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CRACK GROWTH CRITERIA AND CRACK TIP MODELS

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1. Introduction

From being a relatively exclusive field of research up to about the middle of the 1950's, fracture theory has developed into a major research area within the field of mechanics. To some extent this expansion has been driven by the scientific challenge to potential researchers since fracture mechanics naturally supplies very complex and many faceted problems both to the theoretical and to the experimentally orientated scientist. However, it seems that the main factor responsible for the large resources devoted to the subject is the technological need for reliable methods for fracture assessment. As technological systems become more complex and of more sophisticated design the risk of fracture increases, as well as the potential losses should a failure occur. Today, the bulk of fracture research is orientated to a few, rather specific, industrial sectors of which the most important are the aircraft, nuclear energy and military industries.

It is the opinion of the present authors, that several basic unresolved questions remain in fracture mechanics and that attention devoted to these appears to be somewhat stagnating. The main object of the paper is to review a number of these questions in light of some of the more recent findings and also to focus on some not so recent results which perhaps have not received the general attention they deserve. It is not the intention to present complete solutions to the questions here but rather to draw the attention to some short-comings of the present knowledge. The discussion will be confined to cases with straight symmetrical cracks (mode I) in two-dimensional bodies under plane strain conditions. Occasionally, anti-plane shear (mode III) analogues of these problems are discussed because of the simpler mathematical formulation which may help to bring insight even in the more complicated situations.

More complex problems like three-dimensional, mixed mode and plane stress cases have many features in common with the mode I plane strain case and in those respects the discussion is relevant even for these. However, there are several additional complexities and in our opinion it is unlikely that much progress can be made with the more complex cases before fuller understanding of the simpler ones has been obtained. For the same reason fatigue and stress corrosion cracking is excluded from explicit discussion. Linearized geometry description will be assumed unless large deformations effects are explicitly discussed.

The ultimate aim of fracture theory is to formulate criteria for the initiation and the continuation of crack growth that are suitable for use within a continuum mechanics model of a body. One basic item towards the development of such criteria is the modelling of the crack tip region. Several questions immediately arise. Can the constitutive behaviour assumed for bulk deformation of the body be extrapolated to the tip region or is special constitutive modelling necessary? One may ask if it is really necessary to bother about the crack tip behaviour at all. In many commonly used fracture models very little regard to the crack tip region is taken and to some extent such considerations are deliberately avoided.

2. General Aspects on Crack Tip Modelling

The concept of a crack tip is a mathematical idealization. In a real body there is a region of relatively small size within which processes of material degradation operate which finally may lead to the creation and separation of new surfaces. These micro-processes may be of several different kinds depending on the material properties and the particular conditions prevailing. Various models have been proposed and analysed. The mechanism of void initiation, growth and coalesence has been suggested as appropriate for ductile crack growth in metals while in materials like concrete, micro-crack formation and growth is considered to be the most important mechanism.

The term *physical modelling* will here be reserved to mean models where void growth, micro-crack formation etc. are modelled in such detail that the development of individual voids is followed. In some cases even modelling on the dislocation level is done. In contrast, by the term *continuum modelling* is meant models where the degradation processes are not followed in detail but rather through constitutive assumptions about continuum stresses and strains. In this latter category models with displacement discontinuities in cohesive zones will also be included.

There are several reasons why continuum modelling is preferred to physical modelling. The most obvious one is that physical models easily become very complicated. To model a complete structure on such a fine scale that a single micro-crack can be described easily exhausts the capacity of even the largest computers. Instead, when physical modelling is attempted, only the region in which the micro-processes are active is modelled. The size and the boundary conditions of this region is taken from a continuum solution where the crack tip region constitutive behaviour is the same as for the surrounding material. For this approach to be relevant the solution outside the crack-tip region must not depend too much on the processes within the region. This is one of the requirements for *autonomy* to prevail. This concept was introduced by Barenblatt [1] and has later been discussed by Broberg [2]-[3]. Autonomy, will be discussed more extensively below.

There is also a more fundamental reason why physical modelling cannot provide all the answers to the problem of how to describe a crack tip. It has been pointed out by Goodier [4] among others that even for a model on the micro-scale, fracture criteria are still needed to predict when the final micro-separation occurs. If void growth is modelled with continuum plasticity models ligament instability may be predicted. In order to predict separation some mechanism for this must be introduced into the constitutive relation. Likewise, metal physicists use concepts like the Griffith criterion in order to predict when a micro-crack becomes unstable. The rather paradoxical situation may occur that one has to use fracture criteria on the micro-level of precisely the same nature as those on the macro-level for which one originally used the physical model.

This does of course not imply that physical modelling is unnecessary. It often happens that the fracture criterion on the micro-level is much simpler than the one on the macro-level. It may for example be possible to apply linear elastic fracture mechanics to describe the growth of micro-cracks in a local environment. If autonomy is violated in that the growth of individual voids or micro-cracks influence the outer state appreciably there is not much choice but to use physical modelling. We will however restrict the subsequent discussion to continuum modelling.

The viewpoint will be taken that the effects of the various micro-processes operating can adequately be represented by appropriate continuum constitutive modelling and thus that details of the micro-behaviour are not important on the higher level. This is obviously an assumption of central character and is also one that the present authors do not claim expertise in delineating. We feel that this is certainly an area where further research is needed.

In the following the concept of a process region will be used. A process region is a region wherein the special constitutive assumptions needed to take account of micro-behaviour of the discussed type becomes significant. Obviously depending on the nature of these constitutive assumptions the border between the process region and the normal material may be diffuse in consistence with what might be found in a real body. The different constitutive assumptions may somewhat loosely be divided into three categories.

a) No particular constitutive modelling is made. Conventional material models developed for normal structural calculations are used everywhere. The process region will thus be of an infinitesimally small size. A particular fracture criterion must be supplied to determine the motion of the crack tip. The strains at the tip approach infinity and thus also the stresses for hardening materials.

b) Separation of surfaces is admitted in the constitutive model if some measure, usually the maximum principal stress, exceeds a critical value. During the separation process there is a traction between the surfaces gradually dropping to zero as the displacement discontinuity increases. The fracture criterion is specified once a cohesive law has been adopted. These models give a surface process region if the material outside a cohesive zone behaves in a normal fashion.

c) Softening behaviour is introduced into the constitutive laws. This can be done in several ways. One class of models that has achieved some popularity, e.g. Bui and Ehrlacher [5], is the damage accumulation type of models. Here the micro-damage is assumed to be describable by one or more internal state variables. The rate of these is governed by the stresses and/or strains. The state variables enter as parameters in the constitutive laws. If the equations are suitably constructed all stresses and strains remain finite and the crack growth conditions can be formulated directly in terms of strains or stresses. If the process region is defined as a region where damage has occurred it is seen that this is in general a volume.

Other possibilities for introducing softening effects are through non-local constitutive models (Eringen et al [6]). Here the stress at a point is assumed to depend on a functional of the strains in some finite neighbourhood. Also in this case non-singular stress and strain states may result and thus the fracture criterion can be formulated simply.

Models leading to volume process regions commonly result in complicated computations. In a numerical treatment stability problems easily occur. One may of course also combine models of type b) and c). This has not been attempted according to the knowledge of the authors.

Models of type c) are not further discussed here since it is believed that the important features of having process regions with finite dimensions are well demonstrated by type b) models. In many cases cohesive zone models can be used as approximations for volume process regions and the considerable extra complications with type c) models may not be worth the effort. However, no such comparison is so far available in the literature so this remains an open question.

3. Cohesive Zone Models

Cohesive zone models are well-known and seem to originate from a suggestion by Prandtl [7]. It is assumed that the crack surfaces start to separate if some measure of the stress or strain state exceeds a critical level. Usually this is taken as the normal stress perpendicular to the prospective crack plane in a mode I case, but there is no reason why the conditions for the onset of separation could not equally well be based on a strain measure. Between the partly opened crack surfaces there is a traction p_y that depends in some way on the surface displacement v (Fig. 1), for example of the form (1).

$$g\left[\frac{\partial^{k_1}p_y(x_1)}{\partial t^{k_1}}, \frac{\partial^{k_2}v(x_1)}{\partial t^{k_2}}\right]=0, \quad 0\le k_1\le m_1, \quad 0\le k_2\le m_2. \tag{1}$$

Here t denotes time and the m's are the numbers of the respective derivatives that appear in the law. In most applications the simpler form (2) is used.

$$p_y = f_C(v). \tag{2}$$

Here f_C is a decreasing function which goes to zero at some value v_m of the displacement. Even more general forms than (1) may be used when the intention is to model complicated micro-processes. Non-local formulations, i.e. when p_y depends on a spatial functional of v, are examples of such generalizations. However, the practical difficulties involved in determining a suitable separation law in a particular case are probably so large that in most cases simple forms are preferred.

Once the separation law is specified the fracture problem is reduced to the solution of a boundary-value problem and the remaining difficulties are of a computational nature. These are in most cases considerable especially if the surrounding material behaves non-linearly or inertia effects are important. The motion of the front ($x_1 = a^+(t)$) and the trailing ($x_1 = a^-(t)$) ends of the cohesive zone are not given a priori but are part of the solution. This together with the negative stiffnesses associated with a f_C that is decreasing with increasing v may become prohibitive to a successful numerical solution. These difficulties are likely to be most pronounced for very small zones. Large gradients then occur and a high resolution in the numerical scheme is needed. It is thus of great interest to investigate when the cohesive zone can be approximated by a point-sized process region and how the fracture criterion should then be formulated. This is a question of autonomy which is discussed in more detail below.

The main difficulty is how to obtain the separation law. With the viewpoint taken here that the cohesive zone reflects complicated micro-processes it seems that physical modelling ought to be the main tool if one desires anything but the simplest type of laws. Direct observation appears unrealistic. Hillerborg et al [8] suggests that for fracture of concrete the separation law can be taken from observations of the stress-strain behaviour of uniaxial tension tests of uncracked specimens.

If no physical arguments for a specific cohesive law are at hand it might be the wisest to choose a simple few-parameter description, for example a linearly varying f_C which can be described by two parameters, the maximum traction p_{max} and v_m. Alternatively one of these parameters can be replaced by the specific separation energy γ_S per unit area. In the linearly decreasing case γ_S equals $p_{max}\, v_m$. In order to determine these parameters fracture tests can be conducted as in common fracture testing. The increased complexities in order to determine say two parameters are however considerable compared to the case when only one parameter is to be evaluated.

It may be of some interest to calculate the energy flow to the cohesive region. Letting T be the total power supplied to the cohesive zone per unit thickness, this is easily obtained as

$$T = T_m + T_t = 2\left[\int_{a^-}^{a^+} p_y \dot{v} dx_1 - \int_{a^-}^{a^+} h_y dx_1\right]. \tag{3}$$

Here h_y is the x_2-component of the heat-flux vector h_i defined as positive directed out from the cohesive region. It can firstly be observed that the thermal flux T_t is not directly coupled to the mechanical power T_m. T_t together with the dissipation in the cohesive zone determines the temperature which of course is of importance for the mechanical properties. The thermal flux can however not be used directly in the mechanical process. This point is trivial in context with cohesive zones but is more involved when dealing with singular crack tips, since then the separation of energies may sometimes be difficult to perform.

It is often preferred to expressed the energy flow to the tip region in terms of energy per unit crack advance. That this is not a uniquely defined quantity in the cohesive zone case is already clear from the fact that the crack tip velocity cannot be unambiguously defined. Choose, however, for definiteness the velocity of the front end $\dot{a}^+$ as characterizing velocity. Introducing the coordinate transformation (4) the mechanical energy flow per unit front end advance γ_m can be written as (5).

$$x_1' = x_1 - a^+(t) \tag{4}$$

$$\begin{aligned}\gamma_m &= -2\int_{a^- - a^+}^{0} p_y \frac{\partial v}{\partial x_1'} dx_1' + \frac{2}{\dot{a}^+}\int_{a^- - a^+} p_y \dot{v} dx_1' \\ &= \gamma_S - 2\left[\int_{v(a^-)}^{v_m} p_y dv \frac{1}{\dot{a}^+}\int_{a^- - a^+}^{0} p_y \dot{v} dx_1'\right],\end{aligned} \tag{5}$$

$$\gamma_S = 2\int_0^{v_m} p_y dv\ . \tag{6}$$

The quantity γ_S is the specific separation energy per unit area and it is seen that in general γ_m differs from γ_S by a time-dependent part as was noted by e.g. Schapery [9] and Freund [10]. This underlines the fact that a cohesive zone model is in general not compatible with a criterion of a prescribed energy flux to the tip region as has been discussed by Rice [11]. It is only under conditions of steady-state crack growth that the energy flow per unit area of crack advance is equal to γ_S since then the second and third terms in Eq. (5) vanish. This fact is however of minor interest since the presence of a cohesive zone alters the fields around the tip region and the energy flow as calculated from a case without a zone in general will be different. Since thus the boundary value problem in any case has to be solved with a cohesive zone, knowledge of the energy flow becomes redundant.

The only situation when there can be a possibility of equivalence between the cohesive zone description and an energy criterion is the limiting case when the zone size tends to zero. Willis [12] showed that in the quasi-static case and in the case of steady-state crack growth in linearly elastic materials these two descriptions become equivalent as the zone size decreases to zero. Apart from analyses of elastic materials there seem not to exist any well-established results of this character. The most important conclusion is that however small the cohesive zone there is always a finite energy flow to it. This observation is likely to be true for all finite size crack tip models and the question is whether such a requirement should also be put on crack-tip models of type a) i.e. with zero dimensions.

4. General Aspects of Singular Crack Tip Models

The attention will now be focussed on singular crack tip models i.e. when conventional non-softening laws are used to model the entire problem. For many material models the fields can be written in the following way (see Fig. 2 for notation).

$$\sigma_{ij}(r,\phi,t) = S_{ij}(r,\phi;\ Q_\sigma(t),\dot{a}) + \hat{S}_{ij}(r,\phi,t)\ , \tag{7}$$

$$\varepsilon_{ij}(r,\phi,t) = E_{ij}(r,\phi;\ Q_\varepsilon(t),\dot{a}) + \hat{E}_{ij}(r,\phi,t)\ , \tag{8}$$

$$u_i(r,\phi,t) = U_i(r,\phi;Q_u(t),\dot{a}) + \hat{U}_i(r,\phi,t)\ , \tag{9}$$

Here Q_σ, Q_ε and Q_u are scalar time functions depending on the entire problem. S_{ij} , E_i and U_i are *problem independent* functions that dominate the fields as $r \to 0$.

$$\lim_{r \to 0}(S_{ij}\hat{S}_{ij})/(S_{kl}S_{kl}) = 0\ , \tag{10}$$

$$\lim_{r \to 0}(E_{ij}\hat{E}_{ij})/(E_{kl}E_{kl}) = 0\ , \tag{11}$$

$$\lim_{r \to 0}(U_{i,j},\hat{U}_{i,j})/(U_{k,l}U_{k,l}) = 0\ . \tag{12}$$

The strength-parameters Q_σ, Q_ε and Q_u are interdependent through the constitutive law. In history-dependent materials the relation between these may become complicated so that a reduction to a one parameter description is impractical.

In many cases the crack-tip functions S_{ij} etc. are of the following simple form

$$S_{ij} = r^{-\alpha}\Sigma_{ij}(\phi,\dot{a})Q_\sigma(t)\ , \qquad 0<\alpha<1\ , \tag{13}$$

$$E_{ij} = r^{-\beta}\varepsilon_{ij}(\phi,\dot{a})Q_\varepsilon(t)\ , \qquad 0<\beta<1\ , \tag{14}$$

$$U_i = r^{-\beta+1}\Omega_i(\phi,\dot{a})\ . \tag{15}$$

For such cases Strifors [13] has pointed out that the particle velocities and accelerations can be expressed as

$$\dot{u}_i \rightarrow -\dot{a}\frac{\partial u_i}{\partial x_1} \quad , \quad \text{as } r \rightarrow 0 \, , \tag{16}$$

$$\ddot{u} \rightarrow \dot{a}^2 \frac{\partial^2 u_i}{\partial x_1^2} \quad , \quad \text{as } r \rightarrow 0 \, . \tag{17}$$

The crack-tip is then moving under locally steady-state conditions. This simplifies the calculation of the energy-flow to the crack-tip and also means that conclusions about the singular behaviour can be obtained from steady-state solutions.

It should be pointed out that there are cases when the fields in the crack tip vicinity cannot be written under the form given by Eqs. (7)-(17). The most notable is the elastic perfectly plastic material model which is discussed below.

A concept of central importance in fracture mechanics is that of autonomy (c.f. Barenblatt [1], Broberg [3]). A (Fig. 3) crack growth initiation and propagation problem is considered which is modelled by a particular set of constitutive relations. This model will be termed I in the following. It is assumed that for model I the fields are given by Eqs. (7)-(9). Let $C_I(t)$ be a curve that encloses a region around the tip where the fields are given to a prescribed relative accuracy δ by the first terms of Eqs. (7)-(9).

In addition a model II of the same problem is considered which is equal to model I except for a region in the vicinity of the crack tip where the constitutive relations are different from those of model I. This may for example be due to modelling of decohesive processes. This region is enclosed by a curve C_p which is assumed to be wholly contained within C_I. Because of this different behaviour the fields outside C_p will also differ from that of model I

$$\sigma_{ij}(r,\phi,t) = \Delta\sigma_{ij}(r,\phi,t) + S_{ij}(r,\phi;\, Q_\phi,\dot{a}) + \hat{S}_{ij}(r,\phi,t) \, ; \tag{18}$$

$$\varepsilon_{ij}(r,\phi,t) = \Delta\varepsilon_{ij}(r,\phi,t) + E_{ij}(r,\phi;\, Q_\varepsilon,\dot{a}) + \dot{E}_{ij}(r,\phi,t) \, , \tag{19}$$

$$u_i(r,\phi,t) = \Delta u_i(r,\phi,t) + U_i(r,\phi;\, Q_u,\dot{a}) + \hat{U}_i(r,\phi,t) \, . \tag{20}$$

A curve $C_{II}(t)$ is now defined for model II outside which the fields are given to the prescribed relative accuracy δ at the time t by the last two terms of Eqs. (18)-(20). Obviously C_{II} will depend on the assumed constitutive behaviour both within and outside C_p and in some cases such a curve may not exist. Assume however for the moment that C_{II} exists.

Autonomy is now said to prevail if $\Delta\phi_{ij}$,$\Delta\varepsilon_{ij}$ and Δu_i outside C_p and conversely the state within C_p depend within the relative accuracy δ only on the properties within C_p and on the time-histories of Q_σ, Q_ε, Q_u and $\dot{a}$. If autonomy is at hand it enables us to express the state within C_p in terms of parameters of model I. This is very important since in most cases one is ignorant of the behaviour within C_p or wants to avoid the more elaborate modelling.

Autonomy will obviously prevail if C_{II} is wholly contained within C_I since then the state of the annular region between C_{II} and C_I is given by the singular terms of Eqs. (7)-(9) only. Intuitively we would expect this to be the case if the region within C_p is sufficiently small and the governing equations of the material outside C_p are elliptic. For quasi-static behaviour and a linearly elastic material outside C_p, Broberg [2] has shown that the assumption of autonomy is satisfied if C_p is fully contained within C_I, and C_{II} does not reach the outer boundary. These requirements are obviously less restrictive than those mentioned above. It is possible that a similar behaviour may be found for other material models but no results of a general nature seem to exist. This is for obvious reasons a problem area which is certainly worthy of further research pursuits.

By the definition of autonomy the state within C_p will be determined by a functional dependent on the time-histories of the strength parameters Q and the velocity $\dot{a}$. For a stationary crack tip the complete time-histories must in general be considered while for a growing crack only a part of the time-history needs to be considered. Suppose that the crack tip is at a_0 at a certain time instant t_0. The state of the region within C_p will give effects on the fields outside C_p that are not considered in model I. When autonomy prevails however these effects can be neglected at sufficiently far distances ahead of a_0. Let us denote the distance needed for these effects to be neglected by l_{min} which is in most cases smaller than the extent of C_p. l_{min} will depend on the material properties and on the size of C_p. Thus the state within C_p at the time instant t corresponding to the crack tip position a_0+l_{min} will not depend on the time-histories of the Q's and a before t_0. The time-histories of the parameters can be represented by a Taylor series. The state of the crack tip at the time t is then expressible by a function h of the time-derivatives of the Q's and $\dot{a}$. Regarding the velocity of the tip as a state-variable we can then write

$$\dot{a}(t) = h\left[\frac{d^{k_1}Q_\sigma}{dt^{k_1}}, \frac{d^{k_2}Q_\varepsilon}{dt^{k_2}}, \frac{d^{k_3}Q_u}{dt^{k_3}}, \frac{d^{k_4}\dot{a}}{dt^{k_4}}\right], \quad 0 \le k_i \le \infty . \tag{21}$$

A certain derivative can be neglected if its contribution to the time-history between t_0 and t is small. If for example a derivative of Q_σ satisfies

$$\left|\frac{1}{k_1!}\left[\frac{l_{min}}{\dot{a}_{min}}\right]^{k_1}\frac{d^{k_1}Q_\sigma}{dt^{k_1}}\right| \ll |Q_\sigma(t)| , \tag{22}$$

the derivative can be neglected. $\dot{a}_m$ is the mean velocity during the considered time interval. If C_p and thus l_{min} become infinitesimally small all derivatives with $k_i>0$ can be neglected and $\dot{a}$ becomes a function of the momentary values of the Q parameters. Furthermore if the relation between these parameters is time-independent Eq. (22) can be stated in terms of one of the parameters, say Q_σ.

An important special case is when the material both inside and outside C_p is not explicitly time-dependent and inertia effects can be neglected. The crack growth equation can then be formulated in terms of one strength parameter, say Q_σ. If furthermore Q_σ is monotoneously increasing, the time derivatives can be replaced by derivatives with respect to Q_σ. Since the singularity under these assumptions does not depend on $\dot{a}$, the crack growth equation (21) reduces to Eq. (23) even if l_{min} does not vanish

$$\frac{da}{dQ_\sigma} = h(Q_\sigma) . \tag{23}$$

Integration with respect to Q_σ then gives Eq. (24) which is of the form frequently used in quasi-static fracture mechanics.

$$a - a_0 = H(Q_\sigma) . \tag{24}$$

The concept of autonomy used here is somewhat different from that introduced by Barenblatt [1] and Broberg [2]. These authors considered mainly stationary cracks under monotonic loading. In their case autonomy implies that the state of region within C_p is always the same at crack growth initiation.

It should be realized that the previous discussion is purely heuristic and the conclusions have only been strictly verified for elastic cases (c.f.[2],[12]). It seems worthwhile to investigate these questions more extensively. One way of performing such studies is to model the process region by a simple cohesive zone model. By solving the same problems both with and without this zone one can study the conditions under which autonomy in the sense implied here develops and also how the crack growth equation (21) should be formulated.

In the discussion nothing has been said about the nature of the processes within C_p. In one way the discussion may be seen as leading to the conditions that have to be satisfied if some special modelling of the top region is to be avoided. It is analogous to the concept of small-scale yielding. Thus, the concept of autonomy may be applied on different levels. Only the existence of an autonomous region with respect to the particular model chosen is needed. However, the growth equation (21) can be formulated in a simpler manner if autonomy regions on a finer scale can be detected.

The existence of an autonomy region is obviously a necessary requirement for any criterion based on a singularity strength parameter. The use of a path-independent integral is not in general a remedy for loss of autonomy. Suppose that an integral is path-independent for the material behaviour assumed outside the curve C_p. If autonomy does not prevail the fields within C_p are not given by a single parameter and there is no reason why the chosen path-independent quantity should be a measure that is appropriate for characterizing the state within c_p.

If C_p is too large so that no autonomy develops, one way to proceed is to use more appropriate modelling. An example is furnished by the development of the theory of quasi-static fracture in metals. For very small plastic zones, linear elastic fracture mechanics works well since autonomy exists with respect to the elastic model. When the plastic zone grows bigger, theories essentially based on deformation type plasticity are often applied since a region may exist where the fields are accurately described with this constitutive assumption. This model works fairly well until, due to crack growth, the effects of unloading become important and the autonomy concept breaks down. Instead, one can consider modelling of the growing crack with aid of incremental plasticity models and base the growth equation on the singular description associated with those models. It may well be that the region within C_l is now so small that effects of the decohesive process prohibit a region of autonomy with respect to the incremental plastic model. The next step would then be modelling of the process region by either a cohesive zone or a volume process region. However, the complexities rapidly increase as more complicated models are used. Even the continuum plasticity is not without severe problems in a proper modelling of hardening characteristics etc. Also effects of large deformations become more prominent as the scale is decreased.

For these reasons other routes instead of more detailed modelling are often preferred. In most of these the fracture criterion is based on some quantity that is not a direct measure of a singularity. Such criteria are discussed below.

5. Energy Flow to the Crack-Tip

Since long the flow of energy to the crack tip has been considered important for crack growth studies. Numerous articles have been devoted to the discussion of this concept. Strifors [13]-[14] considers in depth the energy and entropy balances at a moving singular tip under general three-dimensional and large deformation conditions. Earlier, Kostrov and Nikitin [15] derived some of the results under small deformation assumptions. From [15] the following result is obtained for the power T supplied to a moving singular tip

$$T = \lim_{C\to 0}\oint_C \left[\left(e+\frac{1}{2}\dot{u}_i\dot{u}_i\right)\rho\dot{a}\delta_{ij}+\sigma_{ij}\dot{u}_i-h_j\right]n_j dc\ . \tag{25}$$

C is a curve enclosing the tip (see Fig. 4), c is a line-coordinate along this curve, e is the internal energy per unit mass, ρ the density and h_j the heat flux vector. δ_{ij} is the Kronecker symbol.

Utilizing the results for asymptotically steady crack motion Eqs. (16)-(17), Eq. (25) takes the following form for the energy flow per unit crack advance.

$$\gamma = \lim_{C\to 0}\frac{1}{\dot{a}}\oint_C \left[\left(e+\frac{1}{2}\dot{a}^2 u_{i,1}u_{i,1}\right)\rho\dot{a}\delta_{1j}-\dot{a}\sigma_{ij}u_{i,1}-h_j\right]n_j dc\ . \tag{26}$$

It is immediately clear from Eq. (26) that a finite non-trivial value for γ is obtained only if the integrand behaves as $0(r^{-1})$ for $r\to 0$. It is furthermore noted that γ comprises the total energy flow including the thermal flux. In accordance with the previous discussion it is desirable to single out the mechanical part of γ i.e. γ_m. In order to do this some constitutive assumptions are made which are somewhat more restrictive than entirely necessary for obtaining the desired result. For the sake of clarity it is assumed tha mechanical power can be divided into one elastic and one inelastic part.

$$\sigma_{ij}\dot{\varepsilon}_{ij} = \sigma_{ij}\dot{\varepsilon}_{ij}^{(e)} + \sigma_{ij}\dot{\varepsilon}_{ij}^{(ie)} = \dot{w}^{(e)} + \dot{w}^{(ie)} = \dot{w} \ . \qquad (27)$$

It is assumed that the elastic energy is wholly reversible and that the nonelastic energy is purely dissipative i.e. converted into heat. The equation of local entropy balance (28) can be found in any textbook on thermodynamics

$$\rho\dot{s} = \left(\frac{h_i}{\theta}\right)_{,i} + \rho\dot{s}^{(i)} , \qquad (28)$$

s is the entropy per unit mass, θ the temperature and $\dot{s}^{(i)}$ the internal entropy production rate per unit mass. Under the assumptions made above, the internal entropy production consists of two parts

$$\rho\dot{s}^{(i)} = \frac{\dot{w}^{(ie)}}{\theta} - h_i\frac{\theta_{,i}}{\theta^2} > 0 \qquad (29)$$

Using Eqs. (28)-(29) together with the local energy balance (30) the expression (31) for the internal energy is obtained

$$\rho\dot{e} = \sigma_{ij}\dot{\varepsilon}_{ij} - h_{i,i} \ , \qquad (30)$$

$$\rho\dot{e} = \sigma_{ij}\dot{\varepsilon}_{ij} + \rho\theta\dot{s} - \dot{w}^{(ie)} = \sigma_{ij}\dot{\varepsilon}_{ij}^{(e)} + \rho\theta\dot{s} \ . \qquad (31)$$

With this partitioning of the internal energy it is reasonable to single out the mechanical part of T as follows

$$T_m = \lim_{C\to 0}\oint\left[\left(w^{(e)} + \frac{1}{2}\rho\dot{u}_i\dot{u}_i\right)\dot{a}\delta_{1j} + \sigma_{ij}\dot{u}_i\right]n_j dc \ . \qquad (32)$$

Again utilizing Eqs. (16)-(17) we obtain the mechanical energy flow per unit crack advance γ_m.

$$\gamma_m = \lim_{C\to 0}\oint_C\left[\left(w^{(e)} + \frac{1}{2}\rho\dot{a}^2 u_{i,1}u_{i,1}\right)\delta_{1j} - \sigma_{ij}u_{i,1}\right]n_j dc \ . \qquad (33)$$

In order to get meaningful results for γ and γ_m it must be required that Eqs. (26) and (33) give unique results regardless of how the limiting process is done. This requires an asymptotic path-independence and Strifors [13] shows that under the assumption of regular boundary conditions on the crack surface this requires that the divergence of the integrand within the parentheses then must be $0(r^{-2})$ as $r\to 0$. For Eq. (26) this results in the following condition

$$\rho\dot{a}e_{,1} + \rho\dot{a}^3 u_{i,1}u_{i,11} - \dot{a}\sigma_{ij,j}u_{i,1} - \dot{a}\sigma_{ij}\varepsilon_{ij,1} - h_{j,j} = 0(r^{-2}) \,. \tag{34}$$

Observing the steady-state results for the time-derivatives of u_i it is immediately obtained from the equation of motion and the local form of the energy balance Eq. (30) that Eq. (34) is satisfied. Performing the same operation on Eq. (33) results in

$$w_{,1}^{(e)} + \rho\dot{a}^2 u_{i,1}u_{i,11} - \sigma_{ij,j}u_{i,1} - \sigma_{ij}\varepsilon_{ij,1} = 0(r^{-2}) \,. \tag{35}$$

From the equations of motion and Eq. (27) it is found that this condition is not satisfied unless $\sigma_{ij} = \partial w^e / \partial e_{ij} + O(r^{-2})$. Thus the material must behave elastically as $r \to 0$ in order to give a unique value of γ_m. This result was obtained in [13]-[14] for more general conditions by consideration of the entropy balance. Thus if the material allows for energy dissipation as the crack tip is approached no unique value for the mechanical energy flow can be obtained and this precludes the use of an energy criterion within the considered model. In fact materials that do not behave elastically in the limit do not seem to give even a non-zero total energy flow γ because of the nature of the singular solution. A more formal approach to the energy balance has recently been given by Lidström [16], whose results do not, however, alter the general conclusions advanced here.

Nakamura et al [17] have derived an expression for the energy flow to the tip in terms of mechanical quantities only. Corresponding to Eq. (25) their results read as

$$T^1 = \lim_{C \to 0} \oint_C \left[\left(w + \frac{1}{2}\rho\dot{u}_i\dot{u}_i \right) \dot{a}\delta_{1j} + \sigma_{ij}\dot{u}_i \right] n_j dc \,. \tag{36}$$

$$w = \int_0^t \dot{w} dt \,. \tag{37}$$

The result (36) is obtained in the same way as Eq. (25) if the heat flux is set to zero. It might, however, be of interest to compare the expressions for the general case. If the local energy balance Eq. (30) is integrated with respect to time and the result substituted for e in Eq. (25), the difference between the two integrals can be written as

$$T - T' = \lim_{C \to 0} \oint_C \left[(-\int_0^t h_{i,i} dt)\dot{a}\delta_{1j} - h_j \right] n_j dc \,. \tag{38}$$

Clearly, if h_j does not behave as $0(r^{-1})$, the heat flux will not contribute to the integrals and they will be equal, including the case when the material behaves asymptotically elastic. In cases when h_j is $0(r^{-1})$ they seem not in general to yield equivalence results. This point may, however, be of mostly academic interest since then an energy balance criterion seems doubtful anyway.

It is obvious from Eq.(34) that Eq. (36) is asymptotically path-independent as was noted in [17]. Both Eq. (26) and Eq. (36) are strictly path-independent under steady-state conditions and this property has been utilized by Freund and Hutchinson [18] for evaluation of the energy flow to the tip of a crack moving steadily in an elastic visco-plastic material.

Because of the limiting process involved in calculating γ its value is completely determined by the strength and the form of the singularity. Thus, the fracture criterion can equally well be formulated in terms of the Q parameters as in Eq. (21) and no additional advantage has been obtained by using the energy flow expression. In many cases, however, the form of the singularity leads to a zero energy flow. One may ask what physical relevance should be attached to a fracture criterion based on a singularity with such a form. Consider again the models I and II discussed previously and let model II contain a small but finite process region of size C_p while model I has a singular crack tip field which admits no non-zero value of γ. In model II a finite energy flow to the region within C_p of say γ_S would then be expected. Across the curve C_{II} the energy flow is then generally not equal to γ_S. Since we do not require that the fields within C_{II} coincide for model I and II, the only requirement on model I would then be that the flow of energy across C_{II} is the same in both models. The vanishing of γ as the crack tip is approached is thus of no consequence and a growth equation of the form (21) can be used. This will however be dependent on a number of time-derivatives of the Q's. The dependence on derivatives cannot be expected to vanish unless C_p, and thereby C_{II} and l_{min}, tends to zero. As C_p tends to zero, model I will successively become a better approximation of model II and the curve C_{II}, where an energy flow larger than γ_S is required, will shrink. In the limit models I and II coincide and the energy flow becomes γ_S. Based on these arguments the following conjecture is made. It is probably only possible to wholly neglect the dependence on derivatives of Q and $\dot{a}$ in the fracture equation (21) when the singularity is such that a non-zero energy flow to the crack tip results. The practical implications of this observation are unknown. It may however explain to some extent the difficulties in obtaining a unique relation between the strength parameter and the crack tip velocity frequently encountered in dynamic fracture mechanics.

6. Fields at Stationary Crack Tips in Different Materials

Crack tip fields for model I and different non-linear materials have gained much interest during the last decade. Problems with enormous analytical difficulties are often posed. One indication of this is that several problems have not yet been solved while contradictory solutions have been proposed for a few other problems. Unfortunately many of the solutions only serve as academic exercises, particularly so for asymptotic fields when moderate strain rates and hardening rates of practical significance are considered. We refer to the variety of cases where the region of validity for the solution is extremely small. Here a few asymptotic crack tip solutions are examined with respect to the extension of the region where these can be assumed to approximate the exact solution within a certain error δ. Crack tip fields for mode I, plane strain are examined but in many cases for which estimates cannot be found, the mode III counterparts have been examined. It is assumed that the conclusions regarding the validity of asymptotic solutions are due to the material behaviour, rather than the crack mode. Special interest is attached in the fact that the ellipticity of the governing equations seems to be very weak for common metallic materials. The implication is that the process region is autonomous only when the plastic zone is autonomous.

A common feature for the asymptotic stress and strain fields is that they can be expanded in separable terms. This is something that can be expected when the governing equations are elliptic but it may not be so for hyperbolic equations and as a consequence the strain field cannot be determined from an asymptotic analysis for perfectly plastic materials. For all cases reviewed, the asymptotic solution for stresses can be expanded in the form (see Fig. 2 for notations).

$$\sigma_{ij} = Q_\sigma^{(1)} R_\sigma^{(1)}(r) \Sigma_{ij}^{(1)}(\phi) + Q_\sigma^{(2)} R_\sigma^{(2)}(r) \Sigma_{ij}^{(2)}(\phi) + \cdots \tag{39}$$

and for cases of hardening materials strains can be written in the form

$$\varepsilon_{ij} = Q_\varepsilon^{(1)} R_\varepsilon^{(1)}(r) \overset{(1)}{\mathcal{E}_{ij}}(\phi) + Q_\varepsilon^{(2)} R_\varepsilon^{(2)} \overset{(2)}{\mathcal{E}_{ij}}(\phi) + \cdots , \tag{40}$$

where the terms $R_\sigma^{(1)}(r)/R_\sigma^{(1)}(r_0) \gg R_\sigma^{(2)}(r)/R_\sigma^{(2)}(r_0) \gg ..$ and $R_\varepsilon^{(1)}(r)/R_\varepsilon^{(1)}(r_0) \gg R_\varepsilon^{(2)}(r)/R_\varepsilon^{(2)}(r_0) \gg \cdots$ for any fixed ratio $r/r_0<1$ if r_0 is sufficiently small. $Q_\sigma^{(n)}$ and $Q_\varepsilon^{(n)}$ are intensity factors.

Rice [19] showed that the Prandtl slip line field provides the limiting stress state as $r \to 0$ for a stationary crack in a plate of an *elastic perfectly plastic* material. Contained plastic yielding and plane strain was assumed. One finds that all strain components remain finite and the stresses are constant, as the crack tip is approached from ahead in a sector $-\pi/4<\phi<\pi/4$. In the sectors $\pi/4<\phi<3\pi/4$ and $-3\pi/4<\phi<-\pi/4$, r^{-1} singularities result for the shear strain while all other strain components remain finite. The stress field is of a centered fan type. The remaining parts adjacent to the crack surface are constant stress sectors of the same type as the one ahead of the crack tip. The solution yields a displacement discontinuity $[u_2]$ at the crack tip

$$[u_2] = \xi(1+\nu)\frac{K_I^2}{E\sigma_Y} , \tag{41}$$

where ξ is a numerical constant. Using a finite element method Levy et al. [20] estimated ξ to be 0.33 at small scale yielding. The numerical calculations showed that the solution in [19] covers a substantial part of the plastic zone. Note that the angular distribution is left undetermined by the asymptotic analysis.

An asymptotic stress and strain field for a static crack in a *linearly hardening* material was given by Hutchinson [21]. As one might expect an Irwin-Williams type of singularity ($r^{-1/2}$ dependent stresses and strains) is obtained as for a linearly elastic material. A simple change of notation is made to the solution for the linearly elastic material, i.e. $E \to (EE_t)^{1/2}$ where E_t is the tangent modulus at plasticity and Poisson's ratio ν is put to 1/2.

It is readily noted that the limiting solution for $E_t \to 0$ does not reduce to the solution for the perfectly plastic material. In numerical calculations the region of validity for the numerical solution is observed to decrease with decreasing tangent modulus E_t. Simultaneously the solution in a region with an outer limit at say one fourth of the distance to the elastic plastic boundary and an inner limit very close to the tip, coincides with the asymptotic solution for the perfectly plastic material. Thus a solution that is well approximated by the solution for the perfectly plastic material imbeds an asymptotic square-root singular solution for which one may assume that the region of validity vanishes as the perfectly plastic limit is reached. Since metallic materials of practical interest have hardening rates E_t, typically less than $0.01E$, it might be interesting to quantitatively examine the region of dominance for the asymptotic solution. The elastic plastic mode I problem poses extreme analytical difficulties, but since some information might be obtained from the corresponding, analytically simpler mode III problem, this is instead chosen for examination. The comparison is further encouraged since the same behaviour of the near tip field has been observed at the transition from linearly hardening to perfectly plastic materials for both the mode I and mode III problems.

A solution for a static mode III crack in a body of a perfectly plastic material and a small scale of plastic yielding was first given by Hult and Mcclintock [22]. The result was later extended by Rice [23] for a strip geometry and large scale yielding. The shear strain $\varepsilon_{\phi z}$ is $O(r^{-1})$ whereas ε_{rz} remains finite in a centered fan slip line field, which covers the entire plastic zone.

If we introduce a hodograph with $r=\partial F/(2\partial\varepsilon_e)$ on the crack surface, effective strain $\varepsilon_e=[4/3(\varepsilon_{rz}^2+\varepsilon_{\phi z}^2)]^{1/2}$, effective stress $\sigma_e=[3(\sigma_{rz}^2+\sigma_{\phi z}^2)]^{1/2}$ and assume that proportional loading prevails in the near tip region then the following equation (cf.[23]) results

$$\sigma_e\varepsilon_e\frac{d\varepsilon_e}{d\sigma_e}\frac{\partial^2 F}{\partial\varepsilon_e^2}+\varepsilon_e\frac{\partial F}{\partial\varepsilon_e}-\lambda^2F=0 \quad \text{for} \quad \lambda=1,3,5,\ldots \quad . \tag{42}$$

The stress-strain relationship

$$\varepsilon_{ij}=\varepsilon_{ij}^e+\eta\left(\frac{\sigma_e}{\sigma_Y}-1\right)\frac{s_{ij}}{\sigma_e} \tag{43}$$

is assumed at plasticity, i.e. when $\sigma_e>\sigma_Y$. Here e_{ij}^e are the elastic strain components, η is a hardening parameter which equals $3/2\sigma_Y(d\varepsilon_e^p/d\sigma_e)$ where ε_e^p is the effective plastic strain. σ_Y is the yield stress and s_{ij} is the stress deviator.

Equation (42) may be integrated in a straight-forward manner. However, the resulting implicit expressions for r become very lengthy already for $\lambda\geq3$. Thus, we write along the crack surface as follows

$$r = \frac{9}{2}\frac{\mu a_0}{\eta^2}\left[\log\left(1+\frac{2\eta}{3\varepsilon_e}\right)-\left(1+\frac{3\varepsilon_e}{2\eta}\right)^{-1}\right]+a_1 f\left(\frac{3\varepsilon_e}{2\eta}\right), \tag{44}$$

where $f(\zeta)\rightarrow\zeta^{-4}$ when $\zeta\rightarrow\infty$ and $\mu=[\sigma_Y/(9G)](\sigma_Y/G+2\eta)$. G is here the shear modulus. a_0 and a_1 are arbitrary constants. Inversion of (44) under the assumption that a_0 is non-zero gives

$$\varepsilon_e = \left(\frac{r}{\mu a_0}\right)^{-1/2} - \frac{4\eta}{9} + \left(\frac{67\eta^2}{81} + \frac{a_1}{6\mu a_0}\right)\left(\frac{r}{\mu a_0}\right)^{1/2} \cdots \tag{45}$$

If the assumptions of small scale yielding are invoked it can be shown that the solution is obtained by putting $a_0 = R_p$ and $a_1 = 0$. After matching of the plastic zone to the linearly elastic remote field one finds that

$$R_p = \frac{3}{2\pi}\left(\frac{K_{III}}{\sigma_Y}\right)^2. \tag{46}$$

For small scale yielding, mode III case and assumed proportional loading it can be shown [23] that the shape of the plastic zone, for all values of η, is circular and its radius equals R_p. Now the region of dominance for the asymptotic solution

$$\varepsilon_e = \left(\frac{r}{\mu a_0}\right)^{-1/2} \tag{47}$$

can be estimated in relation to $a_0 = R_p$ which at small scale yielding is the radius of the plastic zone but generally is just a load parameter related to the plastic zone size. Thus we obtain

$$r < \frac{9}{16}\delta^2\sigma_Y\frac{\sigma_Y + 2G\eta}{(G\eta)^2}R_p \tag{48}$$

For a material with a rather high hardening rate given by $\eta=50\sigma_Y/G$, if a largest error δ of 10% is allowed, the crack surface strain according to (45) can be approximated by the asymptotic term (47) only for a distance from the crack tip

$$r < 2.3\,10^{-4}R_p\,. \tag{49}$$

The conclusion must be that the practical significance of the asymptotic solution even for high hardening rates is very limited, for small scale yielding and probably also for many cases of well-contained plastic yielding, since the region of validity at small scale yielding is extremely small, certainly when it is compared to the process region. It should be emphasized that this might not be true for cases of a very large scale of yielding, for which large strains in the crack tip neighborhood might promote the development of an asymptotic one term field. If this is the case, then the conclusion must be that near tip fields at small scale yielding cannot be directly compared with those at large scale yielding.

Further it is noted by insertion of (47) and (48) into (43) that the effective stress, σ_e, is asymptotically independent of the hardening rate and about $7.7\sigma_Y$ at the border of the region where the single term approximation is valid. It is suggested that stress strain relationships taken from tensile tests are of very little significance and should be used with caution since one might question the reliability of the material behaviour assumed, and of the extrapolation of experimental results to these high stress levels.

The dominating singular term in a series expansion for a *power law hardening* material was given by Hutchinson [21] and Rice and Rosengren [24]. At plasticity the stress strain relationship

$$\varepsilon_{ij} = \eta(\sigma_e/\sigma_Y)^n(s_{ij}/\sigma_e) \ , \tag{50}$$

was considered, where η and n are hardening rate parameters. The von Mises yield criterion and its associated plastic flow rule was chosen. The asymptotic radial dependence of the so-called HRR-type is, both for mode I and mode III, given by

$$R_\sigma = r^{-1}/(1+n) \ , \tag{51}$$

for stresses and

$$R_\varepsilon = r^{-n}/(1+n) \ , \tag{52}$$

for strains. The angular functions attached to these functions of r were numerically estimated for different hardening rates n.

In order to examine the relation of the asymptotically dominating field to higher order terms in a series expansion for a near tip solution, we again choose the simpler mode III case for inspection and refer to Eq. (42). A more general stress strain relationship for plasticity is then assumed as

$$\varepsilon_{ij} = \varepsilon_{ij}^e + \eta\left(\frac{\sigma_e}{\sigma_Y} - 1\right)^n \frac{s_{ij}}{\sigma_e} \ . \tag{53}$$

After insertion into (40) a complex series expansion results. For finite values of G and η and $n>2$ we find after some calculations that the leading terms of the expansion for ε_e are given as

$$\varepsilon_e = \left(\frac{r}{a_0}\right)^{-n/(n+1)} - \frac{2n}{n+1}\frac{(2\eta)^{1/n}}{3^{1/n}}\left(\frac{r}{a_0}\right)^{-(n-1)/(n+1)} + \cdots \ , \tag{54}$$

where a_0 is a length parameter related to the remote load. The linear extension R_p of the plastic zone at small scale yielding for the material given by (53) is $R_p = 3^{(n+1)/n}(2\eta)^{-1/n}G\sigma^{-1}a_0$. In relation to R_p the region where the one term approximation is valid is given by

$$r < \left[\frac{n+1}{2n}\delta\right]^{n+1} \frac{\sigma_Y}{2G\eta} R_p \ . \tag{55}$$

Assuming a fairly high hardening rate by choosing the parameters $\eta=1$, $n=5$ and a ratio $G/\sigma_Y = 100$, one finds that the region where the single term asymptotic solution (52) is valid is given by

$$r < 2.3\ 10^{-10} R_p \ . \tag{56}$$

We conclude that the asymptotic solution is of minimal practical significance also for this material. An important step in fracture analyses is to include rate effects. This has been done not only for running cracks for which rate effects play an important role but also for stationary cracks for which rapid changes of remote load may cause high strain rates. Many investigations refer to *visco plastic* materials according to the so-called Perzyna model, which for perfectly plastic materials can have the form

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^{e} + \Omega \left[\frac{\sigma_e}{\sigma_Y} - 1\right]^m \frac{s_{ij}}{\sigma_e} \quad \text{for} \quad \sigma_e \geq \sigma_Y \ ,$$

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^{e} \qquad \text{for} \quad \sigma_e < \sigma_Y \ , \tag{57}$$

where Ω is a viscosity parameter. For stationary cracks it was shown by Riedel and Rice [25] that the dominating singularity is of HRR-type (cf.(51) and (52)) if the exponent m is greater than one. A square-root singular behaviour as for the linearly elastic material dominates the near tip field for m less than or equal to one. In the latter case it seems reasonable to use the intensity of this term as fracture parameter. This is however only obvious if the region where the $r^{-1/2}$ singular term, in an expansion of the linear elastic near tip field, dominates, and is considerably larger than the process region. At present there does not seem to be any general result clarifying this quantitatively. It is an important subject for further research. A visco plastic material of the Bodner-Partom type was investigated by Achenbach, Nishimura and Sun [26]. They were able to show that the asymptotic behaviour for a stationary crack is, as regards the particle velocity, $0(r^{-1/2})$ with $r\ logr$ and r for second order terms. They also showed, by choosing as example a titanium material, that the dominance of the $r^{-1/2}$ term is very limited. The boundary value problem considered was a cracked plate which was instantaneously loaded at small scale yielding conditions. The near tip stress field was monitored at a very small distance from the crack tip, i.e. 10^{-5} of the crack length. It was found that the stress field initially is $0(r^{-1/2})$ but very soon, after about 10^{-6}s, the $r\log(r)$ term becomes significant. Furthermore, after about 4 10^{-6}s even the two term solution $r^{-1/2}$ and $r\log(r)$ in combination rapidly gives a very poor description of the particle velocity field, e.g. the error is as large as 23% at 5 10^{-6}s. Since the analysis was made for an instantaneously applied load the conclusion for practical cases is that the $r^{-1/2}$-term is only significant if the rise time for the full remotely applied load is less than 10^{-6}s. For a practical case this means that if say K_{Ic}=100MPa$\sqrt{m}$ then $\dot{K}_I$ has to be larger than 10^8MPa$\sqrt{m/s}$, which is an extremely high rate.

7. Fields At Moving Crack Tips In Different Materials

The earliest asymptotic solution for a *quasi statically* growing mode I crack in an *elastic perfectly plastic* material was obtained by Slepyan [27]. Influence of inertia was neglected and a Tresca yield criterion and associated flow rule was used. The crack tip field was shown to divide into four angular sectors. A point in the vicinity of the crack tip experiences first a plastic constant stress sector and then a centered fan slip line sector followed by an elastic unloading sector and finally a trailing plastic sector. Strains were shown to vary logarithmically with r.

For Poisson's ratio $\nu=1/2$ all isotropic yield criteria with associated flow rules are equivalent at plane strain and thus the solution is also valid for von Mises materials. The result was later extended for von Mises materials and arbitrary ν by Drugan, Rice and Sham [28]. A fifth angular sector had to be inserted between the centered fan slip line sector and the elastic unloading sector. This fifth sector is plastic with a stress field which cannot be described as a slip line field.

Recently, Ponte Castañeda [29] has considered the problem of a quasistatically growing crack in a *linearly hardening* von Mises material (cf. Eq. (41)). Solutions were sought of the form

$$\dot{u}_i = \dot{a}\frac{\sigma_Y}{E}\left(\frac{r}{R_p}\right)^{\beta} f(\phi,\nu,\eta^*)\,. \tag{58}$$

Here R_p is a length parameter related to the linear extension of the plastic zone and $\eta^* = (1+\eta G/\sigma_Y)^{-1}$. The exponent β was found to decrease with decreasing strain hardening. The general tendency for small values of η^* suggest that β tends to a small but finite value. The latter is inconsistent with the corresponding result found for the perfectly plastic material. It is however argued in [29] that the result for small values of η^* might be anomalous, due to the numerical formulation, which implicitly assumes continuity in velocity and thus cannot deal appropriately with the possibility of discontinuities or rapid changes in the velocity field.

In [29], the mode III case is considered as well. The numerical results strongly suggests that $\beta \sim \sqrt{\eta^*}$ for small values of η^* and it can indeed be shown analytically that this result is valid as $\eta^* \to 0$. Stahle [30] has found that $\beta \simeq -0.83\sqrt{\eta^*}$ in the limit. It can also be shown that even though the general character of the fields is similar to the perfectly plastic counterpart, the result is rather different in the limit $\eta^* \to 0$ as compared to the result for perfect plasticity. Elastic unloading occurs at the angle $\phi=19.71°$ for the perfectly plastic material whereas this angle is 33.2° in the limiting case of a hardening material with $\eta^* \to 0$. It is not clear whether a second order solution can be found through the limit analysis or not. By assuming, *a priori*, the same angular distribution as for the perfectly plastic material, analytical calculations (cf.[30]) show that the particle velocities are given asymptotically by

$$\dot{u}_3 = \left[A \left(\frac{r}{R_p} \right)^{-\sqrt{\eta^*}} - B \left(\frac{r}{R_p} \right)^{\sqrt{\eta^*}} \right] \sin(\phi) \ , \tag{59}$$

as $\eta^* \to 0$. A and B are constants that cannot be determined by the asymptotic analysis. By choosing $R_0 = R_p \exp\{\log(A/B)/(2\sqrt{\eta^*})\}$ the result (59) can, for small values of η^*, be expanded in a logarithmic series:

$$\dot{u}_3 = 2\sqrt{AB} \left[\log\frac{r}{R_0} + \frac{\eta^*}{6} \log^3\frac{r}{R_0} + \cdots \right] \sin\phi \ . \tag{60}$$

One observes that this reduces to the result for a perfectly plastic material (cf.[29]), in the limit as η^* approach zero. Assume for a moment that the exact solution to our boundary value problem is given by

$$\dot{u}_3 = \sqrt{AB} \left[\left(\frac{r}{R_0} \right)^{-\sqrt{\eta^*}} - \left(\frac{r}{R_0} \right)^{\sqrt{\eta^*}} \right] \sin\phi \ . \tag{61}$$

Asymptotically the solution is approximated by

$$\dot{u}_3^s = \sqrt{AB} \left(\frac{r}{R_0} \right)^{-\sqrt{\eta^*}} \sin\phi \ . \tag{62}$$

The inequality

$$\frac{\dot{u}_3^s - \dot{u}_3}{\dot{u}_3} = \left(\frac{r}{R_0} \right)^{2\sqrt{\eta^*}} < 0.1 \tag{63}$$

is chosen for a limit below which u_3^s is considered to dominate the solution. If, for metallic materials, a rather high hardening rate $\eta^* = 0.01$ is chosen, the inequality (63) is fulfilled for

$$r < 10^{-5} R_0 \ . \tag{64}$$

By inspection of (61) one observes that the displacement rate is zero at $r=R_0$. Since the displacement rate at the elastic plastic boundary must be comparatively small, it seems reasonable to assume that R_0 is of the same order as the linear extension of the plastic zone. Thus it is here argued that the range of validity of (62) is extremely small if we are confined to metallic materials. In fact the single term $r^{-\sqrt{\eta^*}}$ (or even $r^{-0.83\sqrt{\eta^*}}$) solution is, under these conditions, of very little practical interest since the extension of the region where the single asymptotic term dominates is hardly larger than the process region and certainly not imbedding it.

Steady state crack growth was investigated for *power law hardening* materials [31]. The solution has however not been verified in any other investigations. An attempt to demonstrate the analytical result numerically a FEM analysis has been made but this does, according to the view of the present authors, not seem to be very reliable for materials with reasonable hardening rates. The reason is that the solution [31] which predicts $\sigma_{ij} \propto [\log(r)]^{1/(n+1)}$ and $\varepsilon_{ij} \propto [\log(r)]^{n/(n+1)}$ does not diverge in any significant way from the result for the perfectly plastic material, e.g. $\sigma_{ij} \propto$ const. and $\varepsilon_{ij} \propto \log(r)$. The character of the secondary term has not, to the knowledge of the present authors, yet been obtained, neither for the mode I nor for the mode III cases. The extent of the region where the solution proposed by [31] dominatess the near tip field can therefore not be estimated. On the other hand one can assume that the field in the outer parts of the plastic zone can be expanded in a logarithmic series for which the leading term is the solution for the perfectly plastic material. Now one may ask "When do these two solutions diverge in any significant way?". The answer can be found if we compare the two solutions for either stresses or strains. We also assume that the two solutions give the identical result at, say $r=R_p$. Then the ratio has reached a factor 2 at $1.1\ 10^{-7}R_p$ and if a minimum factor of 10 is asked for, then this is found at $r=\exp(-10^4)R_p$ which is at a ridiculously small distance from the crack tip.

We now discuss crack tip fields at steady motion where inertia effects are considered. It was shown quite recently (Leighton et al. [32]) for the mode I problem of steady crack motion in a perfectly plastic material for which the effects of inertia were considered that particle velocities are asymptotically independent of the distance from the crack tip. The result is based on the assumption that the hydrostatic stress is bounded. The Tresca yield criterion was used together with von Mises flow rule which is reasonable only for the case $\nu=1/2$. A $\log(r)$ term independent of the angle was ruled out as it was shown that jumps in strain rate contradict the principle of maximum plastic work if this term is retained. At vanishing crack speed the near tip strain field does not reduce to quasistatic near tip field [28] and it is claimed in [32] that the region where the dynamic crack tip field is valid vanishes as the crack propagation speed decrease to zero. It was observed by Achenbach and Dynayevsky [33] that the stress state approaches the modified Prandtl slip line field for the perfectly plastic material.

Turning again to the mode III case, the asymptotic solution for ε_{32} along the x_1-axis straight ahead of the crack tip
for a dynamically propagating crack is

$$\varepsilon_{32} = -\left(\frac{C_2}{\dot{a}} - 1\right)\ln(r)\,, \qquad x_2 = 0\,, \tag{65}$$

due to Slepyan [27]. Here $\dot{a}$ is the crack tip speed and C_2 is the shear wave speed. The result for quasistatic crack growth ($\dot{a}$=0) is c.f.[27].

$$\varepsilon_{32} = 1 - \ln(r) + \frac{1}{2}\ln(r)^2\,, \qquad x_2 = 0\,, \tag{66}$$

The exact solution was obtained by Freund and Douglas [35] by means of the hodograph transform. It reduces to the correct asymptotic behaviour (65) in the crack tip vicinity and the solution reduces to (66) at every point on the crack line as the crack tip speed vanishes. Thus an estimation of the region of validity for the dynamic tip field (65) is possible. It was assumed in [35] that dominance of the solution (65) prevails when the ratio of magnitudes between the asymptotic solution (65) and the exact solution is 1/10. Then, if a=0.2C_2, the region extends to less than $4 \cdot 10^{-6}$ of the distance to the elastic plastic boundary, along the crack plane. If $\dot{a} = 0.3C_2$, which in practice is a rather high crack growth rate according to [35], it appears that the range of validity is less than 1/100 of the distance to the elastic plastic boundary, i.e. less than the extension of a typical process region.

For *dynamic* crack growth in *linearly hardening* materials, results due to Achenbach et al [36] show that the asymptotic solution can be written:

$$\sigma_{ij} = Q\sigma_Y \left(\frac{r}{R_p} \right)^{\beta} \Sigma_{ij}(\phi,\dot{a}) \quad as \quad r \to 0\,. \tag{67}$$

The parameter β and the angular functions Σ_{ij} were estimated numerically. It was noted that the solutions converge to the quasi-static result [29] for vanishing crack tip speed. The results also indicate that the crack tip speed C_2, determined as $(G_t/\rho)^{1/2}$ where G_t is the plastic tangent modulus, is a limit beyond which solutions of the form (67) cannot be found.

Regarding the asymptotic solution (67) one might, due to what was found for the limiting steady state case and for the perfectly plastic case, have strong doubts whether there is a significant region of validity in practical cases defined by, say $\dot{a}$<0.3 C_2 and $E_t < 0.01E$. Unfortunately, results were only obtained for hardening rates higher than this due to numerical difficulties. Recently, Ostlund and Gudmundsson [37], who considered the same problem, also included the effects of reversed plasticity. Their results do not significantly change the conclusions of [36].

It was shown by Lo [38] and Brickstad [39] that the Perzyna visco plastic material model gives a linearly elastic near tip field for m<3. A quantitative analysis was performed in [18] by means of a path independent integral for steady state situations (c.f.(39)). The investigation shows that the strain rates near the tip of a running crack ($\dot{a}$=0.1C_2) are so high in the major part of the plastic zone that for a typical ferrous material elastic strains are dominating. It is thus claimed that the energy release rate in this region is found by applying the square root singular solution for the linearly elastic material. The findings for a linearly elastic material, where the series expansion

$$\varepsilon \propto k_1 r^{-1/2} + k_2 + k_3 r^{1/2} + \cdots \tag{68}$$

should be applied, suggest that the dominating $r^{-1/2}$ term might be valid only in a small fraction of the zone where the elastic strain rates dominate. The prospect of finding an autonomous near tip region is however somewhat better than for the other situations reviewed in the present paper. We note that at a distance less than, say, 1/10 to 1/100 of the plastic zone size the ratio between the $r^{-1/2}$ and the constant term is increased by a factor 10 at the peripheral parts of the plastic zone. It seems that the asymptotic field supplies a few-parameter description of the process region characteristics within the specific class of problems considered. It must however be noted that the frame work of autonomy must be used with caution.

For a moving crack in a visco plastic material of the form (57) with $m>3$, Hui and Riedel [40] (for the quasi static case) and Lo [38] (for the dynamic case) showed that the singular field behaves as

$$\sigma_{ij} \to \left[\frac{\dot{a}}{\Omega E}\right]^{1/m} r^{-1/m}\Sigma_{ij}(\phi) \text{ as } r \to 0 . \tag{69}$$

The most remarkable feature of this singularity is that it does not depend in any way on the outer field and is thus not of use for formulating a growth equation. Again the extension of the zone where the singularity dominates may be very small as is indicated by estimates in [38] and [40]. Yang and Freund [41] recently considered the mode III version of the present problem with a modified constitutive relation. The difference with respect to the relation (57) is that visco plastic relaxation is not permitted so that the effective stress remains constant until the effective strain decreases, whereupon purely elastic loading occurs. As a result an elastic wake exists behind the moving tip unlike the situation for the original equation where the tip is completely surrounded by the visco plastic field. In this way the solution will then contain an arbitrary magnitude constant. Obviously, the question of the significance of this result requires further research especially in clarifying the nature of constitutive visco plastic relations at high strain rates.

We may note that all singular solutions presented in this section with the expection of the visco plastic material with $m<3$, have an asymptotic behaviour that does not permit any energy flow to the tip and that the path-area integrals discussed below cannot be used to determine the singularity strength.

8. Path-Area Integral and Other Singularity Measures

The viewpoint advanced so far in this article is that if some particular constitutive model has been adopted the crack growth equation should be formulated in terms of an amplitude measure of the singularity (Eq. (21)), provided that the process region is sufficiently imbedded in the singular field. In most cases analytical solutions cannot be obtained and there remains the problem of extracting the amplitude parameter from, say, a FEM-solution. It is then important that the chosen way of evaluating the singular behaviour does not depend on the particular way of discretizing the problem. In the authors' experience, it is often difficult to evaluate the amplitude directly from stresses and strains since the singular solution is in many cases valid only over a very small region and the costs of computing may be prohibitive. The use of special elements where the desired displacement behaviour is incorporated in the element formulation can significantly reduce this difficulty. Akin [42] has proposed a method for modifying the shape functions of an arbitrary iso-parametric element so that the displacements behave as $r^{1-\beta}(0<\beta<1)$ for $r \to 0$. In a study by Thesken and Gundmundson [43] this element is compared to conventional elements for elasto-dynamic problems with a stationary crack tip. They find a certain improvement by using the Akin element, but an even better accuracy is obtained when the stress-intensity factor is evaluated by means of a path-area integral.

It appears that the use of a path or a path-area integral is a great advantage when evaluating the singularity strength. Recently Moran and Shih [44] discussed path integrals and path-area integrals extensively. One of their main conclusions is that most of the proposed path-independent integrals are variations of a basic result and can therefore not add any new knowledge about fracture problems. They noted however that differences between the different integrals exist with regard to usefulness in numerical evaluations. To this end they advocated the use of domain integrals. We shall here introduce this class of integrals in a slightly different way from that used in ref. [44].

Let C be a curve enclosing the crack tip and let A denote the area enclosed by it. The crack surfaces are assumed traction-free. Now a weight function $q(x_i)$ is introduced of which should be differentiable within A. Consider now a path-area integral of the following form

$$I = \oint_C \left[\left(w + \frac{1}{2}\rho \dot{u}_i \dot{u}_i \right) \delta_{1j} - \sigma_{ij} u_{i,1} \right] q n_j dc + \int_A Z dA \ . \tag{70}$$

The integrand of the area-integral should be chosen so that I becomes indpendent of the choice of C. Assume for the moment that C is a closed curve not enclosing the tip and that the divergence theorem can be used on the line-integral. It is easily found that I taken over a closed area is zero if Z is chosen as

$$Z = -\left[\left(q \left[w + \frac{1}{2}\rho \dot{u}_i \dot{u}_i \right] \right)_{,1} - (q\sigma_{ij} u_{i,1})_{,j} \right] \tag{71}$$

$$= -\left[\left[\left(w + \frac{1}{2}\rho \dot{u}_i \dot{u}_i \right) \delta_{1j} - \sigma_{ij} u_{i,1} \right] q_{,j} + \left[\left[w + \frac{1}{2}\rho \dot{u}_i \dot{u}_i \right]_{,1} - (\sigma_{ij} u_{i,1})_{,j} \right] q \right]$$

$$= -\left[\left[\left(w + \frac{1}{2}\rho \dot{u}_i \dot{u}_i \right) \delta_{1j} - \sigma_{ij} u_{i,1} \right] q_{,j} + (w_{,1} - \sigma_{ij}\varepsilon_{ij,1} + \rho \dot{u}_i \dot{u}_{i,1} - \rho \ddot{u}_i u_{i,1}) q \right] .$$

In the derivation of the last line the equations of motion and the definition of strain have been used. Since the crack surfaces are traction free, I becomes independent of the choice C when it encloses the tip. We note that under steady-state conditions, since time-derivatives are replaced by x_1-derivatives, the second parenthesis of the last member of Eq. (71) vanishes.

For the particular case when q is constant and equal to unity within A and on C Eq. (70) obtains the form

$$I = \oint_C \left[\left(w + \frac{1}{2}\rho\dot{u}_i\dot{u}_i\right)\delta_{ij} - \sigma_{ij}u_{i,1}\right]n_j dc - \int_A [w_{,1} - \sigma_{ij}\varepsilon_{ij,1} + \rho\dot{u}_{i,1}\dot{u}_i - \rho\ddot{u}_i u_{i,1}]dA\ . \tag{72}$$

This equation and integrals of similar form have been discussed by several authors (Atluri [45], Kishimoto et al [46], Strifors [47] among others). If the crack growth is asymptotically steady-state it is obvious that the area integral vanishes for small areas around the tip. Thus a finite, non-zero value of I can only come from the line integral and this only if the integrand behaves as $O(r^{-1})$ as $r \rightarrow 0$. Thus we cannot expect a finite value of I from Eq. (70) for moving cracks in materials where the singularity is not of the elastic type. This has been pointed out in [44]. Any finite values that are obtained from calculations on moving cracks in rate-independent elastic-plastic materials are due to discretization errors in the numerical procedure.

For stationary cracks there is a wider class of problems where the integral in Eq. (70) can be expected to yield a finite non-zero value since, as seen previously, the singularities for stationary cracks in most cases give the r^{-1}-dependence of energy density. The local steady-state argument used for moving cracks is not relevant for stationary cracks. The area-integral vanishes however for an infinitesimal path for the following reasons. If the fields at the tip can be written as Eqs. (13)-(15) the first two terms of the integrand cancel each other in the limit. The particle velocity and acceleration are non-singular and the strength of the singularity of the displacement gradient is such that the area-integral vanishes for infinitesimal areas. This leads to the condition that the J-integral, which is the first part of I in Eq. (70), is asymptotically path-independent for stationary tips.

Another choice of the weighting function is employed in [44], where it is required that q approach unity at the tip and be zero on the curve C. Then the line-integral vanishes and the remainder (i.e. the second term in Eq. (68)) is the domain integral discussed in [44]. Since q goes to unity at the tip, this domain integral will yield the same value of I as Eq. (70) and the comments made above are relevant also to this case.

It is of interest to investigate if some other choice of the weighting function can give finite non-zero values for cases when energy density has a weaker singularity than r^{-1}. Suppose that q is chosen so that the resulting integrand of the line integral is precisely $O(r^{-1})$ as $r \rightarrow 0$. Then the line-integral gives a finite value. It is however seen that the first part of the integrand of the area-integral then behaves as $O(r^{-2})$ for $r \rightarrow 0$ and the area-integral is divergent as $\ln r$ for $r \rightarrow 0$. Thus it seems that the present type of integrals are of no use in determining the singularity strength for moving crack tips in materials that do not behave asymptotically elastic and one is instead forced to use direct observation of stress or stain components. In fact it does not seem to be possible to construct a path or path-area integral that remains path independent and at the same time yields bounded, non-trivial values for cases when the energy is not $O(r^{-1})$.

Other singularity measures have been suggested for use in fracture criteria, most notably the crack tip opening displacement (CTOD) and the crack tip opening angle (CTOA). The problem with these is that, except for some special cases, they cannot be unambiguously related to the singularity strength and therefore tend to depend very much on the calculation model (e.g. the mesh division in a FEM analysis). The problem is equivalent to that of extracting the singularity strength from calculated displacement values and there seems to be no special reason to attach any particular significance to the CTOD or the CTOA, especially since these are difficult to observe directly in experimentation.

9. Initiation of Quasi-Static Crack Growth in Elasto-Plastic Materials

Numerous articles have been written on the subject of quasi-static crack growth initiation and propagation in materials that can be described by an elasto-plastic constitutive model. Consider first the initiation of growth. Since in this case the integral I (Eq.(70)) gives a non-trivial value, which in most cases is a unique measure of the singularity strength, it seems appropriate to base a criterion on this quantity. In common practice the J-integral, which is the first part of (70), is used. If the integration path is taken in the near vicinity of the tip J will coincide with I even in situations where the loading system is non-proportional. It has been verified by several studies (c.f.Shih[48]) that under monotonic one-parameter loading the state in the entire structure of commonly used specimens is near proportional loading so that path-independence of J is obtained even for more remote integration paths. However, in many situations of practical interest, several independent load systems may act on the body and the loading state may be far from proportional. The safest course is to use the complete expression for I. A particular problem occurs for non-hardening materials (c.f.Broberg[2]). Here the angular variations of the strains are not given by the asymptotic solution as discussed above and a unique characterization with the J-integral is not possible in a strict sense.

In practice empirical or semi-empirical procedures are often used to evaluate J. In most cases these procedures are based on the assumption of proportional loading and may be unreliable in more complex loading situations. Some progress in this direction has been reported by Sönnerlindh and Kaiser [49], who find good agreement between an approximate procedure and FEM results for non-proportional loading of single-edged notched specimens.

Consider now experiments performed on different geometries of the same material under uniform environmental conditions but varying loading conditions. If inaccuracies in the experimental procedure and material scatter can be neglected, a properly calculated singularity parameter should be constant at crack growth initiation for the different experiments. Systematic differences in the results will then be due to loss of autonomy with respect to the chosen model. One reason for this is that the process region, as previously defined, is too large. Another reason may be that the bulk properties of the material are not modelled accurately enough. The distinction between these causes is diffuse and according to our definition of autonomy both can be considered to give loss of autonomy.

As a particular example consider inaccuracy in the modelling of strain hardening. Different assumptions about the hardening properties lead to different singularities which in principle are not comparable. If, however, significant discrepancies from the assumed hardening only occur for high values of strain the zone with deviating material behaviour may be so small that autonomy with respect to the chosen models prevails. It appears more important to model the material properties at low values of strain than at high values. The procedure of converting hardening materials into perfectly plastic materials using the mean value of yield strength and ultimate strength, which is common in engineering type fracture criteria, seems questionable in light of these arguments.

When the assumption of autonomy with respect to the singularity strength parameter fails, one possible way is to consider other terms in a series representation of the fields around the tip. The success of such a procedure depends on how well separated the contributions from different terms are, and can not be expected to work well for materials with a low degree of hardening. Furthermore, the fracture criterion will then contain several parameters which considerably complicates evaluation and prediction.

In many investigations it is is assumed that crack growth starts when some measure of strain or stress at a fixed distance in front of the tip reaches a critical value. This distance is chosen with regard to the physical properties of the material, such as the spacing between inclusions. Whether the criterion is based on stress or strain depends on the anticipated fracture mechanism. For a ductile fracture the criterion is based on a strain measure, such as the effective plastic strain, while for a cleavage failure the stress σ_2 is usually chosen.

These criteria are special cases of a more general type which we here term activation zone criteria. The crack growth is assumed to be governed by some functional of stress and/or strain evaluated over a field volume, the activation zone. These models have some similarity with the process region models in that the crack growth behaviour is governed by the state in a finite volume as opposed to criteria based on the singularity parameters. The activation zone models do, however, differ markedly from process region models since the possible processes in the activation zone are assumed not to affect the constitutive behaviour of the material. Another difference is of course that the activation volume is fixed and its determination is not a part of the solution as is the case for process region models.

Clearly, if the activation zone is contained within the curve C_l defined above, an activation zone criterion is equivalent to a singularity based criterion. If it is larger than the region where the singular field dominates, the fracture criterion will obviously be dependent on the non-singular parts of the fields. Whether it is meaningful to assume the presence of a large activiation zone and at the same time neglect all possible changes of the material behaviour seems doubtful to the present authors.

The activation zone models are mostly not used as alternatives to singularity parameter models. More common is to assume the crack tip field to be determined by the singularity parameter and to use an activation zone criterion on a smaller scale in order to predict the dependence of the critical singularity parameter on other variables such as temperature or physical material properties. One advantage with such a procedure is that fracture toughness can be connected to conventional material properties, such as fracture strain, at least in a formal way. This can also be achieved by use of process region models at the expense of considerably more complicated calculations. It should be pointed out that the general argument given above against activation zone models is equally relevant for the case when the activation zone is wholly contained within C_l, since in the near tip region it is most likely that the constitutive behaviour is different from the assumed one.

If a singularity parameter criterion is adopted and material parameters such as yield stress, hardening moduli etc. are kept fixed it is immaterial whether the crack growth criterion is based on Q_σ or Q_ε since these are then uniquely related. If, on the other hand, the fracture behaviour is studied as function of a quantity such as temperature that affects the constitutive properties it can be of interest to base the fracture criterion on different singularity parameters. A reasonable assumption would be that cleavage fracture is governed by Q_σ and ductile fracture by Q_ε. It is assumed that the crack starts to grow if one of the criteria (73) and (74) are satisfied.

$$Q_\sigma(J;\sigma_Y,\ldots\ldots)\geq Q_\sigma^{crit} \ , \tag{73}$$

$$Q_\varepsilon(J;\sigma_Y,\ldots\ldots)\geq Q_\varepsilon^{crit} \ . \tag{74}$$

Depending on the chosen constitutive model, different parameters will enter the expressions for Q_σ and Q_ε. By varying some quantity (e.g. temperature) the critical value of J can be found as function of this quantity. It is howeer necessary that the type of singularity remains the same, otherwise no comparisons are possible without introducing a finite length into the model.

10. Quasi-Static Crack Growth in Elasto-Plastic Materials

At moving tips in elasto-plastic materials the character of the singularity is considerably different from that of a stationary tip. Since the asymptotic behaviour is such that the terms in the integrand of the J-integral do not have r^{-1} dependence, this integral cannot be used to calculated the singularity strength from say a FEM computation. As far as the authors know there is no general method for doing this. Although recently the understanding of singularities at moving crack tips has been considerably advanced, it is ony for one particular case that definite results for the connection of singularity strength and other loading parameters has been obtained. Rice and coworkers ([50],[51] and [28]) derived the following result for growth under small-scale yielding conditions in an elastic perfectly plastic material with yield stress σ_Y. The strength of the outer elastic field is given by the value of J equal to $K_I^2(1-\nu^2)/E$.

$$u_2(\phi=\pi)=\chi r\frac{\sigma_Y}{E}\ln\frac{r_l}{r} \text{ as } r\rightarrow 0 \ , \tag{75}$$

$$r_t=\lambda \frac{EJ}{\sigma_Y^2} \exp \left[1+\frac{\zeta}{\chi} \frac{E}{\sigma_Y^2} \frac{\delta J}{\delta a}\right] . \tag{76}$$

The dimensionless constant χ is given by the asymptotic solution, while the constants λ and ζ must be determined by a complete solution, and this was done for the small-scale yielding case from FEM-calculations in [51]. The parameter r_t characterizes the crack surface displacement uniquely and it is reasonable to state the growth equation in terms of r_t. Rice et al [51] investigated the criterion that r_t is maintained constant during growth and gave results for J as a function of crack growth.

Extension of these results to cases with more extensive yielding appears cumbersome. Rice et al [51] conjecture that the functional form of the singularity remains the same for many cases although the parameters λ and ζ will depend on the particular problem and will not in general be constants. Furthermore, the complete asymptotic strain distribution contains functions of ϕ that are problem dependent so that in a strict sense a one-parameter description of the crack tip state can never be obtained for a perfectly plastic material. It is also not clear which parameter should be chosen to measure of the outer field.

In the analysis of refs.[50]-[51] no reference is made to the definition of J and this parameter is mainly chosen for convenience since its meaning in small-scale yielding is well-defined. Rice et al [51] discuss different definitions of J. They define J_D as the deformation theory value, i.e. essentially the J-value that would result if the crack tip remained stationary and were monotonically loaded to the actual load level, while J_F is defined as the value that would result from a formal evaluation of the line integral taken around a remote contour for a growing crack. In [51] the relative merits of these different definitions with respect to their use in Eq. (75) is discussed in a qualitative manner, but this discussion does not give conclusive evidence that any one of these measures is appropriate for characterizing the singularity strength. In conclusion, there are few arguments for a unique problem independent J-Δa relation in perfectly plastic materials that is valid for all scales of yielding. Indeed the analysis of [51] indicates a pronounced geometry dependence for low values of $\partial J_R/\partial a$ where J_R denotes the resistance as measured in J. A discussion of these questions has also been made by Broberg [3].

For hardening materials the J-integral has also been suggested as a suitable parameter, but on rather different grounds as compared to the approach for perfectly plastic material. The basic arguments were given by Hutchinson and Paris [52]. They argued heuristically that, subject to certain restrictions, a singular field of the deformation theory type prevails outside the zone near the tip where the unloading effects make the deformation theory assumption non-valid. Using the terminology used in this paper, in model I the stress state is essentially one of radial loading so that a deformation theory is applicable. Model II is the same problem solved for an incremental plasticity law. Hutchinson and Paris argue that the extent of C_p, i.e. where loading path effects are important, is of the same order as the amount of crack growth. Furthermore, in order to ensure that a deformation theory field develops, they require that the strains produced in the outer parts of the plastic zone by the increase of the external loading dominate over the strains due to the increase in crack length. These requirements lead to two criteria for application of the so-called J-controlled crack growth.

$$\Delta a < \omega_1 \min(a, b) \tag{77}$$

$$\frac{dJ}{da} > \omega_2 \frac{J}{b}\,, \tag{78}$$

where b is the remaining ligament of the body.

The values of the constants ω_1 and ω_2 have been subject to discussion and several experimental investigations have been devoted to the determination of these. However, the geometries tested are not very different and also differences in materials between the different investigations make any more definite conclusions difficult.

We have not been able to find any analytical or numerical work that verifies that a deformation theory singularity actually is maintained during crack growth up to the limits indicated by (77) and (78). As discussed previously the existence of an observable deformation theory field is not necessarily required for autonomy with respect to a deformation type model to prevail. This aspect seems, however, not to have received any attention. Some of the results discussed previously regarding singular fields in hardening materials, cast some doubt on the possibility of autonomy with respect to the deformation theory model, since the extent of the region where the hardening singularity prevails is extremely small at least for small-scale yielding in materials with normal hardening behaviour.

Another question that so far has not been extensively studied is which measure of the external field should be used to charcterize the strength of the singularity. Even if a deformation theory field is maintained it remains to relate it to the outer loading. Clearly if such a singularity prevails and the loading outside the plastic zone is proportional then the J-integral evaluated for the actual state around a path enclosing the plastic zone will be a unique measure of the state within the zone. If the assumptions of Hutchinson and Paris [52] are satisfied then J_F, as defined previously, is an appropriate measure. In the case when the deformation theory field does not prevail the question becomes more complicated. It is then not ascertained that either J_F or J_D, as defined above, are appropriate measures of the singularity strength.

In practice often J_D is used. It follows from the previous discussion that its use is only justified for the hardening case provided it coincides with J_F. If the hardening singularity does not develop, neither the use of J_F or J_D has been shown to describe the near tip state except for small-scale yielding.

An example has been analysed numerically in order to determine whether there are any significant differences between these measures. A center crack tensile configuration was studied and the finite element model of one quarter of it is shown in Fig. 5. The material was assumed to be linearly hardening with a tangent modulus of 0.01 E which is a fairly high value for metallic materials. The material is assumed to harden isotropically and follow the Mises flow criterion and the associated flow rule. The FEM-calculations were performed with ABAQUS[53] in the following way. First J_D was calculated from runs with stationary cracks in the interval $0.475 \leq a/d \leq 0.525$. J_D was evaluted as function of the nominal load with the virtual crack extension force method as well as from a line integral evaluation. The integration path was taken outside the plastic zone boundary and as a result $J_D(\sigma_0, a)$ was obtained. A $J_R-\Delta a$ relation of the following form was now assumed

$$J_R = J_C + c_J \Delta a \ . \tag{79}$$

From Eq. (79) together with $J_D(\sigma_0, a)$ the load $\sigma_0(a)$ was obtained as function of crack length assuming an initial length a_0.

In the second step calculations were performed where the crack was advanced through the mesh by the nodal relaxation technique. The relation between load and crack length ($\sigma_0(a)$) obtained in the previous step was assumed for these calculations. The J_F value was then calculated by using line integral evaluation for the growing crack. The results from one such simulation of crack growth are shown in Fig. 6. The scale is non-dimensionalized by divided J by $J_0 = (a\sigma_y^2/2.5E)$. a is chosen as the mean crack length $0.5d$. With this scaling one unit on the J-axis corresponds to the ASTM-limit for small scale yielding. The particular $J_R-\Delta a$ curve assumed for this run and also the $J_D-\Delta a$ curve is shown in the figure. Also shown are the elastically calculated J_E values for the actual load and crack length. The most remarkable feature is the approximate equality of J_D and J_F, which is also found from the rigid plastic solution in [51]. It should be pointed out that in [51] other configurations were considered for which J_D deviates substantially from J_F. An isolated example of this kind does not permit any far-reaching conclusions, but at least it is demonstrated for this case that the current practice of using J_D is not unfounded. For the question of whether J_F or J_D is a suitable measure of the singularity strength, the present analysis does not furnish any answers. This would require analyses of different configurations with a much finer mesh, so that the near tip field could be resolved.

In conclusion there remains a lot of research to be done before the theory of quasi-static crack growth in elastic-plastic materials can be said to be well understood. Of particular importance are methods for relating the singularity strength to external loading. Equally important is to obtain a good estimate of the extent of the region where the singularity is dominant in order to judge its physical significance. A particular issue of interest is the effect of the hardening. The results discussed here indicate that for normal degrees of hardening the material could as well be considered as perfectly plastic at least for moderate levels of yielding. For many problems of contained plasticity an analysis along the lines of ([50]-[51]) appears to be more relevant than an analysis founded on the assumption of pronounced hardening.

Obviously there is a legitimate engineering interest in obtaining simple methods capable of estimating the loading bearing capacity much less accurately than what is aimed for in scientific studies. Several such methods do exist, e.g. options 1 and 2 of the R-6-method [54]. The main problems with methods of this kind is that they are essentially empirical methods, i.e., they have been constructed more or less directly on experimental observations. The results of these experiments are expressed in terms of a few easily calculated variables. Then by bounding the experimental points in the space of these variables, this space is divided into a safe and non-safe region. Obviously this kind of method works well if the object under consideration is sufficiently similar to the objects that were the basis for the method. The predictive capacity of such methods for situations far from the experimental basis is potentially smaller than for methods with a firmer theoretical foundation.

11. Dynamic Crack Growth Initiation and Propagation

The term dynamic crack growth, encompasses situations in which inertia effects or strain rate effects on the material behaviour become important. This can be the case at initiation of crack growth if the applied loading is rapid enough. Such effects also become important for rapid growth of cracks and possible arrest even if the loading is of quasi-static nature.

The main problems of analysing dynamic crack growth are of the same nature as for quasi-static growth. Usually, a model with a point-sized process region is used. The strength of singularity for the chosen constitutive model has to be evaluated in the analytical treatment and then to be correlated to experimental results. All of the difficulties connected with this procedure that have been mentioned for the quasi-static case remain, and additional ones due to inertia and strain rate effects are added.

The problem of initiation of growth from a stationary crack has mostly been analysed assuming purely elastic behaviour. Methods to take transient dynamic loading into account are now well established and the calculations are routine. However, the experimental studies performed have not been successful in demonstrating the applicability of dynamic elastic fracture theory. Results from different investigators performing experiments on the same material show little coherence (c.f.Nilsson[55]). These discrepancies can partly be ascribed to weaknesses in the experimental techniques. The main problem is that of determining when crack growth occurs. Despite the efforts devoted to this problem a reliable method still does not exist, with the possible exception of costly high speed photography. Another possible cause of the discrepancies is that linear elastic theory is not sufficient. Some analyses for particular cases of non elastic material behaviour have been performed (c.f.Nakamura et al[56]-[57], Little et al[58]-[59]), but more systematic approaches are lacking. The forms of the singular fields are known, since these will coincide with the quasi static results for stationary cracks, but the extent of the region where the singular solution dominates may be extremely small. In principle the path area integrals may be used to calculate the strength of the singularity, but the numerical resolution needed to get accurate results may be prohibitive in many cases.

For rate independent elastic plastic material models the only difference compared with the quasi-static situation is the presence of inertia effects which, according to the preceding discussions, ought not to influence the singular field. Thus assuming a correctly performed and accurate analysis, observed differences in the critical quantities are due to rate and history effects in the process region. These are however difficult to distinguish from other rate dependencies.

For rate dependent materials rate effects influence the states in the tip vicinity. Assuming the visco plastic constitutive relation (57) the following relations for appropriately normalized intensity factos Q_σ and Q_ε can easily be derived.

$$J \rightarrow \frac{Q_\sigma Q_\varepsilon (1-\nu^2)}{E} \quad \text{as } \nu \rightarrow 0\,, \tag{80}$$

$$\dot{Q}_\varepsilon = \Omega \left[\frac{Q_\sigma}{E \sigma_Y} \right]^m . \tag{81}$$

Calculating $J(t)$ the intensity factors Q_σ and Q_ε can be obtained by solving (80) and (81) and obviously the time histories of these will in general be different. Intuitively, one would expect Q_σ to be larger than for the corresponding rate independent problem, and Q_ε to be smaller. If a fracture criterion is based on Q_σ the critical outer fracture parameter will decrease with increasing rates and the reverse if the criterion is based on Q_ε.

Little et al [58] have performed small scale yielding calculations assuming a constitutive relation similar, but not equivalent, to the Perzyna model. In the present authors' opinion their finite element mesh was probably too crude for resolving the singular fields properly, but still their analysis show the expected behaviour. In [59] it was further shown that history effects apart from the instantaneous loading rate may be of importance. The main conclusions from the reasoning presented here and in [58]-[59] seem to be substantiated by experimental findings. For ductile fracture mechanisms, i.e. corresponding to a criterion based on Q_ε, the fracture toughness seems to increase with increasing loading rate (c.f. Costin and Duffy[60]). For cleavage type fracture, toughness is generally decreasing with the loading rate ([60], Wilson et al[61]).

In comparison to the relatively scarce analyticaly work on dynamic crack growth initiation, the problem of dynamic crack growth *propagation* has attracted many investigators. However, non linear analyses of properly performed experiments have not been made except in a few cases. The analytical approaches can presently only be viewed as giving qualitative indications of the fracture behaviour.

Unlike the situation at initiation, inertia effects influence the state at the tip of a running crack. The form of the singularity for a dynamically moving mode I crack in a rate indpendent elastic perfectly plastic material has not yet been completely clarified. Numerical analyses of the complete small-scale yielding problem for this case, assuming steady state conditions, have been performed by Lam and Freund [62] for the mode I case and by Freund and Douglas [35] for the mode III case. In the latter case the complete analytical solution for ε_{32} is known and it remains only to determine a length parameter R_P. This was done in [35] from finite element calculations. By assuming that a constant value for ε_{32} or ε_f should be maintained at a fixed distance in front of the tip, a relation between the applied stress intensity factor K_{III} and the velocity could be obtained. It was found that the ratio (K_{III}/K_{IIIC}), where K_{IIIC} is the corresponding quasi-static critical value, increases with the tip velocity, slowly for small velocities and then a rather rapid increase at velocity levels that depended on ε_f. Since the asymptotic solution for the dynamic case is not of the same form as the quasi-static result for $\dot{a} \to 0$, a similar comparison based on the singularity parameter is not possible. However, the general trend that K_{III} increases with increasing velocity is preserved also for a growth equation based on the singularity strength. For the mode I case no analytical solution is known and the authors of [62] had to rely on direct numerical results. In [62] the growth law was instead based on the crack surface displacement at a fixed distance behind the tip. Similar observations as for the mode III case were made. As noted by these authors, the finite element mesh used was probably too crude to resolve the singular field, but judging from the mode III results the general conclusion would most probably remain.

For rapidly moving crack tips rate effects are probably always important. These are mostly modelled by use of elastic visco-plastic materials. By an elegant approximate method Freund and Hutchinson [18] were able to derive a relation between γ and the applied J-value for small scale yielding, steady state, mode I growth in visco plastic materials that admits an elastic singular field (e.g. $m<3$ in the Perzyna type model (57)). This analysis was subsequently refined with the aid of numerical analyses (Freund et al[63], Mataga et al [64]).

Adopting the criterion that a constant value of γ should be maintained during growth these authors derived relations between the applied elastic energy release rate J and the velocity $\dot{a}$ for a case corresponding to $m=1$ in (57). Typically the applied J decreases with increasing velocity for low speeds, reaches a minimum and increases for high velocity. The shape of the curves depends on a dimensionless parameter which in prinicple is a measure of how the total energy consumption is divided into plastic dissipation and fracture energy. It is claimed that the suggested criterion is suitable for cleavage crack propagation.

For ductile crack growth the criterion of a critical plastic strain at a fixed distance from the tip is preferred in [64]. Results are presented for the mode III case, showing a monotonically increasing J_C with velocity. These curves differ from the rate independent ones in that they increase more rapidly for low velocities. One may question if it is realistic to assume the critical distance to remain unaffected by velocity in view of substantial changes of the fields that occur for high velocities. If, for example, the criterion that the strains in the singular field (plastic or total) should be maintained constant, the velocity dependence of J_C would be identical to that of assuming a constant singular stress field, since both stresses and strains are controlled by the inner stress-intensity factor. For low velocities the behaviour would then be the same as assuming γ to be a constant, while a more rapid increase would result for high velocities.

The question of deciding the proper fracture criteria for rapidly propagating cracks certainly needs more study. Even if it appears from the continuum solutions that the singular elastic field for rapidly moving cracks has a relatively larger extension than the singular fields for quasi statically moving cracks, it may also be that the process region effects may be more pronounced because of temperature increase and diminishing wave velocities. In [64] it is conjectured that γ may be an increasing function of the velocities and this would explain some of the reported experimental results. This may well be true but it should be remembered that the analyses discussed so far are only for the small-scale yielding case. Wide-spread plasticity is often encountered in crack propagation experiments with similar results as for the quasi-static case. This is, for example, evident in the work of Brickstad [39] who performed numerical visco plastic analyses of experiments, of which some clearly could be classified as large scale yielding. γ was calculated by nodal relaxation and, in view of the relatively crude mesh used, it is questionable whether the resolution was good enough to obtain the singular energy flow. Still, Brickstad was able to show that γ calculated in this way correlated well with the velocity for tests that had experienced quite different degrees of yielding. The investigation [39] is so far the only completed one of this kind, but similar work is underway within the HSST-project [65]. This type of research is extremely valuable in order to obtain a sound basis for dynamic fracture thoery.

12. Concluding Remarks

In this article we have tried to outline the current knowledge of basic problems of fracture theory and to stimulate further research devoted to several unresolved issues. The exposition is admittedly far from complete, several important subjects such as stability conditions, coupled thermomechanical analysis, statistical aspects etc. have been left out of the discussion. Our account of the vast experimental material available in the literature is almost negligible. This is partly due to our limited knowledge of the experimental work and partly due to the difficulty in finding experimental results that either verify or disprove the analytical models conclusively. Experimental research is extremely important and it is only through properly conducted combined analytical and experimental investigations that real progress in fracture theory can be achieved. It is important to remember that experimental work should not be aimed at developing "the best procedure" of producing toughness numbers. Indeed, as long as one can find reports where such concepts as the specimen dependence of fracture toughness are readily accepted, fracture mechanics has failed its purpose.

The main emphasis in this article has been conditions under which the singular crack tip field obtained from continuum solutions can be used to formulated fracture criteria. Not much in the way of providing conclusive answers have been given, instead it seems that further research is needed especially for clarifying the interaction of fracture processes and bulk deformation.

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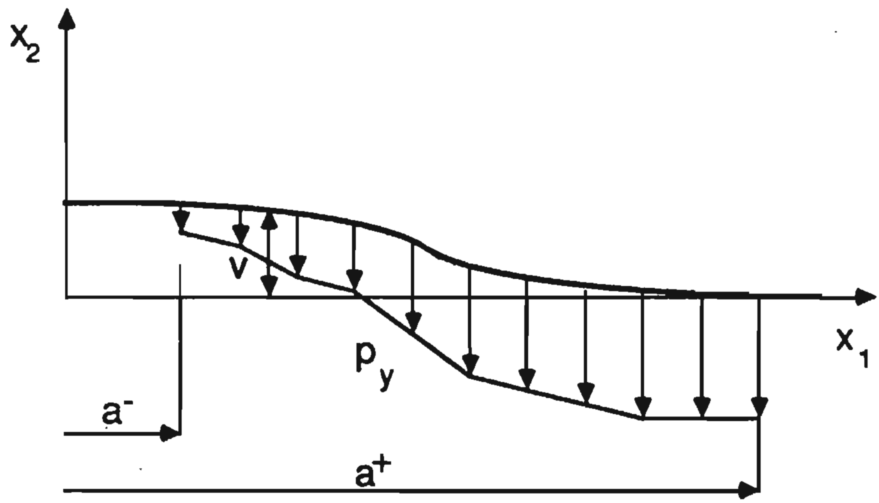

Figure 1: A cohesive zone model

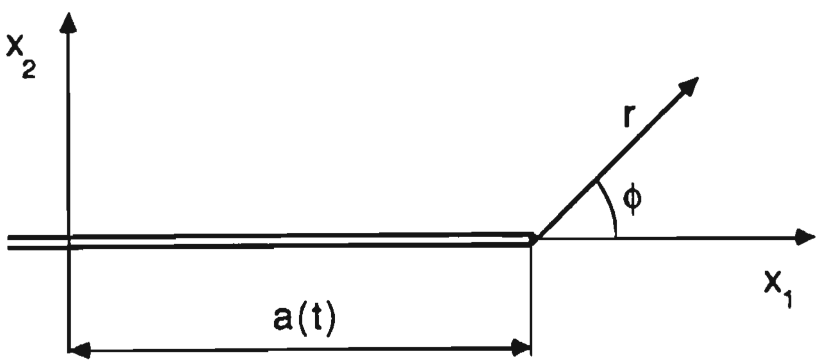

Figure 2: Coordinate definitions

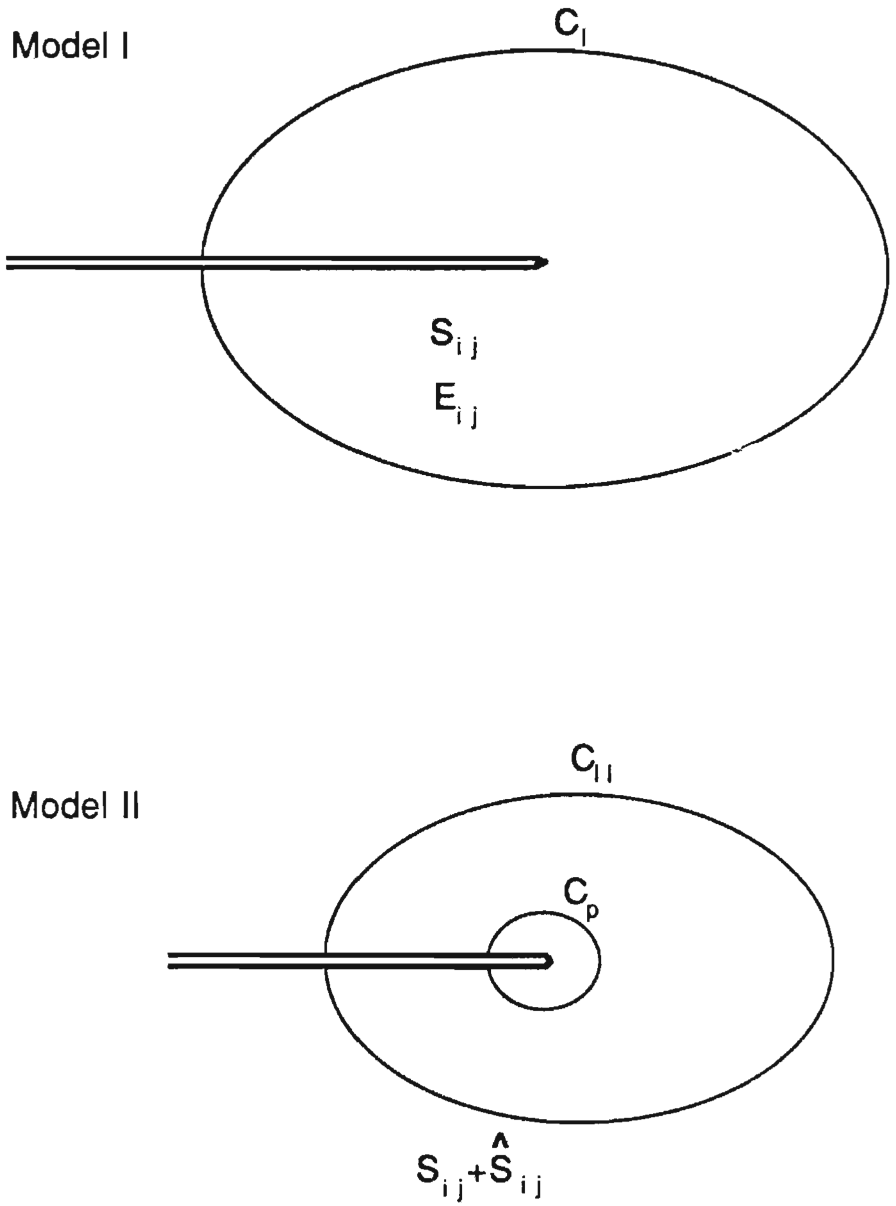

Figure 3: Schematics of model I and II

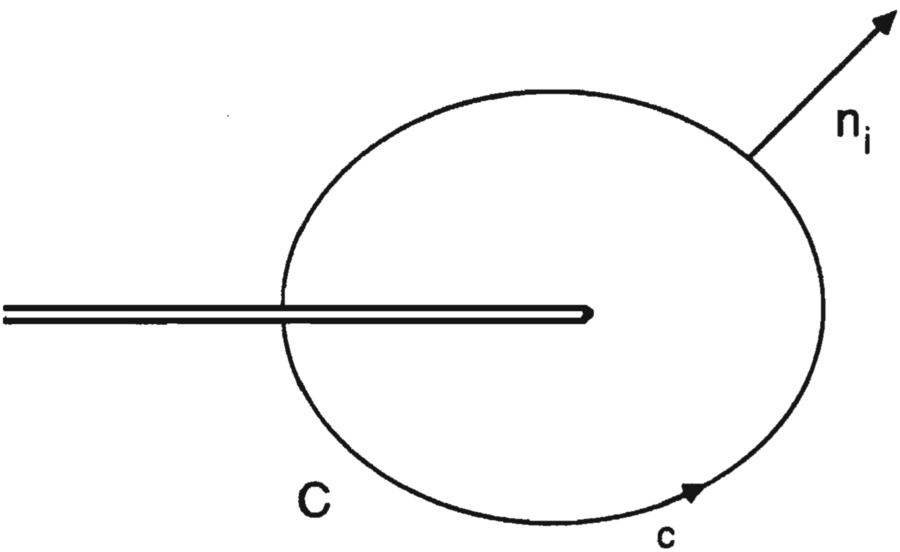

Figure 4: Integration contour for γ

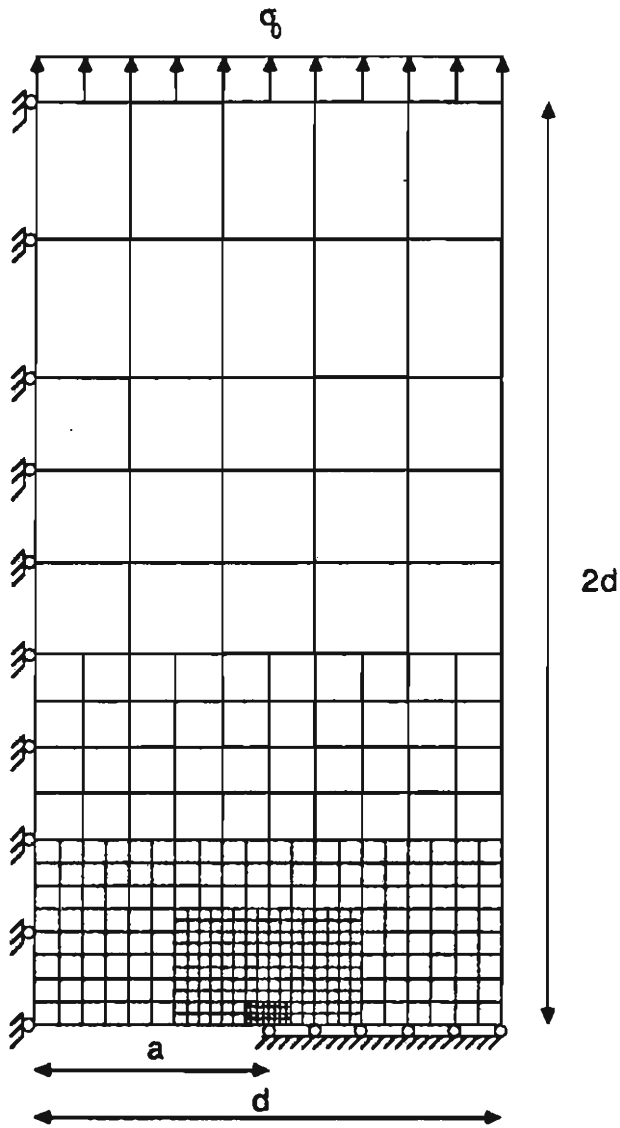

Figure 5: Finite element mesh for example problems

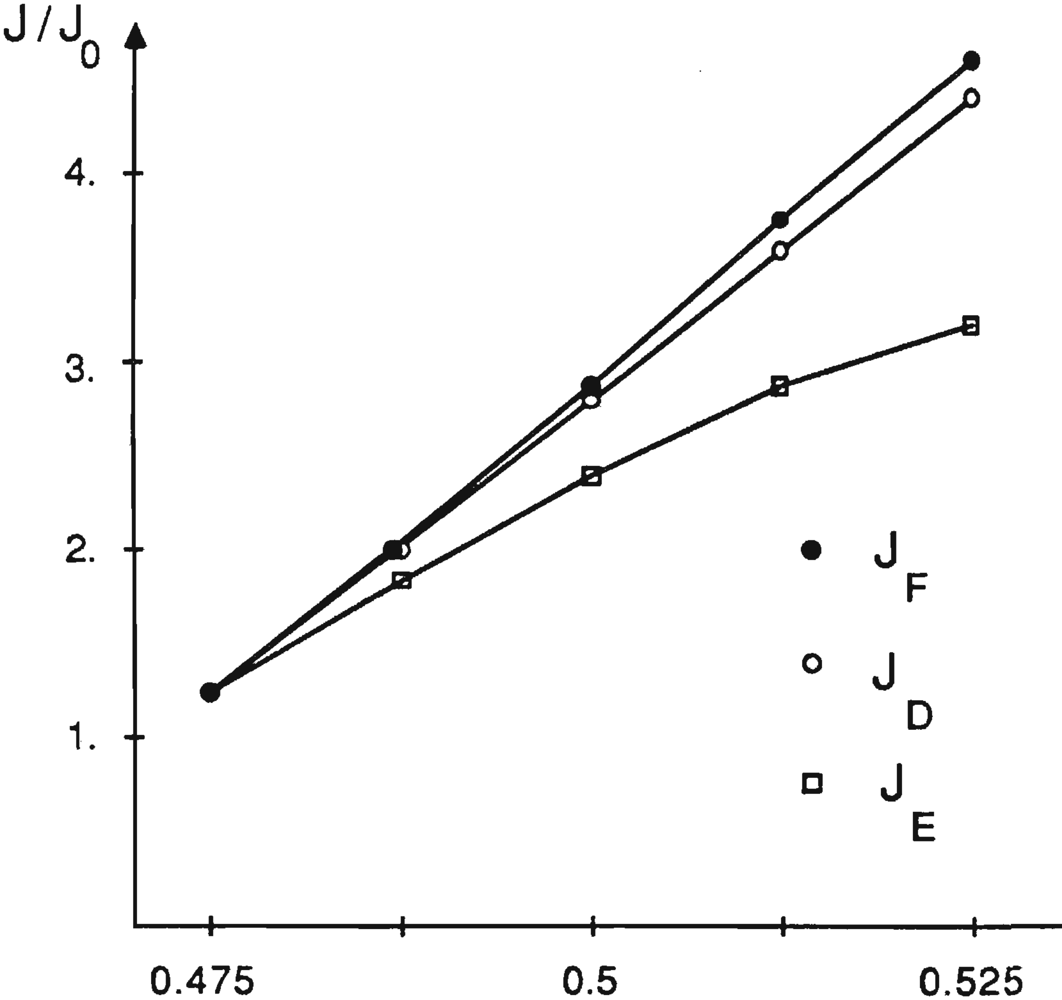

Figure 6: Nondimensional J as function of non-dimensional crack length