

# LUND UNIVERSITY

Exact and asymptotic dispersion relations for homogenization of stratified media with two phases

Sjöberg, Daniel

2005

Link to publication

Citation for published version (APA):

Sjöberg, D. (2005). Exact and asymptotic dispersion relations for homogenization of stratified media with two phases. (Technical Report LUTEDX/(TEAT-7133)/1-11/(2005); Vol. TEAT-7133). [Publisher information missing].

Total number of authors: 1

#### General rights

Unless other specific re-use rights are stated the following general rights apply:

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights. • Users may download and print one copy of any publication from the public portal for the purpose of private study

or research.

- You may not further distribute the material or use it for any profit-making activity or commercial gain
   You may freely distribute the URL identifying the publication in the public portal

Read more about Creative commons licenses: https://creativecommons.org/licenses/

#### Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

#### LUND UNIVERSITY

**PO Box 117** 221 00 Lund +46 46-222 00 00

# Exact and asymptotic dispersion relations for homogenization of stratified media with two phases

**Daniel Sjöberg** 

Department of Electroscience Electromagnetic Theory Lund Institute of Technology Sweden





Daniel Sjöberg

Department of Electroscience Electromagnetic Theory Lund Institute of Technology P.O. Box 118 SE-221 00 Lund Sweden

> Editor: Gerhard Kristensson © Daniel Sjöberg, Lund, April 6, 2005

#### Abstract

Using exact dispersion relations for electromagnetic wave propagation in layered, periodic media, consisting of two phases, we derive explicit asymptotic solutions for small wavenumbers. These solutions are compared to the numerical solutions of the exact dispersion relations, and applications to homogenization problems are discussed. The results can be used as test cases for homogenization techniques intended for finite scale homogenization, that is, where the wavelength is not assumed infinitely large compared to the microscale.

# 1 Introduction

Many interesting problems in wave propagation depend on the dispersion relation, *i.e.*, a relation between the frequency  $\omega$  and the wavenumber k. The problem that has motivated the research presented in this paper is homogenization, where the goal is to model a material with a complicated microstructure with a homogeneous medium, making it easy to treat the wave propagation. The problem is to compute the properties of the fictitious homogeneous medium, which is usually done in the limit of infinitely long wavelength. Many ways have been designed to compute the homogenization, but the one that most directly concerns the dispersion relation is the effective mass approximation, where the effective permittivity is computed by differentiating the dispersion relation  $\omega(k)$  at k = 0. This technique has recently gotten considerable attention from mathematicians, see for instance [2, 3, 8, 10]. For a more thorough review of homogenization techniques, we refer to the books [4, 7, 14].

Some recent contributions [3, 10, 11, 15, 16] have shown that it is possible to do some homogenization even when the wavelength is not infinitely long. This corresponds to studying the dispersion relation for  $k \neq 0$ . In this paper, we first show that the homogenized permittivity is linked to the dispersion relation through a very simple formula in the case of nonmagnetic, stratified media. We then use the exact expressions for the dispersion relation for piecewise constant media given in [18] (maybe more easily accessible in [12, Appendix A]), to find the asymptotic solution when  $k \to 0$ . This provides explicit expressions for the homogenized permittivity up to a given order in k, although we only give a few terms. The results can be used as test cases for more general homogenization approaches.

Wave propagation in onedimensional periodic structures is described by Hill's equation [6, p. 178], with Mathieu's equation as a special case. More information on Mathieu functions can be found in, for instance, [1, Ch. 20] but we do not need much of the deep properties of these functions.

### 2 Basic equations

We study layered media, periodic in the z-direction with period a. The Floquet-Bloch theorem [5, 9] then states that the typical wave can be written (ignoring the

x and y dependence)

$$\boldsymbol{E}(z) = \mathrm{e}^{\mathrm{i}k_z z} \widetilde{\boldsymbol{E}}(z) \tag{2.1}$$

where  $\widetilde{E}$  is periodic in z. This implies pseudoperiodic boundary conditions,  $E(z + a) = e^{ik_z a} E(z)$ . The wavenumber  $k_z$  is called the Bloch wave number, and is a free parameter in the range  $k_z \in [-\pi/a, \pi/a]$ .

Inserting (2.1) into Maxwell's equations, the dispersion relation is determined from the eigenproblem

$$\nabla \times \boldsymbol{E} = i\omega\mu_0 \boldsymbol{H} \tag{2.2}$$

$$\nabla \times \boldsymbol{H} = -\mathrm{i}\omega\epsilon_0\epsilon(z)\boldsymbol{E} \tag{2.3}$$

where we assumed an isotropic, nonmagnetic medium depending only on z. The fields are given in SI units,  $\mu_0$  is the permeability of vacuum, and  $\epsilon_0$  is the permittivity of vacuum. The eigenvalue  $\omega$  depends on the wavenumber  $k_z$  through the pseudoperiodic boundary conditions.

We apply a Fourier transform in x and y to write

$$(\partial_z \hat{\boldsymbol{z}} + i\boldsymbol{k}_{xy}) \times \boldsymbol{E} = i\omega\mu_0 \boldsymbol{H}$$
(2.4)

$$(\partial_z \hat{\boldsymbol{z}} + i\boldsymbol{k}_{xy}) \times \boldsymbol{H} = -i\omega\epsilon_0\epsilon(z)\boldsymbol{E}$$
(2.5)

where the Fourier variable  $\mathbf{k}_{xy}$  can take any value in  $\mathbb{R}^2$ , although we are particularly interested in small values in homogenization problems. Eliminating  $\mathbf{H}$  implies

$$(\partial_z \hat{\boldsymbol{z}} + i\boldsymbol{k}_{xy}) \times [(\partial_z \hat{\boldsymbol{z}} + i\boldsymbol{k}_{xy}) \times \boldsymbol{E}] = c^{-2}\omega^2 \epsilon(z)\boldsymbol{E}$$
(2.6)

where  $c = 1/\sqrt{\epsilon_0 \mu_0}$  is the speed of light in vacuum. The left hand side is

$$\begin{aligned} (\partial_{z}\hat{\boldsymbol{z}} + \mathrm{i}\boldsymbol{k}_{xy}) \times [(\partial_{z}\hat{\boldsymbol{z}} + \mathrm{i}\boldsymbol{k}_{xy}) \times \boldsymbol{E}] &= (\partial_{z}\hat{\boldsymbol{z}} + \mathrm{i}\boldsymbol{k}_{xy}) \times (\partial_{z}\hat{\boldsymbol{z}} \times \boldsymbol{E} + \mathrm{i}\boldsymbol{k}_{xy} \times \boldsymbol{E}) \\ &= \partial_{z}^{2}\hat{\boldsymbol{z}} \times (\hat{\boldsymbol{z}} \times \boldsymbol{E}) + \partial_{z}\hat{\boldsymbol{z}} \times (\mathrm{i}\boldsymbol{k}_{xy} \times \boldsymbol{E}) + \mathrm{i}\boldsymbol{k}_{xy} \times (\partial_{z}\hat{\boldsymbol{z}} \times \boldsymbol{E}) + \mathrm{i}\boldsymbol{k}_{xy} \times (\mathrm{i}\boldsymbol{k}_{xy} \times \boldsymbol{E}) \\ &= -\partial_{z}^{2}[\mathbf{I}_{3} - \hat{\boldsymbol{z}}\hat{\boldsymbol{z}}] \cdot \boldsymbol{E} + \mathrm{i}k_{xy}\partial_{z}[\hat{\boldsymbol{z}} \times (\hat{\boldsymbol{k}}_{xy} \times \boldsymbol{E}) + \hat{\boldsymbol{k}}_{xy} \times (\hat{\boldsymbol{z}} \times \boldsymbol{E})] + k_{xy}^{2}[\mathbf{I}_{3} - \hat{\boldsymbol{k}}_{xy}\hat{\boldsymbol{k}}_{xy}] \cdot \boldsymbol{E} \end{aligned}$$

$$(2.7)$$

where we use a hat to indicate unit vectors. Writing  $\boldsymbol{E} = E_z \hat{\boldsymbol{z}} + E_k \hat{\boldsymbol{k}}_{xy} + E_{\perp} \hat{\boldsymbol{z}} \times \hat{\boldsymbol{k}}_{xy}$ implies  $\hat{\boldsymbol{z}} \times (\hat{\boldsymbol{k}}_{xy} \times \boldsymbol{E}) + \hat{\boldsymbol{k}}_{xy} \times (\hat{\boldsymbol{z}} \times \boldsymbol{E}) = \hat{\boldsymbol{k}}_{xy} E_z + \hat{\boldsymbol{z}} E_k$ , and we have

$$-\mathrm{i}k_{xy}\partial_z E_k + (c^{-2}\omega^2\epsilon(z) - k_{xy}^2)E_z = 0$$
(2.8)

$$\partial_z^2 E_k - ik_{xy}\partial_z E_z + c^{-2}\omega^2 \epsilon(z)E_k = 0$$
(2.9)

$$\partial_z^2 E_{\perp} + (c^{-2}\omega^2 \epsilon(z) - k_{xy}^2) E_{\perp} = 0$$
(2.10)

where (2.10) is Hill's equation, see for instance [6, p. 178]. A special case is Mathieu's equation, when  $\epsilon(z) = \epsilon_1 + \epsilon_2 \cos(2\pi z/a)$ . Combining the first and second equation results in

$$\partial_z \left[ \frac{c^{-2} \omega^2 \epsilon(z)}{c^{-2} \omega^2 \epsilon(z) - k_{xy}^2} \partial_z E_k \right] + c^{-2} \omega^2 \epsilon(z) E_k = 0$$
(2.11)

and we remind of the pseudoperiodic boundary conditions  $E_k(z+a) = e^{ik_z a} E_k(z)$ and  $E_{\perp}(z+a) = e^{ik_z a} E_{\perp}(z)$ .

## 3 Implications for homogenization

This investigation is motivated by the problem of homogenization, where we extract effective material parameters from a given geometry by considering propagation of waves with long wavelength, *i.e.*, small wavenumbers. In a physical sense, the effective permittivity is defined by the relation

$$\langle \boldsymbol{\epsilon} \cdot (\mathbf{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\boldsymbol{E}) \rangle = \boldsymbol{\epsilon}_{\mathrm{eff}} \cdot \langle \mathbf{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\boldsymbol{E} \rangle$$
 (3.1)

where the mean value is taken over the unit cell and the field  $\boldsymbol{E}$  is pseudoperiodic, *i.e.*,  $\boldsymbol{E}(\boldsymbol{x} + \boldsymbol{a}) = e^{i\boldsymbol{k}\cdot\boldsymbol{a}}\boldsymbol{E}(\boldsymbol{x})$  where  $\boldsymbol{a}$  is a lattice vector. In our case, the field  $\boldsymbol{E}$  is the solution to the equations at the end of Section 2. Note that this definition is not a strict mathematical one, but is rather based on physical intuition. However, as is shown in [15, 16], it is mathematically motivated in our case.

Starting with the  $E_{\perp}$ -field, we can use equation (2.10) to find

$$\left\langle \epsilon(z) \mathrm{e}^{-\mathrm{i}k_{z}z} E_{\perp} \right\rangle = \frac{1}{c^{-2}\omega^{2}} \left\langle \mathrm{e}^{-\mathrm{i}k_{z}z} (k_{xy}^{2} - \partial_{z}^{2}) E_{\perp} \right\rangle = \frac{1}{c^{-2}\omega^{2}} \left\langle (k_{xy}^{2} + k_{z}^{2} - \partial_{z}^{2}) \mathrm{e}^{-\mathrm{i}k_{z}z} E_{\perp} \right\rangle$$
$$= \frac{k_{xy}^{2} + k_{z}^{2}}{c^{-2}\omega^{2}} \left\langle \mathrm{e}^{-\mathrm{i}k_{z}z} E_{\perp} \right\rangle \quad (3.2)$$

where we used that  $e^{-ik_z z} \partial_z E_{\perp} = (\partial_z + ik_z) e^{-ik_z z} E_{\perp}$  and that the mean value of the derivative of a periodic function,  $\partial_z e^{-ik_z z} E_{\perp}$ , is zero.

For  $E_k$  and  $E_z$ , matters are a bit more messy. These fields are described by equations (2.8) and (2.9), and using the same technique as above we can prove

$$\left\langle \epsilon(z) \mathrm{e}^{-\mathrm{i}k_z z} E_k \right\rangle = \frac{k_z^2}{c^{-2} \omega^2} \left\langle \mathrm{e}^{-\mathrm{i}k_z z} E_k \right\rangle - \frac{k_{xy} k_z}{c^{-2} \omega^2} \left\langle \mathrm{e}^{-\mathrm{i}k_z z} E_z \right\rangle \tag{3.3}$$

$$\left\langle \epsilon(z) \mathrm{e}^{-\mathrm{i}k_z z} E_z \right\rangle = \frac{k_{xy}^2}{c^{-2} \omega^2} \left\langle \mathrm{e}^{-\mathrm{i}k_z z} E_z \right\rangle - \frac{k_{xy} k_z}{c^{-2} \omega^2} \left\langle \mathrm{e}^{-\mathrm{i}k_z z} E_k \right\rangle \tag{3.4}$$

As is explained in [16], it is sufficient to consider only field components orthogonal to the propagation direction, since these are the only ones concerned with the wave propagation. Since the wave vector is  $k_{xy}\hat{k}_{xy} + k_z\hat{z}$ , the orthogonal polarization is proportional to  $k_z\hat{k}_{xy} - k_{xy}\hat{z}$ , which corresponds to a combination of  $E_k$  and  $E_z$ proportional to  $k_zE_k - k_{xy}E_z$ . This combination satisfies

$$\left\langle \epsilon(z) \mathrm{e}^{-\mathrm{i}k_z z} (k_z E_k - k_{xy} E_z) \right\rangle = \frac{k_{xy}^2 + k_z^2}{c^{-2} \omega^2} \left\langle \mathrm{e}^{-\mathrm{i}k_z z} (k_z E_k - k_{xy} E_z) \right\rangle \tag{3.5}$$

Thus, for both polarizations we have the conclusion

$$\epsilon_{\text{eff}} = \frac{k_{xy}^2 + k_z^2}{c^{-2}\omega^2} \tag{3.6}$$

which is also used in [17] and [14, pp. 227–228]. With the asymptotic dispersion relations (5.2) and (5.3) derived in Section 5, this can be used to obtain explicit results for the effective permittivity. We remark that this result is different from

the effective mass approximation, where the effective permittivity is obtained by differentiating the dispersion relation as

$$(\epsilon_{\text{eff}}^{-1})_{ij} = \frac{1}{2} \frac{\partial^2 [c^{-2} \omega(\boldsymbol{k})^2]}{\partial k_i \partial k_j}$$
(3.7)

This formula is based on the idea that the group velocity should be the same for waves propagating in the heterogeneous medium and in the fictitious homogeneous medium. The results are identical at the origin  $(k_{xy} = k_z = 0)$ , which is the classical homogenization regime.

## 4 Piecewise constant media

With  $\epsilon(z)$  being piecewise constant, say,  $\epsilon_1$  for  $0 < z < a_1$  and  $\epsilon_2$  for  $a_1 < z < a$ , we have the following equations for  $E_{\perp}$  and  $E_k$ :

$$\begin{cases} E'' + (c^{-2}\omega^2\epsilon_1 - k_{xy}^2)E = 0 & 0 < z < a_1 \\ E'' + (c^{-2}\omega^2\epsilon_2 - k_{xy}^2)E = 0 & a_1 < z < a \end{cases}$$
(4.1)

In the following, we use the notation  $a_2 = a - a_1$  to indicate the thickness of material 2 when appropriate. Introducing  $k_1^2 = c^{-2}\omega^2\epsilon_1 - k_{xy}^2$  and  $k_2^2 = c^{-2}\omega^2\epsilon_2 - k_{xy}^2$ , the general solution is

$$E(z) = \begin{cases} A e^{ik_1 z} + B e^{-ik_1 z} & 0 < z < a_1 \\ C e^{ik_2 z} + D e^{-ik_2 z} & a_1 < z < a \end{cases}$$
(4.2)

Looking at the original equations, it is clear that we must require  $E_{\perp}$ ,  $\partial_z E_{\perp}$ ,  $E_k$ , and  $\frac{c^{-2\omega^2\epsilon(z)}}{c^{-2\omega^2\epsilon(z)-k^2}}\partial_z E_k$  to be continuous across the boundaries. Taking into account the pseudoperiodicity  $E_{\perp}(z+a) = e^{ik_z a} E_{\perp}(z)$ , the conditions for  $E_{\perp}$  implies

$$(A+B)e^{ik_{z}a} = Ce^{ik_{2}a} + De^{-ik_{2}a}$$
(4.3)

$$Ae^{ik_1a_1} + Be^{-ik_1a_1} = Ce^{ik_2a_1} + De^{-ik_2a_1}$$
(4.4)

$$(ik_1A - ik_1B)e^{ik_za} = ik_2Ce^{ik_2a} - ik_2De^{-ik_2a}$$
 (4.5)

$$ik_1 A e^{ik_1 a_1} - ik_1 B e^{-ik_1 a_1} = ik_2 C e^{ik_2 a_1} - ik_2 D e^{-ik_2 a_1}$$
 (4.6)

This system of equations has nontrivial solutions only if the following determinant is zero:  $ik_{a}a = ik_{a}a = -ik_{a}a$ 

$$0 = \begin{vmatrix} e^{ik_{z}a} & e^{ik_{z}a} & -e^{ik_{2}a} & -e^{-ik_{2}a} \\ e^{ik_{1}a_{1}} & e^{-ik_{1}a_{1}} & -e^{ik_{2}a_{1}} & -e^{-ik_{2}a_{1}} \\ k_{1}e^{ik_{z}a} & -k_{1}e^{ik_{z}a} & -k_{2}e^{ik_{2}a} & k_{2}e^{-ik_{2}a} \\ k_{1}e^{ik_{1}a_{1}} & -k_{1}e^{-ik_{1}a_{1}} & -k_{2}e^{ik_{2}a_{1}} & k_{2}e^{-ik_{2}a_{1}} \end{vmatrix}$$
(4.7)

It is shown in [18] that this is equivalent to the condition (see also [12] and [6, pp. 181–186])

$$0 = \cos(k_2 a) - \cos(k_1 a_1) \cos(k_2 a_2) + \frac{1}{2} \left(\frac{k_1}{k_2} + \frac{k_2}{k_1}\right) \sin(k_1 a_1) \sin(k_2 a_2)$$
(4.8)

where  $a = a_1 + a_2$ . Requiring  $E_k$  and  $\frac{c^{-2\omega^2\epsilon(z)}}{c^{-2\omega^2\epsilon(z)}-k_{xy}^2}\partial_z E_k$  to be continuous implies

$$(A+B)e^{ik_z a} = Ce^{ik_2 a} + De^{-ik_2 a}$$
(4.9)

$$Ae^{ik_1a_1} + Be^{-ik_1a_1} = Ce^{ik_2a_1} + De^{-ik_2a_1}$$
(4.10)

$$\left(\frac{\mathrm{i}\epsilon_1}{k_1}A - \frac{\mathrm{i}\epsilon_1}{k_1}B\right)\mathrm{e}^{\mathrm{i}k_z a} = \frac{\mathrm{i}\epsilon_2}{k_2}C\mathrm{e}^{\mathrm{i}k_2 a} - \frac{\mathrm{i}\epsilon_2}{k_2}D\mathrm{e}^{-\mathrm{i}k_2 a} \tag{4.11}$$

$$\frac{i\epsilon_1}{k_1} A e^{ik_1 a_1} - \frac{i\epsilon_1}{k_1} B e^{-ik_1 a_1} = \frac{i\epsilon_2}{k_2} C e^{ik_2 a_1} - \frac{i\epsilon_2}{k_2} D e^{-ik_2 a_1}$$
(4.12)

with the corresponding determinant condition

$$0 = \begin{vmatrix} e^{ik_{z}a} & e^{ik_{z}a} & -e^{ik_{2}a} & -e^{-ik_{2}a} \\ e^{ik_{1}a_{1}} & e^{-ik_{1}a_{1}} & -e^{ik_{2}a_{1}} & -e^{-ik_{2}a_{1}} \\ \frac{\epsilon_{1}}{k_{1}}e^{ik_{z}a} & -\frac{\epsilon_{1}}{k_{1}}e^{ik_{z}a} & -\frac{\epsilon_{2}}{k_{2}}e^{ik_{2}a} & \frac{\epsilon_{2}}{k_{2}}e^{-ik_{2}a} \\ \frac{\epsilon_{1}}{k_{1}}e^{ik_{1}a_{1}} & -\frac{\epsilon_{1}}{k_{1}}e^{-ik_{1}a_{1}} & -\frac{\epsilon_{2}}{k_{2}}e^{ik_{2}a_{1}} & \frac{\epsilon_{2}}{k_{2}}e^{-ik_{2}a_{1}} \end{vmatrix}$$
(4.13)

which can be simplified to [12, 18]

$$0 = \cos(k_2 a) - \cos(k_1 a_1) \cos(k_2 a_2) + \frac{1}{2} \left( \frac{\epsilon_1 k_2}{\epsilon_2 k_1} + \frac{\epsilon_2 k_1}{\epsilon_1 k_2} \right) \sin(k_1 a_1) \sin(k_2 a_2) \quad (4.14)$$

# 5 Asymptotic solutions

When looking for small eigenvalues  $\omega$ , it is reasonable to assume the dimensionless quantities  $k_{xy}a$ ,  $k_za$ , and  $c^{-1}\omega a$  to be small. This allows for a Taylor expansion of the determinant conditions, and we use the program Maple to handle the lengthy calculations necessary in this section. Observe that we explicitly include the scale ain this section, in order to obtain dimensionless quantities in the series involved.

The procedure is as follows: we assume that  $k_{xy}a$  and  $k_za$  both scale as a generic parameter ka, and that the solution  $\omega$  can be written as

$$(c^{-1}\omega a)^2 = \alpha_0 (ka)^2 + \alpha_1 (ka)^4 + \mathcal{O}((ka)^6)$$
(5.1)

We then substitute this together with  $k_{1,2}^2 = c^{-2}\omega^2 \epsilon_{1,2} - k_{xy}^2$  in the exact dispersion relations (4.8) and (4.14), and let Maple compute the series expansion with respect to ka. Solving the dispersion relation for each level of ka, we find the following asymptotic solution, where we write  $f_1 = a_1/a$  and  $f_2 = a_2/a$  to indicate the volume fractions of the materials,

$$(c^{-1}\omega a)^{2} = \frac{(k_{xy}a)^{2} + (k_{z}a)^{2}}{f_{1}\epsilon_{1} + f_{2}\epsilon_{2}} - \frac{1}{12} \frac{(\epsilon_{1} - \epsilon_{2})^{2}(f_{1}f_{2})^{2}}{(f_{1}\epsilon_{1} + f_{2}\epsilon_{2})^{3}} \left[ (k_{xy}^{2}a)^{2} + (k_{z}a)^{2} \right]^{2} + O((ka)^{6})$$
(5.2)

for (4.8), and

$$(c^{-1}\omega a)^{2} = (f_{1}\epsilon_{1}^{-1} + f_{2}\epsilon_{2}^{-1})(k_{xy}a)^{2} + (f_{1}\epsilon_{1} + f_{2}\epsilon_{2})^{-1}(k_{z}a)^{2} - \frac{1}{24}\frac{(\epsilon_{1} - \epsilon_{2})^{2}(f_{1}f_{2})^{2}}{(\epsilon_{1}\epsilon_{2})^{2}(f_{1}\epsilon_{1} + f_{2}\epsilon_{2})^{3}} \left[ (f_{1}\epsilon_{1} + f_{2}\epsilon_{2})^{2}(k_{xy}a)^{2} - \epsilon_{1}\epsilon_{2}(k_{z}a)^{2} \right]^{2} + O((ka)^{6})$$
(5.3)



**Figure 1**: Definition of the wave vector points used in figures 2, 3, and 4.  $\Gamma$  is the origin, X corresponds to  $(k_z a, k_{xy} a) = (\pi, 0)$ , M corresponds to  $(k_z a, k_{xy} a) = (\pi, \pi)$ , and L corresponds to  $(k_z a, k_{xy} a) = (0, \pi)$ . Since there is no welldefined period in x and y, the limits for  $k_{xy}$  are arbitrary and have been chosen for symmetry.

for (4.14). The second term, proportional to  $(ka)^4$ , in these expressions is the dominating contribution to the spatial dispersion for large wavelengths. We see that this term is zero in both cases if  $\epsilon_1 = \epsilon_2$ , corresponding to a homogeneous material.

In (5.2), there is no formal difference between the wavenumber  $k_{xy}$ , concerned with propagation in the xy-plane, and the wavenumber  $k_z$ , concerned with propagation in the z-direction. Thus, (5.2) is an isotropic dispersion relation.

But in (5.3), there is clearly a difference between  $k_{xy}$  and  $k_z$ , and the dispersion relation is anisotropic. Furthermore, it is seen that the second term in (5.3) can be forced to zero by choosing  $k_z = k_{xy}(f_1\epsilon_1 + f_2\epsilon_2)/\sqrt{\epsilon_1\epsilon_2}$ . This means there exists directions of propagation where the spatial dispersion is minimal.

#### 5.1 Numerical illustrations

In this subsection, we compare numerical solutions to the dispersion relations (4.8) and (4.14) with the asymptotic solutions (5.2) and (5.3). The plots in figures 2, 3, and 4, show  $c^{-1}\omega a$  as a function of  $k_z a$  and  $k_{xy} a$  in the meaning that these parameters are varied linearly between the points  $\Gamma$ , X, M, and L, depicted in Figure 1.

The results are given for three different contrasts. In each case, we choose  $f_1 = f_2 = 1/2$  and  $\epsilon_1 = 1$ , and choose  $\epsilon_2 = 2$ , 10, and 50. It can be seen that the asymptotic solution is very good for small wave numbers (large wavelengths), but may fail miserably for large contrasts and high wavenumbers. Higher order terms are necessary in these cases.

The isotropy of the asymptotic dispersion relation (5.2) causes the dashed blue curves in the top part of the figures to be symmetric. It is interesting to note, that the exact solution (solid red curves) is *not* symmetric.



**Figure 2**: Dispersion relations for  $f_1 = f_2 = 1/2$ ,  $\epsilon_1 = 1$ , and  $\epsilon_2 = 2$ . The solid red curve in the upper diagram is the exact solution (4.8), and the dashed blue curve is the asymptotic solution (5.2). The solid red curve in the lower diagram is the exact solution (4.14), and the dashed blue curve is the asymptotic solution (5.3).

### 5.2 Homogenized permittivity

As described in Section 3, the effective permittivity is given by

$$\epsilon_{\text{eff}} = \frac{k_{xy}^2 + k_z^2}{c^{-2}\omega^2} \tag{5.4}$$

for both polarizations. We insert the asymptotic dispersion relations (5.2) and (5.3) in this formula to obtain

$$\epsilon_{\text{eff}} = \left[ (k_{xy}a)^2 + (k_za)^2 \right] \left\{ \frac{(k_{xy}a)^2 + (k_za)^2}{f_1\epsilon_1 + f_2\epsilon_2} - \frac{1}{12} \frac{(\epsilon_1 - \epsilon_2)^2 (f_1f_2)^2}{(f_1\epsilon_1 + f_2\epsilon_2)^3} \left[ (k_{xy}^2a)^2 + (k_za)^2 \right]^2 + O((ka)^6) \right\}^{-1} = f_1\epsilon_1 + f_2\epsilon_2 + \frac{1}{12} \frac{(\epsilon_1 - \epsilon_2)^2 (f_1f_2)^2}{f_1\epsilon_1 + f_2\epsilon_2} \left[ (k_{xy}^2a)^2 + (k_za)^2 \right] + O((ka)^4) \quad (5.5)$$



**Figure 3**: Dispersion relations for  $f_1 = f_2 = 1/2$ ,  $\epsilon_1 = 1$ , and  $\epsilon_2 = 10$ . The spike in the solid red curve in the lower diagram is due to numerical difficulties in solving (4.14) for  $\omega$ .

for the  $E_{\perp}$  field, and

$$\epsilon_{\text{eff}} = \left[ (k_{xy}a)^2 + (k_za)^2 \right] \left\{ (f_1\epsilon_1^{-1} + f_2\epsilon_2^{-1})(k_{xy}a)^2 + (f_1\epsilon_1 + f_2\epsilon_2)^{-1}(k_za)^2 - \frac{1}{24} \frac{(\epsilon_1 - \epsilon_2)^2(f_1f_2)^2}{(\epsilon_1\epsilon_2)^2(f_1\epsilon_1 + f_2\epsilon_2)^3} \left[ (f_1\epsilon_1 + f_2\epsilon_2)^2(k_{xy}a)^2 - \epsilon_1\epsilon_2(k_za)^2 \right]^2 + O((ka)^6) \right\}^{-1} \\ = \frac{(k_{xy}a)^2 + (k_za)^2}{(f_1\epsilon_1^{-1} + f_2\epsilon_2^{-1})(k_{xy}a)^2 + (f_1\epsilon_1 + f_2\epsilon_2)^{-1}(k_za)^2} \\ + \frac{1}{24} \frac{(\epsilon_1 - \epsilon_2)^2(f_1f_2)^2}{(\epsilon_1\epsilon_2)^2(f_1\epsilon_1 + f_2\epsilon_2)^3} \frac{\left[ (f_1\epsilon_1 + f_2\epsilon_2)^2(k_{xy}a)^2 - \epsilon_1\epsilon_2(k_za)^2 \right]^2 \left[ (k_{xy}a)^2 + (k_za)^2 \right]}{\left[ (f_1\epsilon_1^{-1} + f_2\epsilon_2^{-1})(k_{xy}a)^2 + (f_1\epsilon_1 + f_2\epsilon_2)^{-1}(k_za)^2 \right]^2} \\ + O((ka)^4) \quad (5.6)$$

for the  $E_k$  and  $E_z$  fields. The dominating terms correspond to classical homogenization results. For polarizations parallel to the material interfaces, the effective material is simply the arithmetic average of the permittivity. For polarizations orthogonal to the interfaces, we obtain the harmonic average, which is easiest seen by setting  $k_z = 0$  in (5.6), corresponding to a dominating  $E_z$  component. The first term for the effective permittivity in (5.6) is the classical formula for the effective permittivity for propagation of extraordinary rays in uniaxial media [13, p. 340], with the arithmetic average and the harmonic average as the principal values of the permittivity matrix.



**Figure 4**: Dispersion relations for  $f_1 = f_2 = 1/2$ ,  $\epsilon_1 = 1$ , and  $\epsilon_2 = 50$ . Note that the asymptotic solution (dashed blue curve) only gives imaginary solutions in a large interval.

The terms proportional to  $(ka)^2$  in the equations above do not lend themselves easily to interpretation. However, they both share the property of being proportional to  $(\epsilon_1 - \epsilon_2)^2 (f_1 f_2)^2$ . This means they are small whenever the contrast is small, or the volume fraction of one material is small. Thus, spatial dispersion can be expected to be more important for composite materials with high contrast and sizable volume fractions, than for almost homogeneous materials with low contrast.

## 6 Conclusions

The exact and asymptotic versions of the dispersion relations for layered media presented in this paper can be used to check homogenization procedures intended for finite scale homogenization. In particular, the formulas in [16] can be further investigated using these results, and further information on the validity range of homogenization results may be obtained. Also, further comparisons should be made to the classical results from the effective mass approximation, where the homogenized matrix is found from the Hessian matrix of  $\omega(\mathbf{k})$  at  $\mathbf{k} = \mathbf{0}$ .

# References

 M. Abramowitz and I. A. Stegun, editors. *Handbook of Mathematical Functions*. Applied Mathematics Series No. 55. National Bureau of Standards, Washington D.C., 1970.

- [2] G. Allaire and C. Conca. Bloch wave homogenization and spectral asymptotic analysis. J. Math. Pures Appl., 77, 153–208, 1998.
- [3] G. Allaire, Y. Capdeboscq, A. Piatnitski, V. Siess, and M. Vanninathan. Homogenization of periodic systems with large potentials. *Arch. Rational Mech. Anal.*, **174**, 179–220, 2004. doi:10.1007/s00205-004-0332-7.
- [4] A. Bensoussan, J. L. Lions, and G. Papanicolaou. Asymptotic Analysis for Periodic Structures, volume 5 of Studies in Mathematics and its Applications. North-Holland, Amsterdam, 1978.
- [5] F. Bloch. Uber die Quantenmechanik der Electronen in Kristallgittern. Z. Phys., 52, 555–600, 1928.
- [6] L. Brillouin. Wave propagation in periodic structures. Dover Publications, New York, 1953.
- [7] D. Cioranescu and P. Donato. An Introduction to Homogenization. Oxford University Press, Oxford, 1999.
- [8] C. Conca and M. Vanninathan. Homogenization of periodic structures via Bloch decomposition. SIAM J. Appl. Math., 57(6), 1639–1659, 1997.
- [9] G. Floquet. Sur les équations différentielles linéaries à coefficients périodique. Ann. École Norm. Sup., 12, 47–88, 1883.
- [10] S. S. Ganesh and M. Vanninathan. Bloch wave homogenization of scalar elliptic operators. Asymptot. Anal., 39(1), 15–44, 2004.
- [11] S. S. Ganesh and M. Vanninathan. Bloch wave homogenization of linear elasticity system. ESAIM: Control, Optimisation and Calculus of Variations, 2005. Accepted for publication.
- [12] M. Gerken and D. A. B. Miller. Multilayer thin-film structures with high spatial dispersion. Appl. Opt., 42(7), 1330–1345, March 2003.
- [13] L. D. Landau, E. M. Lifshitz, and L. P. Pitaevskii. Electrodynamics of Continuous Media. Pergamon, Oxford, second edition, 1984.
- [14] G. W. Milton. The Theory of Composites. Cambridge University Press, Cambridge, U.K., 2002.
- [15] D. Sjöberg. Homogenization of dispersive material parameters for Maxwell's equations using a singular value decomposition, 2005. Accepted for publication, preprint available as technical report TEAT-7124 at www.es.lth.se/teorel.
- [16] D. Sjöberg, C. Engström, G. Kristensson, D. J. N. Wall, and N. Wellander. A Floquet-Bloch decomposition of Maxwell's equations, applied to homogenization. *Multiscale Modeling and Simulation*, 2005. Accepted for publication, preprint available as technical report TEAT-7119 at www.es.lth.se/teorel.

- [17] K. Sølna and G. W. Milton. Bounds for the group velocity of electromagnetic signals in two phase materials. *Physica B*, 279, 9–12, 2000.
- [18] A. Yariv and P. Yeh. Electromagnetic propagation in periodic stratified media.
   II. Birefringence, phase matching, and x-ray lasers. J. Opt. Soc. Am., 67(4), 438–448, 1977.