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Feasible Coordination of Multiple Homogeneous or Heterogeneous Mobile Vehicles with Various Constraints

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Abstract— We consider the problem of feasible coordination control for multiple homogeneous or heterogeneous mobile vehicles subject to various constraints (nonholonomic motion constraints, holonomic coordination constraints, equality/inequality constraints etc). We develop a general framework involving differential-algebraic equations and viability theory to describe and determine coordination feasibility for a coordinated motion control under heterogeneous vehicle dynamics and various constraints. A heuristic algorithm is proposed for generating feasible trajectories for each individual vehicle. We show several application examples and simulation experiments on multi-vehicle coordination under various constraints to validate the theory and the effectiveness of the proposed algorithm and control schemes.

I. INTRODUCTION

In the active research field of mobile robot motion planning and control, multi-vehicle coordination and cooperative control has been and will remain an attractive research topic, motivated by an increasing number of practical applications requiring multiple robots or vehicles to cooperatively perform coordinated tasks. A fundamental problem in multi-vehicle coordination is to plan feasible motion schemes and trajectories for each individual vehicle which should satisfy both kinematic or dynamic requirement for all vehicles, and inter-vehicle geometric constraints that define a given coordination task.

Typically, an individual vehicle is subject to various kinematic motion constraints which limit possible motion directions. In the seminal paper by Tabuada *et al.* [1], the problem of motion feasibility was studied in the context of multi-agent formation control. Via the tools of differential geometry, feasibility conditions were derived for a group of mobile agents to maintain formation specifications (described by strict equality constraints) in each agent's motions. Recently, the motion feasibility problem in multi-vehicle formation and cooperative control has resumed its interests in the control and robotics community. The paper [2] discusses coordination control with dynamically feasible vehicle motions, and solves a rigid formation shape maintenance task and formation reconfiguration problem. Our recent work [3] investigates the formation and coordination feasibility with heterogeneous systems modelled by control affine nonlinear systems with drift terms (which include

fully-actuated systems, under-actuated systems, and non-holonomic vehicles). The paper [4] discussed cooperative transport of a buoyant load using two autonomous surface vehicles via feasibility coordination approach. More recently, the work by Colombo and Dimarogonas [5] extends the motion feasibility condition in [1] to multi-agent formation control systems on Lie groups.

Coordinating multiple vehicles often involve various types of inter-vehicle constraints, typically described by equality or inequality functions of inter-vehicle geometric variables. For example, a practical coordinated motion may be described by some inequality constraints that require a bounded inter-vehicle distance between mobile vehicles; i.e., a lower bound to guarantee collision avoidance, and an upper bound to avoid communication loss due to excessively long ranges. As another example, multi-robotic visibility maintenance control is also often modelled by certain inequality constraints [6]. All these practical coordination control scenarios call for a general framework for multi-vehicle coordination planning and control under various constraints. We remark that the above referenced papers [1]–[5] only discussed formation or coordination control for multiple vehicles with *strict equality* functions. This paper will focus on a more general problem in multi-vehicle coordination control that also includes inequality constraints, or a mix of equality and inequality constraints. Our tools to solve feasible coordination problem of multiple vehicles with various constraints are an interplay of differential geometry for nonlinear control [7], viability theory [8] and differential-algebraic equations and inequalities.

In this paper, a synthesis of coordination control under various constraints will be provided, which include vehicles' kinematic constraints (often modelled by nonholonomic motion constraints) and inter-vehicle constraints (which include holonomic formation constraints, inequality functions or a mix of various constraints). We will also devise a heuristic algorithm to solve the proposed feasibility equations and inequalities that generate feasible trajectories for all vehicles to achieve a coordination task.

II. PRELIMINARIES AND PROBLEM FORMULATION

In this section we introduce some standard notions and tools of differential geometry and nonlinear control from [9], [10] which will be used in the main part of this paper.

A. Distribution, codistribution and vehicle models

A distribution $\Delta(x)$ on $\mathbb{R}^n$ is an assignment of a linear subspace of $\mathbb{R}^n$ at each point x. Given a set of k vector fields

$X_1(x), X_2(x), \cdots, X_k(x)$, we define the distribution as

$$\Delta(x) = \text{span}\{X_1(x), X_2(x), \cdots, X_k(x)\}.$$

A vector field X belongs to a distribution Δ if $X(x) \in \Delta(x)$, $\forall x \in \mathbb{R}^n$, and we assume all distributions have constant rank.

A codistribution assigns a subspace to the dual space, denoted by $(\mathbb{R}^n)^\star$. Given a distribution Δ, for each x consider the annihilator of Δ, which is the set of all covectors that annihilates all vectors in $\Delta(x)$ (see [7, Chapter 1])

$$\Delta^\perp = \{\omega \in (\mathbb{R}^n)^\star | \langle \omega, X \rangle = 0, \ \forall X \in \Delta\}. \quad (1)$$

In this paper, we model each individual vehicle's dynamics by the following control-affine form

$$\dot{p}_i = f_{i,0} + \sum_{j=1}^{l_i} f_{i,j} u_{i,j}, \quad (2)$$

where $p_i \in \mathcal{C}_i \in \mathbb{R}^{n_i}$ is the state of vehicle i ($\mathcal{C}_i$ denotes the configuration for vehicle i, for which we embed $\mathcal{C}_i$ in $\mathbb{R}^{n_i}$ where n_i denotes the dimension of state space for vehicle i), $f_{i,0}$ is a smooth drift term, and $u_{i,j}$ is the *scalar* control input associated with the smooth vector field $f_{i,j}$, and l_i is the number of vector field functions.

B. Viability theory and set-invariance control

In this paper, we will treat coordination tasks with inequality constraints, and a key tool to address inequality constraint is the viability theory and set-invariance control [8], [11]. Consider a control system described by a differential equation $\dot{x}(t) = f(x(t), u(t))$. A subset $\mathcal{F}$ enjoys the viability property for the system $\dot{x}(t)$ if for every initial state $x(0) \in \mathcal{F}$, there exists at least one solution to the system starting at $x(0)$ which is viable in the time interval $[0, \bar{t}]$ in the sense that $\forall t \in [0, \bar{t}], x(t) \in \mathcal{F}$. A key result in the set-invariance analysis, the celebrated Nagumo theorem, is stated as follows (see e.g. [11] or [8]).

Theorem 1: (**Nagumo theorem**) Consider the system $\dot{x}(t) = f(x(t))$, and assume that, for each initial condition in a set $\mathcal{X} \subset \mathbb{R}^n$, it admits a globally unique solution. Let $\mathcal{F} \subset \mathcal{X}$ be a closed and convex set. Then the set $\mathcal{F}$ is positively invariant for the system if and only if

$$f(x(t)) \in T_{\mathcal{F}}(x), \ \forall x \in \mathcal{F}. \quad (3)$$

where $T_{\mathcal{F}}(x)$ denotes the *contingent cone* of $\mathcal{F}$ at x.
For the definition of contingent cone and generalizations of the Nagumo theorem, see [8].

The condition in Theorem 1 is only meaningful when $x \in \text{bnd}(\mathcal{F})$, where $\text{bnd}(\mathcal{F})$ denotes the boundary of $\mathcal{F}$. Therefore, the condition in (3) can be equivalently stated by

$$f(x(t)) \in T_{\mathcal{F}}(x), \ \forall x \in \text{bnd}(\mathcal{F}). \quad (4)$$

The above condition clearly has an intuitive and geometric interpretation: if at $x \in \text{bnd}(\mathcal{F})$, the derivative $\dot{x} = f(x(t))$ points inside or is tangent to $\mathcal{F}$, then the trajectory $x(t)$ remains in $\mathcal{F}$.

Now we consider a viable set $\mathcal{F}$ parameterized by an inequality associated with a continuously differentiable func-

tion $g(x): \mathbb{R}^n \to \mathbb{R}$,

$$\mathcal{F} = \{x | g(x) \leq 0\}. \quad (5)$$

In this way, the calculation of $T_{\mathcal{F}}(x)$ is simplified to be

$$T_{\mathcal{F}}(x) = \{v \in x | \langle v, \nabla g(x) \rangle \leq 0\}, \quad (6)$$

for any $g(x) = 0$ and $T_{\mathcal{F}}(x) = \mathbb{R}^n$ when $g(x) < 0$. For the set $\mathcal{F}$ defined in (5), a consequence of Nagumo theorem is the following lemma on a controlled-invariant set.

Lemma 1: (**Set-invariance in control**, [11]) Consider a set $\mathcal{F}$ parameterized by an inequality of a continuously differentiable function $g(x)$: $\mathcal{F} = \{x | g(x) \leq 0\}$. Then the set $\mathcal{F}$ is positively invariant under the dynamic control system $\dot{x}(t) = f(x(t), u(t))$ if $\dot{x}(t) \in T_{\mathcal{F}}(x)$ of (6), or equivalently

$$\langle \nabla g(x), f(x(t), u(t)) \rangle \leq 0, \ \forall x : g(x(t)) = 0. \quad (7)$$

C. Problem formulation

Consider a group of n vehicles, whose kinematic equations are described by the control-affine systems (2) with possibly different kinematics and/or drift terms. We assign the vehicle group with a coordination task, described by inter-vehicle geometric equality or inequality constraints that incorporate formation, flocking or other cooperative tasks. Two key problems to be addressed in this paper are the following:

- Determine whether a group of homogeneous or heterogeneous vehicles can perform a coordination task with various constraints;
- If the coordination task with various constraints is feasible, determine feasible motions that generate trajectories for an n-vehicle group to perform the task.

III. FORMULATION OF COORDINATION CONSTRAINTS

A. Motion constraints arising from vehicle kinematics

In this subsection we follow the techniques in [3], [10] to formulate vehicle's kinematic constraints using (affine) codistributions. A vehicle's kinematics modelled by a nonlinear control-affine system (2) with drifts can be equivalently described by the following *affine* distribution

$$\Delta_i = f_{i,0} + \text{span}\{f_{i,1}, f_{i,2}, \cdots, f_{i,l_i}\}. \quad (8)$$

For the system (2) with drifts, one can obtain a corresponding transformation with equivalent constraints via the construction of covectors

$$\omega_{i,j}(p_i)\dot{p}_i = q_{i,j}, \quad j = 1, \cdots, n_i - l_i, \quad (9)$$

where the term $q_{i,j}$ is due to the existence of the drift term $f_{i,0}$. We collect all the *row covectors* $\omega_{i,j}$ as $\Omega_{K_i} = [\omega_{i,1}^\top, \ \omega_{i,2}^\top, \ \cdots, \ \omega_{i,n_i-l_i}^\top]^\top$, and similarly define $T_{K_i} = [q_{i,1}, q_{i,2}, \cdots, q_{i,n_i-l_i}]^\top$. By doing this, one can write (9) as

$$\Omega_{K_i}\dot{p}_i = T_{K_i}, \quad (10)$$

where the subscript K stands for *kinematics*. Furthermore, we collect all the kinematic constraints for all the n vehicles in a composite form $\Omega_K = [\Omega_{K_1}^\top, \Omega_{K_2}^\top, \cdots, \Omega_{K_n}^\top]^\top, T_K = [T_{K_1}^\top, T_{K_2}^\top, \cdots, T_{K_n}^\top]^\top$. For ease of notation, we collect all of the vehicles' states together, denoting them by the

composite state vector $P = [p_1^\top, p_2^\top, \cdots, p_n^\top]^\top$. Thus, the overall kinematic constraint for all the vehicles can be stated compactly as $\Omega_K(\dot{P}) = T_K$. We remark that if the drift term $f_{i,0}$ satisfies $f_{i,0} \in \text{span}\{f_{i,1}, f_{i,2}, \cdots, f_{i,l_i}\}$, then one can choose a control u_i to cancel the drift, and the affine control system with a drift term (2) can be reduced to a drift-free control system.

B. Motion constraints arising from coordination tasks

In this section we formulate motion constraints from coordination tasks using distributions/codistributions.

1) Coordination with equality constraints: We follow the formulation in [12] (also treated in [3]) and describe the (possibly time-varying) equality constraints encoded in a network. We assume a networked multi-vehicle control system modelled by an undirected graph $\mathcal{G}$, in which we use $\mathcal{V}$ to denote its vertex set and $\mathcal{E}$ to denote the edge set. The vertices consist of n homogeneous or heterogeneous vehicles, each modelled by the general dynamical equation (2) with possibly different dynamics. The graph consists of m edges, each associated with one or multiple inter-vehicle constraints describing a coordination task. A family of (possibly time-varying) equality constraints Φ is indexed by the edge set, denoted as $\Phi_\mathcal{E}(P, t) = \{\Phi_{ij}\}_{(i,j)}$ with $(i,j) \in \mathcal{E}$. For each edge (i,j), $\Phi_{ij}(p_i, p_j, t)$ is a continuously differentiable vector function of p_i and p_j defining the coordination constraints between the vehicle pair i and j. The constraint for edge (i,j) is enforced if $\Phi_{ij}(p_i, p_j) = 0$. To satisfy an equality constraint for edge (i,j), it should hold that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_{ij}(p_i, p_j, t) = \frac{\partial \Phi_{ij}}{\partial p_i}\dot{p}_i + \frac{\partial \Phi_{ij}}{\partial p_j}\dot{p}_j + \frac{\partial \Phi_{ij}}{\partial t} = 0. \quad (11)$$

A coordination task is maintained if $\Phi_\mathcal{E}(P, t) = 0$ is enforced for all the edges. Coordination feasibility requires that if the constraint is satisfied at time $t_0 = 0$, then it should be satisfied for all time $t \geq t_0$ under suitable coordination controls. Thus, one can obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(P, t) = \frac{\partial \Phi}{\partial P}\dot{P} + \frac{\partial \Phi}{\partial t} = 0. \quad (12)$$

Now we group all the constraints for all the edges by writing down a compact form $T_E = -[\cdots, (\frac{\partial \Phi_{ij}}{\partial t})^\top, \cdots]^\top$. By employing the fact that partial derivative (vectors) and differentials (co-vectors) are dual in $\mathbb{R}^n$, we identify a codistribution matrix Ω_E associated with the Jacobian $\partial \Phi / \partial P$ using the nominal dual coordinate basis from the differential $[\mathrm{d}P]$, from which one can reexpress equation (12) as

$$\Omega_E(\dot{P}) = T_E. \quad (13)$$

where the subscript E stands for *equality* constraints. For time-invariant equality constraint, one has $T_E = 0$. In summary, the vector field $\dot{P}$ defined by the above equation (13) represents possible motions for all the vehicles that respect the coordination equality constraint.

2) Coordination with inequality constraints: Now we consider a feasible coordination problem involving *inequality*

constraints. A family of inequality coordination constraints $\mathcal{I}_\mathcal{E} = \{\mathcal{I}_{ij}\}_{(i,j)}$ is indexed by the edge set $\mathcal{E}$, and each edge (i,j) is associated with a function $\mathcal{I}_{ij}(p_i, p_j)$ [1] which is assumed continuously differentiable. The constraints for the edge (i,j) are enforced if $\mathcal{I}_{ij}(p_i(t), p_j(t)) \leq 0 \; \forall t$. Now we consider the subset of *active* constraints among all edges,

$$\chi(P) = \{(i,j), \; i,j = 1, 2, \cdots, n \mid \mathcal{I}_{ij}(p_i, p_j) = 0\}. \quad (14)$$

We remark that the set $\chi(P)$ is a dynamic set along time as it involves the edge set with active constraints when the condition $\mathcal{I}_{ij}(p_i, p_j) \leq 0$ is about to be violated. An inequality constraint for edge (i,j) is maintained if

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_{ij} = \frac{\partial \mathcal{I}_{ij}}{\partial p_i}\dot{p}_i + \frac{\partial \mathcal{I}_{ij}}{\partial p_j}\dot{p}_j \leq 0, \; \forall (i,j) \in \chi(P). \quad (15)$$

At any point in time, all the active constraints in the edge set $\chi(P)$ generate a codistribution

$$\Omega_I = [\cdots, \Omega_{I,ij}^\top, \cdots]^\top, \; \forall (i,j) \in \chi(P), \quad (16)$$

where the subscript I stands for *inequality* constraints, and $\Omega_{I,ij}$ is obtained by the Jacobian of the vector function $\mathcal{I}_{ij}$ using the nominal coordinate basis $[\mathrm{d}p_i, \mathrm{d}p_j]$ associated with the active constraint $\mathcal{I}_{ij}(p_i, p_j) = 0$. Based on the Nagumo theorem and Lemma 1, to guarantee the validity of the inequality constraints, the control input $u(t) = [u_1(t)^\top, \cdots, u_n(t)^\top]^\top$ for each vehicle should be designed such that $\Omega_I \dot{P}(P, u(t)) \leq 0, \forall (i,j) \in \chi(P)$.

IV. COORDINATION FEASIBILITY AND MOTION GENERATION

A. Coordination feasibility with inequality task constraints

We first state the following theorem on a feasible coordination for an n-vehicle group with kinematic constraint and inequality constraints in a coordination task.

Theorem 2: The coordination task with inequality constraints has feasible motions if the following mixed (in)equalities have solutions

$$\begin{aligned} \Omega_K \dot{P} &= T_K, \\ \Omega_I \dot{P} &\leq 0, \; \forall (i,j) \in \chi(P), \end{aligned} \quad (17)$$

where $\chi(P)$ denotes the set of active constraints among all the edges.

B. Coordination feasibility of multiple vehicles with both equality and inequality task constraints

We now consider coordination tasks with both equality and inequality constraints. The following theorem determines coordination feasibility with various constraints.

Theorem 3: The coordination task with both equality and inequality constraints has feasible motions if the following

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[1] In this paper we only consider time-invariant functions $\mathcal{I}_{ij}(p_i, p_j)$ in the inequality constraint. Extensions to time-varying inequality constraints are also possible, by employing the temporal viability regulation theory [13].

mixed equations and inequalities have solutions

$$\Omega_K \dot{P} = T_K,$$
$$\Omega_E \dot{P} = T_E,$$
$$\Omega_I \dot{P} \leq 0, \quad \forall (i,j) \in \chi(P), \tag{18}$$

where $\chi(P)$ denotes the set of active constraints among all the edges.

Proofs of the above theorems follow from the analysis in Section III and are omitted here. Extensions to the leader-follower coordination case are available; see [14].

C. Generating vehicle's motion and trajectory for a feasible coordination

The feasibility conditions presented in Theorems 2 and 3 involve the determination of the existence of solutions for an algebraic equation and a mixed inequality with equations. Algorithm 1 presents a heuristic approach to determine coordination feasibility and motion generation for the multi-vehicle coordination control under both equality and inequality constraints. The algorithm does not generate all feasible motion directions, but only a set of feasible motions depending on the choice of virtual input w_l. Generalizations of the algorithm and selections of optimal motion directions will be in future research. Note also when a feasible motion is determined with a set of virtual input w_l, the actual control input u_i can be readily calculated via each vehicle's kinematic equations.

V. APPLICATION EXAMPLES: COORDINATING MULTIPLE VEHICLES WITH DISTANCE AND HEADING CONSTRAINTS

A. Typical vehicle kinematics and coordination constraints

In this section, we consider several application case studies to illustrate the proposed coordination theory and algorithms. Two types of vehicles, a unicycle-type vehicle and a car-like vehicle, will be considered in the examples. The unicycle vehicle is described by

$$\dot{x}_i = v_i \cos(\theta_i),$$
$$\dot{y}_i = v_i \sin(\theta_i), \tag{19}$$
$$\dot{\theta}_i = u_i,$$

where the state variable is $p_i = [x_i, y_i, \theta_i]^\top \in \mathbb{R}^2 \times \mathbb{S}^1 \in \mathbb{R}^3$. The kinematic constraint for a unicycle-type vehicle can be equivalently stated by the annihilating codistribution $\Omega_{K_i} = \Delta_i^\perp = \text{span}\{\sin(\theta_i)dx_i - \cos(\theta_i)dy_i\}$.

Further consider a car-like vehicle (see e.g. [15]), whose kinematic equation is described by

$$\dot{x}_i = u_{i,1}\cos(\theta_i),$$
$$\dot{y}_i = u_{i,1}\sin(\theta_i),$$
$$\dot{\theta}_i = u_{i,1}(1/l_i)\tan(\phi_i),$$
$$\dot{\phi}_i = u_{i,2}, \tag{20}$$

with the state variables $p_i = (x_i, y_i, \theta_i, \phi_i) \in \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1$, where (x_i, y_i) are the Cartesian coordinates of the rear wheel, θ_i is the orientation angle of the vehicle body with

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Algorithm 1: Coordination feasibility checking and motion generation.

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1 Initialization: Ω_{K_i}, T_{K_i}, Ω_E, T_E, $\chi(P)$, Ω_I;
2 Construct the overall kinematic codistribution matrix Ω_K and the vector T_K.
3 while *Running* **do**
4   *Solve equality* $\begin{bmatrix} \Omega_K \\ \Omega_E \end{bmatrix} \dot{P} = \begin{bmatrix} T_K \\ T_E \end{bmatrix}$
5   **if** *Solution does not exist* **then**
6   Return: No solution;
7   Condition checking STOP.
8   **else**
9   Calculate a special solution to the above equality constraint equation, denoted by $\bar{K}$;
10   Determine κ basis vectors of Null $\left(\begin{bmatrix} \Omega_K \\ \Omega_E \end{bmatrix} \right)$, denoted by $K_1, K_2, \cdots, K_\kappa$.
11   **end**
12   **if** $\chi(P) = \emptyset$ *(No active inequality constraint)* **then**
13   Feasible motions $\dot{P} = \bar{K} + \sum_{l=1}^{\kappa} K_l w_l$, where w_l is a set of virtual inputs that activate the associated vector field K_l;
14   Return: a set of feasible motions $\dot{P} = \bar{K} + \sum_{l=1}^{\kappa} K_l w_l$ (according to different choices of w_l).
15   **else**
16   **for** $l = 1, 2, \cdots, \kappa$ **do**
17   Calculate and obtain the codistribution matrix Ω_I for active equality constraints, with $\forall (i,j) \in \chi(P)$;
18   **if** $\Omega_I(\bar{K} + K_l w_l) \leq 0$ *for certain w_l* **then**
19   Return: A feasible motion $\dot{P} = \bar{K} + K_l w_l$.
20   **end**
21   **end**
22   **end**
23   **if** $l > \kappa$ **then**
24   Return: Feasible solution not found. Try again.
25   **end**
26 end

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respect to the x axis, ϕ_i is the steering angle, and l_i is the distance between the midpoints of the two wheels. The model (20) describes kinematic motions for a typical rear-wheel-driving car, which is subject to two non-holonomic motion constraints (rolling without slipping sideways for each wheel, respectively). In an equivalent compact form, one can write

$$\dot{p}_i = [\dot{x}_i, \dot{y}_i, \dot{\theta}_i, \dot{\phi}_i]^\top = f_{i,1} u_{i,1} + f_{i,2} u_{i,2}, \tag{21}$$

with $f_{i,1} = [\cos(\theta_i), \sin(\theta_i), (1/l_i)\tan(\phi_i), 0]^\top$ and $f_{i,2} = [0, 0, 0, 1]^\top$. The distribution generated by the two vector fields $f_{i,1}$ and $f_{i,2}$ is described by $\Delta_i = \text{span}\{f_{i,1}, f_{i,2}\}$, which can be equivalently stated by the annihilating co-
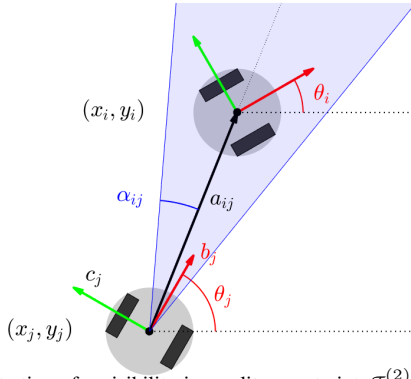
Fig. 1. Illustration of a visibility inequality constraint, $\mathcal{I}_{ij}^{(2)}$, bounding the direction b_j to the blue cone defined by a_{ij} and the half apex angle α_{ij}.

distribution: $\Omega_{K_i} = \Delta_i^\perp = \text{span}\{\sin(\theta_i + \phi_i)dx_i - \cos(\theta_i + \phi_i)dy_i - l_i\cos(\phi_i)d\theta_i, \sin(\theta_i)dx_i - \cos(\theta_i)dy_i\}$. Consider two of the previously defined vehicles in the form (10) sub-indexed i and j respectively, as illustrated in Fig. 1. A common coordination task may include a simple inter-vehicle distance equality constraint,

$$\Phi_{ij}^{(1)} : \frac{1}{2}(x_i - x_j)^2 + \frac{1}{2}(y_i - y_j)^2 - \frac{1}{2}d_{ij}^2 = 0, \quad (22)$$

for some $d_{ij} > 0$, which generates a codistribution matrix

$$\Omega_{E,ij}^{(1)} = [(x_i - x_j)(dx_i - dx_j) + (y_i - y_j)(dy_i - dy_j)]. \quad (23)$$

In a practical setting, however, it may be useful to consider a distance constraint in terms of a two-sided inequality,

$$\mathcal{I}_{ij}^{(1)} : \frac{1}{2}(d_{ij}^-)^2 \leq \frac{1}{2}(x_i - x_j)^2 + \frac{1}{2}(y_i - y_j)^2 \leq \frac{1}{2}(d_{ij}^+)^2, \quad (24)$$

with $d_{ij}^-, d_{ij}^+ > 0$, and codistribution matrix given by $\Omega_{I,ij}^{(1)} = \Omega_{E,ij}^{(1)}$ if the right inequality becomes active, or $\Omega_{I,ij}^{(1)} = -\Omega_{E,ij}^{(1)}$ if the left inequality becomes active.

Another useful inequality constraint often encountered in visibility maintenance control in multi-robotic systems is the so-called visibility constraint, see e.g. [6]. This constraint seeks to constrain the heading angle of vehicle j such that vehicle i always resides in a cone of visibility (see Fig. 1). Instead of defining this constraint with the arctangent function as in [6], we consider an equivalent inequality constraint using cosine apex angle, thereby avoiding issues with the range of the arctangent function. Let $a_{ij} := [x_i - x_j, y_i - y_j]$, $b_j := [\cos(\theta_j), \sin(\theta_j)]$, $c_j := [-\sin(\theta_j), \cos(\theta_j)]$, and form the inequality constraint

$$\mathcal{I}_{ij}^{(2)} : \cos(\alpha_{ij})\langle a_{ij}, a_{ij}\rangle^{1/2} \leq \langle a_{ij}, b_j\rangle. \quad (25)$$

The associated codistribution matrix can be derived as

$$\Omega_{I,ij}^{(2)} = \frac{\langle a_{ij}, c_j\rangle}{\sqrt{\langle a_{ij}, a_{ij}\rangle}}\left(\frac{1}{\langle a_{ij}, a_{ij}\rangle}\left\langle a_{ij}, \begin{bmatrix} dx_i - dx_j \\ dy_j - dy_i \end{bmatrix}\right\rangle + d\theta_j\right). \quad (26)$$

when the inequality constraint (25) becomes active.

Remark 1: It should be noted that the constraint (26) may become singular due to the division by $\langle a_{ij}, a_{ij}\rangle$, a corner

case to be revisited and addressed in the examples.

B. Coordinating a unicycle and a car-like vehicle

Consider a two-vehicle group, one described by the unicycle equation and the other by a car-like dynamics. The two vehicles assume a task to cooperatively maintain a constant distance d_{12} and a heading or visibility inequality constraint.

The joint codistribution matrix from both kinematic constraint and distance equality constraint can be obtained as (using the dual space basis $[dx_1, dy_2, \cdots, d\phi_2, d\theta_2]$): $\Omega = [\sin(\theta_1)dx_1 - \cos(\theta_1)dy_1, \sin(\theta_2 + \phi_2)dx_2 - \cos(\theta_2 + \phi_2)dy_2 - l_2\cos(\phi_2)d\theta_2, \sin(\theta_2)dx_2 - \cos(\theta_2)dy_2, (x_1 - x_2)(dx_1 - dx_2) + (y_1 - y_2)(dy_1 - dy_2)]$. The solution to the algebraic equation $\Omega(\dot{P}) = T = 0$ is obtained as $\dot{P} = \sum_{l=1}^3 w_l K_l$ with $K_1 = [0,0,1,0,0,0,0]^\top, K_2 = [0,0,0,0,0,0,1]^\top$, and

$$K_3 = \begin{bmatrix} \cos(\theta_1)\left(\cos(\theta_2)(x_1 - x_2) + \sin(\theta_2)(y_1 - y_2)\right) \\ \sin(\theta_1)\left(\cos(\theta_2)(x_1 - x_2) + \sin(\theta_2)(y_1 - y_2)\right) \\ 0 \\ \cos(\theta_2)\left(\cos(\theta_1)(x_1 - x_2) + \sin(\theta_1)(y_1 - y_2)\right) \\ \sin(\theta_2)\left(\cos(\theta_1)(x_1 - x_2) + \sin(\theta_1)(y_1 - y_2)\right) \\ \frac{1}{l_2}\tan\phi_2\left(\cos(\theta_1)(x_1 - x_2) + \sin(\theta_1)(y_1 - y_2)\right) \\ 0 \end{bmatrix}$$

By following Algorithm 1, the coordination feasibility and motion generation result is given in the following lemma.

Lemma 2: Consider a two-vehicle group consisting of a unicycle-type vehicle and a car-like vehicle, with a coordination task of maintaining a constant inter-vehicle distance d_{12} and a heading/visibility constraint described as above. By using the above derived control solutions:

- The inter-vehicle distance is preserved with the above-derived control for any w_l.
- If initially the heading/visibility inequality is satisfied, then a feasible control always exists (with the possible choice of w_l) that preserves both distance equality and heading/visibility inequality constraints.

C. Multiple homogeneous vehicles with mixed constraints

Now we consider multiple unicycle models described by (19), with one leader vehicle $p_1(t) \in \mathbb{R}^2 \in \mathbb{R}^3 \times \mathbb{S}^1$ and two followers $p_2(t), p_3(t) \in \mathbb{R}^2 \times \mathbb{S}^1 \in \mathbb{R}^3$. The kinematics yield an annihilating co-distribution $\sin(\theta_i)dx_i - \cos(\theta_i)dy_i = 0$, resulting in an $\Omega_K \in \mathbb{R}^{3 \times 9}$ with $T_K = [0,0,0]^\top$. The leader is constrained to follow an arbitrary reference trajectory in terms of two continuous control inputs $v_{1,r}(t), u_{1,r}(t)$. These time-varying speeds are incorporated as two equality constraints, with $\cos(\theta_1)dx_1 + \sin(\theta_1)dy_1 = v_{1,r}(t)$, $d\theta_1 = u_{1,r}(t)$, represented in the standard compact matrix form with $\Omega_E \in \mathbb{R}^{2 \times 9}$ with $T_E = [v_{1,r}(t), u_{1,r}(t)]^\top \in \mathbb{R}^2$.

In order for the followers to maintain visibility of the leader, we pose two inequality constraints in the form (25), enforcing $\mathcal{I}_{12}^{(2)}$ and $\mathcal{I}_{13}^{(2)}$ with a maximum heading angle of $\Delta\theta_{12} = \Delta\theta_{13} = 0.4$ (rad). The annihilating codistribution matrices $\Omega_{I,12}^{(2)}$ and $\Omega_{I,13}^{(2)}$ are given in (26), which are omitted here for brevity. As was noted in Remark 1, these matrices are well defined when the distance between the vehicles is

non-zero. To eliminate the possibility of singular solutions, a distance inequality constraint is posed in the form (24) as $\mathcal{I}_{12}^{(1)}$ and $\mathcal{I}_{13}^{(1)}$, with $d_{12}^- = d_{13}^- = 1$ and $d_{12}^+ = d_{13}^+ = 2$. We note there exists a direction $\langle a_{1j}, b_j \rangle = 0$, at which the motion solution becomes singular when activating any distance constraint $\mathcal{I}_{1j}^{(1)}$. This caveat is conveniently avoided by the posed heading inequality constraint, effectively bounding $\langle a_{1j}, b_j \rangle \geq d_{1j}^- \cos(\alpha_{1j}) = 0.92$. Consequently, any feasible motion found by Algorithm 1 satisfying the posed inequality constraints gives rise to non-singular, well-defined solution control flows. Combining the constraints yields $\Omega_I = [(\Omega_{I,12}^{(1)})^\top (\Omega_{I,13}^{(1)})^\top (\Omega_{I,12}^{(2)})^\top (\Omega_{I,13}^{(2)})^\top] \in \mathbb{R}^{6 \times 9}$, of which at most four constraints may be active at any point in time (the distance upper and lower bound cannot be met simultaneously). This complex system with one leader vehicle (with predefined constrained speeds) and two following unicycles always has feasible coordination motions in all possible combinations of these constraints when checked with Algorithm 1. To show the found solutions in practice, a simulation was run with the three vehicles, recomputing the virtual inputs $w_i \in \mathbb{R}$ each time an inequality constraint was activated. We consider a leader vehicle reference trajectory $v_{1r}(t) = 2\sin(t)$, $u_{1r}(t) = 2\cos(2t)$, which is followed perfectly when incorporated through time-varying equality constraints, as demonstrated in Fig. 2. Furthermore, the two-dimensional trajectories of the leader (red) and followers (blue, green) are depicted in Fig. 2, along with the distance inequality constraints $\{\mathcal{I}_{12}^{(1)}, \mathcal{I}_{13}^{(1)}\}$, and the cosine angle inequality constraints $\{\mathcal{I}_{12}^{(2)}, \mathcal{I}_{13}^{(2)}\}$ for maintaining visibility, which are met at all times.

More application examples and demonstrations can be found in the online arXiv version [14] and the accompanying video.

VI. Conclusions

In this paper, we discuss the coordination control problem for multiple mobile vehicles subject to various constraints (nonholonomic motion constraints, holonomic formation constraints, equality or inequality constraints, among others). Using tools from differential geometry, distribution/codistributions for control-affine systems and viability theory, we have developed a general framework to determine whether feasible motions exist for a multi-vehicle group that meet both kinematic constraints and coordination constraints with a mix of inequality and equality functions. A heuristic algorithm is proposed to find feasible motions and trajectories for a group of homogeneous or heterogeneous vehicles to achieve a coordination task. Several case study examples and simulation experiments are also provided to illustrate the proposed theory and coordination control schemes.
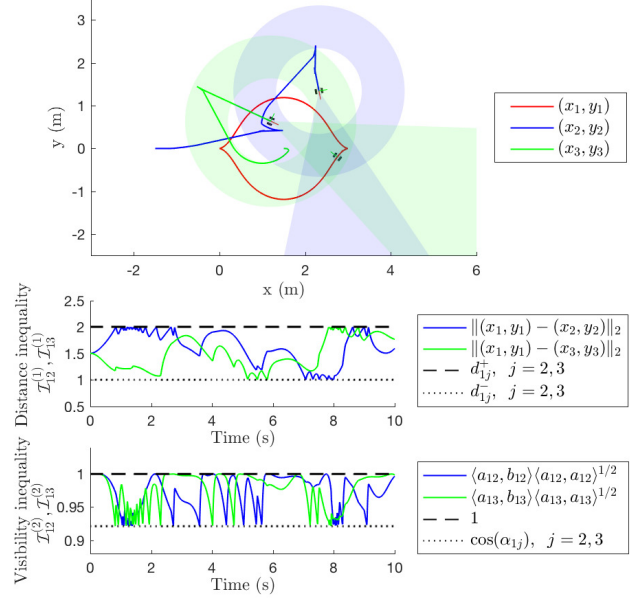

Fig. 2. *Top*: Two-dimensional plot of the three unicycle vehicles, with the leader (red) and the two followers (blue and green). *Center*: Inequality-constrained distance between the leader and the followers. *Bottom*: Inequality-constrained cosine angle between the vectors a_{ij} and b_j for visibility maintenance of the leader by the followers.

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