be more important for composite materials with high contrast and sizable volume fractions, than for almost homogeneous materials with low contrast.

6 Conclusions

The exact and asymptotic versions of the dispersion relations for layered media presented in this paper can be used to check homogenization procedures intended for finite scale homogenization. In particular, the formulas in [16] can be further investigated using these results, and further information on the validity range of homogenization results may be obtained. Also, further comparisons should be made to the classical results from the effective mass approximation, where the homogenized matrix is found from the Hessian matrix of $\omega(\boldsymbol{k})$ at $\boldsymbol{k} = \boldsymbol{0}$.

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