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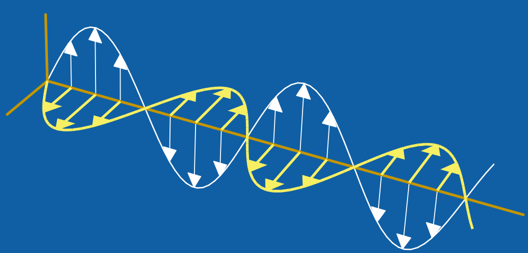

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Abstract

A decomposition of the field of the wave equation in three dimensions is suggested. This wave splitting decomposes the field into two components such that in free space simple propagation properties are obtained. The inhomogeneous region is assumed to have a phase velocity variation that varies only in the radial coordinate r from the origin. The suggested wave splitting is a generalization of the wave splitting concept in one spatial dimension. Specifically, the propagation properties in free space in the three-dimensional wave splitting are very analogous to the corresponding propagation properties in free space in one dimension. The three-dimensional decomposition is illustrated in an example of scattering by a sound-soft sphere of radius a.

1 Introduction

In a series of papers Corones *et al.* [3, 4] introduced the concept of wave splitting as a tool to solve one-dimensional wave propagation problems in the time domain. This approach applies to acoustic, electromagnetic and elastodynamic wave propagation, including propagation in dissipative [5, 8], dispersive [2] and viscoelastic media [1, 7]. The wave splitting method is of special importance for the solution of the corresponding one-dimensional inverse scattering problem. Attempts to generalize the wave splitting ideas to higher spatial dimensions have been made by Weston [10, 11] by means of integral operator identities in the time domain. Weston's method is based upon mean value integrals over the field. The wave splitting technique that is introduced in this paper, however, uses an expansion of the field in spherical harmonics and the two wave splitting techniques are therefore different.

As an introduction to wave splitting, consider the scalar wave equation in one spatial dimension with varying phase velocity $c(z)$

$$\left(\partial_z^2 - \frac{1}{c^2(z)}\partial_t^2\right) u(z,t) = 0.$$

One successful wave splitting technique [4] is to transform the field $u(z,t)$ into plus and minus components by the transformation

$$u^{\pm}(z,t) = \frac{1}{2}\left[u(z,t) \mp c(z)\partial_t^{-1} u_z(z,t)\right], \tag{1.1}$$

where

$$\partial_t^{-1} u_z(z,t) = \int_{-\infty}^{t} u_z(z,t')\,dt'.$$

In free space, with $c(z) = c_0$, the plus and the minus fields are of the form

$$\begin{cases} u^+(z,t) = f(t - z/c_0) \\ u^-(z,t) = g(t + z/c_0), \end{cases}$$

i.e., right and left going waves, respectively. The transformation in (1.1) is interpreted as a generalization of left and right going waves at locations where $c(z)$ varies.

The wave splitting concept in three dimensions is more complicated. There is no longer a discrete set of directions in which the wave can propagate. The directions of propagation are continuous even in free space. Several interesting new ideas have, however, been suggested [10, 11], which generalize the splitting of the one-dimensional wave in (1.1) to higher dimensions. The phase velocity c is still a function of only one variable, which is perpendicular to the bounding surface of the profile.

In this paper the phase velocity is assumed to vary with the radial coordinate from the origin, *i.e.*, , $c(\boldsymbol{r}) = c(r)$, where $r = |\boldsymbol{r}|$. Thus, it is still a one-dimensional variation and the variation is perpendicular to the bounding surfaces. This geometry can also be treated with integral operator techniques in the time domain [11]. However, the wave splitting presented in this paper differs from the one suggested in Ref. 11.

Traditionally, the three-dimensional scattering problem is Fourier transformed and solved at a fixed frequency. However, the analysis developed in this paper does not make use of any fixed frequency results, although several specific time domain results can be related to corresponding frequency results.

Of primary interest for the analysis in this paper is the following relation

$$u(z,t) = u^+(z,t) + u^-(z,t), \tag{1.2}$$

obtained from (1.1). This trivial identity is generalized in Section 3. Specifically, operators Γ^+ and γ are found such that

$$u = \Gamma^+ w^+ + \gamma w^-,$$

and such that $w^\pm$ are functions of the single argument $t \mp r/c_0$ in free space, *i.e.*, where $c(r) = c_0$. This generalization is then taken as the starting point for the wave splitting for the radially symmetric geometry considered in this paper. The details are found in Section 4. The plane wave solution of the wave equation is analyzed in Section 5. In Section 6 the new transformation of the wave field is also used to define a radiating solution and its relation to the radiation field is derived. The paper is closed with two appendices, one illustrating the analysis with an explicit treatment of the case of a sound-soft sphere, and a second providing some mathematical details about the form of the reflection operator.

2 Formulation of the problem

The basic equation in this paper is the three-dimensional scalar wave equation. Assume that the field $U(\boldsymbol{r},t)$ satisfies

$$\left(\nabla^2 - \frac{1}{c^2(r)}\partial_t^2\right) U(\boldsymbol{r},t) = 0. \tag{2.1}$$

The phase velocity $c(r)$ is assumed to vary only with $r = |\boldsymbol{r}|$. Moreover, $c(r)$ is a constant, $c(r) = c_0$, outside the radius $r = a$. The analysis becomes simpler if $c(r)$ is continuous everywhere and continuously differentiable inside the sphere of radius a. Furthermore, in order to have well behaved properties at the origin of the scatterer, assume that $c'(0) = 0$.

A change of dependent variable

$$u(\boldsymbol{r}, t) = rU(\boldsymbol{r}, t),$$

in (2.1) gives

$$\left(\nabla^2 - \frac{2\hat{\boldsymbol{r}}}{r} \cdot \nabla - \frac{1}{c^2(r)} \partial_t^2\right) u(\boldsymbol{r}, t) = 0.$$

The scattering problem is reduced to a one-dimensional problem by an expansion of $u(\boldsymbol{r}, t)$ in spherical harmonics $Y_{lm}(\hat{\boldsymbol{r}})$

$$\begin{cases} v_{lm}(r,t) = \displaystyle\int_{|\boldsymbol{r}'|=r} u(\boldsymbol{r}', t) Y_{lm}(\hat{\boldsymbol{r}}') \, d\Omega_{\hat{\boldsymbol{r}}'} \\ u(\boldsymbol{r}, t) = \displaystyle\sum_{lm} v_{lm}(r,t) Y_{lm}(\hat{\boldsymbol{r}}), \end{cases} \tag{2.2}$$

where $d\Omega_{\hat{\boldsymbol{r}}'}$ is the surface measure of the unit sphere.

The function $v_{lm}(r, t)$ satisfies the one-dimensional wave equation in the radial coordinate

$$\left(\partial_r^2 - \frac{1}{c(r)^2} \partial_t^2 - \frac{l(l+1)}{r^2}\right) v_{lm}(r,t) = 0. \tag{2.3}$$

Note that this equation is independent of m and any dependence of m in $v_{lm}(r, t)$ is then just a constant multiplicative factor. Thus, the index m is suppressed in the subsequent analysis.

It is convenient to introduce the travel time coordinate transformation

$$\begin{cases} x(r) = \dfrac{1}{\tau} \displaystyle\int_0^r \frac{1}{c(r')} \, dr' \\ s = \dfrac{t}{\tau} \\ \tau = \displaystyle\int_0^a \frac{1}{c(r')} \, dr', \end{cases} \tag{2.4}$$

which maps the inhomogeneous region, $(r \le a)$, to the interval $[0, 1]$. The quantity τ is the time it takes for the wave front to travel radially from the origin to the edge of the inhomogeneous region $r = a$. Outside the sphere $(r \ge a)$: $x(r) = 1 + (r - a)/c_0\tau$. The transformation Eq. (2.4) is performed for convenience and an analogous treatment can be made with the use of the physical coordinates r and t.

The wave equation, Eq. (2.3), transforms into

$$\left(\partial_x^2 - \partial_s^2 + A(x)\partial_x + C(x)\right) v_l(x, s) = 0, \tag{2.5}$$

where

$$
\begin{cases}
v_l(x,s) = v_{lm}(r,t) \\
A(x) = -\tau c'(r) \\
C(x) = -\dfrac{l(l+1)}{q^2} \\
q(x) = \dfrac{r}{c(r)\tau}.
\end{cases}
\tag{2.6}
$$

The quantity $q(x)$ is the time (scaled) it takes for the wave to travel from the origin to r in a homogeneous medium with phase velocity $c(r)$ everywhere in space. Outside the sphere ($r \geq a$): $x = 1 + q - q_1$, where $q_1 = q(1) = a/c_0\tau$. The functions $A(x)$ and $C(x)$ are according to the assumptions continuous in the interval $0 < x < 1$. The assumption of a well behaved scatterer at the origin implies that $A(0) = 0$. Since $C(x) \neq 0$ everywhere, ($l \neq 0$), Eq. (2.5) differs from the wave equation in the one-dimensional case even outside the sphere $r = a$, where $A(x) = 0$. Only the $l = 0$ value gives an equation similar to the wave equation in the one-dimensional case.

3 The operators $\Gamma_l^{\pm}(q)$ and $\gamma_l(q)$

The starting point for a generalization of the one-dimensional splitting in this paper is to generalize Eq. (1.2). Suppose there are operators $\Gamma^{\pm}$, such that

$$
v_l(x,s) = \left\{\Gamma_l^+ f_l(x,\cdot)\right\}(s) + \left\{\Gamma_l^- g_l(x,\cdot)\right\}(s),
\tag{3.1}
$$

holds. In free space, $A(x) = 0$, the functions f_l and g_l are required to be functions of $s \mp x$ only. The operators $\Gamma_l^{\pm}$ are assumed to commute with differentiation with respect to s. That such operators exist is shown in this section.

Let $A(x) = 0$ and insert Eq. (3.1) in Eq. (2.5). Identify the operators in front of f_l and g_l. This implies that the operators satisfy

$$
\left(\partial_q^2 \mp 2\partial_q\partial_s - \frac{l(l+1)}{q^2}\right)\Gamma_l^{\pm} = 0.
\tag{3.2}
$$

A formal change of variable $\zeta = \mp\frac{\partial_s^{-1}}{2q}$ transforms Eq. (3.2) into

$$
\zeta^2\partial_\zeta^2\Gamma_l^{\pm} + (2\zeta - 1)\partial_\zeta\Gamma_l^{\pm} - l(l+1)\Gamma_l^{\pm} = 0.
$$

It is clear that the solution $\Gamma_l^{\pm}$ is proportional to the function ${}_2F_0(-l, l+1;;\zeta)$ [9]. The arbitrary constant is determined by requiring that $\Gamma_l^{\pm}(\zeta)$ are the identity operator for large q, *i.e.*, as $x \to \infty$. The operators $\Gamma_l^{\pm}$ have the formal series representations

$$
\Gamma_l^{\pm}(q) = \sum_{k=0}^{l} \frac{(l-k+1, 2k)}{k!}(\pm 2q\partial_s)^{-k},
\tag{3.3}
$$

where, $l = 0, 1, 2, \ldots$, and (a, n) is the Appell's symbol defined as: $(a, 0) = 1$ and $(a, n) = a(a+1) \cdot \ldots \cdot (a+n-1)$. From Eqs. (3.2) and (3.3) it is possible to derive the following operator equality:

$$\frac{1}{2}\Gamma_l^+ \, \partial_s^{-1}\partial_q\Gamma_l^- - \frac{1}{2}\Gamma_l^- \, \partial_s^{-1}\partial_q\Gamma_l^+ + \Gamma_l^+ \, \Gamma_l^- = 1, \tag{3.4}$$

where 1 is the identity operator.

It can be shown from Eq. (3.3) that the operators $\Gamma_l^{\pm}$ have the following integral representations

$$\big(\Gamma_l^{\pm}(q)f(\cdot)\big)(s) = \partial_s \int_{-\infty}^{s} P_l(1 \pm \frac{s-s'}{q})f(s')\, ds', \tag{3.5}$$

$$\big(\partial_q\Gamma_l^{\pm}(q)f(\cdot)\big)(s) = \mp\partial_s \int_{-\infty}^{s} P_l'(1 \pm \frac{s-s'}{q})\frac{s-s'}{q^2}f(s')\, ds',$$

where $P_l(x)$ is the Legendre polynomial of order l. From Eq. (3.3) it is seen that the operators $\Gamma_l^{\pm}(q)$ have a singular behavior at $q = 0$. A combination of the operators $\Gamma_l^{\pm}(q)$ that is regular at the origin $q = 0$ is

$$\big(\gamma_l(q)f(\cdot)\big)(s) = \frac{1}{2}\left\{\big(\Gamma_l^-(q)f(\cdot)\big)(s) - (-1)^l\big(\Gamma_l^+(q)f(\cdot)\big)(s-2q)\right\}.$$

This regular combination has the specific integral representation

$$\big(\gamma_l(q)f(\cdot)\big)(s) = \partial_s\frac{1}{2}\int_{s-2q}^{s} P_l(1 - \frac{s-s'}{q})f(s')\, ds', \tag{3.6}$$

$$\big(\partial_q\gamma_l(q)f(\cdot)\big)(s) = \partial_s\frac{1}{2}\int_{s-2q}^{s} P_l'(1 - \frac{s-s'}{q})\frac{s-s'}{q^2}f(s')\, ds' + (-1)^l f'(s-2q).$$

The operator γ_l can be proven to satisfy

$$\left(\partial_q^2 + 2\partial_q\partial_s - \frac{l(l+1)}{q^2}\right)\gamma_l = 0,$$

i.e., the same equation as for Γ_l^-, and an operator equality similar to Eq. (3.4) can be derived

$$\Gamma_l^+ \, \partial_s^{-1}\partial_q\gamma_l - \gamma_l \, \partial_s^{-1}\partial_q\Gamma_l^+ + 2\Gamma_l^+ \, \gamma_l = 1. \tag{3.7}$$

The operators Γ_l^+ and γ_l, defined in Eqs. (3.3) or (3.5) and (3.6), respectively, were found under the assumption $A(x) = 0$ (constant phase velocity). However, once these operators are defined the proceeding derivation to obtain Eqs. (3.4) and (3.7) does not depend on the assumption of constant phase velocity. Equation (3.7) is therefore an operator identity of general validity with Γ_l^+ and γ_l defined as in

Eqs. (3.5) and (3.6), respectively. The merits of this identity is used in the next section.

To summarize the main result of this section, operators Γ_l^+ and γ_l exist such that

$$v_l(x,s) = \left\{\Gamma_l^+ f_l(x,\cdot)\right\}(s) + \left\{\gamma_l g_l(x,\cdot)\right\}(s), \tag{3.8}$$

where, in free space, f_l and g_l are functions of $s \mp x$ only. Moreover, in free space, the spatial derivative of $v_l(x,s)$ is

$$\partial_x v_l(x,s) = \left\{\left(\partial_q \Gamma_l^+ - \partial_s \Gamma_l^+\right) f_l(x,\cdot)\right\}(s) + \left\{\left(\partial_q \gamma_l + \partial_s \gamma_l\right) g_l(x,\cdot)\right\}(s), \tag{3.9}$$

since f and g are functions of $s \mp x$ only and the operators Γ_l^+ and γ_l commute with differentiation with respect to s. In a matrix notation Eqs. (3.8) and (3.9) can be written as

$$\begin{pmatrix} v_l(x,s) \\ \partial_x v_l(x,s) \end{pmatrix} = \begin{pmatrix} \Gamma_l^+ & \gamma_l \\ \partial_q \Gamma_l^+ - \partial_s \Gamma_l^+ & \partial_q \gamma_l + \partial_s \gamma_l \end{pmatrix} \begin{pmatrix} f_l(x,s) \\ g_l(x,s) \end{pmatrix},$$

which formally can be inverted with the identity in Eq. (3.7). Thus, in free space the relation between the functions $\begin{pmatrix} f_l \\ g_l \end{pmatrix}$ and $\begin{pmatrix} v_l \\ \partial_x v_l \end{pmatrix}$ is

$$\begin{pmatrix} f_l(x,s) \\ g_l(x,s) \end{pmatrix} = \begin{pmatrix} \gamma_l + \partial_s^{-1}\partial_q \gamma_l & -\partial_s^{-1}\gamma_l \\ \Gamma_l^+ - \partial_s^{-1}\partial_q \Gamma_l^+ & \partial_s^{-1}\Gamma_l^+ \end{pmatrix} \begin{pmatrix} v_l(x,s) \\ \partial_x v_l(x,s) \end{pmatrix}. \tag{3.10}$$

This equation will be the starting point in the definition of the generalized wave splitting transformations for an inhomogeneous region defined in the next section.

4 Transformations

Equation (3.10) was derived under the assumption of constant phase velocity c. As a generalization of the wave splitting concept in one-dimension this equation is now taken as the definition of the wave splitting for a radially varying phase velocity $c(r)$. Specifically, the solution $v_l(x,s)$ of the original PDE

$$\left(\partial_x^2 - \partial_s^2 + A(x)\partial_x + C(x)\right) v_l(x,s) = 0.$$

is used to define four new functions $w_l^{\pm}(x,s)$ and $v_l^{\pm}(x,s)$.

$$\begin{pmatrix} w_l^+(x,s) \\ w_l^-(x,s) \end{pmatrix} = \begin{pmatrix} \gamma_l + \partial_s^{-1}\partial_q \gamma_l & -\partial_s^{-1}\gamma_l \\ \Gamma_l^+ - \partial_s^{-1}\partial_q \Gamma_l^+ & \partial_s^{-1}\Gamma_l^+ \end{pmatrix} \begin{pmatrix} v_l(x,s) \\ \partial_x v_l(x,s) \end{pmatrix}. \tag{4.1}$$

$$\begin{pmatrix} v_l^+(x,s) \\ v_l^-(x,s) \end{pmatrix} = \begin{pmatrix} \Gamma_l^+ & 0 \\ 0 & \gamma_l \end{pmatrix} \begin{pmatrix} w_l^+(x,s) \\ w_l^-(x,s) \end{pmatrix}. \tag{4.2}$$

These two equations are taken as definitions on the two new pairs of fields, $w_l^{\pm}(x,s)$ and $v_l^{\pm}(x,s)$, respectively. Since the operator identity in Eq. (3.7) holds even in a region of non-constant phase velocity c, the following important property of $v_l^{\pm}$ can be derived.

$$v_l^+(x,s) + v_l^-(x,s) = \left(\Gamma_l^+ w_l^+(x,\cdot)\right)(s) + \left(\gamma_l w_l^-(x,\cdot)\right)(s) = v_l(x,s). \tag{4.3}$$

Note that this identity holds everywhere, even for a region with varying phase velocity $c(r)$. Another independent way of verifying Eq. (4.3) is to note that Eq. (4.3) is equivalent to the following integral identities

$$\int_{1-s}^{1} P_l(s+y)(s+2y)P_l(y)\,dy = 2s$$
$$\int_{-1}^{1-s} P_l(s+y)(s+2y)P_l(y)\,dy = 0,$$

which can be derived by other means.

By construction the functions $w_l^{\pm}(x,s)$ have the property that if $c(r)$ is constant in an interval then

$$(\partial_x \pm \partial_s) w_l^{\pm}(x,s) = 0,$$

in that interval. Thus, the solutions $w_l^{\pm}(x,s)$ outside the sphere $r = a$ behave as outgoing and incoming waves, respectively. For $x \geq 1$

$$\begin{cases} w_l^+(x,s) = w_l^+(1, s-x+1) \\ w_l^-(x,s) = w_l^-(1, s+x-1). \end{cases} \tag{4.4}$$

Any solution $v_l(x,s)$ to Eq. (2.5) can be transformed into the functions $w_l^{\pm}(x,s)$, which have simple propagation properties in a region where the phase velocity $c(r)$ is constant. The solution $v_l(x,s)$ can then be recovered from the functions $w_l^{\pm}(x,s)$ by an additional transformation. On the other hand, in a region where $c(r)$ is constant, solutions $v_l^{\pm}(x,s)$ to Eq. (2.5) are given from arbitrary functions $f_l(s)$ and $g_l(s)$ by

$$\begin{pmatrix} v_l^+(x,s) \\ v_l^-(x,s) \end{pmatrix} = \begin{pmatrix} \Gamma_l^+ & 0 \\ 0 & \gamma_l \end{pmatrix} \begin{pmatrix} f_l(x-s) \\ g_l(x+s) \end{pmatrix}.$$

It is obvious that the splitting presented in this section is different from the one-dimensional splitting in (1.1). This difference holds even for the $l = 0$ case, in which the underlying differential equation is the same. This is due to the definition of the regular operator γ_l.

Physical arguments imply that the field $u(\boldsymbol{r}, t) = 0$ at the origin $r = 0$. The spherical projections $v_l(x,s)$ must also vanish at the origin $x = 0$, which implies that $v_l^+(0,s) = 0$ and $w_l^+(0,s) = 0$, since $v_l^-(0,s) = 0$ by construction. The boundary condition $v_l^+(0,s) = 0$ simply says that there are no sources at the origin.

5 Plane wave solution

The transformations introduced in the previous section are now applied to a general plane wave solution, *i.e.*,

$$U_0(\boldsymbol{r}, t) = f(t - \frac{z+a}{c_0}). \tag{5.1}$$

This is a solution to the wave equation (2.1) in the region where $c(r) = c_0$, which is the region outside the spherical inhomogeneity, *i.e.*, $r \geq a$ or $x \geq 1$. In order to simplify the analysis the function $f(t)$ is assumed to be smooth and, furthermore, $f(t) = 0, \quad t < 0$. Thus, the wave front impinges on the sphere $r = a$ at a time $t \geq 0$. The plane wave solution in (5.1) has the following projection on the spherical harmonics:

$$v_{lm}(r, t) = rN_l\delta_{m,0} \int_{-1}^{1} f(t - \frac{\eta r + a}{c_0}) P_l(\eta)\, d\eta,$$

where

$$N_l = 2\pi\sqrt{\frac{2l+1}{4\pi}}. \tag{5.2}$$

Lengthy calculations show that the transformations in Eqs. (4.1) and (4.2) give

$$\begin{cases} w_l^+(x, s) = 0 \\ w_l^-(x, s) = 2(-1)^l N_l \delta_{m,0} c_0 g(s + x - 1), \end{cases} \tag{5.3}$$

$$\begin{cases} v_l^+(x, s) = 0 \\ v_l^-(x, s) = rN_l\delta_{m,0} \displaystyle\int_{-1}^{1} f(t - \frac{\eta r + a}{c_0}) P_l(\eta)\, d\eta, \end{cases} \tag{5.4}$$

where $g(s)$ is the anti-derivative of the plane wave $f(t)$, more precisely, $g(s) = \int_{-\infty}^{s\tau} f(t')\, dt'$. The zero contribution in the function $w_l^+(x, s)$ (and $v_l^+(x, s)$) indicates that the solution $U_0(\boldsymbol{r}, t)$ defined in (5.1) contains no outgoing part. Note also that $v_l(x, s) = v_l^+(x, s) + v_l^-(x, s)$.

If formally the plane wave is chosen as a delta function, *i.e.*, $f(t) = \delta(t)$, Eqs. (5.3) and (5.4) become ($H(s)$ is the Heaviside function, *i.e.*, $H(s) = 1$ for $s > 0$, zero otherwise)

$$\begin{cases} w_l^+(x, s) = 0 \\ w_l^-(x, s) = 2(-1)^l N_l \delta_{m,0} c_0 H(s + x - 1), \end{cases} \tag{5.5}$$

$$\begin{cases} v_l^+(x, s) = 0 \\ v_l^-(x, s) = N_l\delta_{m,0} c_0 \left[H(s + x - 1) - H(s - x + 1 - 2q_1)\right] P_l\left(\dfrac{s - q_1}{x - 1 + q_1}\right). \end{cases} \tag{5.6}$$

Equations (5.3) and (5.4) can also be obtained obtained from Eqs. (5.5) and (5.6) by a convolution with $f(t)$.

6 Radiating solution

The evolution of the solution $u(\boldsymbol{r}, t)$ to the scattering problem is specified by prescribing the solution in the past, $t \to -\infty$. In this paper $u(\boldsymbol{r}, t)$ is specified as $u(\boldsymbol{r}, t) = rU_0(\boldsymbol{r}, t) = rf(t - (z + a)/c_0)$ as $t \to -\infty$, where the function $f(t)$ is the plane wave solution in Section 5. Due to causality, the solution $u(\boldsymbol{r}, t)$ equals $rf(t - (z + a)/c_0)$ for all $t < 0$.

The behavior of the solution $u(\boldsymbol{r}, t)$ as $t \to \infty$ is of primary interest in this section, especially its far field behavior ($r \to \infty$). Following Friedlander [6], it is appropriate to define the radiation field $F(\hat{\boldsymbol{r}}, t)$ as

$$F(\hat{\boldsymbol{r}}, t) = \lim_{r\to\infty} \left\{ u(r\hat{\boldsymbol{r}}, t + \frac{r-a}{c_0}) \right\}.$$

Friedlander [6] shows that this is a well defined quantity.

The radiation field expressed in the spherical projections $v_l(x, s)$ of Eq. (2.2) is

$$F(\hat{\boldsymbol{r}}, t) = \lim_{r\to\infty} \left\{ \sum_{l=0}^{\infty} \frac{N_l}{2\pi} v_l(x, s + x - 1) P_l(\cos\theta) \right\},$$

where $x = 1 + (r - a)/c_0\tau$, $s = t/\tau$, θ is the angle between the z-axis and the $\hat{\boldsymbol{r}}$-direction and the normalization factor N_l is defined in Eq. (5.2).

A decomposition of the field $v_l(x, s)$ in plus and minus parts gives two contributions to the radiation field $F(\hat{\boldsymbol{r}}, t)$. Due to Eq. (4.4) and the support of the prescription of the solution in the past, *i.e.*, $u(\boldsymbol{r}, t) = rf(t - (z + a)/c_0)$ as $t < 0$, the only minus part of the solution outside the sphere $r = a$ at any time s is the field $v_l^-(x, s)$ in Eq. (5.4). The minus part contribution of the field in the radiation field is thus due to the prescribed field $f(t)$. It can be proved that this field contributes to the radiation field only in the forward direction, $\theta = 0$, *i.e.*, in the positive z-direction.

The plus part of the field reflects the presence of the inhomogeneity. This part contributes to the radiation field $F(\hat{\boldsymbol{r}}, t)$

$$\begin{aligned} F(\hat{\boldsymbol{r}}, t) &= \lim_{r\to\infty} \left\{ \sum_{l=0}^{\infty} \frac{N_l}{2\pi} v_l^+(x, s + x - 1) P_l(\cos\theta) \right\} \\ &= \lim_{r\to\infty} \left\{ \sum_{l=0}^{\infty} \frac{N_l}{2\pi} \left\{ \Gamma_l^+(q) w_l^+(x, \cdot) \right\} (s + x - 1) P_l(\cos\theta) \right\} \\ &= \lim_{r\to\infty} \left\{ \sum_{l=0}^{\infty} \frac{N_l}{2\pi} \left\{ \Gamma_l^+(q) w_l^+(1, \cdot) \right\} (s) P_l(\cos\theta) \right\}, \end{aligned}$$

where the result of Eq. (4.4) has been used to express the solution $w_l^+(x, s)$ in terms of its value at $x = 1$.

Assume that the following linear relation between $w_l^+(x,s)$ and $w_l^-(x,s)$ at $x=1$ exists

$$w_l^+(1,s) = \frac{1}{2}(-1)^l \left[w_l^-(1,s-2q_1) - w_l^-(1,s-2)\right] + \int_0^s R_l(s-s')w_l^-(1,s')\,ds'. \tag{6.1}$$

where $q_1 = q(1) = a/c_0\tau$. The mathematical details for this assumption is found in Appendix B. The kernel $R_l(s)$ is a continuous function of s except for possible finite jump discontinuities. Since the only minus contribution to the total field $u(\boldsymbol{r},t)$ outside the sphere $r=a$ is due to the prescribed plane wave $f(t)$ described in Section 5 the function $w_l^-(1,s)$ is given by Eq. (5.3) and Eq. (6.1) can be written as

$$w_l^+(1,s) = c_0 N_l\left[g(s-2q_1) - g(s-2) + 2(-1)^l \int_0^s R_l(s-s')g(s')\,ds'\right], \tag{6.2}$$

where the function $g(s)$ is the anti-derivative of the plane wave $f(t)$, *i.e.*, $g(s) = \int_0^{s\tau} f(t')\,dt'$.

The two time delayed terms in Eq. (6.2) give divergent contributions to the radiation field $F(\hat{\boldsymbol{r}},t)$. They diverge as

$$\sum_l (2l+1)P_l(x)P_l(y),$$

where $-1 \le x \le 1$ and $y > 1$. The physical interpretation of these two terms are not fully understood. It is conjectured that they contribute only in the forward direction, $\theta = 0$, and to the alteration of the wave front. The radiation field due to the convolution part is ($t \ge 0$)

$$F(\hat{\boldsymbol{r}},t) = \lim_{r\to\infty}\left\{\sum_{l=0}^{\infty} \frac{N_l}{2\pi}\left\{\Gamma_l^+(q)\left[R_l(\cdot) * w_l^-(1,\cdot)\right](\cdot)\right\}(s)P_l(\cos\theta)\right\}. \tag{6.3}$$

The operators $\Gamma_l^+(q)$ have the identity operator as the limit as $q \to \infty$ (see Section 3, *e.g.*, Eq. (3.3)). Note that this limit is not uniform in l. However, whenever the kernel $R_l(s)$ (or appropriate parts of it) contributes to the convergence such that the operators $\Gamma_l^+(q)$ could be replaced by the identity inside the sum, Eq. (6.3) simplifies to (for the appropriate part of $R_l(s)$)

$$F(\hat{\boldsymbol{r}},t) = \sum_{l=0}^{\infty} \frac{N_l}{2\pi} \int_0^s R_l(s-s')w_l^-(1,s')\,ds' P_l(\cos\theta).$$

With the plane wave solution of Section 5 inserted the result is ($t \ge 0$)

$$F(\hat{\boldsymbol{r}},t) = 2c_0 \sum_{l=0}^{\infty} \frac{2l+1}{2} \int_0^s R_l(s-s')g(s')\,ds' P_l(-\cos\theta).$$

With the special case of $f(t) = \delta(t)$ this equation simplifies even further to $(t = s/\tau \geq 0)$

$$F(\hat{\boldsymbol{r}}, t) = 2c_0 \left\{ \sum_{l=0}^{\infty} \frac{2l+1}{2} \int_0^s R_l(s')\, ds' P_l(-\cos\theta) \right\}.$$

This expression can be viewed as a Fourier-Legendre series of the impulse response $F(\hat{\boldsymbol{r}}, t)$ with Fourier coefficients $R_l(s)$.

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Appendix A Wave propagation outside a sound-soft sphere

This appendix illustrates the analysis given in this paper with an explicit example. A simple example is that of wave propagation outside a sound-soft sphere of radius a. The phase velocity profile for the sound-soft sphere does not satisfy the continuity requirements assumed in Section 2 everywhere. It is, however, obvious that the wave splitting presented in Section 4 applies to the region outside the sphere. The travel time coordinate transformation and q simplify to

$$\begin{cases} x(r) = r/a \\ s = tc_0/a \\ q(x) = x, \end{cases} \tag{A.1}$$

and the wave equation and the boundary condition are

$$\begin{cases} \big(\partial_x^2 - \partial_s^2 + C(x)\big) v_l(x, s) = 0, \quad x \geq 1 \\ v_l(1, s) = 0, \end{cases}$$

where

$$C(x) = -\frac{l(l+1)}{x^2}.$$

The incident wave is assumed to be a plane wave as in Section 5, *i.e.*, $U_0(\boldsymbol{r}, t) = f(t - \frac{z+a}{c_0})$. The projected fields are (see Eqs. (5.3) and (5.4)):

$$\begin{cases} W_l^+(x,s) = 0 \\ W_l^-(x,s) = 2(-1)^l N_l \delta_{m,0} a \int_0^{s+x-1} g(s')\, ds', \quad x \geq 1 \\ V_l^+(x,s) = 0 \\ V_l^-(x,s) = r N_l \delta_{m,0} \int_{-1}^1 g(s - \eta x - 1) P_l(\eta)\, d\eta, \quad x \geq 1, \end{cases}$$

where $N_l = 2\pi\sqrt{\frac{2l+1}{4\pi}}$ and $g(s) = f(as/c_0)$.

At $x = 1$ the boundary condition $v_l(1,s) = 0$ implies (since the assumption $v_l(x,s) = V_l(x,s)$ and $\partial_x v_l(x,s) = \partial_x V_l(x,s)$ for all $s < 0$ and all $x \geq 1$ imply that $v_l^-(x,s) = V_l^-(x,s)$ for all s and all $x \geq 1$)

$$v_l^+(1,s) = -v_l^-(1,s) = -V_l^-(1,s),$$

or (use Eqs. (3.5) and (4.2))

$$y_l(s) + \int_{-\infty}^s P_l'(1 + s - s') y_l(s')\, ds' = -\int_{-1}^1 g(s - \eta - 1) P_l(\eta)\, d\eta, \tag{A.2}$$

where

$$w_l^+(1,s) = N_l \delta_{m,0} a y_l(s).$$

Notice that Eq. (A.2) is a Volterra equation of the second kind for the asymptotic field $w_l^+(x,s) = w_l^+(1, s - x + 1)$. Furthermore, $y_l(s) = w_l^+(1,s) = 0$ for $s < 0$. Equation (A.2) is particularly useful in the study of the short time behavior of the field.

A simple numerical example is shown in Figure 1. The incident plane wave $f(t)$ is chosen as

$$f(t) = \begin{cases} \sin^2(\pi c_0 t/2a), & 0 < t < 2a/c_0 \\ 0, & \text{otherwise.} \end{cases}$$

This choice gives a plane wave with a width of one sphere diameter. The results shown agree with the corresponding Fourier transform solution results.

In the examination of the long time behavior of the solution it is appropriate to study the resolvent equation to Eq. (A.2)

$$L_l(s) + P_l'(1 + s) + \int_0^s P_l'(1 + s - s') L_l(s')\, ds' = 0,$$

or

$$\int_0^s P_l(1 + s - s') L_l(s')\, ds' = 1 - P_l(1 + s).$$

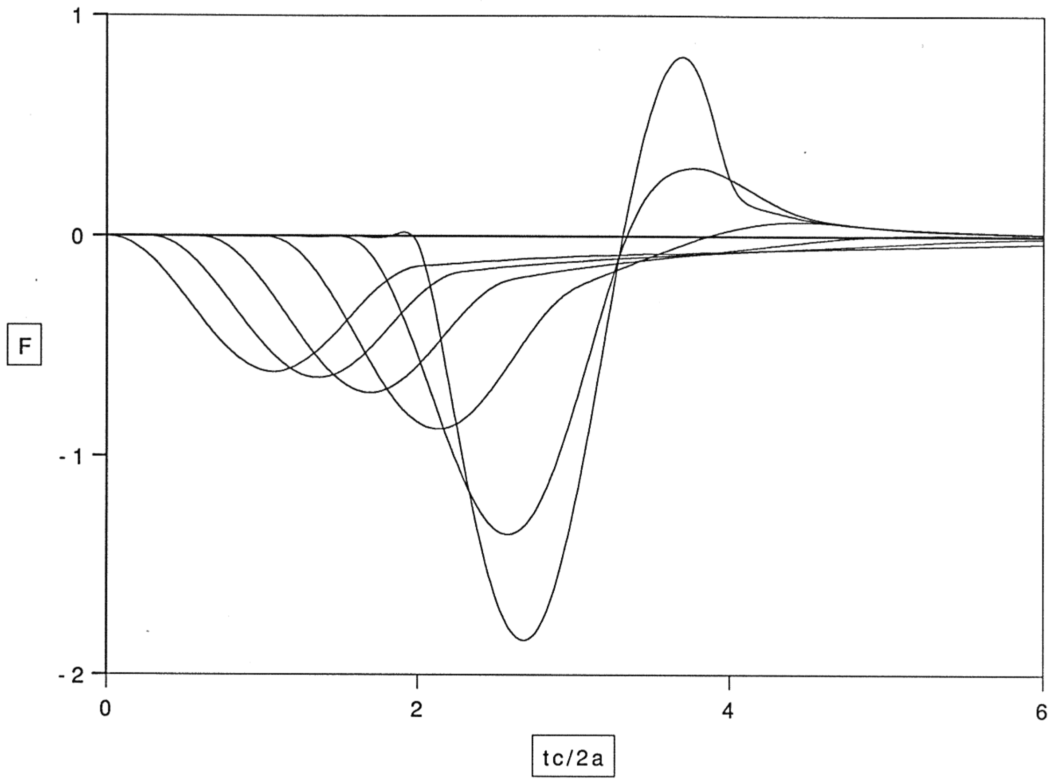

Figure 1: The radiation field $F(\hat{\boldsymbol{r}}, t)$ for a sound-soft sphere of radius a. The incident plane wave is $f(t) = \sin^2(\pi c_0 t/2a),\ 0 < t < 2a/c_0,\ f(t) = 0$, otherwise. The scale on the time axis is in travel time units (see Eq. (A.1)). The radiation field is shown at the angles $\theta = 0, 30, 60, 90, 120, 150, 180$ degrees (the largest response for $\theta = 0$).

Since $P_l'(1+s)$ is a polynomial in s of degree $l-1$ the resolvent kernel $L_l(s)$ is

$$L_l(s) = \sum_{k=1}^{l} \frac{\lambda_{l,k}^l}{\Phi_l'(\lambda_{l,k})} e^{\lambda_{l,k} s},$$

where $L_0(s) = 0$ and $\lambda_{l,1}, \cdots, \lambda_{l,l}$ are the zeroes of the polynomial [9]

$$\begin{aligned}\Phi_l(s) =& s^l \sum_{n=0}^{l} \frac{(l-n+1, 2n)}{n!} (2s)^{-n} \\ =& s^l {}_2F_0(-l, l+1;; \frac{-1}{2s}) = \frac{(l+1, l)}{2^l} {}_1F_1(-l; -2l; 2s).\end{aligned}$$

The resolvent $L_l(s)$ is a real function of s, since the zeroes $\lambda_{l,k}$ are either negative real numbers or form pairs of conjugate complex numbers (with negative real part).

The resolvent $L(s)$ can be written as

$$L_l(s) = -\frac{1}{2} \sum_{k=1}^{l} e^{\lambda_{l,k}(s+1)} {}_2F_0(-l, l+1;; \frac{1}{2\lambda_{l,k}}).$$

The solution to Eq. (A.2) for $s > 0$ then is

$$
\begin{aligned}
y_l(s) = &- \int_{-1}^{1} g(s-\eta-1) P_l(\eta)\, d\eta \\
&- \int_0^s L_l(s-s') \int_{-1}^{1} g(s'-\eta-1) P_l(\eta)\, d\eta\, ds', \quad s > 0.
\end{aligned}
$$

Define the radiation field as in Section 6 and the plus part of the field contributes to the radiation field with

$$
\begin{aligned}
F(\hat{\boldsymbol{r}}, s) =& \lim_{x\to\infty} \left\{ \sum_{l=0}^{\infty} \frac{N_l}{2\pi} v_l^{+}(x, x+s-1) P_l(\cos\theta) \right\} \\
=& a \lim_{x\to\infty} \left\{ \sum_{l=0}^{\infty} \frac{2l+1}{2} \partial_s \int_0^s P_l(1 + \frac{s-s'}{x}) y_l(s')\, ds' P_l(\cos\theta) \right\}, \quad s > 0,
\end{aligned}
$$

or in terms of the resolvent $L(s)$ for $s > 0$

$$
\begin{aligned}
F(\hat{\boldsymbol{r}}, s) =& - a \lim_{x\to\infty} \Bigg\{ \sum_{l=0}^{\infty} \frac{2l+1}{2} \partial_s \int_0^s P_l(1 + \frac{s-s'}{x}) \Bigg\{ \int_{-1}^{1} g(s'-\eta-1) P_l(\eta)\, d\eta \\
&+ \int_0^{s'} L_l(s'-s'') \int_{-1}^{1} g(s''-\eta-1) P_l(\eta)\, d\eta\, ds'' \Bigg\}\, ds'\, P_l(\cos\theta) \Bigg\} \\
=& - a \lim_{x\to\infty} \Bigg\{ \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} \partial_s \int_0^s P_l(1 + \frac{s-s'}{x}) \int_{-1}^{1} d\eta \int_0^{2\pi} d\varphi P_l(\cos\gamma) \\
& \left\{ g(s'-\eta-1) + \int_0^{s'} L_l(s'-s'') g(s''-\eta-1)\, ds'' \right\} ds' \Bigg\},
\end{aligned}
\tag{A.3}
$$

where

$$
\cos\gamma = \eta\cos\theta + \sqrt{1-\eta^2}\sin\theta\cos\varphi.
$$

To obtain the last equality use the addition theorem for the Legendre functions to derive the identity

$$
P_l(\cos\theta) P_l(\cos\theta') = \frac{1}{\pi} \int_0^{\pi} P_l\big(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\varphi\big)\, d\varphi.
$$

Note that the angle γ is always real in the φ integration in Eq. (A.3), and the integration in the variables η and φ over the unit sphere in Eq. (A.3) is effectively over the illuminated part of the sphere $r = a$.

Appendix B Derivation of the form of the reflection kernel

In this appendix the form of the reflection kernel, introduced in Eq. (6.1), is derived. The model problem is given by Eq. (2.5). For the sake of proving a stronger result, which should be useful in future work, the assumptions made in Section 2 will be relaxed and to avoid heavy notation the index l is suppressed in this appendix.

Define the following differential operator L

$$[Lv](x,s) = v_{xx}(x,s) - v_{ss}(x,s) + A(x)v_x(x,s) + B(x)v_s(x,s) + C(x)v(x,s), \tag{B.1}$$

where $A(x)$ and $C(x)$ are smooth functions except at the boundaries $x = 0, 1$ and defined by Eq. (2.6). The function $B(x)$ is also assumed to be a smooth function in $0 < x < 1$ and models the losses of the medium.

The basic problem is

$$\begin{cases} [LV](x,s) = 0, \quad 0 < x < 1, \quad s < S \\ V(0,s) = 0, \quad s < S \\ \begin{cases} V(x,0) = 0 \\ V_s(x,0) = 0 \end{cases} ,0 < x < 1 \\ W^-(1,s) = H(s)s^2/2, \quad s < S, \end{cases} \tag{B.2}$$

where $H(s)$ is the Heaviside function and W^- is

$$\begin{aligned} W^-(x,s) =& V(x,s) + \int_{-\infty}^{s} F^-(x,s-s')V(x,s')\,ds' \\ &+ \int_{-\infty}^{s} G^-(x,s-s')V_x(x,s')\,ds', \end{aligned}$$

where $F^-(x,s)$ and $G^-(x,s)$ are defined as

$$\begin{aligned} F^-(x,s) &= \frac{1}{q}(1+\frac{s}{q})P_l'(1+\frac{s}{q}) \\ G^-(x,s) &= P_l(1+\frac{s}{q}). \end{aligned}$$

Eq. (B.2) is assumed to have a unique solution $V(x,s)$ and by causality $V(x,s) = 0$ for $s < 1-x$. Above the characteristic curve $s = 1-x$, $V(x,s)$ is assumed to be a classical solution except on the characteristic curves $s = 1-x$ and $s = 1+x$.

Propagation of singularity arguments and the boundary value of $W^-(x,s)$ at $x = 1$ imply that $V(x,s)$ and the first partial derivatives of $V(x,s)$ are continuous across the characteristic curve $s = 1-x$, *i.e.*,

$$\begin{cases} V(x, 1-x+0) = 0 \\ V_s(x,1-x+0) = V_x(x,1-x+0) = 0 \end{cases} ,0 < x < 1. \tag{B.3}$$

$U(x,s) = V_{ss}(x,s)$ satisfies at $x = 1$

$$\begin{cases} U(1,s) + \frac{1}{2}G^-(1,s) + \int_0^s F^-(1,s-s')U(1,s')\,ds' \\ \qquad\qquad + \int_0^s G^-(1,s-s')U_x(1,s')\,ds' = 1, \quad s > 0 \\ \qquad\qquad U(1,0^+) = V_{ss}(1,0^+) = V_{xs}(1,0^+) = V_{xx}(1,0^+) = \frac{1}{2}, \end{cases} \tag{B.4}$$

and the value of $U(x,s)$ at the characteristic $s = 1-x$ is

$$U(x, 1-x+0) = \frac{1}{2}\exp\left\{-\frac{1}{2}\int_1^x (A(x') + B(x'))\,dx'\right\}, \quad 0 < x < 1. \tag{B.5}$$

Theorem B.1. *Let $V(x,s)$ be the solution of Eq.* (B.2) *($U(x,s) = V_{ss}(x,s)$). If $v(x,s)$ satisfies*

$$\begin{cases} [Lv](x,s) = 0, \quad 0 < x < 1, \quad 0 < s < S \\ v(0,s) = 0, \quad s < S \\ \begin{cases} v(x,0) = 0 \\ v_s(x,0) = 0 \quad, 0 < x < 1 \end{cases} \\ w^-(1,s) = g(s), \quad s < S, \end{cases} \tag{B.6}$$

where $g(s) = H(s)g_0(s)$, $g_0 \in C^2(-\infty,\infty)$ and

$$\begin{aligned} w^-(x,s) =& v(x,s) + \int_{-\infty}^s F^-(x,s-s')v(x,s')\,ds' \\ &+ \int_{-\infty}^s G^-(x,s-s')v_x(x,s')\,ds'. \end{aligned}$$

Then

$$\begin{aligned} v(x,s) =& \partial_s \int_0^{s+x-1} U(x,s-s')g(s')\,ds' \\ =& U(x,s)g(0^+) + \int_0^{s+x-1} U(x,s-s')g'(s')\,ds' \\ =& U(x,1-x+0)g(s+x-1) \\ &+ \int_0^{s+x-1} U_s(x,s-s')g(s')\,ds', \quad s > 1-x, \end{aligned} \tag{B.7}$$

and $v(x,s) = 0$ for $s < 1-x$, $0 < x < 1$.

Proof: It is easy to show that Eq. (B.7) indeed is a solution to the PDE in Eq. (B.1) for $s > 1-x$. Propagation of singularity arguments and Eqs. (B.3) and (B.4) show that

$$v(x,1-x+0) = \frac{1}{2}g(0^+)\exp\left\{-\frac{1}{2}\int_1^x \big(A(x') + B(x')\big)\,dx'\right\}, \quad 0 < x < 1.$$

It is then straightforward to show that the remaining conditions on the solution in Eq. (B.6) are satisfied. Uniqueness then shows that Eq. (B.7) is the solution. QED.

Now define (*cf.* the definition of w_l^+ in Eq. (4.1))

$$\begin{aligned} w^+(x,s) =& \frac{1}{2}\Bigg\{ v(x,s) + (-1)^l v(x,s-2q) + \int_{s-2q}^{s} F^+(x,s-s')v(x,s')\,ds' \\ & + \int_{s-2q}^{s} G^+(x,s-s')v_x(x,s')\,ds' \Bigg\}, \end{aligned} \tag{B.8}$$

where $F^+(x,s)$ and $G^+(x,s)$ are defined as

$$\begin{cases} F^+(x,s) = -\frac{1}{q}(1-\frac{s}{q})P_l'(1-\frac{s}{q}) \\ G^+(x,s) = -P_l(1-\frac{s}{q}). \end{cases}$$

Evaluate Eq. (B.8) at $x=1$ and express it in $w^-(1,s) = g(s)$ and $U(1,s)$. The result is

$$w^+(1,s) = \frac{1}{2}H(s-2q_1)(-1)^l w^-(1,s-2q_1) + \int_0^s \tilde{R}(s-s')w^-(1,s')\,ds', \quad s>0,$$

where $q_1 = a/c_0\tau$ and

$$\begin{aligned} \tilde{R}(s) =& \frac{1}{2}\left(U_s(1,s) - U_x(1,s)\right) \\ & + \frac{1}{2}H(s-2q_1)(-1)^l\left(U_s(1,s-2q_1) + U_x(1,s-2q_1)\right) \\ & + \frac{1}{4}H(2q_1-s)\big(F^+(1,s) + G_s^+(1,s)\big) \\ & + \frac{1}{2}\int_0^s H(s'-s+2q_1)\left[F^+(1,s-s')U_s(1,s') + G_s^+(1,s-s')U_x(1,s')\right]\,ds'. \end{aligned} \tag{B.9}$$

In free space, when $c(r) = c_0$ inside the sphere $r=a$ and $B(x)=0$, the solution $U(x,s)$ is (cf Eqs. (5.5) and (5.6) with $q_1 = 1$)

$$U(x,s) = \frac{1}{2}\left[H(s+x-1) - H(s-x-1)\right]P_l(\frac{1-s}{x}) \quad 0<x<1.$$

With this explicit expression of the free space solution it is not hard to prove that in free space

$$\tilde{R}(s) = -\frac{1}{2}\delta(s-2)(-1)^l,$$

and thus $w^+(1,s) = 0$ as it should be.

The boundary condition at $x=0$ implies that $U(x,s)$ is discontinuous along the characteristic curve $s = 1+x$. To determine the jump discontinuity along this characteristic it is convenient to study the behavior of the solution in a small region $0 < x < \epsilon$ at the origin. The value of ϵ is taken small enough such that all variations in the phase velocity $c(r)$ can be ignored. This implies that the function $A(x) = 0$ and, furthermore, assume that $B(x) = B$, where B is a constant. Moreover, the coordinate $x = q$ inside the region $0 < x < \epsilon$. The solution in the region $0 < x < \epsilon$ is thus the solution of

$$v_{xx}(x,s) - v_{ss}(x,s) + Bv_s(x,s) - \frac{l(l+1)}{x^2}v(x,s) = 0,$$

The solution that is finite at the origin is proportional to

$$\begin{aligned}&\exp\left[\frac{Bs}{2}\right]\Bigg\{\left[H(s+x-1) - H(s-x-1)\right]P_l(\frac{1-s}{x})\\ &+\frac{1}{2}B\int_{1-x}^{\min(1+x,s)} P_l(\frac{1-u}{x})I_1(\frac{B\sqrt{s^2-u^2}}{2})\left(s^2-u^2\right)^{-\frac{1}{2}}u\,du\Bigg\}, \quad 0 < x < \epsilon,\end{aligned}$$

where $I_1(x)$ is the modified Bessel function of the first kind of order one. This solution can be found by Laplace transformation. The appropriate constant is then determined from the jump discontinuity at the characteristic $s = 1-x$. The constant is

$$\frac{1}{2}\exp\left\{-\frac{1}{2}\int_1^{\epsilon}(A(x') + B(x'))\,dx'\right\}\exp\left[\frac{B(\epsilon-1)}{2}\right],$$

as seen from Eq. (B.5) and the solution in the region $0 < x < \epsilon$ is

$$\begin{aligned}U(x,s) =&\frac{1}{2}\exp\left\{-\frac{1}{2}\int_1^{\epsilon}(A(x') + B(x'))\,dx'\right\}\exp\left[\frac{B(s-1+\epsilon)}{2}\right]\\ &\Bigg\{\left[H(s+x-1) - H(s-x-1)\right]P_l(\frac{1-s}{x})\\ &+\frac{1}{2}B\int_{1-x}^{\min(1+x,s)} P_l(\frac{1-u}{x})I_1(\frac{B\sqrt{s^2-u^2}}{2})\left(s^2-u^2\right)^{-\frac{1}{2}}u\,du\Bigg\},\\ &0 < x < \epsilon.\end{aligned}$$

The jump discontinuity at the characteristic $s = 1+x$ is in the limit $\epsilon \to 0$

$$[U(x,1+x)] = -\frac{1}{2}(-1)^l\exp\left\{\frac{1}{2}\int_x^1[A(x') - B(x')]\,dx'\right\}\exp\left\{\int_0^1 B(x')\,dx'\right\},$$

with the explicit value at $x = 1$

$$[U(1,2)] = -\frac{1}{2}(-1)^l\exp\left\{\int_0^1 B(x')\,dx'\right\}.$$

This jump discontinuity at $s = 2$ gives a delta function contribution in the expression of $\tilde{R}(s)$ in Eq. (B.9). The final expression for the relation between $w^+(1, s)$ and $w^-(1, s)$ is therefore

$$\begin{aligned} w^+(1,s) =& \frac{1}{2}H(s-2q_1)(-1)^l w^-(1,s-2q_1) - \frac{1}{2}H(s-2)(-1)^l w^-(1,s-2) \\ &+ \int_0^s R(s-s')w^-(1,s')\,ds', \quad s > 0, \end{aligned}$$

where $R(s)$ is a piecewise continuous function of s.

References

[1] E. Ammicht, J. P. Corones, and R. J. Krueger. Direct and inverse scattering for viscoelastic media. *J. Acoust. Soc. Am.*, **81**, 827–834, 1987.

[2] R. S. Beezley and R. J. Krueger. An electromagnetic inverse problem for dispersive media. *J. Math. Phys.*, **26**(2), 317–325, 1985.

[3] J. P. Corones, M. E. Davison, and R. J. Krueger. Direct and inverse scattering in the time domain via invariant imbedding equations. *J. Acoust. Soc. Am.*, **74**(5), 1535–1541, 1983.

[4] J. P. Corones, M. E. Davison, and R. J. Krueger. Wave splittings, invariant imbedding and inverse scattering. In A. J. Devaney, editor, *Inverse Optics*, pages 102–106, Bellingham, WA, 1983. Proc. SPIE 413, SPIE.

[5] J. P. Corones, M. E. Davison, and R. J. Krueger. Dissipative inverse problems in the time domain. In W.-M. Boerner, editor, *Inverse Methods in Electromagnetic Imaging*, volume 143, pages 121–130, Reidel Dordrecht, 1985. NATO ASI series, Series C.

[6] F. G. Friedlander. On the radiation field of pulse solutions of the wave equation. *Proc. Roy. Soc.*, **Series A 269**, 53–65, 1962.

[7] A. Karlsson. Inverse scattering for viscoelastic media using transmission data. *Inverse Problems*, **3**, 691–709, 1987.

[8] G. Kristensson and R. J. Krueger. Direct and inverse scattering in the time domain for a dissipative wave equation. Part 1: Scattering operators. *J. Math. Phys.*, **27**(6), 1667–1682, 1986.

[9] W. Magnus, F. Oberhettinger, and R. P. Soni. *Formulas and Theorems for the Special Functions of Mathematical Physics.* Springer-Verlag, New York, 1966.

[10] V. H. Weston. Factorization of the wave equation in higher dimensions. *J. Math. Phys.*, **28**, 1061–1068, 1987.

[11] V. H. Weston. Factorization of the wave equation in a non-planar stratified medium. *J. Math. Phys.*, **29**, 36–45, 1988.