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ON THE FORMATION OF NECKS AT CRACK TIPS UNDER PLANE STRESS CONDITIONS

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SUMMARY

The conditions for necking are well defined through the theory of Hill [1]. These conditions are applied to elasto plastic linearly hardening Tresca and v Mises materials for different hardening characteristics. The results suggest that necking developes for all hardening materials. Below a certain hardening the stresses normal to the necking region and just outside it decrease during the necking process. The plastic region is generally found to be extended outside the necking region but for a Tresca material with a certain hardening the Dugdale solution is found.

1. INTRODUCTION

The theory of necking in thin plates was developed by Hill [1]. Generally two necessary and sufficient conditions have to be fulfilled. The first one can be written:

$$d\sigma_e/d\varepsilon_e < (\sigma_1+\sigma_2)/2 \qquad (1)$$

for v Mises materials and

$$d\sigma_e/d\varepsilon_e < \sigma_1 \quad \text{if } \sigma_1\sigma_2 > 0 \text{ and } |\sigma_1| > |\sigma_2|$$

$$d\sigma_e/d\varepsilon_e < 0 \quad \text{if } \sigma_1\sigma_2 < 0 \qquad (2)$$

for Tresca materials. σ_1 and σ_2 are the in-plane principal stresses, σ_e the effective stress and ε_e the effective strain. The second condition is

$$|\sigma_1+\sigma_2| \leqslant \sqrt{3}\sigma_e \qquad (3)$$

for v Mises materials whereas no complement to condition (2) is needed for Tresca materials, c.f. [1].

The necking process leads to a non-uniform thickness but by defining the stress and strain as averaged over the thickness the problem can still be considered as one in plane stress. The theory incorporates the progressive thinning of the plate through the equations of equilibrium

$$\partial(h\sigma_{ij})/\partial x_i = 0 \tag{4}$$

where h is the thickness.

It follows from the condition of zero volume change during plastic flow that whenever necking (i.e. a discontinuity of displacements parallel to the plate surface) is allowed there must be a discontinuity in thickness. Since the thickness reduction is small outside the necking region, h is assumed to be constant, equal to the original plate thickness, except for the necking region.

When a thin plate containing a straight and sharp crack subjected to mode I loading some amount of plasticity is always found at the crack tip and the strains are very often observed to localize, forming a necking region ahead of the crack tip. In some cases the plastic region consists exclusively of the necking region and therefor the upper and lower boundaries of the neck are elastic. Generally, however the plastic region is extended outside the necking region.

To explore the implications of Hill's conditions, the stress and strain state at the neck boundary is here examined. Fig. 1 shows a necking region of length l ahead of the crack. It is assumed that the plate thickness $h \ll l$ and that the height of the necking region (i.e. its extension in the y-direction) is of the order of h. Obviously $\sigma_z = 0$ (plane stress definition) and $\tau_{xy} = 0$, $\gamma_{xy} = 0$ in the necking region and at its boundary (symmetry condition). Remaining stresses and strains are called σ_x^*, σ_y^*, ε_x^*, ε_y^* and ε_z^* for the necking region and σ_x, σ_y, ε_x, ε_y and ε_z for the region just outside. The conditions of continuity are

$$\sigma_y h = \sigma_y^* h^* \quad \text{and} \quad \varepsilon_x = \varepsilon_x^* \tag{5}$$

where h and h* are the thicknesses of the plate outside and inside the neck.

It is assumed that the changes of strains outside the necking region are much smaller than the changes of ε_y^* and that especially the change $\Delta\varepsilon_x$ of ε_x can be neglected. Thus, since $\varepsilon_x^* = \varepsilon_x$ one can put

$$\Delta\varepsilon_x^* = 0 \tag{6}$$

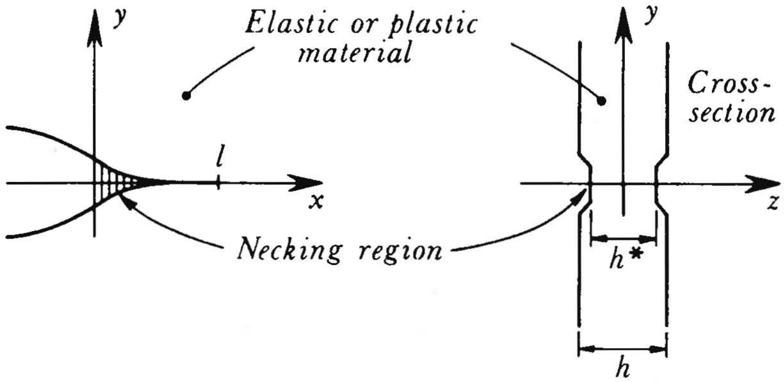

Figure 1. Mode I crack. The crossection shows the reduction in thickness at the neck. The length of the neck l is supposed to be much larger than the plate thickness h.

For v Mises materials this gives as a consequence of the normality rule for plastic strain increments (see Fig. 2)

$$\sigma_x^* = \sigma_y^*/2 \tag{7}$$

which agrees with Hill's condition (3). For Tresca materials

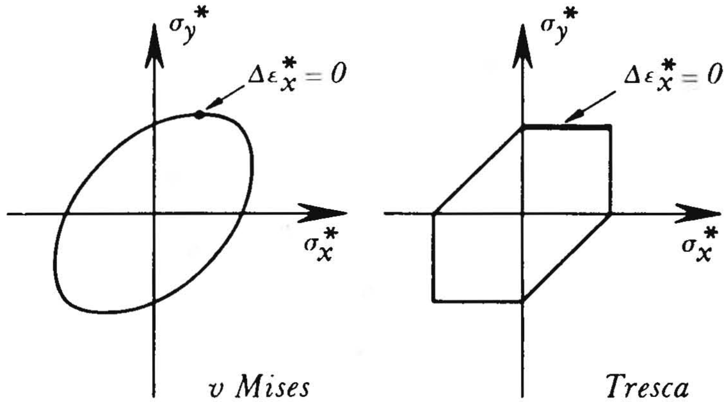

Figure 2. Yield surfaces for v Mises and Tresca materials in plane stress.

Eq. (6) implies (see Fig. 2)

$$0 < \sigma_x^* < \sigma_e \quad \text{and} \quad \sigma_y^* = \sigma_e \tag{8}$$

The continuity of stress across the neck boundary (5) gives

$$(\sigma_y+\Delta\sigma_y)h = (\sigma_y^*+d\sigma_y^*/d\varepsilon_e\cdot\Delta\varepsilon_e)h^*(1+\Delta\varepsilon_z^*) \approx$$

$$\approx \sigma_y^* h^*+d\sigma_y^*/d\varepsilon_e\cdot\Delta\varepsilon_e h^*+\sigma_y^*\Delta\varepsilon_z^* h^* \tag{9}$$

From the incompressibility of plastic strain increments and (6) follows that

$$\Delta\varepsilon_z^* = -\Delta\varepsilon_y^* \qquad (10)$$

and hence for v Mises materials where $\Delta\varepsilon_e = 2\Delta\varepsilon_y^*/\sqrt{3}$ and $\Delta\sigma_y^* = 2\Delta\sigma_e/\sqrt{3}$ use of condition (1) gives

$$\Delta\sigma_y < 0 \qquad (11)$$

i.e. σ_y (though not necessarily σ_y^*) cannot increase during necking. The same result is obtained for a Tresca material where $\Delta\varepsilon_e = \Delta\varepsilon_y^*$, $\Delta\sigma_y^* = \Delta\sigma_e$ and $\sigma_x^* < \sigma_y^*$. It should be noted that Hill's conditions do not impose any restrictions on the stresses parallel to the necking region immediately outside the neck boundary. Thus, in general, $\sigma_x \neq \sigma_x^*$ and consequently generally $\sigma_x \neq \sigma_y/2$ for both Tresca and v Mises materials.

2. MODELS WITH PLASTIC ZONES OF DUGDALE TYPE

In the Dugdale model [2] the influence of plate thinning at the neck is usually neglected, i.e. the stress σ_y is considered to be continuous across the necking region instead of Eq. (5). For a perfectly plastic Tresca material, a solution is found such that the plate remains elastic outside the necking region. However, this leads to the rather unrealistic consequence that the tractions on the neck boundaries remain constant during the whole necking process. In view of the cross sectional thinning in the necking region a perfectly plastic material would give decreasing tractions at the elastic boundaries of the neck towards the crack tip and in the case of ductile fracture σ_y would decrease to zero as the thickness is reduced to almost zero at the crack tip. Low strain hardening would slow down the reduction of σ_y and in the limit when $d\sigma_e/d\varepsilon_e \to \sigma_Y$ where σ_Y is the yield stress, σ_y would remain constant along the entire neck boundary. For strain hardening rates $d\sigma_e/d\varepsilon_e > \sigma_Y$ plasticity is extended outside the necking region since then necking will not occur as long as $\sigma_e < d\sigma_e/d\varepsilon_e$.

As regards v Mises materials the hardening rate $d\sigma_e/d\varepsilon_e \geqslant \sqrt{3}\sigma_Y/2$ would maintain σ_y constant during the necking process. If we assume that a necking region occurs then $\sigma_y^* = 2\sigma_x^*$ inside the necking region, i.e. $\sigma_y^* = 2\sigma_e^*/\sqrt{3}$. However for extremely small scale yielding the Dugdale model gives $\sigma_y \to \sigma_e$ because $\sigma_x \to \sigma_y$ as $y \to \pm 0$ in the elastic material just outside the necking region. Then, since $\sigma_y = \sigma_y^*$ at the neck tip where the necking process is initialized, $\sigma_e = 2\sigma_e^*/\sqrt{3}$ but obviously necking cannot take place if $\sigma_e^* < \sigma_Y$. Consequently necking in v Mises materials is not possible for any strain hardening rate if the plate is assumed to remain elastic outside the necking region.

A more realistic model for a non-hardening or for a hardening rate less than σ_Y for Tresca materials or $\sqrt{3}\sigma_Y/2$ for v Mises materials would be a model where σ_y is decreasing towards the crack tip, for instance $\sigma_y = \sigma_Y(\delta_t - \delta)/\delta_t$ or $\sigma_y = \sigma_Y x/1$ where $\delta/2$ is the displacements of the upper boundary of the neck, $\delta_t = \delta$ at the crack tip and 1 is the length of the necking region. These models which have been studied (for quite another purpose) in [3], are consistent with the assumption of elastic behaviour outside the necking region for Tresca materials, but, when applied on v Mises materials, the yield stress is exceeded outside the necking region along a small portion (about $2/\sqrt{3}-1 \approx 15\%$ or more) in the vicinity of the neck tip (see Fig. 3)

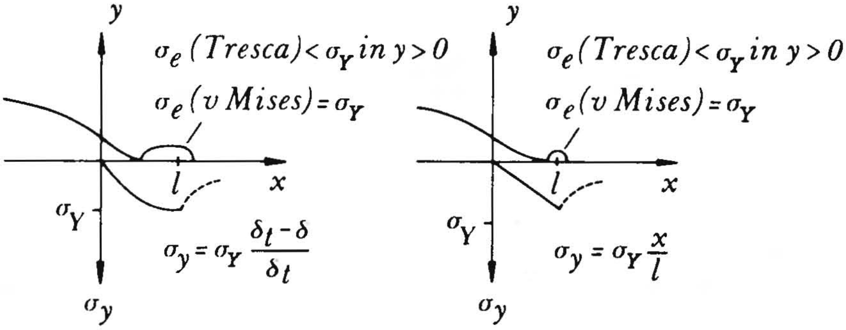

Figure 3. The areas where σ_Y is exceeded by σ_e(v Mises) in the solution of [3] where plasticity in the upper and lower half plane $y \neq 0$ is ignored.

3. SINGULAR NEAR TIP BEHAVIOUR

A near tip solution for a power hardening v Mises material was given by Hutchinson [4] in the form of the dominating singular strains. The solution was found by means of the J-integral, claiming that the strain energy density must have an r^{-1} dependence. However, in such a case σ_e tends to infinity and $d\sigma_e/d\varepsilon_e$ is decreasing as the crack tip is approached, i.e. condition (1) is fulfilled in a certain vicinity of the crack tip. As regards condition (3) the ratio σ_1/σ_2 and the directions of zero extension have been calculated from diagrams of σ_r, σ_ϕ and $\tau_{r\phi}$ provided by [4]. The result show that necking is prohibited within a certain angle ahead of the crack tip. This angle is decreased with decreased hardening and in the non-hardening case condition (3) allows a neck in the direction straight ahead of the crack tip (see Fig. 4). Since then both conditions (1) and (3) are fulfilled necking will actually take place, but in contrast to the Dugdale model it will be embedded in a plastic surrounding. Then, obviously, since necking occurs, the assumption of an r^{-1} dependence of the strain energy density is erroneous in the non-hardening case.

An investigation of Hutchinson's [4] results for the hardening case (power law hardening $\sigma \propto \varepsilon^{1/n}$) shows that necking would result, though at a more complicated mechanism. Large strains will inevitably cause blunting of the crack tip as the load is increased. It is obvious that the ratio σ_1/σ_2 would increase to infinity at the blunted tip if no necking occured and that the area where $\sigma_1/\sigma_2 \geqslant 2$ would increase with the carck tip radius. Since only values of $\sigma_1/\sigma_2 \geqslant 2$ allows necking and $\sigma_1/\sigma_2 = 2$ allows necking in the direction of the x axis one can conclude that necking will start immediately at the crack tip. The same discussion can be applied again, but now considering the tip of the necking region rather than the tip of the crack.

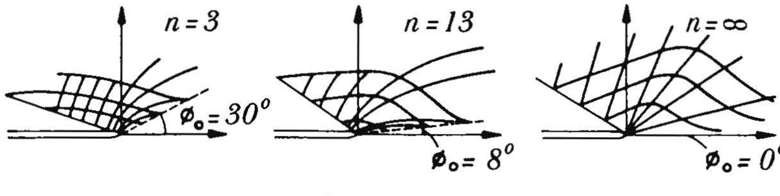

Figure 4. Curves of possible displacement discontinuities. Necking is not allowed within the angle ϕ_o. Since only those curves starting at the crack tip can be considered, necking is possible only in the non-hardening case and then preferably in the direction straight ahead of the crack tip where $d\varepsilon_y/d\varepsilon_e$ assumes its maximum. n is the strain hardening exponent.

Since the singular stress field of a linearly elastic material does not fulfill (3) within an angle of 39° ahead of the crack tip it is here assumed this is the case also for a linearly hardening material and that progressive necking ought to occur for a linearly hardening v Mises material as well.

4. NUMERICAL CALCULATIONS

In order to further elaborate the question whether necking will occur or not the finite element method is employed. The code uses the initial stress approach with an accelerating diagonal matrix operating on the corrective displacements, as described by Zienkiewicz et al. [5], [6]. The stiffness matrix is determined only once and equals the one corresponding to the initially linearly elastic material. The following problem is studied: An infinite plate of a linearly hardening elastic plastic material containing a mode I crack $x \leqslant 0$, $y = 0$ is subject to a remote load which is chosen as the dominating term of the small scale yielding solution. The crack is assumed to be much larger than the size of the plastic zone which in turn is assumed to be much larger than the plate thickness, i.e.

$$a \gg (K/\sigma_Y)^2 \gg h \qquad (12)$$

where K is the stress intensity factor and a is the half crack length. The neck is considered as a line of discontinuous displacements. The mesh consists of 200 8-node isoparametric elements on the upper half plane $y > 0$. The element stiffness is sampled from 4 gaussian points and integration is thus reduced.

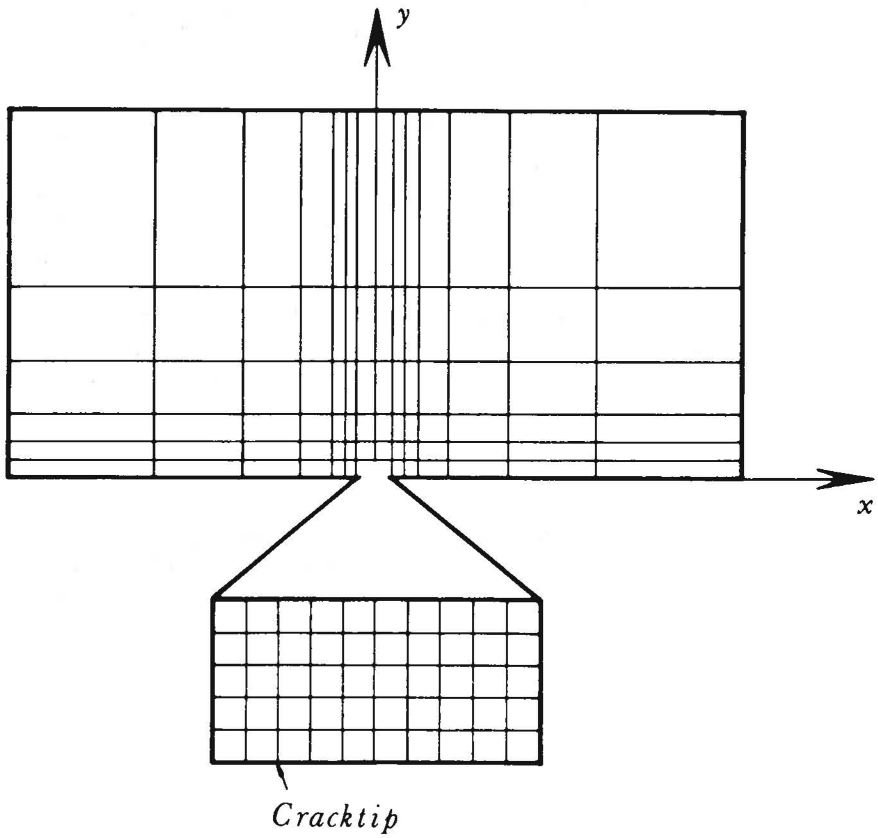

Figure 5. Element mesh. The region in the close surrounding of the neck is divided into quadratic elements with a side length of about 1/10 of the maximum plastic zone length.

In the plastic region hardening occurs under the condition

$$d\sigma_e / d\varepsilon_e^p = \text{const} = H \qquad (13)$$

where $\sigma_e = (3s_{ij}s_{ij}/2)^{1/2}$, $s_{ij} = \sigma_{ij} - \delta_{ij}\sigma_{kk}/3$ and $d\varepsilon_e^p = (2d\varepsilon_{ij}^p d\varepsilon_{ij}^p/3)^{1/2}$ for v Mises materials. The subscript p denotes the plastic components. $\sigma_e = \max(|\sigma_1 - \sigma_2|, |\sigma_1|, |\sigma_2|)$ and $d\varepsilon_e^p = \max(|d\varepsilon_1^p - d\varepsilon_2^p|, |d\varepsilon_2^p - d\varepsilon_3^p|, |d\varepsilon_3^p - d\varepsilon_1^p|)/2$ for Tresca materials where σ_1 and σ_2 are the principal stresses and $d\varepsilon_1^p$, $d\varepsilon_2^p$ and $d\varepsilon_3^p$ are the principal plastic strain increments. The plastic strain increments are supposed to be normal to the flow surface (see Fig. 2) for both Tresca and v Mises materials. Here both materials are handled by an elastic plastic matrix $[D^{ep}]$ due to [6] used in the incremental relation

$$[d\sigma] = [D^{ep}][d\varepsilon] \tag{14}$$

If the hardening rate is large enough, i.e. if $H \geqslant \sqrt{3}\sigma_Y/2$ for v Mises and $H > \sigma_Y$ for Tresca materials the stress component σ_y will remain constant during the whole necking process and the solution is hence independent of the discontinuous displacements of the neck boundaries.

The mechanism of necking is modelled by releasing the node at the tip of the neck as the consistent nodal force reaches an ultimate magnitude corresponding to a stress $\sigma_y = 4H/3$ for v Mises materials and $\sigma_y = H$ for Tresca materials. When released, the node is included in the necking region and the load is then increased in increments until the force on the next node in the plane $y = 0$ reaches its ultimate magnitude. This procedure is continued until the length of the plastic zone is about 1/20 of the total length of the finite element mesh. This technique has been used earlier by [7].

5. RESULTS AND DISCUSSION

Table 1 shows the extents ρ_1, ρ_2 and ρ_3 (c.f. Fig. 6) of the plastic zones for v Mises and Tresca materials for different ratios of H/E where E is the elastic modulus and different ratios of H/σ_Y. All results are numerical except for the Tresca

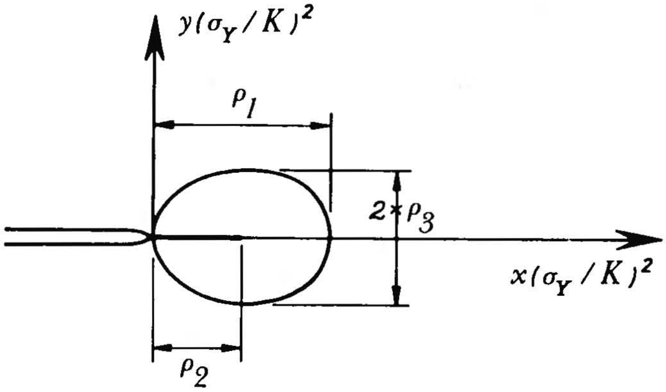

Figure 6. Dimensionless length characteristics of the plastic zone

material with $H/\sigma_Y = 1$ where the analytical Dugdale solution is used. The length of the necking region is seen to decrease with increasing H/σ_Y and decreasing H/E. When $H/E \to 0.0$ the necking region decreases to zero for all ratios of H/σ_Y though these ratios of H/σ_Y can only be fulfilled by a rigid plasic material. The absence of a neckline is obvious for hardening rates larger than $\sqrt{3}\sigma_Y/2$ for v Mises and larger than σ_Y for Tresca materials, since then σ_e at a possible neck would be larger than σ_Y a value which in turn never can be exceeded when H/E = 0.

Table 1

Dimensionless characteristics of the plastic zones

		H/E = 1/10	H/E = 1/40	H/E → 0
v Mises	ρ_1	.33	.33	.36
$H/\sigma_Y = \sqrt{3}/2$	ρ_2	.22	.13	<.03
	ρ_3	.24	.14	.12
v Mises	ρ_1	.30	.32	
$H/\sigma_Y = 9/8$	ρ_2	.06	.07	→.00
	ρ_3	.26	.22	
Tresca	ρ_1	.39	.39	.39
$H/\sigma_Y = 1$	ρ_2	.39	.39	.39
	ρ_3	.00	.00	.00
Tresca	ρ_1	.32	.33	
$H/\sigma_Y = 2/\sqrt{3}$	ρ_2	.15	.07	→.00
	ρ_3	.34	.26	
Tresca	ρ_1	.24	.30	
$H/\sigma_Y = 3/2$	ρ_2	.04	.04	→.00
	ρ_3	.34	.36	

For a v Mises material with the ratio $H/\sigma_Y = \sqrt{3}/2$ a finite length of the necking region is not found within the limits of the finite element mesh. Paradoxically this is the case where the asymptotic near tip field due to [4] allows necking in the direction of the x axis, i.e. where $\sigma_1/\sigma_2 = 2$. Calculation of the stress field in the plastic zone indicates that $\sigma_1/\sigma_2 \geqslant 2$ at the side of the crack and possibly along a very small portion of the x axis in the immediate vicinity of the crack tip (see Fig. 7). If the limit H/E = 0 is excluded but still H/E << 1

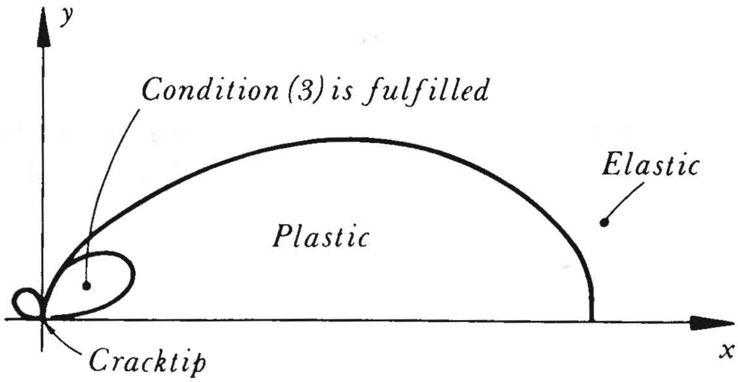

Figure 7. The extent of the region that fulfilles condition (3), i.e. $\sigma_1/\sigma_2 \geqslant 2$, for a non-hardening v Mises material.

the result strongly suggest that necking occurs in a close vicinity of the crack tip with an extension increasing with increasing H/E.

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