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ABSTRACT

INTRODUCTION

Necking is the dominant process at fracture of thin plates. It implies continuous cross-sectional slip and requires a three-dimensional analysis to be fully understood. Dugdale, however presented a solution [1] by using a two-dimensional approach for a mode I crack developing necks in an infinite plate of an elastic-perfectly-plastic material. He assumed that the plate was uniaxially stressed normal to the crack at infinity and that plasticity was confined to a narrow strip ahead of the crack tip.

Drucker and Rice [2] later argued that the Dugdale model is consistent only at plane stress and a Tresca material. For both Tresca and von Mises materials the effective stress exceeds the yield stress in large regions off-side the necking region at plane strain whereas it is less than the yield stress in the entire plane outside the necking regions at plane stress. However the rule of plastic flow is generally violated at plane stress for a von Mises material but not so for a Tresca material. The flow rule by definition meaning that the plastic strain increments are normal to the flow surface, implies that necking is allowed only if the strain increment is zero along the necking region.

However, even in the case of Tresca materials, the realization of a Dugdale type region might be difficult. The difference between the stresses normal and parallel to the crack is constant throughout the crack plane and its extension. Therefor, under a uniaxial stress state at infinity compressive stresses parallel to the crack prevail near the crack surfaces and hence there is a risk for local buckling. Buckling is however not mandatory in reality, since the finite plate thickness-width ratio allows small but finite loads without buckling and, moreover buckling, might be suppressed by the aid of external supports.

A convex yield surface like the one for von Mises materials implies field equations of three different kinds, c.f. Hill [4]. To the left of the maximum B the field equations are hyperbolic and allow necking in two different directions and to the right they are elliptic and do not allow necking at all. As the stress state at B is approached from the left side the angle between the two directions for necking decrease and in the limit the field equations are parabolic and only one direction is allowed. For all stress states between A and C in Fig. 1 the field equations for Tresca materials are parabolic only allowing necking in one direction. It is believed that the yield surface is convex for most real materials and that von Mises materials in this respect provides a better qualitative description of the behaviour at necking in a real material. A further indication of this is that assumption of a Tresca material does not enable an explanation of the fact that necking of a plate without a crack seldom is formed transversly to but rather at an angle about 20° - 40° from the transverse direction.

In an earlier paper [5] necking at small scale of yielding was investigated for Tresca and von Mises materials. It was then found that the necking region is embedded in a diffuse plastic zone for both materials at strain hardening. For an elastic-perfectly-plastic Tresca material the plastic zone is shrunk to a strip containing only the necking region whereas for a von Mises material the necking region is degenerated to infinitesimal length in the crack tip vicinity (see Fig. 2).

In the present paper an investigation of an elastic-perfectly-plastic von

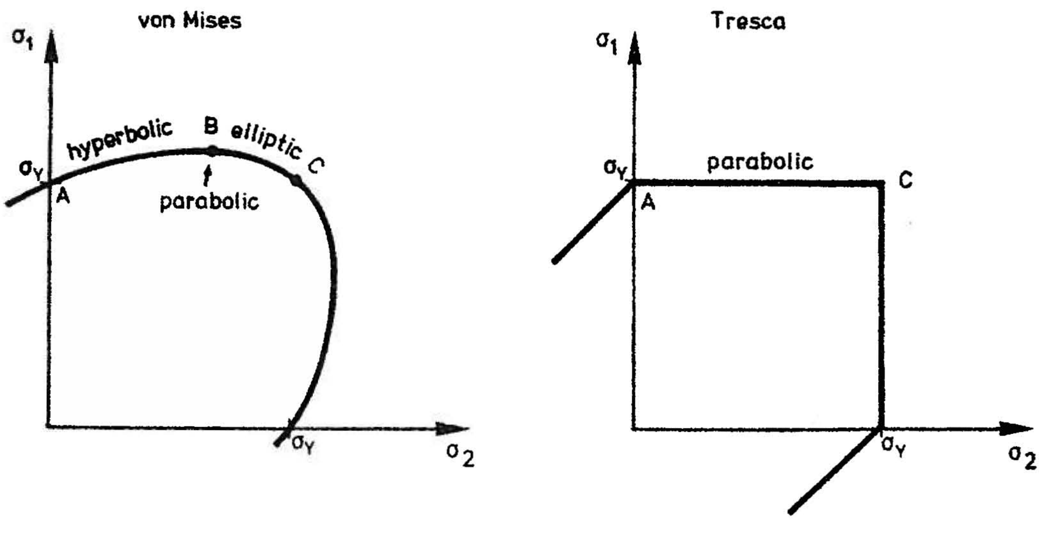

Fig. 1. The yield surfaces for von Mises and Tresca materials. Necking is allowed in two different directions for the stress states from A to B on the yield surface for von Mises materials whereas the stress states from B to C do not allow necking at all. For Tresca materials necking is allowed in one direction for the stress states from A to C.

Mises material at a large scale of yielding is carried out and at the end of the paper some experiments supporting the results are discussed.

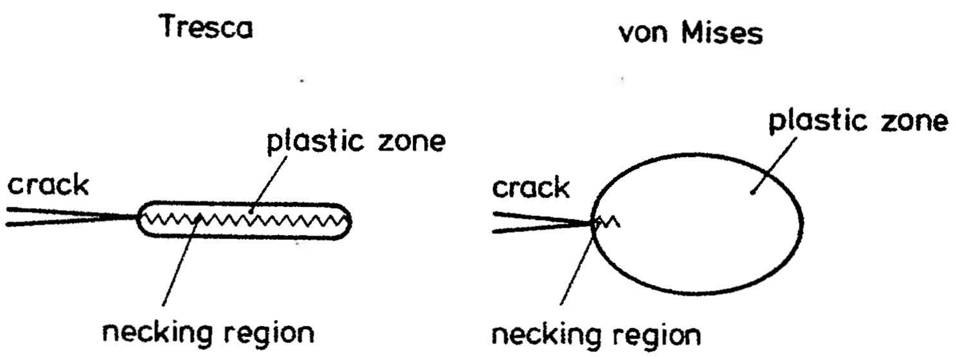

Fig. 2. Plastic zones and necking regions for elastic-perfectly-plastic von Mises and Tresca materials. The length of the necking region is infinitesimal for the von Mises material. For the Tresca material the plastic zone consists exclusively of the necking region.

THE DUGDALE MODEL

Consider an infinite plate of an elastic-perfectly-plastic von Mises material with the yield stress σ_Y. A crack is situated at $|x|<a$, $y=0$ (see Fig. 3). A remote load σ_y^∞ is assumed at infinity. Plasticity is assumed to be confined to necking regions at $a \le |x| < a+d$, $y=0$.

The analytical solution is easily found (see for instance Westergaard [6]). One obtains:

$$\sigma_y^\infty = 2/\pi \, \sigma_y^D \cos^{-1}[a/(a+d)] \qquad (1)$$

and

$$\sigma_x - \sigma_y = \sigma_x^\infty - \sigma_y^\infty \qquad (2)$$

along y=0. The flow rule can be written:

$$d\varepsilon_x^p/(\sigma_x-\sigma_y/2) = d\varepsilon_y^p/(\sigma_y-\sigma_x/2) = d\gamma_{xy}^p/(3\tau_{xy}) \qquad (3)$$

It is now assumed that the elastic strain increment is much less than the plastic counterpart at necking. Thus the condition for necking is that the strain along the neck must be zero [4]. Since only necking straight ahead of

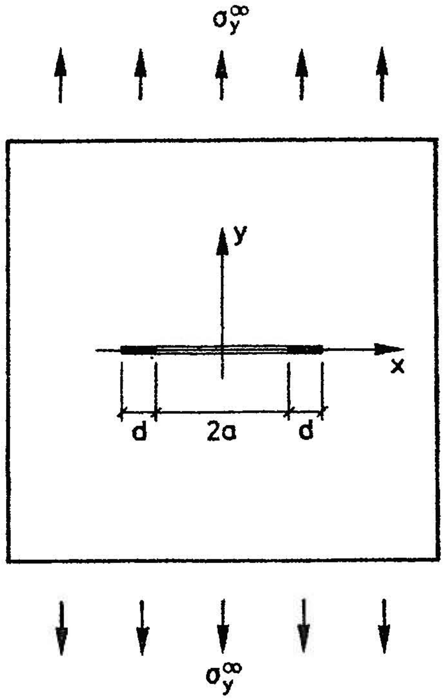

Fig. 3. An infinite plate with a crack. The load σ_y^∞ is remotely applied.

the crack is considered it follows that

$$\sigma_x^D = \sigma_y^D/2 \tag{4}$$

where the superscript D denotes stresses at the necking region. By insertion of stresses at infinity and at the necking region it follows from (2) that

$$\sigma_x^D - \sigma_y^D = \sigma_x^\infty - \sigma_y^\infty \tag{5}$$

which gives

$$\sigma_y^\infty/\sigma_y^D = 1/2 \tag{6}$$

since $\sigma_x^\infty=0$. Using (4) the effective von Mises stress is found to be $\sqrt{3}\sigma_y^D$ at the neck and thus $\sigma_y^D = 2\sigma_Y/\sqrt{3}$ where σ_Y is the yield stress. From (6) one obtains the condition

$$\sigma_y^\infty/\sigma_Y = 1/\sqrt{3} \tag{7}$$

i.e. a Dugdale necking region is possible only for one specific value of σ_y^∞.

A MODIFIED MODEL

It was shown in the previous section that the Dugdale solution is satisfied for a ratio $\sigma_y^\infty/\sigma_Y = 1/\sqrt{3}$. Below this ratio a diffuse plastic zone ought to be developed and a necking region might be embedded in this zone, e.g. at small scale of yielding where $\sigma_y^\infty/\sigma_Y \to 0$, a diffuse plastic zone develops where as the condition for necking, i.e. that the strain along the neck is zero , is fulfilled in a small region in the vicinity of the crack tip [5].

For a ratio $\sigma_y^\infty/\sigma_Y > 1/\sqrt{3}$ the stress $\sigma_y^D > 2\sigma_x^D$ and thus according to (3) the strain along the necking region is less than zero. Then lines of zero extension can be found at $a \leq |x| < a+d$, $y=0$ in two directions slanting symmetrically with respect to the x-axis, indicating that the assumption of Dugdale necking along $y = 0$ must be dropped. This invites for an investigation of a crack forming an angle to the x-axis. A coordinate system ξ and η is introduced with the ξ axis at an angle ψ to the x-axis.

A crack $|\xi| < a$, $\eta=0$ is considered and necking regions are assumed at $a \leq |\xi| < a+d$, $\eta=0$ (see Fig. 4). The load σ_y^∞ is remotely applied. Thus

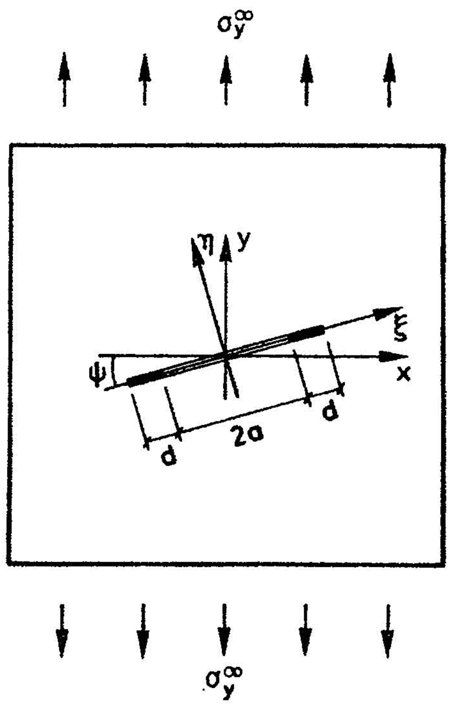

Fig. 4. An infinite plate with a slanted crack. The load σ_y^∞ is remotely applied.

$$\sigma_\xi^\infty = \sigma_y^\infty(1 - \cos 2\phi)/2 \tag{8}$$

$$\sigma_\eta^\infty = \sigma_y^\infty(1 + \cos 2\phi)/2 \tag{9}$$

$$\tau_{\xi\eta}^\infty = \sigma_y^\infty \sin(2\phi)/2 \tag{10}$$

where σ_ξ^∞, σ_η^∞ and $\tau_{\xi\eta}^\infty$ are the remote stresses. The condition that stresses should be finite everywhere gives the following stresses in the necking region:

$$\sigma_\eta^D = \pi/2\ \sigma_\eta^\infty/\cos^{-1}[a/(a+d)] \tag{11}$$

$$\tau_{\xi\eta}^D = \pi/2\ \tau_{\xi\eta}^\infty/\cos^{-1}[a/(a+d)] \tag{12}$$

One also obtains:

$$\sigma_\xi - \sigma_\eta = \sigma_\xi^\infty - \sigma_\eta^\infty \tag{13}$$

for $\eta = 0$ and

$$\sigma_\xi^D = \sigma_\eta^D/2 \tag{14}$$

where the last equation follows from the flow rule and the condition that $d\varepsilon_\xi^P = 0$. Equations (8), (9), (11), (13) and (14) give

$$\cos(2\phi) = [8/\pi\cos^{-1}(a/a+d) - 1]^{-1} \tag{15}$$

Furthermore the yield condition reads

$$\sigma_\xi^{D^2} + \sigma_\eta^{D^2} - \sigma_\xi^D\sigma_\eta^D + 3\tau_{\xi\eta}^{D^2} = \sigma_Y^2 \tag{16}$$

and hence, by use of (11), (12) and (14):

$$\frac{3\pi^2}{16}\cdot\frac{1}{[\cos^{-1}(\frac{a}{a+d})]^2}\cdot[\sigma_\eta^{\infty^2} + 4\tau_{\xi\eta}^{\infty^2}] = \sigma_Y^2 \tag{17}$$

By use of (8), (10) and (15) d/a can be eliminated and one finds after some calculations,

$$\frac{1 + \tan^2 2\phi}{1 + 4\tan^2\phi} = 3\ [\sigma_y^\infty/\sigma_Y]^2 \qquad (18)$$

The fact that ϕ depends on σ_y^∞ implies that the assumed with straight necking regions cannot be realized by a monotonically increased load. However, the model strongly suggests a deviation of the necking regions from a path straight ahead of the tip of a crack oriented along $\phi = 0$. In fact ϕ is observed to increase rapidly with increasing load (see Tab. 1). Thus the crack must be slanted to 20° at a load only about 5% above the load allowing necking straight ahead of the crack tip i.e. $\sigma_y^\infty = \sigma_Y/\sqrt{3}$. For $\sigma_y^\infty/\sigma_Y \to 1.0$ the crack should be slanted 35.3°, the same angle as for necking in a plate without a crack (c.f. Hill [4]).

Table 1.

The angle ϕ for the slanted crack for different ratios of σ_y^∞/σ_Y

σ_y^∞/σ_Y	ϕ
0.5774	0.0
0.5794	10.0
0.6093	20.0
0.7559	30.0
1.0	35.3

NUMERICAL ANALYSIS

To further enlighten the behaviour during necking a finite element analysis is carried out. An elastic-perfectly-plastic von Mises material is assumed, characterized by Young's modulus E, Poisson's ratio $\nu = 0.3$ and the yield stress σ_Y. A plate is assumed to occupy the space $|x| \le 20a$, $|y| \le 20a$ and $|z| \le h/2$ and a crack is situated at $|x| < a$, $y=0$. It is assumed that the plate is large enough to be regarded as infinite even at a large scale of yielding. The load σ_y^∞ is applied at the edges $|x|\le 20a$, $|y|\le 20a$.

Solutions symmetrical with respect x and y for which

$$u(-x,-y) = u(-x,y) = -u(x,y) \tag{19}$$

$$v(-x,-y) = -v(-x,y) = -v(x,y) \tag{20}$$

are looked for. Then only one quarter, $0\le x\le 20a$, $0\le y\le 20a$, of the plate has to be considered. The load σ_y^∞ is applied in increments at $0<x<20a$, $y=20a$. The boundaries at $x=20a$, $0\le y\le 20a$ and the crack surface at $0<x<a$, $y=0$ are traction free. The remaining boundary conditions are

$$\tau_{xy} = 0 \tag{21}$$

$$u = 0 \tag{22}$$

at $x=0$, $0\le y\le 20a$ and

$$\tau_{xy} = 0 \tag{23}$$

$$v = 0 \tag{24}$$

at $a\le x<20a$, $y=0$.

In addition to this the possibility for solutions anti-symmetrical with respect to x is also investigated. For these

$$u(-x,-y) = -u(x,y) \tag{25}$$

$$v(-x,-y) = -v(x,y) \tag{26}$$

Here one half of the plate $0\le x\le 20a$, $|y|\le 20a$ has to be considered. The load σ_y^∞ is applied in increments at $0<x<20a$, $|y|=20a$. The boundaries at $x=20a$, $|y|\le 20a$ and the crack surface at $0<x<a$, $y=0$ are traction free. The remaining boundary conditions are

$$u(y) = -u(-y) \tag{27}$$

$$v(y) = -v(-y) \tag{28}$$

at $x=0$, $|y|\le 20a$.

For the symmetrical case the region $0\le x\le 20a$, $0\le y\le 20a$ is covered by 144 8-node isoparametric elements. A region around the crack tip with an extension of 0.16a times 0.16a is covered by 64 equal square elements. The linear size of these elements in the crack tip vicinity is thus 0.02a.

For the anti-symmetrical case the region $0 \le x \le 20a$, $|y| \le 20a$ is covered by 288 8-node isoparametric elements. The mesh is symmetric with respect to y=0 and the upper part $y \ge 0$ equals the one described for the symmetrical case.

A code [7] based on a modified Newton-Raphson iteration procedure proposed by Nayak and Zieňkiewicz [8] has been used for the non-linear analysis. This method produces numerically stable solutions for both elastic-perfectly-plastic as well as strain softening materials. Element stiffnesses are calculated numerically at 2 by 2 integration points. The load is applied in 15-20 equal increments. Iteration for equilibrium is performed at each load increment.

RESULTS

The cross sectional slip takes place at planes inclined about 45° to the x-y plane (see Fig. 5). Therefore the width of the necking region is finite and approximately equals the plate thickness. Within the FEM approximation strain localize in a band with a width depending on the element size. Due to the limited representation of large strain gradients the strain at the necking region is distributed over one or two element hights depending on the orientation of the element with respect to the neck.

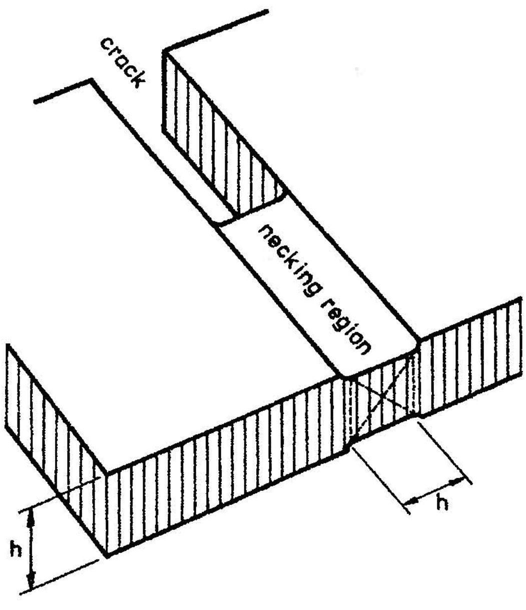

Fig. 5. Necking is a matter of cross sectional slip at planes inclined 45° to the x-y plane. It is observed that a region with a width equal to the plate thickness is involved in the necking process.

At the load $\sigma_y^\infty/\sigma_Y=0.3$ a diffuse plastic zone is developed for the symmetrical case. The plastic zone contains a small region straight ahead of the crack tip with an elliptic stress state inhibiting necking. Regions with a hyperbolic stress state implying two sets of characteristic curves along which the condition for necking is fulfilled are developed symmetrically above and below the crack plane. This is in accordance with what was reported for the case of small scale yielding [5]. Fig. 6 shows these regions and the characteristic curves. One observes by comparing the present solution with [5] that the region with an elliptic stress state is more extensive at small scale yielding. Fig. 6 also shows the distribution of the cross sectional strain ε_z which is observed to be rather smoothly distributed with a maximum straight ahead of the crack tip but without a distinct necking region.

Figure 7 shows the distribution of the strain ε_z and the characteristic curves for a load $\sigma_y^\infty/\sigma_Y=0.4$. A clear localization of strain forming a neck is visible in the direction straight ahead of the crack tip. The characteristic curves indicate that necking would be allowed at almost any direction diverging radially out from the crack. It is also noted that the stress state is hyperbolic in the entire plastic zone. At a larger load $\sigma_y^\infty/\sigma_Y=0.5$ (see Fig.8) strains are still large straight ahead of the crack tip but the angular distribution near the crack tip is rather smooth up to an angle about 45° to

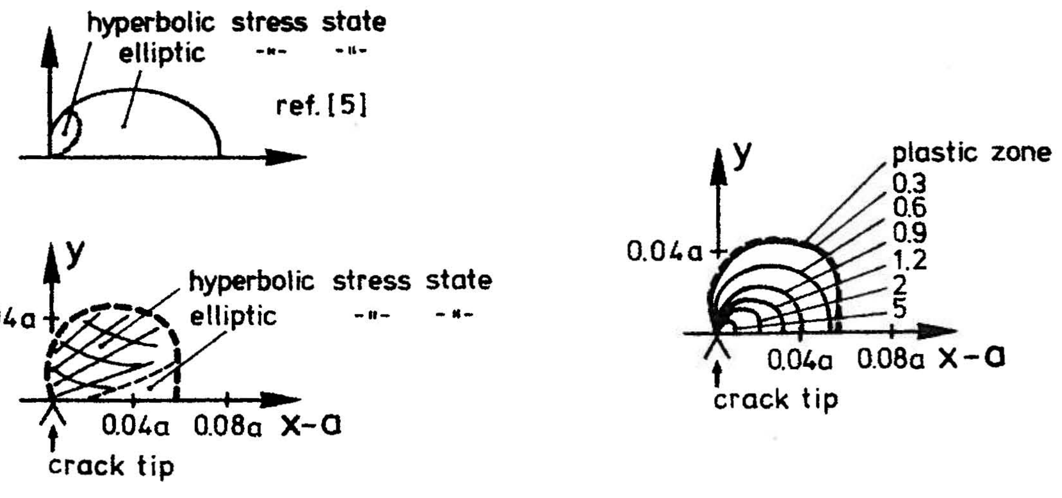

Fig. 6. Characteristic curves along which the condition for necking is fulfilled and curves for constant strain at a load $\sigma_y^\infty/\sigma_Y=0.3$. The resulting strain is independent of the material parameters E and σ_Y and is given as $\varepsilon_z E/\sigma_Y$.

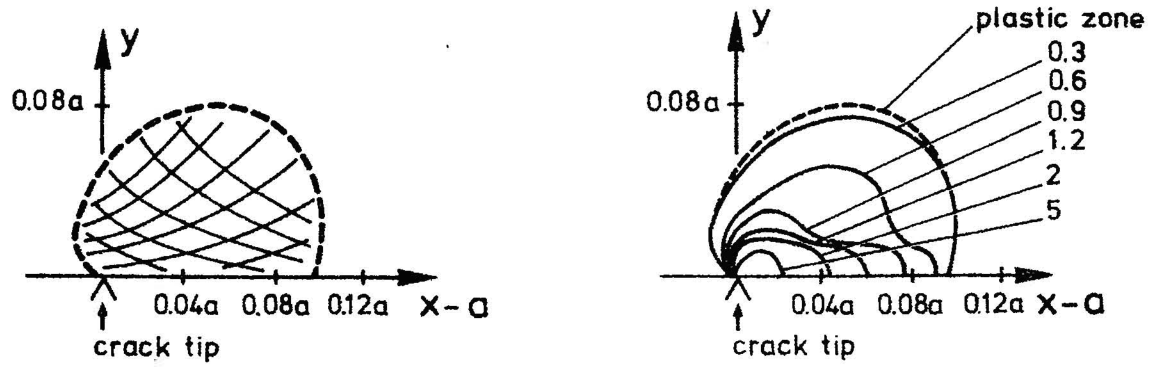

Fig. 7. Same as Fig.6 for σ_y^∞/σ_Y=0.4. A neck is formed straight ahead of the crack tip.

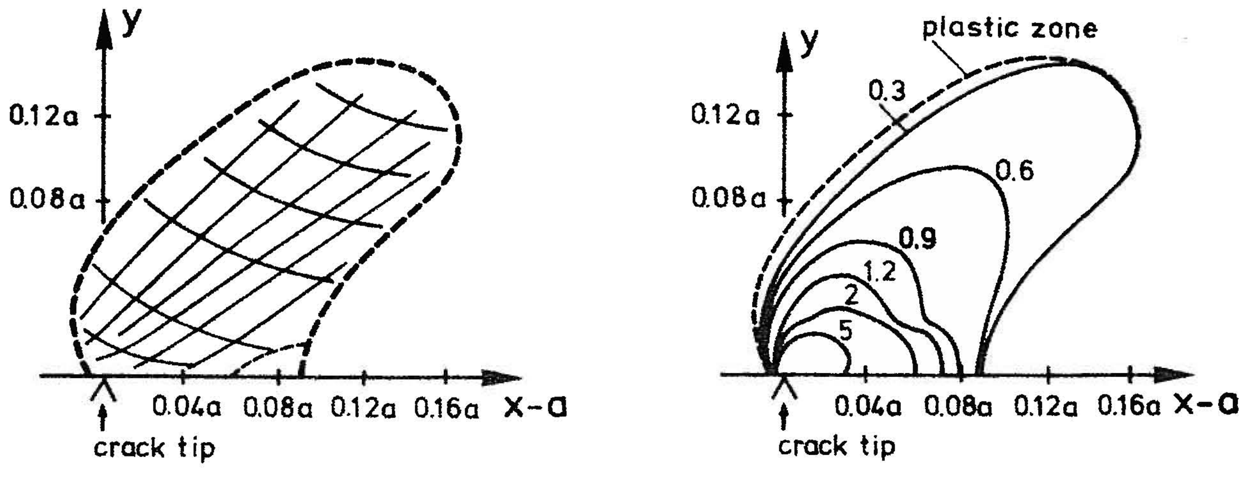

Fig. 8. Same as Fig.6 for σ_y^∞/σ_Y=0.5. Strains are observed to increase in a direction out of the crack plane. The angle between the x-axis and a straight line from the crack tip to the remote tip of the plastic zone is 40°.

the x-axis. At a larger distance from the crack tip strains are concentrated in a region directed 40° to the x-axis even though a necking region is not visible. The characteristic curves show that the condition for necking is fulfilled at an angle of 33° to the x-axis at the end of the plastic zone remote from the crack tip. Near the x-axis where the angle decreases to zero. Surprisingly the stress state is again elliptic in a few elements at the front of the plastic zone near the x-axis.

Finally Fig. 9 shows the characteristic curves and the strain distribution for $\sigma_y^{\infty}/\sigma_Y$=0.6. Here the plastic zone extends outside the region with the uniform mesh. A meaningful continuation to larger loads would require remodelling of the mesh. This has not been done.

The strain distribution is rather similar to the one shown in Fig. 8. A necking region is not visible. The width of the band of concentrated strain is almost the same as in Fig. 8. Only the length of the band is increased. The width is about 3 times the element height and it is thus assumed that the strain distribution is independent of the element size. Characteristic curves show that the condition for necking is fulfilled in a direction about 35° to the x-axis in the banded part of the plastic zone.

Since there could be some doubt as regards the uniqueness of the symmetrical solution the anti-symmetrical case was also investigated but the differences were negligible. Note that solutions fulfilling boundary conditions

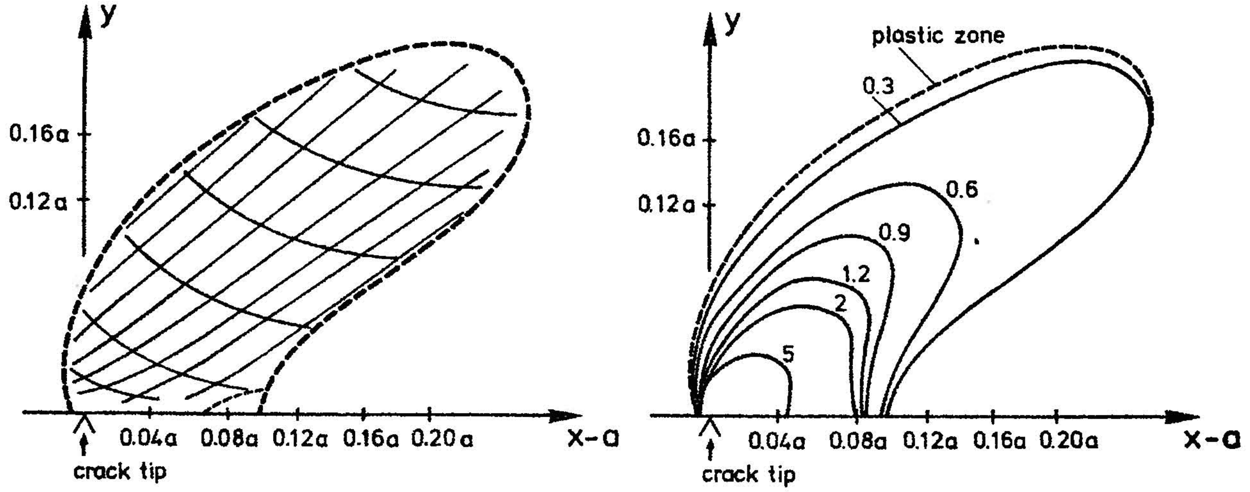

Fig. 9. Same as Fig.6 for $\sigma_y^{\infty}/\sigma_Y$=0.6. Strains is localized in the same direction as in Fig. 8.

for symmetrical solutions also fulfill the boundary conditions for anti-symmetrical solutions. Thus if the symmetrical solutions are stable they will result even if the boundary conditions (27) and (28) are used. Even when the elastic-perfectly-plastic material is replaced with a slightly softening material symmetrical solutions are obtained.

EXPERIMENTS

An investigation of necking in thin sheets of aluminum SIS 4007-14 and SIS 4007-18 has been carried out [3]. The height b of the sheets was 320mm for all specimens. Different widths w from 50mm to 320mm were examined. Thicknesses h were 1mm, 2mm and 3mm. 0.35mm wide cracks were cut out by a fine saw to lengths from 20mm to 120mm from a 2.4mm drill hole. The crack edges were sharpened with a razor blade (see Fig. 10). A Mohr-Federhaff tensile test machine was used to perform the tests at a low strain rate.

At these experiments necking generally started and proceeded in a direction independent of the original crack length and plate dimensions but dependent on the rolling direction and the plate thickness. This suggests that the angle of necking at a crack for a homogeneous plate should equal the angle for necking in an homogeneous plate without a crack [9].

In a few cases, especially for the thinner plates (1mm and 2mm) fracture started straight ahead of the crack tip to about 0.2a - 0.3a and then continued at an angle to the x-axis as described above.

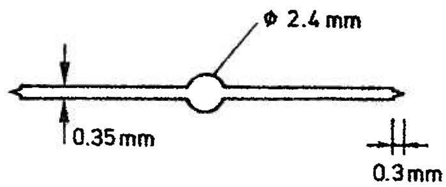

Fig. 10. Manufactured crack profile.