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Fundamental bounds on MIMO antennas

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Abstract

Antenna current optimization is often used to analyze the optimal performance of antennas. Antenna performance can be quantified in *e.g.*, minimum Q-factor and radiation efficiency. The performance of MIMO antennas is more involved and, in general, a single parameter is not sufficient to quantify it. Here, the capacity of an idealized channel is used as the main performance quantity. An optimization problem in the current distribution for optimal capacity, measured in spectral efficiency, given a fixed Q-factor and radiation efficiency is formulated as a semi-definite optimization problem. A model order reduction based on characteristic and energy modes is employed to improve the computational efficiency. The performance bound is illustrated by solving the optimization problem numerically for rectangular plates and spherical shells.

1 Introduction

Wireless communication in modern systems utilize multiple input multiple output (MIMO) networks and antennas [21, 24]. These systems consist of two sets of antennas, one transmitting, and one receiving. Normally, one of these sets is situated in a location where space allocation is not an issue, such as a base station. However, the other set is usually contained within a small device, such as a mobile phone, where design space is limited [28]. Naturally, antenna designs aim at maximizing performance in such an environment. However, there is little knowledge of how the performance depends on size, Q-factor, and radiation efficiency restrictions. Having this knowledge *a priori* would enable designers to optimize their antenna designs more efficiently. There has been efforts to bound MIMO antennas performance for spherical surfaces [8, 11] and through information-theoretical approaches [18, 20, 25]. However, in order to create tight bounds for a design region the bounding surface must be arbitrary.

Antenna current optimization can be used to determine physical bounds for antennas of arbitrary shape [15]. These physical bounds are found by maximizing a certain performance parameter by freely placing currents in the design space. By having total control of the current distribution an optimal solution can be reached. While these currents might not necessarily be realizable they provide an upper bound for the considered problem. Construction of such physical bounds are made possible by the ability to formulate convex optimization problems [1] for the performance quantity of interest. The performance of simple antennas can be quantified in *e.g.*, the Q-factor, gain, directivity, and efficiency [13]. MIMO antennas, on the other hand, are more complex and a single parameter is insufficient to determine their performance. The maximization of their main performance quantity, capacity, cannot be formulated as a classical quadratic programming problem. As such, it is a challenging problem to construct physical bounds for MIMO systems. However, it is still possible to utilize antenna current optimization and semi-definite programming [1] to maximize a given performance quantity, such as capacity, with restrictions on, *e.g.*, the Q-factor and radiation efficiency.

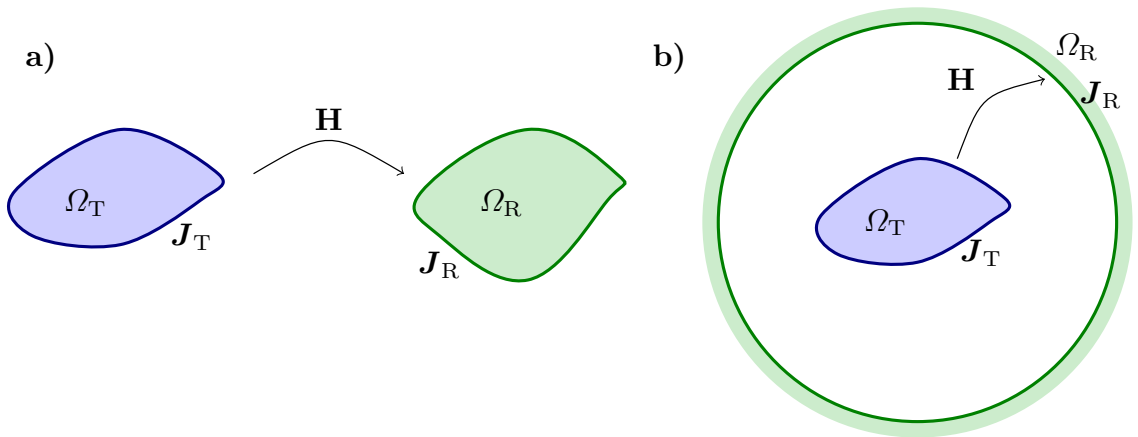


Figure 1: Illustration of the MIMO system model with transmitter region Ω_T and receiver region Ω_R . Part (a) shows the classical MIMO setup with spatially separated regions. Part (b) illustrates the idealized case when the receiver region entirely surrounds the transmitter. The system in (b) is utilized in this paper to determine performance bounds on MIMO antennas confined to the region Ω_T .

In this letter a method for constructing a performance bound on capacity for arbitrary shaped MIMO antennas using current optimization is presented. Transmitting at maximum capacity is formulated as a convex optimization problem in the current distribution on the MIMO antenna. The problem is constrained by the radiation efficiency and Q-factor. These are expressed as quadratic forms in the current density, where the stored energy in [26] is used. This leads to a convex optimization problem that maximizes the capacity in terms of spectral efficiency for a fixed signal-to-noise ratio (SNR), radiation efficiency and Q-factor. The convex optimization problem is a semi-definite program [1] expressed in the covariance matrix of the current distribution. In order to bound the performance of a MIMO antenna, rather than a specific channel, the receiver antenna set has been characterized as the spherical modes in the far-field. This leads to an idealized channel in terms of spherical modes [11], which can be thought of as a direct line of sight channel where all radiation is received. Considering such a channel has the benefit of reducing computational complexity. This is further reduced by a model order reduction of the method of moments (MoM) impedance matrix characterizing the antenna.

2 MIMO model

A classical MIMO system is modeled as [24]

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}, \quad (2.1)$$

where \mathbf{x} is a $N \times 1$ matrix of the input signals, \mathbf{y} is a $M \times 1$ matrix of the output signals, \mathbf{n} is a $M \times 1$ matrix of additive noise, and \mathbf{H} is the $M \times N$ channel matrix. The channel matrix models how power is transmitted from the input signals to the

output signals, this includes the receiving and transmitting antennas and the wave propagation between them [24].

Fig. 1a displays a classical MIMO setup where two sets of antennas form a channel. Analysis of such systems depend greatly on external factors, such as, scattering phenomena, channel characterization, and antenna location [24]. However, to investigate performance bounds for MIMO antennas we must limit the degrees of freedom to a single antenna. This implies that \mathbf{H} in (2.1) should model the channel between an arbitrary antenna and an idealized receiver, corresponding to Fig. 1b. The transmitting antenna is modeled with its current distribution using a MoM approximation [13] such that each basis function corresponds to an element of \mathbf{x} . The receiver is modeled with the radiated spherical modes, where each mode is an element in \mathbf{y} [11, 14]. This leads to a MIMO system of infinite dimension as N increases with mesh refinement and M increases with the number of included spherical modes. In numerical evaluation N and M are chosen sufficiently large to ensure convergence.

The transmitted signals are modeled as the MoM current elements $\mathbf{I} = \mathbf{T}\mathbf{x}$, where the matrix \mathbf{T} maps the transmitted signals \mathbf{x} to the current distribution on the antenna \mathbf{I} . The covariance matrix of the transmitted signal is $\mathbf{P} = \frac{1}{2}\mathcal{E}\{\mathbf{x}\mathbf{x}^H\}$, where $\mathcal{E}\{\cdot\}$ denotes the temporal average [24]. With this matrix we can calculate the average transmitted power,

$$P = \frac{1}{2}\mathcal{E}\{\mathbf{I}^H\mathbf{R}\mathbf{I}\} = \frac{1}{2}\mathcal{E}\{\mathbf{x}^H\mathbf{T}^H\mathbf{R}\mathbf{T}\mathbf{x}\} = \frac{1}{2}\text{Tr}\mathcal{E}\{\mathbf{T}^H\mathbf{R}\mathbf{T}\mathbf{x}\mathbf{x}^H\} = \text{Tr}(\widehat{\mathbf{R}}\mathbf{P}), \quad (2.2)$$

where $\widehat{\mathbf{R}} = \mathbf{T}^H\mathbf{R}\mathbf{T}$, and \mathbf{R} is the resistive part of the MoM impedance matrix, $\mathbf{Z} = \mathbf{R} + j\mathbf{X}$ [13]. Since we are concerned with connecting the currents on the antenna structure to the spherical modes [12] in the idealized receiver we express our channel as

$$\mathbf{y} = \mathbf{M}\mathbf{I} + \mathbf{n} = \mathbf{M}\mathbf{T}\mathbf{x} + \mathbf{n} = \widehat{\mathbf{M}}\mathbf{x} + \mathbf{n}, \quad (2.3)$$

where \mathbf{M} denotes the map from the currents to the spherical modes. This is a direct channel between the antenna current distribution and the spherical modes [10]. The capacity, expressed as spectral efficiency (b/(s Hz)), of this channel is given by [24]

$$C = \max_{\text{Tr}(\widehat{\mathbf{R}}\mathbf{P})=P} \log_2 \det \left(\mathbf{1} + \frac{1}{N_0}\widehat{\mathbf{M}}\mathbf{P}\widehat{\mathbf{M}}^H \right), \quad (2.4)$$

where $\mathbf{1}$ is the $M \times M$ identity matrix, and N_0 is the noise power. The noise is modeled as white complex Gaussian noise. The optimal energy allocation in this channel for capacity maximization is given by the water-filling solution [24]. Alternatively, the optimal solution for this problem can be solved by a semidefinite optimization program,

$$\begin{aligned} & \text{maximize} && \log_2 \det(\mathbf{1} + \gamma\widehat{\mathbf{M}}\mathbf{P}\widehat{\mathbf{M}}^H) \\ & \text{subject to} && \text{Tr}(\widehat{\mathbf{R}}\mathbf{P}) = 1 \\ & && \mathbf{P} \succeq 0, \end{aligned} \quad (2.5)$$

where the unit transmitted power is considered, and $\gamma = P/N_0$ is the total SNR. Maximizing the capacity of this channel corresponds to focusing the radiation of the antenna to the orthogonal spherical modes.

The solution to (2.5) is unbounded and increases as mesh refinement and the number of spherical modes are increased if the SNR is scaled with the number of channels in $\widehat{\mathbf{M}}$ [24]. Here, we consider the case of a fixed SNR where the solution only depends on the SNR [5, 6]. The solution to (2.5) can be made more realistic by adding constraints on the losses or Q-factor of the transmitting antenna [11, 22]. The Ohmic losses are calculated as

$$P_\Omega = \frac{1}{2} \mathcal{E} \{ \mathbf{I}^H \mathbf{R}_\Omega \mathbf{I} \} = \frac{1}{2} \mathcal{E} \{ \mathbf{x}^H \mathbf{T}^H \mathbf{R}_\Omega \mathbf{T} \mathbf{x} \} = \text{Tr}(\widehat{\mathbf{R}}_\Omega \mathbf{P}), \quad (2.6)$$

where $\widehat{\mathbf{R}}_\Omega = \mathbf{T}^{-H} \mathbf{R}_\Omega \mathbf{T}$, and \mathbf{R}_Ω is the loss matrix of the antenna [13]. The stored electric energy is

$$W_e = \frac{1}{4\omega} \mathcal{E} \{ \mathbf{I}^H \mathbf{X}_e \mathbf{I} \} = \frac{1}{4\omega} \mathcal{E} \{ \mathbf{x}^H \mathbf{T}^H \mathbf{X}_e \mathbf{T} \mathbf{x} \} = \frac{1}{2\omega} \text{Tr}(\widehat{\mathbf{X}}_e \mathbf{P}), \quad (2.7)$$

where $\widehat{\mathbf{X}}_e = \mathbf{T}^H \mathbf{X}_e \mathbf{T}$, and \mathbf{X}_e is the electric reactance matrix [13]. The stored magnetic energy W_m is similarly defined by the magnetic reactance matrix \mathbf{X}_m as $W_m = \frac{1}{2\omega} \text{Tr}(\widehat{\mathbf{X}}_m \mathbf{P})$, where $\widehat{\mathbf{X}}_m = \mathbf{T}^H \mathbf{X}_m \mathbf{T}$.

With these constraints in hand we can formulate our optimization problem. We note that the solution is independent of the power P , so it is sufficient to consider the case $P = 1$ giving

$$\begin{aligned} & \text{maximize} && \log_2 \det(\mathbf{1} + \gamma \widehat{\mathbf{M}} \mathbf{P} \widehat{\mathbf{M}}^H) \\ & \text{subject to} && \text{Tr}((\widehat{\mathbf{X}}_e + \widehat{\mathbf{X}}_m) \mathbf{P}) \leq 2Q \\ & && \text{Tr}(\widehat{\mathbf{X}} \mathbf{P}) = 0 \\ & && \text{Tr}(\widehat{\mathbf{R}}_\Omega \mathbf{P}) \leq 1 - \eta \\ & && \text{Tr}(\widehat{\mathbf{R}} \mathbf{P}) = 1 \\ & && \mathbf{P} \succeq \mathbf{0}, \end{aligned} \quad (2.8)$$

where η is the radiation efficiency, and self-resonance is enforced. Here, the problem has been normalized to dissipated power, including losses. The consequence of this is that the Q-factor considered includes losses in its calculation. It is possible, and sometimes advantageous, to normalize to different quantities such as the radiated power. Equation (2.8) is a semi-definite optimization problem which has a unique solution [1]. However, the problem is non-trivial due to the large number of unknowns for realistic antenna problems. For example a rectangular plate of size $\ell \times \ell/2$ discretized into 64×32 rectangular elements has $N = 4000$ unknowns. This size is not a problem for convex optimization of type G/Q and Q [2, 12, 13]. However, the semi-definite relaxation has close to $N^2/2 = 8 \cdot 10^6$ unknowns, making the problem much more computationally demanding. Moreover, the logarithm used in the definition of capacity is more involved than the simple quadratic functions in G/Q and Q type problems [12, 13]. Here, the number of unknowns is reduced by

expansion of the currents in characteristic, energy, and efficiency modes [13], with similar results.

The expansion includes only the dominating modes and as such constitutes a model order reduction. This implies a change of basis $\mathbf{I} \approx \mathbf{U}\tilde{\mathbf{I}}$, where \mathbf{U} maps between the old and the new currents. This reduces the number of unknowns to the included modes $N_1 \ll N$. With this approximation the stored energy, for example, is calculated as

$$\mathbf{I}^H \widehat{\mathbf{X}}_e \mathbf{I} \approx \tilde{\mathbf{I}}^H \mathbf{U}^T \widehat{\mathbf{X}}_e \mathbf{U} \tilde{\mathbf{I}} = \tilde{\mathbf{I}}^H \widetilde{\mathbf{X}}_e \tilde{\mathbf{I}} = \text{Tr}(\widetilde{\mathbf{X}}_e \tilde{\mathbf{I}} \tilde{\mathbf{I}}^H) = \text{Tr}(\widetilde{\mathbf{X}}_e \widetilde{\mathbf{Y}}), \quad (2.9)$$

where $\widetilde{\mathbf{Y}} = \tilde{\mathbf{I}} \tilde{\mathbf{I}}^H$, and $\widetilde{\mathbf{X}}_e = \mathbf{U}^T \widehat{\mathbf{X}}_e \mathbf{U}$. Similarly $\widehat{\mathbf{X}}_m$, and $\widehat{\mathbf{R}}$, are expressed as $\widetilde{\mathbf{X}}_m = \mathbf{U}^T \widehat{\mathbf{X}}_m \mathbf{U}$, and $\widetilde{\mathbf{R}} = \mathbf{U}^T \widehat{\mathbf{R}} \mathbf{U}$. These replace the corresponding matrices in (2.8), with $\widetilde{\mathbf{Y}}$ replacing \mathbf{P} . This reduces the number of unknowns from approximately $N^2/2$ to $N_1^2/2$.

3 Numerical examples

In the following examples the optimization problem (2.8) has been solved for a MIMO system resembling Fig. 1b using the Matlab library CVX [9, 13]. The logarithm in the optimization problem (2.8) was replaced by a root of order M [9]. After the optimization has been carried out the capacity is calculated as normal with the optimized currents. The energy restriction on the number of transmitter modes and the number of spherical harmonic modes in the receiver have been chosen sufficiently large to ensure convergence and varies from example to example. Using too many modes may also result in the solver failing to solve the problem due to its size and must therefore be regulated for each run individually. Since the performance of a MIMO antenna cannot be quantified by a single parameter the optimization was run with different constraints. This illustrates how capacity is bounded by different requirements on the transmitting antennas. The optimization has also been run for a spherical shell circumscribing the antenna.

By performing a singular value decomposition of the channel matrix $\widehat{\mathbf{M}}$ we can see how many channels dominate the information transfer between the plate and the spherical modes, see Fig. 2. Here, we see that there are only a few channels that dominate the rest. This indicates that so long as our model order reduction preserves these channels it produces correct solutions. The channels between the spherical shell and the spherical modes can be seen in Fig. 3. In this configuration there are more good channels, indicating that more channels need to be taken into account in the model order reduction.

In Fig. 4 the capacity has been optimized for a plate of electrical size $\ell = 0.21\lambda$, and is depicted as a function of the Q-factor restriction. We see a cut-off for $Q \leq 12$ where the optimization problem is unable to realize a feasible current distribution for so low Q-factor, *cf.*, the lower bound on the Q-factor [2]. For higher SNR the capacity increases but the cut-off stays the same, since the SNR does not affect the Q-factor.

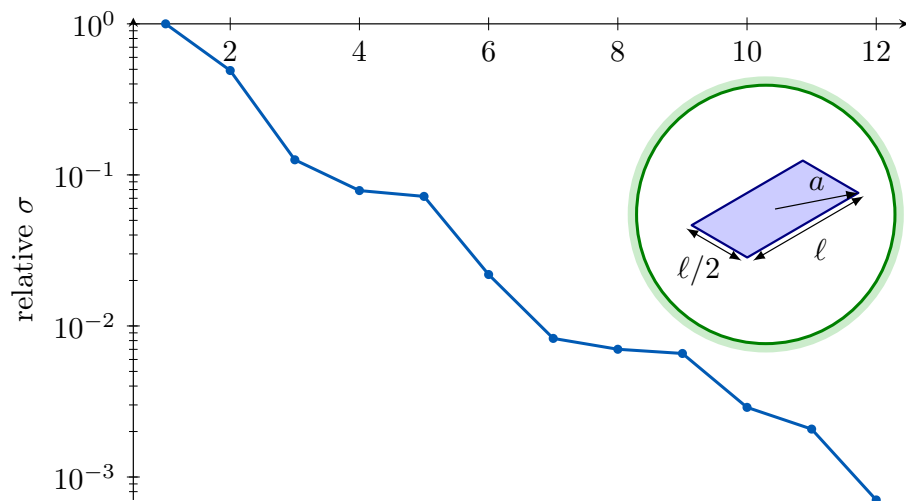


Figure 2: The singular values of the channel matrix $\widehat{\mathbf{M}}$ for a rectangular plate $l \times l/2$ for the wavelength $\ell = 0.21\lambda$.

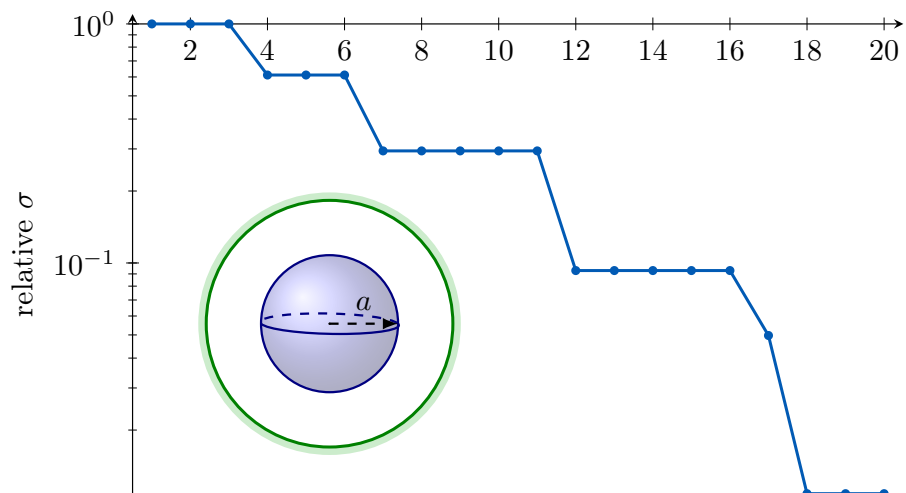


Figure 3: The singular values of the channel matrix $\widehat{\mathbf{M}}$ for a spherical shell $r = a$, where $a = 0.56\ell$, for the wavelength $\ell = 0.21\lambda$.

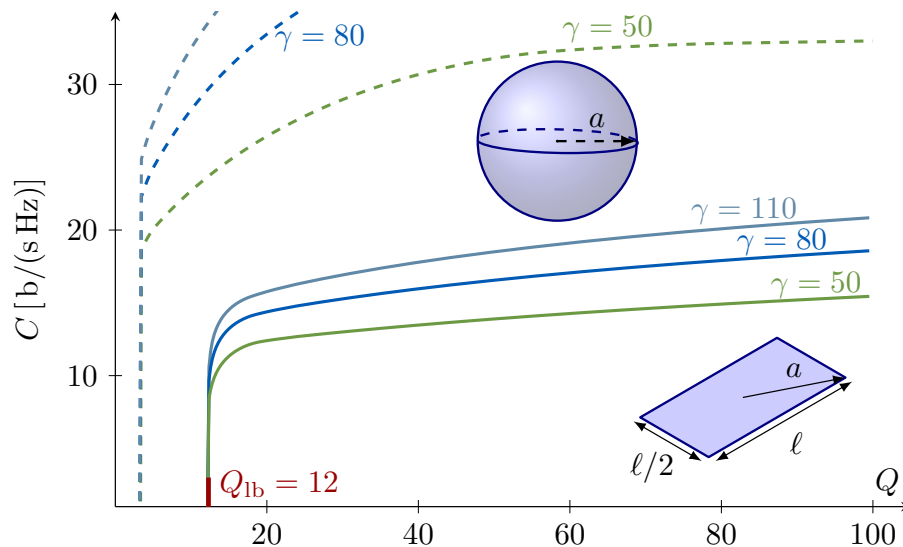


Figure 4: Maximum spectral efficiency achievable for a loss-less rectangular plate of size $\ell \times \ell/2$ for the wavelength $\ell = 0.21\lambda$ given maximum Q-factor on the horizontal axis. The dashed lines show the maximum spectral efficiency achievable for the corresponding circumscribing spherical shell.

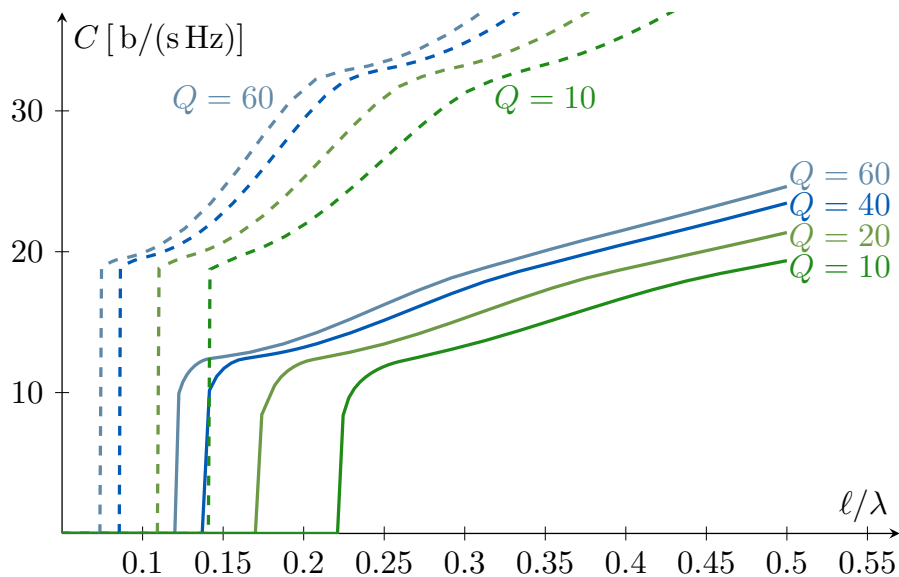


Figure 5: Maximum spectral efficiency achievable for a loss-less rectangular plate of electrical size ℓ/λ for maximum Q-factor with SNR $\gamma = 50$, *cf.*, Fig. 4. The dashed lines show the maximum spectral efficiency achievable for the corresponding circumscribing spherical shell.

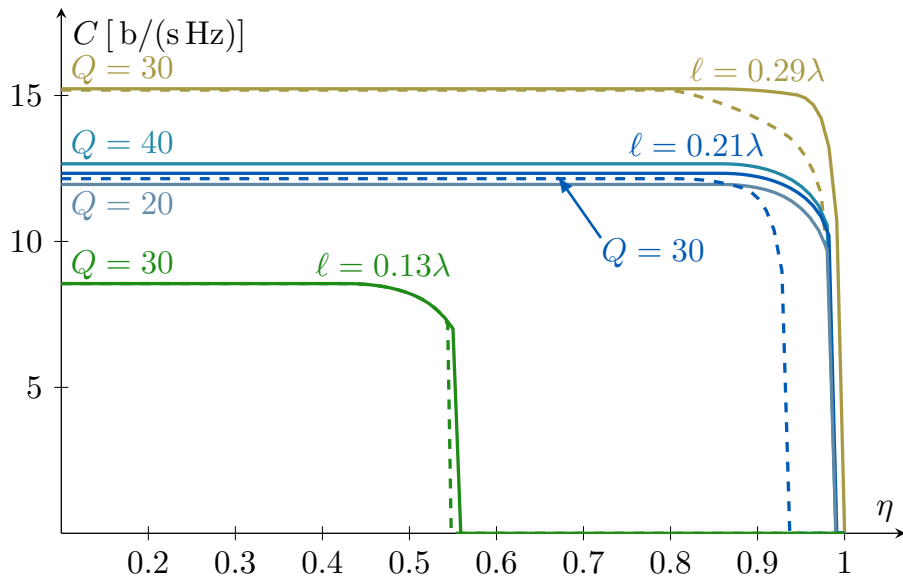


Figure 6: Maximum spectral efficiency achievable for a rectangular plate of electrical size ℓ/λ for minimum radiation efficiency η . The losses are modeled as a resistive sheet with $R = 0.2 \Omega/\square$. The minimum Q-factor is set to 30 for the three main graphs and SNR $\gamma = 50$. Solid lines are optimized without enforcing resonance and dashed lines are optimized with resonance. For $\ell = 0.21\lambda$ the Q-factors [20, 30, 40] are plotted.

We can instead regard the problem with a fixed SNR and investigate how the capacity varies with antenna size, see Fig. 5. Depending on which Q is chosen the solution is only realizable for sizes above a certain cut-off. This cut-off corresponds to the size which has the chosen Q as its minimum achievable Q . Above this size the capacity seems to depend linearly on the antenna size. This is consistent with how capacity scales with the number of antennas included in a MIMO system [24].

In both Fig. 4 and 5 the dashed lines show the optimization problem solved for a spherical shell circumscribing the planar region. We see that the spectral efficiency achievable by a planar antenna is much less than that of the sphere.

Setting an radiation efficiency requirement on the optimization may restrict which modes are realizable. Fig. 6 illustrates how capacity varies as a function of radiation efficiency. We see that the capacity is unaffected until some cut-off value where the solution is no longer realizable. For electrical sizes $\ell = 0.21\lambda$ and 0.29λ this occurs when the radiation efficiency requirements is high, above 90%. However, for smaller sizes, such as $\ell = 0.13\lambda$, we see that this cut-off occurs at lower radiation efficiencies. The optimization problem has been solved both with and without enforcing resonance. When resonance is enforced, showed in dashed lines, we see that the cut-off occurs at lower radiation efficiencies, this is due to self-resonant currents being inherently less efficient [16]. For the size $\ell = 0.21\lambda$ the Q-factor requirement was varied as well, leading to a slight reduction or increase in capacity. Close to the cut-off efficiency we see a slight decrease in capacity for all cases. This corresponds to the requirement on radiation efficiency limiting the optimization problem. For

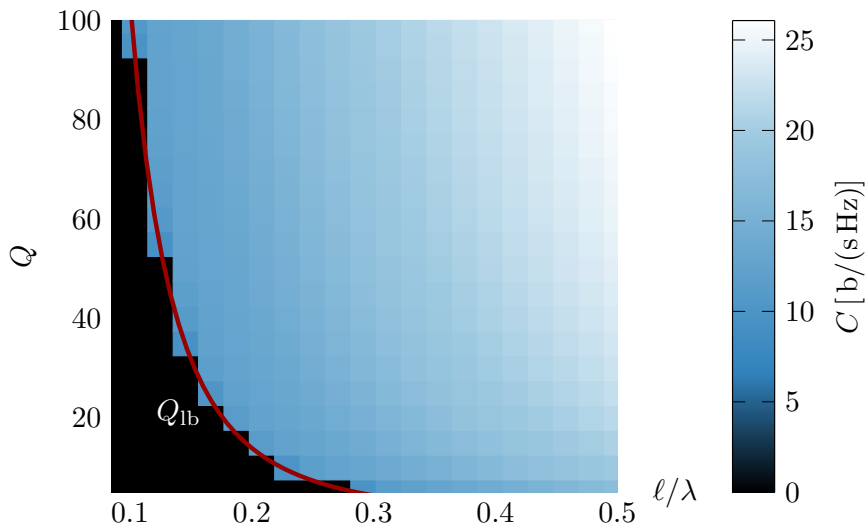


Figure 7: Illustration of the bounding surface of spectral efficiency for a loss less rectangular plate as a function of size and Q-factor with SNR $\gamma = 50$. The red curve shows minimum Q [2].

lower radiation efficiency requirements other constraints limit the optimization and the capacity is unaffected by the bound on radiation efficiency.

In Fig. 7 both the size of the antenna and the Q-factor are varied to create a two dimensional bounding surface. This surface has a sharp cut-off along the minimum Q line [2] seen on the left in Fig. 7. We see that the increase in capacity follows the shape of the minimum Q curve as l/λ and Q are increased. This surface provides a bound on the capacity achievable for MIMO antennas of different sizes and with different bandwidth requirements.

4 Conclusions

In this letter we have presented a framework for constructing performance bounds for MIMO antennas. We simplified the channel problem often considered in communication theory to an idealized channel consisting of a spherical receiver surrounding the antenna region. This enables the formulation of a semi-definite optimization problem that gives a bounding capacity for any antenna that can be constructed within the considered region limited by size, SNR, radiation efficiency, and Q-factor. By utilizing a model order reduction based on energy and characteristic modes the complexity of the problem is reduced such that it is solvable.

These physical boundaries of MIMO antennas represent the ideal solutions possible given complete freedom of current placement within the design area. While the shape of these current distributions are not easily realizable [16], the bounding values provide an upper limit to what is possible for real antenna topologies. Investigating how these bounds are affected by dielectric materials is very interesting and

can be done by including them in the MoM simulation. It remains as future work to explore how these bounds compare to antenna designs and measurements.

Acknowledgment

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Appendix A Antenna parameters

The impedance matrix $\mathbf{Z} = \mathbf{R} + j\mathbf{X}$ is determined from a MoM description of the antenna structure. The impedance matrix is divided into its resistance \mathbf{R} and reactance \mathbf{X} . Moreover, the reactance is decomposed into its magnetic and electric parts, *i.e.*, $\mathbf{Z} = \mathbf{R} + j(\mathbf{X}_m - \mathbf{X}_e)$, where the stored electric and magnetic energies are [3, 26]

$$W_m \approx \frac{1}{8} \mathbf{I}^H \left(\frac{\partial \mathbf{X}}{\partial \omega} + \frac{\mathbf{X}}{\omega} \right) \mathbf{I} = \frac{1}{4\omega} \mathbf{I}^H \mathbf{X}_m \mathbf{I}, \quad (\text{A.1a})$$

$$W_e \approx \frac{1}{8} \mathbf{I}^H \left(\frac{\partial \mathbf{X}}{\partial \omega} - \frac{\mathbf{X}}{\omega} \right) \mathbf{I} = \frac{1}{4\omega} \mathbf{I}^H \mathbf{X}_e \mathbf{I}, \quad (\text{A.1b})$$

respectively, and the dissipated power P_d is given by

$$P_d = \frac{1}{2} \mathbf{I}^H \mathbf{R} \mathbf{I}. \quad (\text{A.2})$$

The Q-factor is defined as the quotient between the time-average stored and dissipated energies [15, 23, 27]

$$Q = \frac{2\omega \max\{W_e, W_m\}}{P_d} = \frac{\max\{\mathbf{I}^H \mathbf{X}_e \mathbf{I}, \mathbf{I}^H \mathbf{X}_m \mathbf{I}\}}{\mathbf{I}^H \mathbf{R} \mathbf{I}}. \quad (\text{A.3})$$

Appendix B Maximum efficiency

To motivate the cut-off values seen in Fig. 6 the maximum efficiency for given Q-factors was investigated. This was evaluated using two optimization problems, one to find the minimum efficiency for a set Q,

$$\begin{aligned} & \text{minimize} && \text{Re Tr}(\widehat{\mathbf{R}}_\Omega \mathbf{P}) \\ & \text{subject to} && \text{Tr}((\widehat{\mathbf{X}}_e + \widehat{\mathbf{X}}_m) \mathbf{P}) = 2Q \\ & && \text{Tr}(\widehat{\mathbf{X}} \mathbf{P}) = 0 \\ & && \text{Tr}(\widehat{\mathbf{R}} \mathbf{P}) = 1 \\ & && \mathbf{P} \succeq \mathbf{0}, \end{aligned} \quad (\text{B.1})$$

and one to find the minimum Q-factor for a certain efficiency,

$$\begin{aligned}
& \text{minimize} && \text{Re Tr}((\widehat{\mathbf{X}}_e + \widehat{\mathbf{X}}_m)\mathbf{P}) \\
& \text{subject to} && \text{Tr}((\widehat{\mathbf{X}}_m - \widehat{\mathbf{X}}_e)\mathbf{P}) = 0 \\
& && \text{Tr}(\widehat{\mathbf{R}}\mathbf{P}) = 1 \\
& && \text{Tr}(\widehat{\mathbf{R}}_\Omega\mathbf{P}) = 1 - \eta \\
& && \mathbf{P} \succeq \mathbf{0}.
\end{aligned} \tag{B.2}$$

These problems can be reformulated so that resonance is not enforced, this results in a higher efficiency limit. The optimization problems (B.1) and (B.2) for the efficiency also arise from semi-definite relaxation [1] of the corresponding problems formulated in the current **I**. Semi-definite relaxation is a technique to solve quadratically constrained quadratic programs (QCQP) and can be applied to many antenna problems [7, 17, 19].

Appendix C Additional shapes

The antenna structure which is bounded can take any shape. In Fig. 8 the maximum spectral efficiency of a circular and triangular plate is presented. The triangular plate is bounded by the rectangular plate presented in the paper. Therefore, the maximum spectral efficiency of the triangular plate is slightly lower than that of the rectangular plate. The circular plate in Fig. 8 circumscribes the rectangular plate and its maximum spectral efficiency is thus higher than that of the plate.

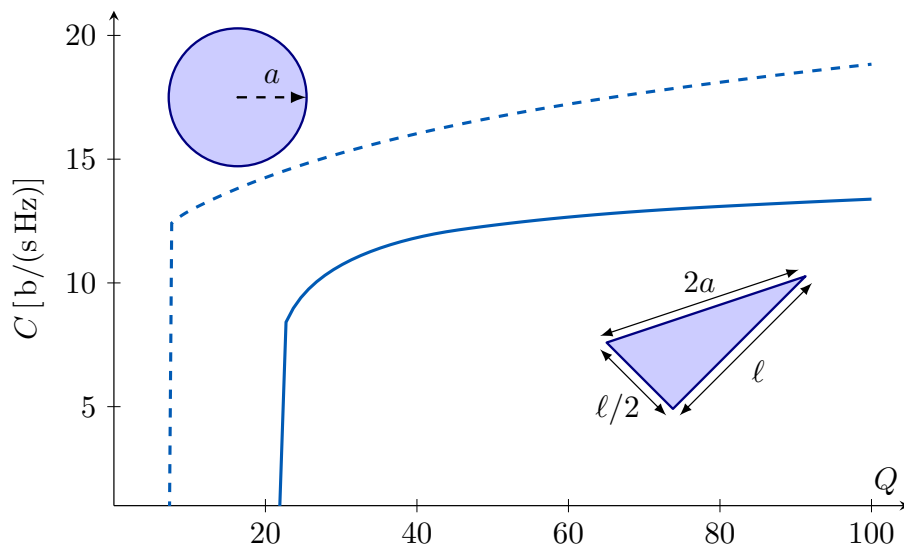


Figure 8: Maximum spectral efficiency achievable for a loss-less triangular and circular plate with $\ell = 0.21\lambda$ given a maximum Q-factor, and SNR $\gamma = 50$.

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