Superconductivity in the presence of magnetic steps

Assaad, Wafaa

2019

Document Version:
Publisher's PDF, also known as Version of record

Citation for published version (APA):
Assaad, W. (2019). *Superconductivity in the presence of magnetic steps*. [Doctoral Thesis (compilation), Faculty of Engineering, LTH]. Mathematics Centre for Mathematical Sciences Lund University Lund.

Total number of authors:
1

Creative Commons License:
Unspecified

Superconductivity in the presence of magnetic steps

Superconductivity in the presence of magnetic steps

by Wafaa Assaad

LUND
UNIVERSITY

DOCTORAL THESIS

which, by due permission of the Faculty of Engineering at Lund University, will be publicly defended on Friday 20th of September, 2019, at 13:15 in the Hörmander lecture hall, Sölvegatan 18A, Lund, for the degree of Doctor of Philosophy in Mathematics.

Faculty opponent
Associate Professor
Virginie Bonnaillie-Noël,
Research Director at CNRS, DMA, ENS Paris.

Organization: LUND UNIVERSITY	Document name: DOCTORAL THESIS
Department of Mathematics Box 118 SE–221 00 LUND Sweden	Date of issue: 2019-09-20
	Sponsoring organization:
Author(s): Wafaa Assaad	

Title: Superconductivity in the presence of magnetic steps

Abstract:
This thesis investigates the distribution of superconductivity in a Type-II planar, bounded, and smooth superconductor submitted to a piecewise-constant magnetic field with a jump discontinuity along smooth curves–the magnetic edge. This discontinuous case has not been treated before in the mathematics literature, where the considered applied magnetic field is usually assumed to be smooth. We examine the behavior of the sample in different regimes of the intensity of the applied magnetic field. When the magnetic field is relatively weak, we prove that superconductivity exists all over the sample. Increasing the magnetic field's intensity to higher levels, superconductivity is shown to vanish in the interior of the sample away from the magnetic edge, and can nucleate near this edge as well as near the boundary. Such a nucleation may not be uniform. Under stronger magnetic fields, superconductivity is confined to the vicinity of the intersection of the magnetic edge with the boundary, when such an intersection exists, before being completely destroyed at a certain stage of the field's intensity. The results show a behaviour of the sample that, according to the intensity-regime, may differ from or resemble to that in the case of smooth/corner domains submitted to uniform magnetic fields. This highlights the particularity of our discontinuous case.
The study is modeled by the Ginzburg–Landau (GL) theory, and the obtained results are valid for the minimizers of the two-dimensional GL functional with a large GL parameter and with a field's intensity comparable to this parameter.

Keywords: Ginzburg–Landau theory, Schrödinger operators with magnetic fields, Superconductivity, functional models, spectral theory, PDE, quantum mechanics

Classification system and/or index termes (if any):

Supplementary bibliographical information:	Language(s): English
ISSN and key title: 1404-0034	ISBN: 978-91-7895-149-9(print) 978-91-7895-150-5(pdf)
Recipient's notes:	Number of pages: xlviii+193	Price:
	Security classification:

Distribution by (name and adress): Department of Mathematics, adress as above.

I, the undersigned, being the copyright owner of the abstract of the above-mentioned thesis, hereby grant to all reference sources permission to publish and disseminate the abstract of the above-mentioned thesis.

Signature_____ Date_____2019-09-20_____

Superconductivity in the presence of magnetic steps

by Wafaa Assaad

LUND
UNIVERSITY

Cover illustration: The figure illustrates the Meissner effect; a superconducting material that reaches its critical temperature becomes a perfect diamagnet and excludes any external magnetic flux. The magnet levitates above the superconductor (©J. Bobroff, F. Bouquet and J. Quilliam/CC BY-SA 3.0).

Acknowledgments

"Check if the bookshelves are steady, I don't want to find you buried under your books one day. […] I started to be jealous of your PC. […] Mom, you should use your calculator so you can finish your thesis earlier.[…] Are you doing maths? where are the numbers?"

– worried family members

I had a dream. A dream that began since I got my Bachelor degree nineteen years ago. I taught in several schools, lived in many countries, raised two beautiful children and yet this dream kept urging me to fulfil it. Now that I am writing the acknowledgements section in my PhD thesis, this dream has finally come true!

Mikael Sundqvist and Ayman Kachmar, soon we will close a chapter of the book that we are writing together. I was blessed to have you both as supervisors and friends. Mikael, your knowledge, thoroughness and confidence when dealing with complex problems gave me scientific maturity and tranquillity. Ayman, your dedication, enthusiasm and efficiency in solving problems are things that I will always admire. Thank you for believing in me, your support helped me reach higher levels in my academic career.

Jacob Stordal Christiansen, apart from your insightful comments which always led to a qualitative improvement of my scientific writing, I appreciate your sharp thinking, sense of humour, and attentive listening during our discussions.

Søren Fournais, thank you for your kindness, care and very useful suggestions during our meetings at Lund university.

My professors and teachers Anders Holst, Erik Wahlén, Eskil Hansen, Frank Wikström, Niels Overgaard, Pelle Pettersson, Philipp Birken, Sandra Pott, Sara

Maad, Sigmundur Gudmundsson, Stefan Diehl, Tomas Persson. I learned a lot from you, and you were so kind to me which makes it hard to leave you now! Claus Führer, also known as my academic guardian father, I consider myself blessed to have known such an exceptional human being.

Eva Lena, your friendship made me feel home in Sweden. Thank you for being strict with me about communicating in Swedish, I will never forget the beautiful moments we shared. I have no doubt that our little palm tree is in great hands!

My dear friends at Lund university Alexia, Anna, Erik, Eskil, Gabriel, James, Julio, Lea, Lena, Linn, Maria, Martin, Mats, Patricia, Peter, Tien, I am taking back home each memory, thought, craziness, laugh, tear, fear, hope, dream we shared. My friends in Women In Mathematics organization, I will always hold the concepts that we all believe in, and will work hard to implement them wherever my path takes me. My friends in Lebanon Rajaa Ibrahim, Ali Assaf, Mahmoud Salami, Nadia Ashab, Dalal Farran, my colleagues and students thank you for your continuous encouragement and help. Dalal Abbass and Ibrahim Mostafa, the seed of love for science you planted in me since I was a child has finally flourished.

بَابَا، مَامَا، عليّ، حسِين، أنتُم عزوَتِي وعيني السَاهِرة. مَا زَرَعتهُ كنتُم أنتُم الرَّاعي والسَّاقي، حَتَّى إذَا أَينَعَ تَخَّيمُ جَانبًا مُكَافئين أَنفُسكُم برُؤيَتِي سَعيدَةً سَاعَة القطَاف. هَذَا العَمَلُ كُلٌّ منكُم سَاهَمَ بقُوَّةٍ (وعَلَى طَريقَتِه) في إتمَامه. هَذَا العَمَلُ هُوَ بَينَ أَيديكُمَ اليَوم، أُهديه إلَيكُم مَع فَائِتِي مَحبَّتي.

My soulmate Raafat, we did it! Now tell me, how can I compensate your patience and your uncountable travels to Sweden to fuel me with strength and love? Would a 2019 Volvo XC90 T8 R-Design be a good start :)?

Nourhane and Ali, my brave kids, the day I got my PhD position I was terribly worried about the consequences of uprooting you from your cosy world to bring you with me to Sweden. That day, the wise words coming from your little mouths enabled me to go that route with no hesitation. While living in Sweden, the three of us have explored together this new world with its ups and downs. There were times when I felt that finishing my thesis and taking care of you were mutually exclusive goals, but I think your presence by my side made these goals possible. You are my rocks!

Popular Summary

A superconductor is a special material that, when cooled below some critical temperature, behaves differently than a conventional conductor. One says that the material passes from a *normal* (ordinary) state to a *superconducting* state. The particular behavior of a superconductor yields astonishing electrical and magnetic properties, which have attracted the interest of physicists and mathematicians for decades, and deserved many Nobel prizes in Science. Moreover, research in superconductivity area has important applications in cutting-edge domains such as medicine (NMR and MRI), transportation (maglev trains) and quantum computers. However, the high cost of implementation due to the very low critical temperatures of the so far known superconductors is still limiting their usage. Therefore, an ultimate goal of scientists is to discover new superconductors with higher critical temperature (for instance room-temperature superconductors). If such a goal is reached, this would be one of the biggest technological revolution of the era.

An amazing electrical property of a superconductor is the total loss of its electrical resistance, once dropping below its critical temperature. In this case, an applied electric current can circulate almost forever in the superconductor without any loss of energy (loss in the form of heat). Another striking magnetic property of a superconductor is the expulsion of exterior magnetic fields; a cooled superconductor creates a shield forbidding an applied magnetic field to penetrate through it (Figure 1). However, if the applied magnetic field is sufficiently strong then it will be able to break this defence-shield and invade the material, forcing the invaded region to transition into a normal state again. This passage to the normal state can be partial (in certain parts of the material) or global (in the whole material), according to the type of the superconductor (Type I vs Type II) and to the intensity H of the applied field. For instance, if we submit an extreme Type II superconductor to a *constant* magnetic field while continuously increasing its intensity, we then observe different *superconductivity* states (Figure 2):
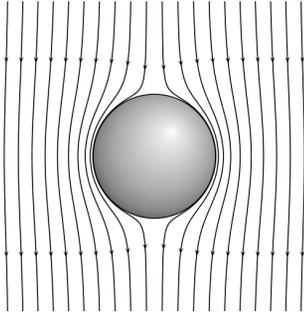
Figure 1: Cooled below its critical temperature, a superconductor expels a weak exterior magnetic field. The arrows represent the magnetic field lines. (©Geek3/CC BY 3.0)
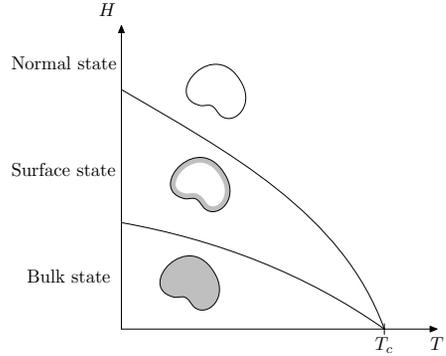
Figure 2: Schematic representation of the different superconductivity states of a generic extreme Type II superconductor submitted to a constant magnetic field. The three surfaces represent a 2-dimensional cross-section of a long smooth wire subjected to increasing intensity values, H, of the magnetic field. The superconductor is below its critical temperature T_c. The grey regions carry superconductivity.

- The bulk (interior) superconductivity state: The whole material is uniformly superconducting.

- The surface (boundary) superconductivity state: Superconductivity disappears from the interior, but is still uniformly distributed along the boundary.

- The normal state: Superconductivity is destroyed in the whole material.

In the description above, we assume that the superconducting sample is a two-dimensional cross-section of a long smooth wire. The aforementioned behavior of the superconductor, in presence of a constant magnetic field, has been intensively explored in the literature. In addition, many publications have addressed the superconductor performance when submitted to a smooth but not necessarily uniform magnetic field. The contribution of this thesis lies in considering a new situation where the applied magnetic field exhibits *discontinuity* jumps along certain curves of the sample–the magnetic edge (Figure 4). In our study, we continuously increase the intensity of the magnetic field and record the superconductivity distribution and strength along the sample. Compared to the constant field case, new superconductivity states appear in the sample, where bulk superconductivity exclusively exists near the magnetic edge as opposed to the whole interior (Figures 4a
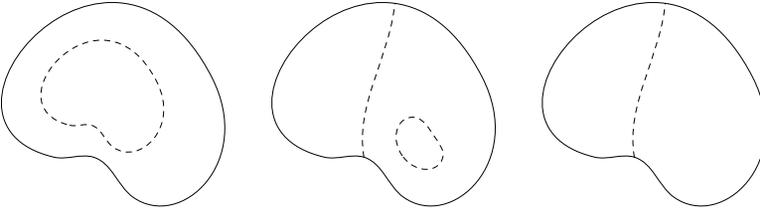
Figure 3: Possible states of a superconducting sample submitted to a discontinuous magnetic field. The dashed curves represent the magnetic edge.

    (a)   (b)   (c)   (d)   (e)
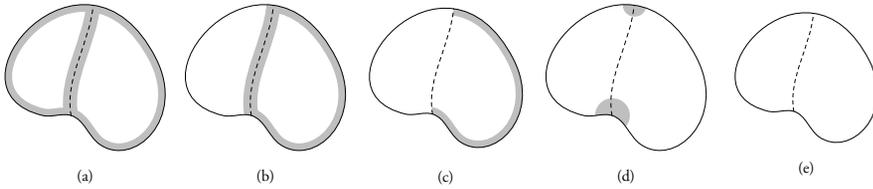
Figure 4: When an extreme Type II superconductor is subjected to our discontinuous magnetic field, new superconductivity states are observed at certain levels of the intensity of the magnetic field. The dark regions are superconducting while the white regions are in a normal state.

and 4b). Also, surface superconductivity can be localized along some parts of the boundary (Figures 4c and 4d). When the field's intensity is increased to higher levels, superconductivity completely disappears and the whole sample switches to the normal state (Figure 4e). This uniquely happens at a specific critical value of the intensity; we show that this normal state persists as long as the intensity is above this critical value. Right before permanently transitioning to the normal state, we prove that superconductivity nucleates near the intersection of the magnetic edge and the boundary (Figure 4d).

    Our theoretical study is modelled by the Ginzburg–Landau theory which is greatly recognized in both physics (quantum mechanics) and mathematics (partial differential equations). As a mathematician, I mainly focus on exploring this theory in the particular case of the discontinuous magnetic field. I am also interested in the potential real-world applications of our findings, especially in light of the recent experiments that made it possible to create such kind of discontinuous fields.

List of publications

[Paper I] W. Assaad and A. Kachmar, *The influence of magnetic steps on bulk superconductivity*, Discrete Contin. Dyn. Syst. Ser. A **36** (2016), 6623–6643.

[Paper II] W. Assaad, A. Kachmar, and M. Persson-Sundqvist, *The Distribution of Superconductivity Near a Magnetic Barrier*, Comm. Math. Phys. **366** (2019), no. 1, 269–332, doi: 10.1007/s00220-019-03284-z.

[Paper III] W. Assaad, *The breakdown of superconductivity in the presence of magnetic steps*, `arXiv:1903.04847` (submitted) (2019).

There exist minor changes between the version of Paper I which is included in this thesis and the published version. The purpose of these changes was to make the presentation consistent with that in Papers II and III. The two other papers are reproduced in their most recently published forms, as of the 20th of September, 2019, with reservations regarding corrected typos and editorial tweaks.

My contribution to Paper I: Sections 3–6.
My contribution to Paper II: Sections 1–3 and the appendices. Moreover, in collaboration with my co-authors, I carried out and verified the analysis presented in the rest of the paper.

Table of contents

Paper III: The breakdown of superconductivity in the presence of magnetic steps

Introduction

Introduction

The fascination of superconductivity is associated with the words perfect, infinite and zero.

<div align="right">

Brian Maple

</div>

1   Superconductivity

Superconductors radically differ from normal materials by the way the electrons or electric currents move through the material. This peculiar way creates unique electrical and magnetic properties of superconductors, distinguishing them from other traditional conductors, making superconductivity one of the biggest discoveries in the 20th century.

The history of superconductivity began in 1911 during experiments conducted by the Dutch physicist H. K. Onnes, three years after he had succeeded to liquefy helium gas. While using the liquid helium to cool the mercury metal to an extremely low temperature, he observed an unexpected behaviour of mercury: it was known that when metals are cooled, their electrical resistance continuously falls until it vanishes at 0 K. That year, Onnes' experiments put an end to this previously held knowledge when, cooled at 4 K, the electrical resistance of the mercury metal suddenly fell to zero. So, mercury became a perfect conductor and once an electric current was applied, this current remained almost forever (estimated decay time of 10^5 years).

It was later discovered that a large category of materials exhibit this electrical behaviour, namely, they admit characteristic critical temperatures under which they pass from the *normal state* to a *superconducting state* where the electrical resistance
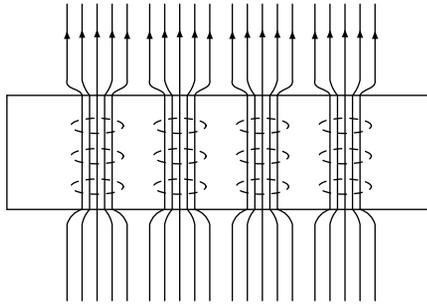
Figure 1: Schematic representation of a superconductor in its mixed state. The magnetic flux penetrates the sample through the vortices. The sample is in a normal state in the vortex core, and superconducting elsewhere.

vanishes. Consequently, any electric current circulating through the material is essentially permanent, and is referred to as 'supercurrent'. These electrical properties are the first hallmark of superconductivity.

The second hallmark is the striking magnetic behaviour of superconductors. Unlike the standard performance of materials, superconducting ones repel weak external magnetic fields (Meissner effect). But if the magnetic field is sufficiently strong, it penetrates the material switching it from the superconductivity state to the normal state. In 1957 and through a famous work [Abr57], A. Abrikosov introduced Type II superconductors which have a more surprising response to applied magnetic fields. Whereas Type I superconductors, the known superconductors before Abrikosov discovery, directly switch between a purely superconducting state and a normal state when the intensity of the applied magnetic field reaches a certain value–the critical field H_C, Type II superconductors undergo several phase-transitions while increasing the field's intensity. We present three main phase-transitions identified by two values of the field's intensity–the critical fields[1] H_{C_1} and H_{C_3}: when the field's intensity is below H_{C_1}, the material is in a perfect superconductivity state and Meissner effect is observed. Between H_{C_1} and H_{C_3}, a mixed state occurs where the applied field partially penetrates the material through vortices (see Figure 1). A. Abrikosov [Abr57] predicted that these vortices form triangular lattices. In 1967, Essman and Trauble [ET67] provided the first image of vortex lattice[2]. The

---

[1]We stick to the notation of the critical fields in the literature. A second critical, H_{C_2}, marking a particular phase-transition, will be introduced later in this introduction, when we become more specific in our presentation (see Section 4).

[2]For vortex lattice images, visit `https://bit.ly/2G6P2VS`
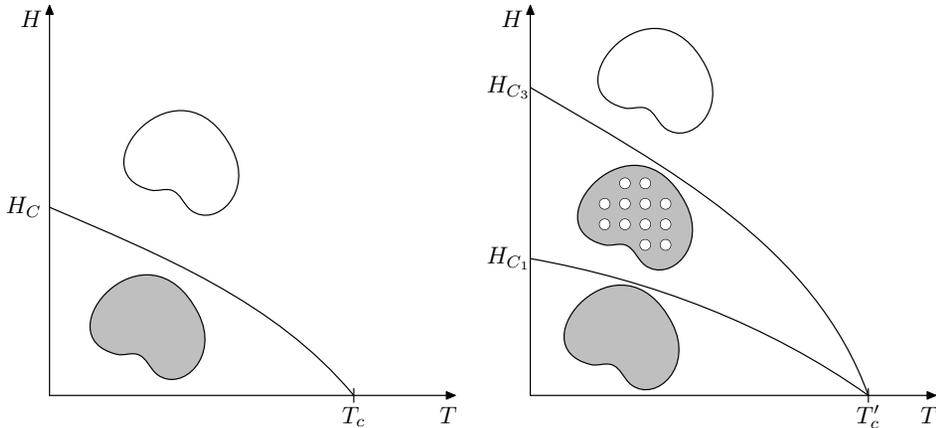
Figure 2: Schematic phase-diagrams illustrating the magnetic response of Type I (left) and Type II (right) superconductors to a constant applied magnetic field, according to the intensity H of the field and the temperature T of the superconductor. T_c and T_c' are the critical temperatures and H_C, H_{C_1} and H_{C_3} are the critical fields determining the different phase-transitions. The samples are assumed to be 2D cross-sections of generic superconducting materials. The grey regions of the superconductor carry superconductivity, while the white regions are in a normal state.

interior of each vortex is in a normal state. While increasing the intensity, the vortex density grows and superconductivity is finally destroyed at H_{C_3} ([SS07]). Above this intensity, the material stays in the normal state. This is illustrated in Figure 2. The above discussion on the phase-transitions is quite informal, and is done for generic superconductors submitted to *constant* magnetic fields. A more careful description of the magnetic behaviour of a Type II superconductor will be provided later in this introduction. One may also refer to the physics literature for more details about this phenomenon (e.g. [LG50, SJG63, SJST69, dG96, Tin96]).

Although Type II superconductors were considered as exotic at the time of their discovery, Abrikosov stated in his Nobel lecture (in 2003) that virtually all new superconducting compounds, discovered since early 1960s up to the time of his lecture, are Type II superconductors.

The electrical and magnetic behaviour of supvotexerconductors induces astonishing properties such as the extremely high current carrying density, the ultra high sensitivity to magnetic fields, the magnetic levitation and the close to speed of light signal transmission, which have widely opened the gate for a huge number of important applications in cutting-edge fields such as medicine (MRI, MEG, MCG, NMR...), transportation (Maglev train), and quantum computers.

However, the expensive cost of implementation and the low critical temperatures of superconductors are still limiting their usage, and physicists are continuously developing new record-high-temperature superconductors, with the ultimate goal of coming up with a room-temperature superconductor (to the best of our knowledge, the latest discovery[3] at the time of writing this thesis was the pressurized hydrogen sulfide [Car15] that reached a superconducting state at $-70°$C, with the caveat of smelling like rotten eggs!). If such ultimate goal is attained, many believe it will be one of the most staggering discovery in the recent history of mankind.

2 Ginzburg–Landau theory

Several physicists had tried to model the superconductivity phenomenon, like the brothers London [LL35], Ginzburg and Landau[GL65] then Bardeen, Cooper and Schrieffer[BCS57]. Our problem is modeled by the Ginzburg–Landau (GL) theory. Aside from being of great recognition in physics with hundreds of works, GL theory has become a large PDE research field with a big amount of contributions in the last decades. It is a macroscopic theory based on the consideration of a complex-valued function ψ—the order parameter–in determining the superconducting state of a material. In 1950, V. Ginzburg and L. Landau introduced this theory as a phenomenological model of superconductivity. Later, it was described as a limit of the Bardeen–Cooper–Schrieffer (BCS) microscopic theory, introduced in 1957, which relates the superconducting state to the existence of Cooper pairs of superconducting electrons. In 1957, Abrikosov used GL theory to explain certain experiments on superconducting alloys, and consequently to present Type-II superconductors.

In addition to their importance in modelling superconductivity phenomenons, Ginzburg–Landau techniques have also been successfully used in the analysis of the models of Bose–Einstein condensates [Aft06], and Gross–Pitaevskii model for superfluidity [TT90, Ser01]. It is not surprising that works related to this model have been awarded many Nobel prizes[4] (Landau 1962, Ginzburg 2003 and Abrikosov 2003).

Performing some reductions and normalisation [Tin96, SS07], the 2D GL

---

[3]In 2019, the U.S. Navy has filed for a patent, claiming building a room-temperature superconductor (`https://bit.ly/2UHjM7P`).

[4]Other Nobel laureates in superconductivity: Bardeen, Cooper, Schrieffer, Esaki, Giaever, Kapista, Wilson, Penzia, Bednorz, Müller [Vid93].

model can describe the state of a superconductor, below its critical temperature, through the following Gibbs energy:

$$\mathscr{E}_{\kappa,H}(\psi,\mathbf{A}) = \int_{\Omega} \left(\left| (\nabla - i\kappa H\mathbf{A})\psi \right|^2 - \kappa^2 |\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 \right) dx$$
$$+ \kappa^2 H^2 \int_{\Omega} \left| \operatorname{curl} \mathbf{A} - B_0 \right|^2 dx. \quad (1)$$

We call this energy the Ginzburg–Landau functional, and explain the different notations in what follows. Ω is an open set of $\mathbb{R}^2$, that we assume to be bounded, smooth and simply connected (unless stated otherwise). Physically, one can view Ω as the cross section of a long cylinder, or a limit domain of a thin film in $\mathbb{R}^3$.

The first variable $\psi \in H^1(\Omega; \mathbb{C})$ is called the order parameter; it reveals the local state of the material. The modulus $|\psi|$ represents the density of the Cooper pairs in the sample ($|\psi| \leq 1$). The sample is in a normal state where $\psi = 0$, and in a superconducting state elsewhere. Both states can coexist in the sample (mixed state). When $|\psi| \equiv 1$, we say that the sample is in a perfect superconducting state.

The second variable $\mathbf{A} \in H^1(\Omega; \mathbb{R}^2)$ is the vector potential of the induced magnetic field $\operatorname{curl} \mathbf{A} = \partial_{x_1} A_2 - \partial_{x_2} A_1$.

B_0 is (the profile of) the applied magnetic field which is a measurable function from Ω to $[-1, 1]$. The parameter H represents the intensity of this field. Finally, the parameter κ is the so-called GL parameter. It is a physical characteristic of the superconductor that depends on its temperature and the nature of the material, and determines its type: if $\kappa < 1/\sqrt{2}$ (respectively $\kappa > 1/\sqrt{2}$) then the superconductor is of Type I (respectively Type II). In some typical situations (depending on the strength of the applied field), the inverse of κ is proportional to the size of vortex cores. We are interested in the London limit $\kappa \to +\infty$, where the vortices become point-like [SS07]. This limit corresponds to extreme Type II superconductors and has been frequently addressed in early works.

The supercurrent, $\mathbf{j}$, is a real vector field given by $\mathbf{j} = \operatorname{Im}(\overline{\psi}(\nabla - i\kappa H\mathbf{A})\psi)$. One can notice that there is no supercurrent ($\mathbf{j} = 0$) circulating in the sample when it is in a normal state ($\psi = 0$), while such a current is generated in the superconducting state.

The ground-state of the superconductor describes its state at the equilibrium. We denote the ground-state energy by

$$E_{g.st}(\kappa, H) = \inf\{\mathscr{E}_{\kappa,H}(\psi, \mathbf{A}) \ : \ (\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2)\}.$$

The only physically meaningful quantities are those that are gauge invariant, such as the density $|\psi|$, the field $\operatorname{curl}\mathbf{A}$, the energy $E_{g.st}$ and the supercurrent $\mathbf{j}$. This means that these quantities do not change under the transformation $(\psi, \mathbf{A}) \mapsto (e^{i\varphi\kappa H}\psi, \mathbf{A} + \nabla\varphi)$, for any $\varphi \in H^2(\Omega; \mathbb{R})$. This gauge invariance allows us to restrict the minimization of the GL functional to the space $H^1(\Omega; \mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$ where

$$H^1_{\mathrm{div}}(\Omega) = \left\{ \mathbf{A} \in H^1(\Omega; \mathbb{R}^2) \ : \ \operatorname{div}\mathbf{A} = 0 \text{ in } \Omega, \ \mathbf{A} \cdot \nu = 0 \text{ on } \partial\Omega \right\}$$

and ν is a unit normal vector of $\partial\Omega$. Consequently, the ground-state energy can be expressed as follows:

$$E_{g.st}(\kappa, H) = \inf\{\mathscr{E}_{\kappa, H}(\psi, \mathbf{A}) \ : \ (\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)\}. \qquad (2)$$

By this restriction, one can make a profit out of the important regularity properties of the space $H^1_{\mathrm{div}}(\Omega)$ (see [FH10, Appendix D] and [Paper I, Appendix B]).

Establishing the existence of a minimizer of $\mathscr{E}_{\kappa, H}$ is standard (see e.g. [FH10, Theorem 10.2.1]), thus the infimum in (2) is actually a minimum. Critical points $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$ of $\mathscr{E}_{\kappa, H}$ are weak solutions of the following Euler–Lagrange equation, called in our context the GL equations:

$$\begin{cases} \left(\nabla - i\kappa H\mathbf{A}\right)^2\psi = \kappa^2(|\psi|^2 - 1)\psi & \text{in } \Omega, \\ -\nabla^\perp\left(\operatorname{curl}\mathbf{A} - B_0\right) = \frac{1}{\kappa H}\operatorname{Im}\left(\overline{\psi}(\nabla - i\kappa H\mathbf{A})\psi\right) & \text{in } \Omega, \\ \nu \cdot (\nabla - i\kappa H\mathbf{A})\psi = 0 & \text{on } \partial\Omega, \\ \operatorname{curl}\mathbf{A} = B_0 & \text{on } \partial\Omega. \end{cases}$$

Here,

$$(\nabla - i\kappa H\mathbf{A})^2\psi = \Delta\psi - i\kappa H(\operatorname{div}\mathbf{A})\psi - 2i\kappa H\mathbf{A}\cdot\nabla\psi - \kappa^2 H^2|\mathbf{A}|^2\psi,$$

and $\nabla^\perp = (\partial_{x_2}, -\partial_{x_1})$ is the Hodge gradient.

For more details on the GL model, one may refer for instance to [SJST69, TT90, Tin96] in the physics literature and to [CHO92, DGP92, BBH94, SS07, FH10] in the mathematics literature.

3   Thesis objectives

Many mathematical contributions were devoted to the study of the GL model in the context of superconductivity (see e.g. [LP99, PK02, BNF07, SS07, FH10, CR14,

CG17]). In these works, the domains were assumed to be (piecewise) smooth, submitted to constant or smooth non-constant applied magnetic fields.

However, various recent physical works considered discontinuous magnetic fields, after the possibility of creating such fields by the present fabrication techniques [FLBP94, STH⁺94, GGD⁺97]. Models with piecewise-constant magnetic fields are analysed in nanophysics [PM93, RP00] such as in quantum transport, and more recently in the study of the transport properties in graphene [GDMH⁺08, ORK⁺08]. The importance of these fields mainly lies in their ability to induce edge currents circulating along the interface of transition between the different values of the magnetic field (see for instance [PM93, RP00, HS08, DHS14, HS15, HPRS16]). We call this interface the *magnetic edge*, and sometimes *the magnetic barrier or the discontinuity edge*.

Despite of that importance, the piecewise-constant magnetic field has been only considered for linear problems in the mathematics literature, and to our knowledge there was no mathematical analysis of this discontinuous case in the context of the non-linear GL functional in superconductivity. So, we wanted to fill this gap; we examined the existence of edge currents by studying the presence of superconductivity near the magnetic edge, in a superconductor submitted to a *piecewise-constant magnetic field*. More generally, we aimed at studying the superconducting state of our sample in various intensity-regimes[5] and, consequently, comparing our findings with existing results in smooth magnetic fields cases.

4   Well-known scenarios

Before presenting our main results, we opt to gather in one section well-known facts about the behaviour of a superconductor in the case of smooth applied magnetic fields ($B_0 \in \mathscr{C}^\infty(\overline{\Omega})$). We are particularly interested in the constant magnetic field case, for a later comparison between this case and our piecewise-constant field case. However, we present some results obtained in the non-constant smooth magnetic field case as well, for the sake of completeness.

Here, we assume that the sample Ω is a 2D smooth, bounded and simply connected domain, and that the GL parameter κ is large.

The case of a constant applied magnetic field.   The sample's behaviour in this case was described above, but the description will be more specific in what follows

---

[5] By intensity-regime, we mean an interval of the applied magnetic field's intensities.
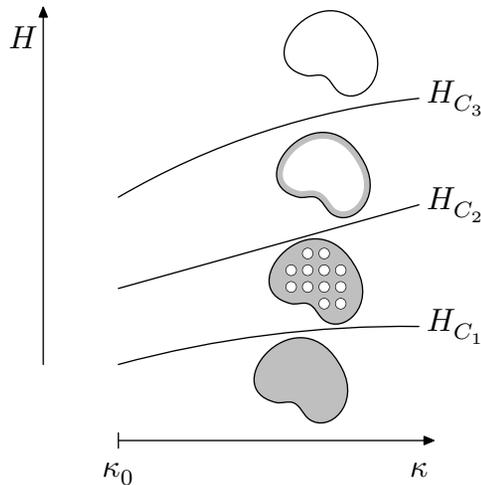
Figure 3: Schematic phase-diagram showing the distribution of superconductivity in the sample Ω submitted to a constant magnetic field, according to the intensity, H, of this field. $\{H_{C_i}(\kappa)\}_i$ are the critical fields. The grey (resp. white) regions of the sample are in a superconducting (resp. normal) state.

(see e.g. [SS07, FH10]). When the magnetic field B_0 is constant (we take $B_0 = 1$), three values of the field's intensity–the critical fields $H_{C_1}(\kappa)$, $H_{C_2}(\kappa)$ and $H_{C_3}(\kappa)$— dependent on κ, identify the following phase-transitions: When $H > H_{C_3}(\kappa)$, the sample is in a *normal state*. Between $H_{C_2}(\kappa)$ and $H_{C_3}(\kappa)$, the *surface superconductivity state* occurs, where superconductivity is (exclusively) localised near the boundary. The regime $H < H_{C_2}(\kappa)$ corresponds to the *bulk superconductivity state*, where superconductivity appears in the interior of the sample. In the constant magnetic field case, the distribution of bulk/surface superconductivity is *uniform* (to leading order). The first critical field $H_{C_1}(\kappa)$ indicates the transition from the state with vortices[6] to the pure superconducting state. We do not focus on this field in our study, and we refer the reader to [SS07] for more information. The aforementioned phase-transitions are illustrated in Figure 3. The identification of critical fields is not easy. In particular, the field $H_{C_2}(\kappa)$ is just loosely defined [FK11]. As κ tends to $+\infty$, the fields $H_{C_2}(\kappa)$ and $H_{C_3}(\kappa)$ are given as follows (see e.g. [FH10]):

$$H_{C_2}(\kappa) = \kappa \qquad \text{and} \qquad H_{C_3}(\kappa) \sim \Theta_0^{-1}\kappa,$$

---

[6]A vortex is described as a quantized amount of vorticity of the superconducting current localised near a point.

where $\Theta_0 \approx 0.59$ is a universal constant, called the de Gennes constant.

The case of a non-vanishing applied magnetic field. We discuss the case when the field B_0 is non-zero everywhere in $\overline{\Omega}$ (see e.g. [LP99, HM01, HM04, Ray09, FH10, Att15a, Att15b]). One distinguishes between two cases:

- The case when $\min_{x \in \overline{\Omega}} |B_0(x)| > \Theta_0 \min_{x \in \partial\Omega} |B_0(x)|$. Here, the scenario is qualitatively similar to that in the constant field case, in the sense that when the intensity H of the field decreases from ∞, the sample passes from the normal state to a superconducting state and the onset of superconductivity starts at the boundary. Under certain assumptions on the minima of $|B_0|_{|\partial\Omega}$, one gets [Ray09]

$$H_{C_3}(\kappa) \sim \frac{\kappa}{\Theta_0 \min_{x \in \partial\Omega} |B_0(x)|}.$$

  In addition, the following definition of $H_{C_2}(\kappa)$ was proposed in [FH10]:

$$H_{C_2}(\kappa) = \frac{\kappa}{\min_{x \in \overline{\Omega}} |B_0(x)|}.$$

- The case when $\min_{x \in \overline{\Omega}} |B_0(x)| < \Theta_0 \min_{x \in \partial\Omega} |B_0(x)|$. There is no surface superconductivity state. More precisely, if we decrease the field's intensity from ∞, the onset of superconductivity starts in the interior, in the vicinity of the minima of $|B_0|$. Consequently, under certain assumptions on the minima of $|B_0|_{|\Omega}$, the definitions of the second and the third critical fields match, and we get the following asymptotics (see e.g. [HM96, FH10, RVN15]):

$$H_{C_2}(\kappa) = H_{C_3}(\kappa) \sim \frac{\kappa}{\min_{x \in \overline{\Omega}} |B_0(x)|}.$$

The case of a vanishing applied magnetic field. Now, we consider the case when the field B_0 is zero along a smooth curve Γ.

In what follows, λ_0 and ζ_1^θ are two spectral quantities such that λ_0 is a real number and ζ_1^θ is a real-valued function of x (see [PK02]).

- When $\Gamma \cap \partial\Omega = \emptyset$, we have [DR13, Att16]

$$H_{C_3}(\kappa) \sim \frac{\kappa^2}{\lambda_0 \min_{x \in \Gamma} |\nabla B_0(x)|},$$

  and the definition of $H_{C_2}(\kappa)$ is not distinguished from that of $H_{C_3}(\kappa)$.

xxix

- When $\Gamma \cap \partial\Omega \neq \emptyset$, the intersection is assumed to be finite and transversal. One gets ([PK02, Att16, Miq16])

$$H_{C_3}(\kappa) \sim \frac{\kappa^2}{\min\left(\lambda_0 \min_{x \in \Gamma \cap \Omega} |\nabla B_0(x)|, \min_{x \in \Gamma \cap \partial\Omega} \zeta_1^{\theta(x)} |\nabla B_0(x)|\right)}.$$

If $\lambda_0 \min_{x \in \Gamma \cap \Omega} |\nabla B_0(x)| < \min_{x \in \Gamma \cap \partial\Omega} \zeta_1^{\theta(x)} |\nabla B_0(x)|$, then the surface superconductivity phenomenon is absent, and $H_{C_2}(\kappa)$ coincides with $H_{C_3}(\kappa)$. While if $\lambda_0 \min_{x \in \Gamma \cap \Omega} |\nabla B_0(x)| > \min_{x \in \Gamma \cap \partial\Omega} \zeta_1^{\theta(x)} |\nabla B_0(x)|$, then surface superconductivity is observed, and a definition of $H_{C_2}(\kappa)$ is naturally given as follows [HK15, KN17]:

$$H_{C_2}(\kappa) = \frac{\kappa^2}{\lambda_0 \min_{x \in \Gamma \cap \Omega} |\nabla B_0(x)|}.$$

Remark. We refer the reader to [AB03, Bon05, BND06, BNF07] for critical fields in the case of domains with corners submitted to constant fields. In this case, the behaviour of the sample in the bulk and the surface superconductivity regimes is similar to that in the case of smooth domains submitted to constant magnetic fields. This behaviour becomes particular at the threshold of the breakdown of superconductivity, where superconductivity is confined to the corners. The scenario occurring at this stage is presented later in this introduction.

5   Main results

We are still considering a bounded, simply connected and smooth domain Ω of $\mathbb{R}^2$. In what follows, we roughly present the case that we treat in this thesis: we divide Ω into two sets Ω_1 and Ω_2 separated by disjoint simple smooth curves, denoted by Γ. We apply on Ω a step magnetic field $B_0 = \mathbb{1}_{\Omega_1} + a\mathbb{1}_{\Omega_2}$, where $a \in [-1, 1) \backslash \{0\}$ is a given constant. Thus, the jump discontinuities of B_0 occur at Γ, referred to as the *magnetic edge* (see Figure 4). In the case when the magnetic edge intersects the boundary, this intersection is assumed to be finite, and also transversal to avoid the presence of cusps which may create technical challenges during the study. For the formal presentation of the case, see [Paper I, Assumption 1.1]. Furthermore, we assume that the GL parameter κ is large and the intensity of the magnetic field has the same order of κ.
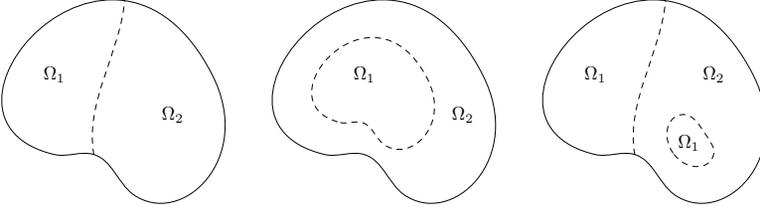
Figure 4: Schematic representation of the set Ω subjected to the step magnetic field B_0. The dashed curves represent the magnetic edge Γ.

The thesis is composed of three reproduced publications: [Paper I], [Paper II] and [Paper III]. In [Paper I], we considered low intensity-regimes and showed that the whole interior of the sample is superconducting. Then, we increased the field's intensity to a certain level, $H_{C_2}(\kappa)$, and proved that superconductivity becomes negligible in the bulk away from the magnetic edge. In [Paper II], we asserted that the bulk keeps superconducting (solely) near the magnetic edge for certain piecewise-constant magnetic fields, when the intensity of the field is near $H_{C_2}(\kappa)$; the different values of the magnetic field interact to *trap* superconductivity there. Such a behaviour is notable, especially when opposed to the uniform distribution of bulk superconductivity in the case of a constant applied magnetic field. Moreover, we examined the state of the sample near the boundary. In certain intensity-regimes, we presented situations where only parts of the boundary are superconducting. Again, this marks a deviation from what occurs in the constant field case, where surface superconductivity is evenly distributed along the boundary. Increasing the field's intensity to higher levels in [Paper III], we investigated the transition of the sample from the superconducting state to the normal state–the breakdown of superconductivity. We considered the interesting case where the magnetic edge intersects the boundary finitely and transversely (additional geometric assumptions were also imposed), and we proved that the aforementioned transition happens at a unique value, $H_{C_3}(\kappa)$, of the field's intensity, which we estimated. Our results showed the localisation of superconductivity near the intersection between the magnetic edge and the boundary, before its breakdown. This behaviour was reminiscent of the case of domains with corners submitted to constant magnetic fields, where superconductivity eventually lives in the vicinity of the corners before disappearing. Hence, a comparison between our discontinuous case and the corner case was done.

Altogether, the three papers showed a behaviour of the sample that, according to the values and the intensity of the discontinuous magnetic field, may resemble

to or differ from that in the case of smooth/corner domains submitted to uniform magnetic fields. This highlights the particularity of the case that we treated. Below, we briefly present the main results of these papers and compare them to some findings in the literature.

5.1 [Paper I]

This paper focuses on the bulk superconductivity. Our results involve an auxiliary function $g : [0, +\infty) \to [-1/2, 0]$, which is continuous, non-decreasing, negative in $[0, 1)$ and vanishing in $[1, +\infty)$. This function was introduced by Sandier and Serfaty in [SS03], and has always played a critical role in the study of bulk superconductivity (see e.g. [AS07, FK11, FK13, Att15b, HK15]).

We established global asymptotic estimates (as $\kappa \to +\infty$) of the ground-state energy $E_{\mathrm{g.st}}(\kappa, H)$:

$$
E_{\mathrm{g.st}}(\kappa, H) = \kappa^2 \int_\Omega g\Big(\frac{H}{\kappa}|B_0(x)|\Big) dx + o(\kappa^2),
$$

and the corresponding order parameter:

$$
\int_\Omega |\psi|^4 \, dx = -2 \int_\Omega g\Big(\frac{H}{\kappa}|B_0(x)|\Big) \, dx + o(1).
$$

These global estimates were deduced from the following local estimates, which describe the strength of superconductivity in any sufficiently regular subdomain D of Ω:

$$
\int_D |\psi|^4 \, dx = -2 \int_D g\Big(\frac{H}{\kappa}|B_0(x)|\Big) \, dx + o(1). \tag{3}
$$

Recalling the definition of B_0 in our case, and the properties of the function g, the previous result implies the following (Figure 5):

- If $H < (1/|a|)\kappa$, then superconductivity exists in the bulk of Ω, but is not uniformly distributed between Ω_1 and Ω_2.

- If $H \geq (1/|a|)\kappa$, then superconductivity is negligible in the whole bulk except near the magnetic edge Γ. In this intensity-regime, the analysis in [Paper I] did not provide information about what happens near Γ and $\partial\Omega$, but we suggested that superconductivity might exists there.
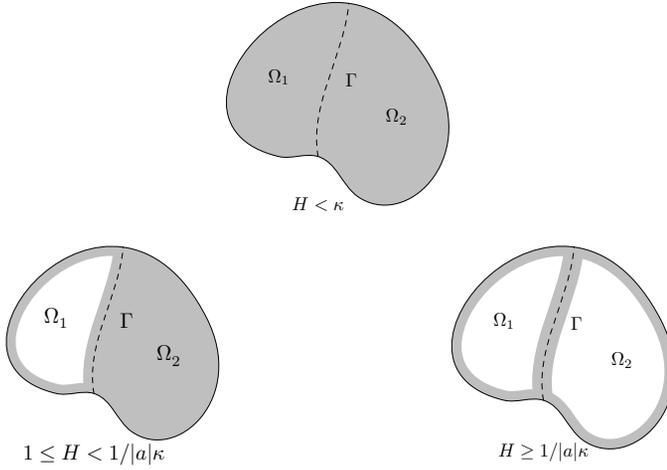
Figure 5: The superconductivity state of the sample Ω submitted to the magnetic field $B_0 = \mathbb{1}_{\Omega_1} + a\mathbb{1}_{\Omega_2}$, according to the field's intensity H. The white regions are in a normal state, while the grey region may carry superconductivity.

See [Paper I, Discussion of Theorem 1.3] for more details.

The result in (3) was sharpened by establishing some (Agmon) estimates showing that superconductivity in $\Omega \setminus \Gamma$ is exponentially small relatively to superconductivity at $\partial\Omega \cup \Gamma$, when $H > (1/|a|)\kappa$.

Earlier results. The findings of [Paper I] are parallel to earlier results obtained in certain cases of smooth applied magnetic fields (e.g. [SS03, SS07]).

Sandier and Serfaty [SS03] considered the unit magnetic field ($B_0 = 1$), and proved that

$$\mathrm{E}_{\mathrm{g,st}}(\kappa, H) = g(H/\kappa)|\Omega|\kappa^2 + o(\kappa^2), \qquad \text{as } \kappa \to +\infty,$$

where g is the auxiliary function alluded to above. In addition, they showed a *uniform distribution* of superconductivity in the bulk of the sample.

Several works have treated the case of a smooth magnetic field ($B_0 \in \mathscr{C}^\infty(\overline{\Omega})$) (e.g. [Att15a, Att15b, HK15]). In [Att15b], the magnetic field is assumed to vanish along a smooth curve Γ. Under certain conditions on B_0, Attar established the following global estimates of the ground-state energy:

$$\mathrm{E}_{\mathrm{g,st}}(\kappa, H) = \kappa^2 \int_\Omega g\left(\frac{H}{\kappa}|B_0(x)|\right) dx + o(\kappa^2), \qquad \text{as } \kappa \to +\infty,$$

xxxiii

and proved that superconductivity is localised near Γ with a length scale κ/H.

5.2 [Paper II]

This contribution aimed at investigating the existence of superconductivity near the magnetic edge, as well as near the boundary. We considered the intensity-regime $H > (1/|a|)\kappa$, where the whole bulk away from Γ is in a normal state. For *negative* values of a, we proved the localisation of bulk superconductivity along Γ, when the intensity is still near $(1/|a|)\kappa$. Such cases imply the existence of an edge current flowing along the magnetic edge.

Our findings are consistent with the existing works about the electron motion near the magnetic edge [Iwa85, RP00, DHS14, HS15, HPRS16]. In the literature, the case $a \in [-1, 0)$ is called the *trapping magnetic steps* [HPRS16], where supercurrents flow along the magnetic edge in the form of snake orbits. Such snake orbits do not seem detectable in the case $a \in (0, 1)$, which is called the *non-trapping magnetic steps*. However as mentioned earlier, the study in the aforementioned references was generally a spectral analysis of relevant linear model operators and, until the present contribution, no estimates for the non-linear GL energy were provided.

Two functions, $e_a : [|a|^{-1}, +\infty) \to (-\infty, 0]$ and $E_{\mathrm{surf}} : [1, +\infty) \to (-\infty, 0]$, are main ingredients in our results.

E_{surf}, referred to as the surface energy, was used in several works that study the surface superconductivity for smooth magnetic fields (see e.g. [Pan02, AH07, HFPS11, FKP13, CR14, CR16a, CG17, HK17]). This is a non-decreasing and continuous function that satisfies

$$E_{\mathrm{surf}} < 0 \text{ in } [1, \Theta_0^{-1}) \,, \text{ and } E_{\mathrm{surf}} = 0 \text{ in } [\Theta_0^{-1}, +\infty) \,. \tag{4}$$

e_a, referred to as the edge energy, is introduced in [Paper II]. It is a non-decreasing continuous function, satisfying:

$$e_a < 0 \text{ in } [|a|^{-1}, \beta_a^{-1}) \,, \text{ and } e_a = 0 \text{ in } [\beta_a^{-1}, +\infty), \tag{5}$$

where β_a is a spectral value in $(0, |a|]$, which satisfies:

$$\beta_a = a \text{ for } a \in (0, 1) \,, \quad \beta_a = \Theta_0 \text{ for } a = -1 \,, \quad |a|\Theta_0 < \beta_a < |a| \text{ for } a \in (-1, 0).$$

We established the following global estimates of the ground-state energy and

the corresponding order parameter ψ: For $b = H/\kappa$,

$$
\mathrm{E}_{\text{g.st}}(\kappa, H) = b^{-1/2}\Big(|\Gamma|e_a(b) + |\partial\Omega_1 \cap \partial\Omega|E_{\text{surf}}(b) +
$$
$$
|\partial\Omega_2 \cap \partial\Omega|\,|a|^{-\frac{1}{2}}E_{\text{surf}}(b|a|)\Big)\kappa + o(\kappa),
$$

and

$$
\int_\Omega |\psi|^4\,dx = -2b^{-1/2}\Big(|\Gamma|e_a(b) + |\partial\Omega_1 \cap \partial\Omega|E_{\text{surf}}(b) +
$$
$$
|\partial\Omega_2 \cap \partial\Omega|\,|a|^{-\frac{1}{2}}E_{\text{surf}}(b|a|)\Big)\kappa^{-1} + o(\kappa^{-1}),
$$

as κ tends to $+\infty$. The terms involving e_a correspond to the contribution of the magnetic edge, while these involving E_{surf} indicate the contribution of the surface.

In fact, we have established more precise estimates, measuring the strength of superconductivity in any patch of the sample: we defined the following distributions in $\mathscr{D}'(\mathbb{R}^2)$,

$$
C_c^\infty(\mathbb{R}^2) \ni \varphi \mapsto \mathscr{T}^b(\varphi)
$$

where

$$
\mathscr{T}^b(\varphi) = -2b^{-\frac{1}{2}}\Big(e_a(b)\int_\Gamma \varphi\,ds_\Gamma + E_{\text{surf}}(b)\int_{\partial\Omega_1\cap\partial\Omega} \varphi\,ds
$$
$$
+ |a|^{-\frac{1}{2}}E_{\text{surf}}(b|a|)\int_{\partial\Omega_2\cap\partial\Omega} \varphi\,ds\Big)
$$

and

$$
C_c^\infty(\mathbb{R}^2) \ni \varphi \mapsto \mathscr{T}_\kappa^b(\varphi) = \int_\Omega |\psi|^4 \varphi\,dx
$$

(note that ψ depends on κ), then we proved that

$$
\kappa\mathscr{T}_\kappa^b \rightharpoonup \mathscr{T}^b \text{ in } \mathscr{D}'(\mathbb{R}^2), \qquad \text{as } \kappa \to +\infty,
$$

in the sense that

$$
\forall\,\varphi \in C_c^\infty(\mathbb{R}^2), \quad \lim_{\kappa\to+\infty}\kappa\mathscr{T}_\kappa^b(\varphi) = \mathscr{T}^b(\varphi).
$$

Consequently, using the properties in (4) and (5), we discussed the distribution of superconductivity according to the value of the magnetic field (i.e. the value of a) and to the intensity H of this field (see [Paper II, Section 1.5]). We present some illustrative plots of this discussion in Figure 6.
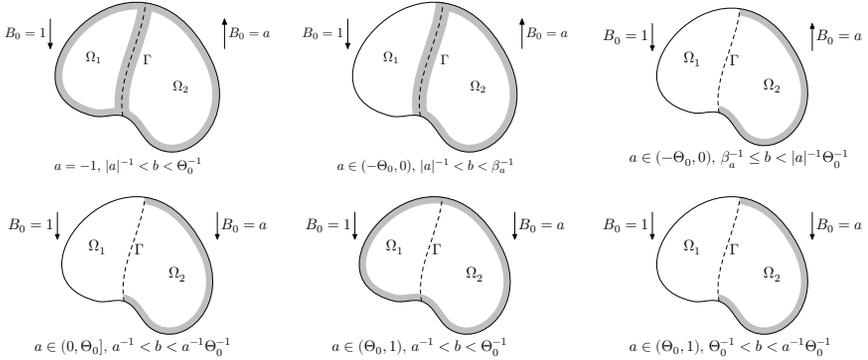
Figure 6: Superconductivity distribution in the set Ω subjected to the magnetic field $B_0 = \mathbb{1}_{\Omega_1} + a\mathbb{1}_{\Omega_2}$, according to the values of a and b, where $b = H/\kappa$. The white regions are in a normal state, while the grey regions carry superconductivity.

Earlier results. In the case of a constant field ($B_0 = 1$) and for $b \in (1, \Theta_0^{-1})$, where $b = H/\kappa$, superconductivity is shown to be confined to the boundary and the ground-state energy is estimated as follows (see e.g. [CR16a, CR16b, CDR17]):

$$E_{\text{g.st}}(\kappa, H) = |\partial\Omega|\kappa b^{-\frac{1}{2}}E_{\text{surf}}(b) + \mathcal{O}(1).$$

Moreover, it is proved that this surface superconductivity is *uniformly distributed* along the boundary [Pan02, AH07, HFPS11, CR14], and *is not affected (to leading order) by the presence of a finite number of corners* (see [CG17]).

In [Paper II, Section 1.5], we compared the behaviour of the sample to that in the constant field case, and showed how the two behaviours are dramatically distinct: With increasing intensities, the constant field case exhibits first a uniform distribution of bulk superconductivity, then this superconductivity disappears uniformly from the bulk to spread evenly along the boundary. On the contrary, our case presents some situations where superconductivity is not evenly distributed in the bulk and/or along the boundary.

In the case of a smooth field $B_0 \in \mathscr{C}^{0,\alpha}(\overline{\Omega})$, [HK17] established the following. Let $(\psi, \mathbf{A})_{\kappa,H}$ be a minimizer of the functional in (1). In a sufficiently narrow neighbourhood of $\partial\Omega$, one can assign to each point x a unique point $p(x) \in \partial\Omega$ such that $\text{dist}(x, p(x)) = \text{dist}(x, \partial\Omega)$. Let $\Omega(b)$ be the set of points of this neighbourhood satisfying $1 < b|B_0(p(x))| < \Theta_0^{-1}$. If $\Omega(b) \neq \emptyset$ then a convergence (in the sense of distributions) of $\kappa|\psi_{\kappa,H}|^4$ to $\mathscr{T}^b$ was established, as κ

tends to $+\infty$. Here

$$C_c^\infty\big(\Omega(b)\big) \ni \varphi \mapsto \mathcal{T}^b(\varphi) = -2b^{-\frac{1}{2}} \int_{\Omega(b)\cap\partial\Omega} |B_0(x)|^{-\frac{1}{2}} E_{\text{surf}}\big(b\,|B_0(x)|\big)\varphi\,ds.$$

This convergence interestingly describes the local behaviour of the sample at the boundary.

Since our step magnetic field is constant in each of Ω_1 and Ω_2, we were allowed to use the results from [HK17] in our study of surface superconductivity. Our essential contribution was to develop a detailed spectral study of new effective operators, while examining superconductivity near the magnetic edge (see [Paper II, Sections 2&3]).

5.3 [Paper III]

We studied superconductivity when the field's intensity is near the threshold $H_{C_3}(\kappa)$, where the transition from the superconducting state to the normal state occurs. This phase transition was extensively examined for smooth applied magnetic fields (see e.g. [SJG63, LP00, HM01, HP03, Bon05, BND06, FH06, FH07, BNF07, Ray09, FH09, FP11, DR13, Att16]).

In the smooth magnetic field case, researchers were investigating the occurrence of a sharp transition, that is whether switching between superconducting and normal states occurs at a unique value of the field's intensity. Such a transition depends on the geometry of the sample and the properties of the magnetic field. It has been established for certain smooth domains submitted to generic smooth fields. However, this result does not hold in certain situations, like in the 2D annuli where the famous Little–Parks effect occurs [LP62, Erd97, FPS15], or in discs submitted to certain non-uniform magnetic fields [FPS15].

In [Paper III], we aimed at checking whether the transition is sharp in our settings. We assumed that the magnetic edge Γ cuts transversely the boundary at a finite number of points, p_j, and we denoted by $\alpha_j \in (0, \pi)$ the angle formed between Γ and $\partial\Omega$ at p_j, measured towards Ω_1 (see Figure 7). Next, we introduced a ground-state, $\mu(\alpha, a)$, corresponding to the Neumann realization of a new Schrödinger operator, $\mathcal{H}_{\alpha,a}$, with a step magnetic field, defined on $\mathbb{R}_+^2$ (see [Paper III, Section 3]). Here $a \in [-1, 1)\backslash\{0\}$, and α is a real parameter which is an angle in $(0, \pi)$. Then under the assumption[7] that $\mu(\alpha_j, a) < |a|\Theta_0$ for any α_j,

---

[7]we provided some examples of pairs (α_j, a) satisfying this condition. The need for such an assumption is explained in [Paper III, Section 1.4].
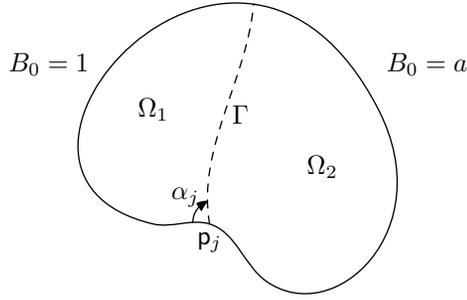
Figure 7: Schematic representation of the set Ω in the case where the magnetic edge Γ cuts the boundary.

we established the aforementioned sharp transition, and we provided asymptotic estimates of the field $H_{C_3}(\kappa)$:

$$H_{C_3}(\kappa) = \frac{\kappa}{\min\limits_{j \in \{1,\dots,n\}} \mu(\alpha_j, a)} + \mathcal{O}(\kappa^{\frac{1}{2}}), \quad \text{as } \kappa \to +\infty. \tag{6}$$

Before its breakdown, superconductivity was shown to be localised near certain intersection points between the magnetic edge and the boundary, called the energetically favourable points.

Earlier results. A complete asymptotics expansion of the third critical field has been established in the literature, for 2D bounded and simply connected domains with piecewise-smooth boundary, submitted to uniform magnetic fields (e.g. [LP99, HM01, HP03, Bon05, FH06, BND06, FH07, BNF07]). In this discussion, we will be satisfied by presenting asymptotics to the leading order of the third critical field, in the case of the constant field $B > 0$.

- *Case of smooth domains subjected to B (the SDUF case):*

$$H_{C_3}^{\text{unif}}(\kappa) = \frac{\kappa}{B\Theta_0} + o(\kappa), \quad \text{as } \kappa \to +\infty \tag{7}$$

- *Case of corner domains subjected to B (the CDUF case):*

$$H_{C_3}^{\text{cor}}(\kappa) = \frac{\kappa}{B\Lambda} + o(\kappa), \quad \text{as } \kappa \to +\infty \tag{8}$$

where $H_{C_3}^{\text{unif}}(\kappa)$ and $H_{C_3}^{\text{cor}}(\kappa)$ are the third critical fields in the SDUF and CDUF cases respectively, and Λ is a spectral value assumed to satisfy $\Lambda < \Theta_0$.

Comparing the asymptotics in (7) and (8), we see how the presence of corners in a domain can prolong the lifespan of superconductivity to the whole interval between $H_{C_3}^{\text{unif}}(\kappa)$ and $H_{C_3}^{\text{cor}}(\kappa)$.

In our step magnetic field case (the SDSF case), the intersection points of the magnetic edge and the boundary play the role of the corners in the CDUF case, in the sense that the presence of such an intersection makes our third critical field, $H_{C_3}(\kappa)$, strictly larger than the field $H_{C_3}^{\text{unif}}(\kappa)$ which corresponds to the constant field $B = |a|$ (though the two fields are of same leading order). Moreover, the eventual nucleation of superconductivity near these intersection points is comparable with that occurring near the corners in the CDUF case (see [BNF07, HK18] and [Paper III, Section 1] for more details).

At this stage, it is worth contrasting this similarity between the SDSF and CDUF cases to the disparity between the two cases observed for lower-level field's intensities (revisit Section 3 or [Paper II, Section 1]). Figure 8 illustrates such a
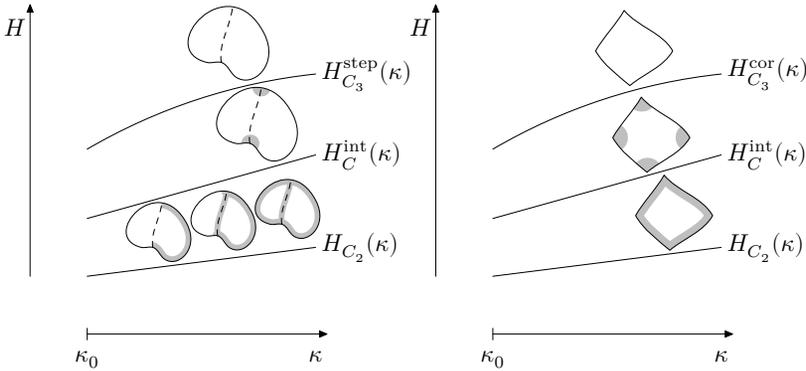
Figure 8: Phase diagrams: the SDSF case to the left and the CDUF case to the right. Only the grey regions carry superconductivity.

comparison between the two cases. The schematic phase-diagrams consider the SDSF case, with the step magnetic field $B_0 = \mathbb{1}_{\Omega_1} + a\mathbb{1}_{\Omega_2}$, and the CDUF case, with the uniform magnetic field $B = |a|$. In each case, the graph shows the distribution of superconductivity in the sample according to the intensity, H, of the applied magnetic field. Considering large κ, we draw critical lines in the (κ, H)-plane that

represent the following:

$$H_{C_2}(\kappa) = \frac{\kappa}{|a|}, \ H_C^{\text{int}}(\kappa) = \frac{\kappa}{|a|\Theta_0}, \ H_{C_3}^{\text{step}}(\kappa) = H_{C_3}(\kappa) \text{ in (6)},$$

$$\text{and } H_{C_3}^{\text{cor}}(\kappa) \text{ as in (8)}.$$

In the SDSF case, the plots between $H_{C_2}(\kappa)$ and $H_C^{\text{int}}(\kappa)$ illustrate different instances of the sample's behaviour, occurring according to the values of H and a.

6   Open questions

Some uncovered points in this thesis deserve a further examination:

- The edge current: in the intensity-regime $\big(H_{C_2}(\kappa), H_{C_3}(\kappa)\big)$, the confinement of bulk superconductivity to the magnetic edge indicates the existence of supercurrents circulating along this edge. A rigorous computation of this edge current is interesting.

- The magnetic wall case: this thesis treats the case of the magnetic field $B_0 = \mathbb{1}_{\Omega_1} + a\mathbb{1}_{\Omega_2}$, where a is a fixed constant in $[-1, 1)\backslash\{0\}$. It will be potentially interesting to study the case $a = 0$, referred to as the *magnetic wall* in physics (see e.g. [HPRS16, RP00]).

- The sample's behaviour in the limiting intensity-regime $H \sim H_{C_2}(\kappa)$: the study in this regime may involve a special linear model (the Abrikosov model). In this case, one may expect the concentration of bulk superconductivity near the magnetic edge.

- Other discontinuous fields: considering more general discontinuous fields may enlarge the scope of the potential applications of such a study.

Bibliography

[AB03]   F. Alouges and V. Bonnaillie. Analyse numérique de la supraconductivité. *C. R. Math.*, 337(8):543–548, 2003.

[Abr57]   A.A. Abrikosov. On the magnetic properties of superconductors of the second type. *Soviet Phys. JETP*, (5):1174–1182, 1957.

[Aft06]   A. Aftalion. *Vortices in Bose–Einstein Condensates*, volume 67. Progress in Nonlinear Partial Differential Equations and their Applications, Birkhäuser-Boston, 2006.

[AH07]   Y. Almog and B. Helffer. The distribution of surface superconductivity along the boundary: On a conjecture of X.B. Pan. *SIAM J. Math. Anal.*, 38(6):1715–1732, 2007.

[AK16]   W. Assaad and A. Kachmar.   The influence of magnetic steps on bulk superconductivity. *Discrete Contin. Dyn. Syst. Ser. A*, 36:6623–6643, 2016.

[AKPS19]   W. Assaad, A. Kachmar, and M. Persson-Sundqvist.   The distribution of superconductivity near a magnetic barrier. *Comm. Math. Phys.*, 366(1):269–332, 2019.

[AS07]   A. Aftalion and S. Serfaty. Lowest landau level approach in superconductivity for the abrikosov lattice close to hc2. *Selecta Math.*, 13(2):183–202, 2007.

[Ass19]   Wafaa Assaad. The breakdown of superconductivity in the presence of magnetic steps. *arXiv:1903.04847*, 2019.

[Att15a]   K. Attar. Energy and vorticity of the Ginzburg–Landau model with variable magnetic field. *Asymptot. Anal.*, 93(1-2):75–114, 2015.

[Att15b]   K. Attar. The ground state energy of the two dimensional Ginzburg–Landau functional with variable magnetic field. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 32(2):325–345, 2015.

[Att16]   K. Attar. Pinning with a variable magnetic field of the two dimensional Ginzburg–Landau model. *Nonlinear Anal.*, 139:1–54, 2016.

[BBH94]   F. Bethuel, H. Brezis, and F. Hélein. *Ginzburg–Landau vortices*, volume 13. Progress in Nonlinear Partial Differential Equations and their Applications, Birkhäuser-Boston, 1994.

[BCS57]   J. Bardeen, L.N. Cooper, and J.R. Schrieffer. Theory of superconductivity. *Phys. Rev.*, 108(2):1175–1204, 1957.

[BND06]   V. Bonnaillie-Noël and M. Dauge. Asymptotics for the low-lying eigenstates of the Schrödinger operator with magnetic field near corners. 7(5):899–931, 2006.

[BNF07]   V. Bonnaillie-Noël and S. Fournais. Superconductivity in domains with corners. *Rev. Math. Phys.*, 19(06):607–637, 2007.

[Bon05]   V. Bonnaillie. On the fundamental state energy for a Schrödinger operator with magnetic field in domains with corners. *Asymptot. Anal.*, 41(3-4):215–258, 2005.

[Car15]   E. Cartlidge. Superconductor high spurs wave of physics: researchers strive to reproduce and understand zero-resistance material that works at a record high temperature. *Nature*, 524(7565):277–278, 2015.

[CDR17]   M. Correggi, B. Devanarayanan, and N. Rougerie. Universal and shape dependent features of surface superconductivity. *Eur. Phys. J. B*, 90(11):231, 2017.

[CG17]   M. Correggi and E. L. Giacomelli. Surface superconductivity in presence of corners. *Rev. Math. Phys.*, 29(2):1750005, 2017.

[CHO92]   S.J. Chapman, S.D. Howison, and J.R. Ockendon. Macroscopic models for superconductivity. *Siam Rev.*, 34(4):529–560, 1992.

[CR14]   M. Correggi and N. Rougerie. On the Ginzburg–Landau functional in the surface superconductivity regime. *Commun. Math. Phys.*, 332(3):1297–1343, 2014.

[CR16a]   M. Correggi and N. Rougerie. Boundary behavior of the Ginzburg–Landau order parameter in the surface superconductivity regime. *Arch. Ration. Mech. Anal.*, 219(1):553–606, 2016.

[CR16b]   M. Correggi and N. Rougerie. Effects of boundary curvature on surface superconductivity. *Lett. Math. Phys.*, 106(4):445–467, 2016.

[dG96]   P.G. de Gennes. *Superconductivity of Metals and Alloys*. Benjamin, New York, 1996.

[DGP92]   Q. Du, M.D. Gunzburger, and J.S. Peterson. Analysis and approximation of the Ginzburg–Landau model of superconductivity. *Siam Rev.*, 34(1):54–81, 1992.

[DHS14]   N. Dombrowski, P.D. Hislop, and E. Soccorsi. Edge currents and eigenvalue estimates for magnetic barrier Schrödinger operators. *Asymptot. Anal.*, 89(3-4):331–363, 2014.

[DR13]   N. Dombrowski and N. Raymond. Semiclassical analysis with vanishing magnetic fields. *J. Spectr. Theory*, 3(3):423–464, 2013.

[Erd97]   László Erdős. Dia-and paramagnetism for nonhomogeneous magnetic fields. *J. Math. Phys.*, 38(3):1289–1317, 1997.

[ET67]   U. Essmann and H. Traeuble. Vortex lattice in high-tc superconductor. *Phys. Lett. A 24, 526*, 1967.

[FH06]   S. Fournais and B. Helffer. On the third critical field in Ginzburg–Landau theory. *Commun. Math. Phys.*, 266(1):153–196, 2006.

[FH07]   S. Fournais and B. Helffer. Strong diamagnetism for general domains and application. *Ann. Inst. Fourier*, 57(7):2389–2400, 2007.

[FH09]   S. Fournais and B. Helffer. On the Ginzburg–Landau critical field in three dimensions. *Comm. Pure Appl. Math.*, 62(2):215–241, 2009.

[FH10]   S. Fournais and B. Helffer. *Spectral Methods in Surface Superconductivity*, volume 77. Springer Science & Business Media, 2010.

[FK11]   S. Fournais and A. Kachmar. Nucleation of bulk superconductivity close to critical magnetic field. *Adv. Math.*, 226(2):1213–1258, 2011.

[FK13]   S. Fournais and A. Kachmar. The ground state energy of the three dimensional Ginzburg–Landau functional. Part I: Bulk regime. *Comm. Partial Differential Equations*, 38(2):339–383, 2013.

[FKP13] S. Fournais, A. Kachmar, and M. Persson. The ground state energy of the three dimensional Ginzburg–Landau functional. Part II: Surface regime. *J. Math. Pures Appl.*, 99(3):343–374, 2013.

[FLBP94] C.L. Foden, M.L. Leadbeater, J.H. Burroughes, and M. Pepper. Quantum magnetic confinement in a curved two-dimensional electron gas. *J. Phys.: Condens. Matter*, 6(10):L127, 1994.

[FP11]   S. Fournais and M. Persson. Strong diamagnetism for the ball in three dimensions. *Asymptot. Anal.*, 72(1-2):77–123, 2011.

[FPS15] S. Fournais and M. Persson-Sundqvist. Lack of diamagnetism and the Little–Parks effect. *Commun. Math. Phys.*, 337(1):191–224, 2015.

[GDMH+08] T.K. Ghosh, A. De Martino, W. Häusler, L. Dell'Anna, and R. Egger. Conductance quantization and snake states in graphene magnetic waveguides. *Phys. Rev. B*, 77(8):081404, 2008.

[GGD+97] A.K. Geim, I.V. Grigorieva, S.V Dubonos, J.G.S Lok, J.C Maan, A.E. Filippov, and Peeters F.M. *Nature (London)*, 390:259, 1997.

[GL65]   V. L. Ginzburg and L. D. Landau. Collected papers of L.D. landau. *Edited by D. Ter. Haar Pergamon Press, Oxford*, 1965.

[HFPS11] B. Helffer, S. Fournais, and M. Persson-Sundqvist. Superconductivity between H_{C_2} and H_{C_3}. *J. Spectr. Theory*, 1(3):273–298, 2011.

[HK15]   B. Helffer and A. Kachmar. The Ginzburg–Landau functional with vanishing magnetic field. *Arch. Ration. Mech. Anal.*, 218(1):55–122, 2015.

[HK17]   B. Helffer and A. Kachmar. Decay of superconductivity away from the magnetic zero set. *Calc. Var. Partial Differ. Equ.*, 56(5):130, 2017.

[HK18]   B. Helffer and A. Kachmar. The density of superconductivity in domains with corners. *Lett. Math. Phys.*, 108(9):2169–2187, 2018.

[HM96]   B. Helffer and A. Mohamed. Semiclassical analysis for the ground state energy of a Schrödinger operator with magnetic wells. *J. Funct. Anal.*, 138(1):40–81, 1996.

[HM01]   B. Helffer and A. Morame. Magnetic bottles in connection with superconductivity. *J. Funct. Anal.*, 185(2):604–680, 2001.

[HM04]   B. Helffer and A. Morame. Magnetic bottles for the Neumann problem: curvature effects in the case of dimension 3 (general case). 37(1):105–170, 2004.

[HP03]   B. Helffer and X.B. Pan. Upper critical field and location of surface nucleation of superconductivity. *Ann. Inst. H.Poincaré Anal. Non Linéaire*, 20(1):145–181, 2003.

[HPRS16]   P.D. Hislop, N. Popoff, N. Raymond, and M. Sundqvist. Band functions in the presence of magnetic steps. *Math. Models Methods Appl. Sci.*, 26(1):161–184, 2016.

[HS08]   P.D. Hislop and E. Soccorsi. Edge currents for quantum Hall systems I: One-edge, unbounded geometries. *Rev. Math. Phys.*, 20(01):71–115, 2008.

[HS15]   P.D. Hislop and E. Soccorsi. Edge states induced by Iwatsuka Hamiltonians with positive magnetic fields. *J. Math. Anal. Appl.*, 422(1):594–624, 2015.

[Iwa85]   A. Iwatsuka. Examples of absolutely continuous Schrödinger operators in magnetic fields. *Publ. Res. I. Math. Sci.*, 21(2):385–401, 1985.

[KN17]   A. Kachmar and M. Nasrallah. On the Ginzburg–Landau energy with a magnetic field vanishing along a curve. *Asymptot. Anal.*, 103(3):135–163, 2017.

[LG50]   L.D. Landau and V.L. Ginzburg. On the theory of superconductivity. *J. Exp. Theor. Phys.*, 20:1064, 1950.

[LL35]   F. London and H. London. The electromagnetic equations of the supraconductor. *Proc.Roy.Soc. A*, 149(866):71–88, 1935.

[LP62]   W.A. Little and R.D. Parks. Observation of quantum periodicity in the transition temperature of a superconducting cylinder. *Phys. Rev. Lett.*, 9(1):9–12, 1962.

[LP99]   K. Lu and X.B. Pan. Estimates of the upper critical field for the Ginzburg–Landau equations of superconductivity. *Physica D*, 127(1):73–104, 1999.

[LP00]   K. Lu and X.B. Pan. Gauge invariant eigenvalue problems in $\mathbb{R}^2$ and $\mathbb{R}^2_+$. *Trans. Amer. Math. Soc.*, 352(3):1247–1276, 2000.

[Miq16]   J.P. Miqueu. *Étude des états fondamentaux du Laplacien magnétique avec annulation locale du champ*. PhD thesis, Université de Rennes 1 sous le sceau de l'Université Bretagne Loire, 2016.

[ORK$^+$08]   L. Oroszlany, P. Rakyta, A. Kormanyos, C.J. Lambert, and J. Cserti. Theory of snake states in graphene. *Phys. Rev. B*, 77(8):081403, 2008.

[Pan02]   X.B. Pan. Surface superconductivity in applied magnetic fields above H_{C_2}. *Comm. Math. Phys.*, 228(2):327–370, 2002.

[PK02]   X.B. Pan and K.H. Kwek. Schrödinger operators with non-degenerately vanishing magnetic fields in bounded domains. *Trans. Amer. Math. Soc.*, 354(10):4201–4227, 2002.

[PM93]   F.M. Peeters and A. Matulis. Quantum structures created by nonhomogeneous magnetic fields. *Phys. Rev. B*, 48(20):15166, 1993.

[Ray09]   N. Raymond. Sharp asymptotics for the Neumann Laplacian with variable magnetic field: case of dimension 2. *Ann. Henri Poincaré*, 10(1):95–122, 2009.

[RP00]   J. Reijniers and F.M. Peeters. Snake orbits and related magnetic edge states. *J. Phys. Condens. Matter*, 12(47):9771, 2000.

[RVN15]   N. Raymond and S. Vu Ngoc. Geometry and spectrum in 2D magnetic wells. 65(1):137–169, 2015.

[Ser01]   S. Serfaty. On a model of rotating superfluids. *ESAIM: COCV*, 6:201–238, 2001.

[SJG63]   D. Saint-James and P.G. de Gennes. Onset of superconductivity in decreasing fields. *Phys. Lett.*, 7(5):306–308, 1963.

[SJST69]   D. Saint-James, G. Sarma, and E. J. Thomas. Type II superconductivity. 1969.

[SS03]   E. Sandier and S. Serfaty. The decrease of bulk superconductivity close to the second critical field in the Ginzburg–Landau model. *SIAM J. Math. Anal.*, 34(4):939–956, 2003.

[SS07]   E. Sandier and S. Serfaty. *Vortices in the Magnetic Ginzburg–Landau Model.* Progress in Nonlinear Partial Differential Equations and their Applications, Birkhäuser-Boston, 2007.

[STH⁺94]   A. Smith, R. Taboryski, L.T. Hansen, C.B. Sørensen, P. Hedegård, and P.E. Lindelof. *Phys. Rev. B*, 50(19):14726, 1994.

[Tin96]   M. Tinkham. *Introduction to superconductivity.* Second edition. McGraw-Hill, New York, 1996.

[TT90]   D.R. Tilley and J. Tilley. *Superfluidity and Superconductivity.* Third edition. Institute of Physics Publishing, Bristol and Philadelphia, 1990.

[Vid93]   G. Vidali. *Superconductivity: The Next Revolution?* Cambridge University Press, 1993.

Paper I

The influence of magnetic steps on bulk superconductivity

Wafaa Assaad,[*] Ayman Kachmar[†]

Abstract

We study the distribution of bulk superconductivity in the presence of an applied magnetic field, supposed to be a step function, modeled by the Ginzburg–Landau theory. Our results are valid for the minimizers of the two-dimensional Ginzburg–Landau functional with a large Ginzburg–Landau parameter and where the intensity of the applied magnetic field is comparable with the Ginzburg–Landau parameter.

1 Introduction and Main results

1.1 Motivation

The Ginzburg–Landau functional models the response of a (Type II) superconducting sample to an applied magnetic field. We focus on samples that occupy a long cylindrical domain and subjected to a magnetic field with direction parallel to the axis of the cylinder. This situation has been analyzed in many papers, see for instance the two monographs [FH10, SS07]. However, in the literature, the focus was on uniform applied magnetic fields. The case of non-uniform smooth magnetic fields has received attention in the papers [HK15, Att15b, Att15a, LP00, PK02, LP99].

In the present paper, we consider the situation when the applied magnetic field is a step function. Such fields might occur in many situations (e.g. [HPRS16, HS15, DHS14, HS08, RP00, PM93]). In particular

---

[*]Lund University, Department of Mathematics, Lund, Sweden
[†]Lebanese University, Department of Mathematics, Hadat, Lebanon

- if a sample is separated into two parts, one can apply on one part a uniform magnetic field from above, and on the other part, a uniform magnetic field from below.

- if a sample is not homogeneous, one can have a variable magnetic permeability. This may lead to a magnetic step function (e.g. [CDG95]).

1.2 The functional

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded and simply connected set and $B_0 : \Omega \to [-1, 1]$ be a measurable function. The Ginzburg–Landau (GL) functional in Ω is

$$\mathcal{E}_{\kappa,H}(\psi, \mathbf{A}) = \int_\Omega \left(\left| (\nabla - i\kappa H \mathbf{A})\psi \right|^2 - \kappa^2 |\psi|^2 + \frac{\kappa^2}{2} |\psi|^4 \right) dx$$
$$+ \kappa^2 H^2 \int_\Omega \left| \operatorname{curl} \mathbf{A} - B_0 \right|^2 dx. \quad (1.1)$$

Here, $\kappa > 0$ is the GL parameter, a characteristic of the superconducting material, $H > 0$ is the intensity of the applied magnetic field, $\psi \in H^1(\Omega, \mathbb{C})$ and $\mathbf{A} = (A_1, A_2) \in H^1(\Omega, \mathbb{R}^2)$. In physics, the domain Ω is the cross section of the sample, the function B_0 is the applied magnetic field, the function ψ is the order parameter and the vector field $\mathbf{A}$ is the magnetic potential. The configuration $(\psi, \mathbf{A})$ is interpreted as follows, $|\psi|^2$ measures the density of the superconducting electron pairs and $\operatorname{curl} \mathbf{A} = \partial_{x_1} A_2 - \partial_{x_2} A_1$ measures the induced magnetic field in the sample.

In this paper, we work under the following assumption on the domain Ω and the function B_0 (Figures 1 and 2):

Assumption 1.1.

1. $\Omega_1 \subset \Omega$ and $\Omega_2 \subset \Omega$ are two disjoint open sets.

2. Ω_1 and Ω_2 have a finite number of connected components.

3. $\partial\Omega_1$ and $\partial\Omega_2$ are piecewise smooth with (possibly) a finite number of corners.

4. $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ is the union of a finite number of disjoint simple smooth curves $\{\Gamma_k\}_{k \in \mathcal{K}}$; we will refer to Γ as the magnetic barrier.

5. $\Omega = (\Omega_1 \cup \Omega_2 \cup \Gamma)°$ and $\partial\Omega$ is smooth.

Figure 1: Schematic representations of the set Ω.
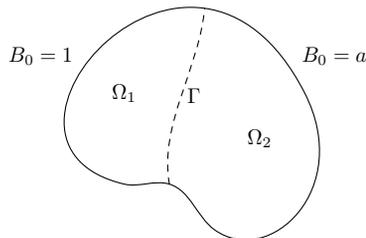
Figure 2: Schematic representation of the set Ω subjected to a step magnetic field B_0, with the magnetic edge Γ.

6. $\Gamma \cap \partial\Omega$ *is either empty or finite.*

7. *For any* $k \in \mathcal{K}$, *if* Γ_k *intersects* $\partial\Omega$ *then the intersection is at two distinct points. This intersection is transversal, i.e.* $T_{\partial\Omega} \times T_{\Gamma_k} \neq 0$ *at the intersection point, where* $T_{\partial\Omega}$ *and* T_{Γ_k} *are respectively unit tangent vectors of* $\partial\Omega$ *and* Γ_k.

8. $B_0 = \mathbb{1}_{\Omega_1} + a\mathbb{1}_{\Omega_2}$, *where* $a \in [-1, 1)\backslash\{0\}$ *is a given constant.*

We introduce the ground-state energy

$$\mathrm{E}_{\mathrm{g.st}}(\kappa, H) = \inf\{\mathscr{E}_{\kappa,H}(\psi, \mathbf{A}) \; : \; (\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)\} \qquad (\text{1.2})$$

where

$$H^1_{\mathrm{div}}(\Omega) = \left\{\mathbf{A} \in H^1(\Omega, \mathbb{R}^2) \; : \; \mathrm{div}\,\mathbf{A} = 0 \text{ in } \Omega, \; \mathbf{A} \cdot \nu_{\partial\Omega} = 0 \text{ on } \partial\Omega\right\}.$$

The functional $\mathscr{E}_{\kappa,H}$ is invariant under the gauge transformations $(\psi, \mathbf{A}) \mapsto (e^{i\varphi\kappa H}\psi, \mathbf{A} + \nabla\varphi)$. This gauge invariance yields (e.g. [FH10])

$$\mathrm{E}_{\mathrm{g.st}}(\kappa, H) = \inf\{\mathscr{E}_{\kappa,H}(\psi, \mathbf{A}) \; : \; (\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2)\}.$$

3

Critical points $(\psi, \mathbf{A}) \in H^1(\Omega, \mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$ of $\mathscr{E}_{\kappa, H}$ are weak solutions of the following GL equations:

$$
\begin{cases}
(\nabla - i\kappa H\mathbf{A})^2 \psi = \kappa^2(|\psi|^2 - 1)\psi & \text{in } \Omega, \\
-\nabla^\perp(\operatorname{curl} \mathbf{A} - B_0) = \frac{1}{\kappa H} \operatorname{Im}(\overline{\psi}(\nabla - i\kappa H\mathbf{A})\psi) & \text{in } \Omega, \\
\nu \cdot (\nabla - i\kappa H\mathbf{A}) = 0 & \text{on } \partial\Omega, \\
\operatorname{curl} \mathbf{A} = B_0 & \text{on } \partial\Omega.
\end{cases}
\tag{1.3}
$$

Here $\nabla^\perp = (\partial_{x_2}, -\partial_{x_1})$ is the Hodge gradient.

1.3 Energy and order parameter asymptotics

The statement of our main results involves a continuous function $g : [0, +\infty) \to [-1/2, 0]$ constructed in [FK13, SS03]. The function g is increasing and satisfies $g(0) = -1/2$, $g(b) = 0$ for all $b \geq 1$, and $-1/2 < g(b) < 0$ for all $b \in (0, 1)$.

Theorem 1.2 (Global asymptotics). *Let $\tau \in (3/2, 2)$ and $0 < c_1 < c_2$ be constants. Under Assumption 1.1, there exist constants $C > 0$ and $\kappa_0 > 0$ such that if*

$$
\kappa \geq \kappa_0 \quad \text{and} \quad c_1 \leq \frac{H}{\kappa} \leq c_2,
\tag{1.4}
$$

then

1. *the ground-state energy in (1.2) satisfies*

$$
-C\kappa^\tau \leq \mathrm{E}_{\mathrm{g.st}}(\kappa, H) - \kappa^2 \int_\Omega g\left(\frac{H}{\kappa}|B_0(x)|\right) dx \leq C\kappa^{3/2}.
\tag{1.5}
$$

2. *for every critical point $(\psi, \mathbf{A})$ of (1.1),*

$$
\int_\Omega |\psi|^4 \, dx + 2 \int_\Omega g\left(\frac{H}{\kappa}|B_0(x)|\right) dx \leq C\kappa^{-2+\tau}.
\tag{1.6}
$$

3. *for every minimizer $(\psi, \mathbf{A})$ of (1.1),*

$$
\left| \int_\Omega |\psi|^4 \, dx + 2 \int_\Omega g\left(\frac{H}{\kappa}|B_0(x)|\right) dx \right| \leq C\kappa^{-2+\tau},
\tag{1.7}
$$

   and

$$
\kappa^2 H^2 \int_\Omega |\operatorname{curl} \mathbf{A} - B_0|^2 \, dx \leq C\kappa^\tau.
$$

Theorem 1.3 (Local asymptotics). *Suppose that the assumptions of Theorem 1.2 are satisfied. Let $D \subset \Omega$ be an open set such that D and $\Omega \setminus \overline{D}$ have piecewise-smooth boundaries (with possibly a finite number of corners). There exist $\kappa_0 > 0$ and a function $err : (\kappa_0, +\infty) \to \mathbb{R}_+$ such that $\lim\limits_{\kappa \to +\infty} err(\kappa) = 0$, and if (1.4) holds and $(\psi, \mathbf{A})$ is a minimizer of (1.1) then*

$$\left| \int_D |\psi|^4 \, dx + 2 \int_D g\left(\frac{H}{\kappa} |B_0(x)| \right) dx \right| \leq err(\kappa).$$

Remark 1.4. The conditions imposed on the set D in Theorem 1.3 particularly aim at excluding the presence of cusps in the boundary of D and that of $\Omega \setminus \overline{D}$. This will prove technically useful later (see Theorem 5.3).

1.4 Discussion of Theorem 1.3

The result in Theorem 1.3 displays the strength of superconductivity in the bulk of Ω. We will use the following notation. Let $\omega \subset \mathbb{R}^2$ be an open set, (f_κ) be a family of functions in $L^\infty(\omega)$, $\alpha \in \mathbb{C}$ and dx be the Lebesgue measure in $\mathbb{R}^2$. By writing

$$f_\kappa dx \rightharpoonup \alpha dx \quad \text{in } \omega, \qquad \text{as } \kappa \to +\infty,$$

we mean that, for every ball B such that $\overline{B} \subset \omega$,

$$\int_B f_\kappa dx \to \alpha |B|, \qquad \text{as } \kappa \to +\infty.$$

Now, we return back to the result in Theorem 1.3. Suppose that $H = b\kappa$ and $b \in (0, +\infty)$ is a constant. We observe that:

1. If $0 < b < 1$, then

   $$|\psi|^4 dx \rightharpoonup -2g(b) \, dx \text{ in } \Omega_1 \quad \text{and} \quad |\psi|^4 dx \rightharpoonup -2g(b|a|) \, dx \text{ in } \Omega_2.$$

   Hence the bulk of Ω carries superconductivity everywhere, but when $0 < |a| < 1$, $0 < -g(b) < -g(b|a|)$ and the strength of superconductivity in Ω_1 is smaller than that in Ω_2.

2. If $1 \leq b < 1/|a|$, then

   $$|\psi|^4 dx \rightharpoonup 0 \text{ in } \Omega_1 \quad \text{and} \quad |\psi|^4 dx \rightharpoonup -2g(b|a|) \, dx \text{ in } \Omega_2,$$
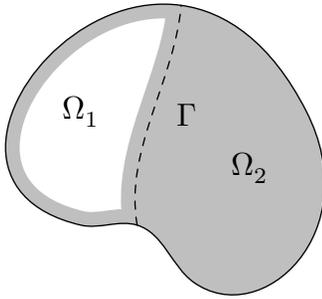
Figure 3: Schematic representation of Ω in the regime $H/\kappa = b$ where $1 \leq b < 1/|a|$. The white region is in a normal state, the dark region may carry superconductivity (the whole interior of Ω_2 is superconducting).
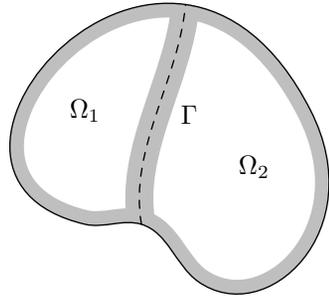
Figure 4: Schematic representation of Ω in the regime $H/\kappa = b$ where $b \geq 1/|a|$. The white region is in a normal state, the dark region may carry superconductivity.

with $g(b|a|) < 0$. In this regime, superconductivity becomes negligible in the bulk of Ω_1 but persists in the bulk of Ω_2 (see Figure 3). Theorem 1.5 below will sharpen this point by establishing that $|\psi|$ is exponentially small in the bulk of Ω_1 (when $b > 1$). However, in light of the analysis in the book of Fournais–Helffer [FH10], the boundary of Ω_1 may carry superconductivity. This point deserves a detailed analysis.

3. If $b \geq 1/|a|$, then superconductivity is negligible in the bulk of Ω_1 and Ω_2 (see Figure 4). However, one might find an interesting behavior near the critical value $b \sim 1/|a|$. In the spirit of the analysis in [FK11], one expects to find superconductivity in the bulk of Ω_2, but with a weak strength. This superconductivity can be evenly distributed and decays as b gradually increases past the value $1/|a|$.

4. (Breakdown of superconductivity [GP99]) If $b \gg 1/|a|$, one expects that $\psi = 0$ and superconductivity is lost in the sample. To this end, the spectral analysis in [HPRS16] must be useful. In the spirit of the book [FH10], this regime is related to the analysis of the third critical field(s) where the transition to the purely normal state occurs.

The interesting case $a = -1$ is reminiscent of the situation of a *smooth* and sign-changing magnetic field analyzed in the paper by Helffer–Kachmar [HK15]. Note that Theorem 1.3 yields that superconductivity is evenly distributed in Ω_1 and Ω_2 as long as $0 < b < 1$ (see Figure 5). In the critical regime $b \sim 1$, one might
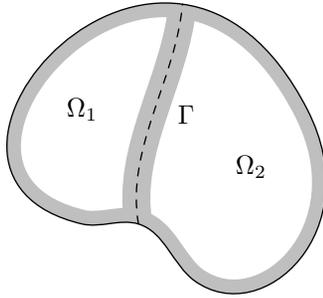
Figure 5: Schematic representation of the set Ω in the regime $H/\kappa = b$ where $a = -1$ and $b > 1$. The white region is in a normal state, the dark region may carry superconductivity.

find that superconductivity is distributed along the curve Γ that separates Ω_1 and Ω_2, in the same spirit of the paper [HK15].

1.5 Exponential decay in regions with larger magnetic intensity

Our last result establishes a regime for the strength of the magnetic field where the order parameter is exponentially small in the bulk of Ω_1. The relevance of this theorem is that, together with Theorem 1.3, display a regime of the intensity of the applied magnetic field such that $|\psi|^2$ is exponentially small in the bulk of Ω_1 while it is of order $\mathcal{O}(1)$ in Ω_2.

Theorem 1.5 (Exponential decay of the order parameter). *Let λ, ε, $c_2 > 0$ be constants such that $0 < \varepsilon < \sqrt{\lambda}$ and $1 + \lambda < c_2$. There exist constants C, $\kappa_0 > 0$ such that, if*

$$\kappa \geq \kappa_0, \quad (1 + \lambda)\kappa \leq H \leq c_2\kappa, \quad (\psi, \mathbf{A})_{\kappa, H} \text{ is a solution of } (1.3),$$

then

$$\int_{\Omega_1 \cap \{\operatorname{dist}(x, \partial\Omega_1) \geq \frac{1}{\sqrt{\kappa H}}\}} \left(|\psi|^2 + \tfrac{1}{\kappa H} |(\nabla - i\kappa H\mathbf{A})\psi|^2 \right)$$
$$\exp\left(2\varepsilon\sqrt{\kappa H}\operatorname{dist}(x, \partial\Omega_1)\right) dx \leq C \int_{\Omega_1 \cap \{\operatorname{dist}(x, \partial\Omega_1) \leq \frac{1}{\sqrt{\kappa H}}\}} |\psi|^2 \, dx.$$

The proof of Theorem 1.5 does not follow the same pattern for similar situations in [FK09, BNF07], as there is a specific difficulty when dealing with the critical points of (1.1). The technical reason behind this is as follows. A necessary ingredient

7

in the proof given in [FK09, BNF07] is the following estimate of the magnetic energy

$$\|\operatorname{curl}\mathbf{A} - B_0\|_{L^2(\Omega)} = o(\kappa^{-1}) \quad (\kappa \to +\infty).$$

For critical points, we have the following estimate from [FH10, Lemma 10.3.2]

$$\|\operatorname{curl}\mathbf{A} - B_0\|_{L^2(\Omega)} \le C\kappa^{-1}\|\psi\|_{L^2(\Omega)}\|\psi\|_{L^4(\Omega)}.$$

To control the L^2- and L^4- norms of ψ, we use Theorem 1.3. But this will give that $\|\psi\|_{L^4(\Omega)} = o(1)$ only for $H \ge |a|^{-1}\kappa$, the necessary condition to get that $g(H\kappa^{-1}) = g(H\kappa^{-1}|a|) = 0$. This condition does not cover all the values of H in Theorem 1.5. As a substitute, we may choose to control the magnetic energy by the estimate in Theorem 1.2 that is valid for *minimizing* configurations only. Consequently, the proof in [FK09, BNF07] applies in our case but for the minimizing configurations.

The exponential decay for critical configurations is obtained by a local argument. Instead of searching for an *energy* estimate of $\|\operatorname{curl}\mathbf{A} - B_0\|_2$, we use the Hölder estimate $\|\mathbf{A} - \mathbf{F}\|_{C^{0,\alpha}} = \mathcal{O}(\kappa^{-1})$. Here $\operatorname{curl}\mathbf{F} = B_0$. When working locally in a region Q_ℓ of diameter $\ell \ll 1$, we may apply a gauge transformation and get that $\|\mathbf{A} - \mathbf{F}\|_{L^\infty(Q_\ell)} = \mathcal{O}(\kappa^{-1}\ell) = o(\kappa^{-1})$. This estimate is sufficient for our needs, as shown later in Section 7.

1.6 Notation

- The letter C denotes a positive constant whose value may change from line to line. Unless otherwise stated, the constant C may depend on the function B_0 and the domain Ω, but is independent of κ, H and the minimizers $(\psi, \mathbf{A})$ of the functional in (1.1).

- Given $\ell > 0$ and $x = (x_1, x_2) \in \mathbb{R}^2$, we denote by

$$Q_\ell(x) = (-\frac{\ell}{2} + x_1, \frac{\ell}{2} + x_1) \times (-\frac{\ell}{2} + x_2, \frac{\ell}{2} + x_2)$$

the square of side length ℓ centered at x.

- Let $a(\kappa)$ and $b(\kappa)$ be two positive functions, we write:

  - $a(\kappa) \ll b(\kappa)$, if $a(\kappa)/b(\kappa) \to 0$ as $\kappa \to +\infty$.

– $a(\kappa) \approx b(\kappa)$, if there exist constants κ_0, C_1 and C_2 such that for all $\kappa \geq \kappa_0$, $C_1 a(\kappa) \leq b(\kappa) \leq C_2 a(\kappa)$.

- The quantity $o(1)$ indicates a function of κ such that $|o(1)| \ll 1$. Any expression $o(1)$ is independent of the critical points $(\psi, \mathbf{A})$ of (1.1). Similarly, $\mathcal{O}(1)$ indicates a function of κ, bounded by a constant independent of the critical points of (1.1).

- Let $n \in \mathbb{N}$, $p \in \mathbb{N}$. We use the following Sobolev spaces:

$$W^{n,p}(\Omega) := \left\{ f \in L^p(\Omega) \mid D^\alpha f \in L^p(\Omega), \text{ for all } |\alpha| \leq n \right\},$$

$$H^n(\Omega) := W^{n,2}(\Omega).$$

1.7 On the proofs and the organization of the paper

The results in this paper can be viewed as generalizations of those in [SS03] already proved for the case $B_0 = 1$. Theorem 1.5 is reminiscent of the exponential bounds in [BNF07]. However, the proofs in this paper are simpler than those in [SS03] and contain new ingredients that we summarize below:

- We took advantage of all the available information regarding the limiting function $g(\cdot)$ proved in [FK13] and [Att15a] ;

- We did not use the *a priori* elliptic estimates, e.g. the L^∞-bound $\|(\nabla - i\kappa H\mathbf{A})\psi\|_\infty \leq C\kappa$. Such an estimate is not known to hold in our case of a non-smooth magnetic field B_0. However, we used the simple energy bound $\|(\nabla - i\kappa H\mathbf{A})\psi\|_2 \leq C\kappa$ together with the regularity of the curl-div system (see Theorem 4.2). This method is already used for the three dimensional problem in [Kac06] ;

- To prove Theorem 1.5, we did not establish *weak* decay estimates as done in [BNF07].

The rest of the paper is divided into six sections and two appendices. Section 2 collects the needed properties of the limiting energy $g(\cdot)$. Section 3 establishes an upper bound of the ground-state energy. Section 4 proves the necessary estimates on the critical points of the functional in (1.1). These estimates are used in Section 5 to establish certain local estimates of the order parameter. In Section 6, we complete the proof of Theorem 1.2 and Theorem 1.3. Section 7 is devoted to the proof

of Theorem 1.5. Finally, the appendices collect standard results that are used throughout the paper.

2 The limiting energies

Let $R > 0$ and $Q_R = (-R/2, R/2) \times (-R/2, R/2)$. We define the following GL energy with the constant magnetic field on $H^1(Q_R)$ by

$$G_{b,Q_R}^\sigma(u) = \int_{Q_R} \left(b |(\nabla - i\sigma A_0)u|^2 - |u|^2 + \frac{1}{2}|u|^4 \right) dx. \qquad (2.1)$$

Here $b \geq 0$, $\sigma \in \{-1, +1\}$ and A_0 is the canonical magnetic potential

$$A_0 = \frac{1}{2}(-x_2, x_1) \qquad \forall x = (x_1, x_2) \in \mathbb{R}^2, \qquad (2.2)$$

which satisfies curl $A_0 = 1$.

We introduce the two ground-state energies

$$
\begin{aligned}
m_0(b, R, \sigma) &= \inf_{u \in H_0^1(Q_R)} G_{b,Q_R}^\sigma(u), \\
m(b, R, \sigma) &= \inf_{u \in H^1(Q_R)} G_{b,Q_R}^\sigma(u).
\end{aligned}
\qquad (2.3)
$$

Notice that $G_{b,Q_R}^{+1}(u) = G_{b,Q_R}^{-1}(\overline{u})$. As an immediate consequence, we observe that

$$\inf_{u \in \mathcal{V}} G_{b,Q_R}^{+1}(u) = \inf_{u \in \mathcal{V}} G_{b,Q_R}^{-1}(u) \quad \text{where} \quad \mathcal{V} \in \{H_0^1(Q_R), H^1(Q_R)\}, \qquad (2.4)$$

and the values of $m_0(b, R, \sigma)$ and $m(b, R, \sigma)$ are independent of $\sigma \in \{-1, 1\}$.

In the rest of the paper, we will denote these two values by $m_0(b, R)$ and $m(b, R)$ respectively, hence

$$m_0(b, R, \sigma) = m_0(b, R) \quad \text{and} \quad m(b, R, \sigma) = m(b, R) \quad (\sigma \in \{-1, 1\}). \quad (2.5)$$

We cite the following result from [Att15a] (also see [FK13, SS03]).

Theorem 2.1.

1. For all $b \geq 1$ and $R > 0$, we have $m_0(b, R) = 0$.

2. *For all $b \in [0, +\infty)$, there exists a constant $g(b) \leq 0$ such that*

$$g(b) = \lim_{R \to +\infty} \frac{m_0(b, R)}{R^2} = \lim_{R \to +\infty} \frac{m(b, R)}{R^2} \quad \text{and} \quad g(0) = -\frac{1}{2}.$$

3. *The function $[0, +\infty) \ni b \mapsto g(b)$ is continuous, non-decreasing, vanishes on $[1, +\infty)$ and $g(\cdot) < 0$ on $[0, 1)$.*

4. *There exist constants C and R_0 such that, for all $R \geq R_0$ and $b \in [0, 1]$,*

$$g(b) \leq \frac{m_0(b, R)}{R^2} \leq g(b) + \frac{C}{R} \quad \text{and} \quad g(b) - \frac{C}{R} \leq \frac{m(b, R)}{R^2} \leq g(b) + \frac{C}{R}.$$

3 Energy Upper Bound

The aim of this section is to prove:

Proposition 3.1. *Under the assumption of Theorem 1.2, there exist positive constants C and κ_0 such that if (1.4) holds, then the ground-state energy $E_{g.st}(\kappa, H)$ in (1.2) satisfies*

$$E_{g.st}(\kappa, H) \leq \kappa^2 \int_\Omega g\left(\frac{H}{\kappa}|B_0(x)|\right) dx + C\kappa^{3/2}.$$

Before writing the proof of Proposition 3.1, we introduce some notation. If $D \subset \Omega$ is an open set, we introduce the local energy of the configuration $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\text{div}}(\Omega)$ in the domain $D \subset \Omega$ as follows

$$\mathcal{E}_0(\psi, \mathbf{A}; D) = \int_D \left(|(\nabla - i\kappa H\mathbf{A})\psi|^2 - \kappa^2|\psi|^2 + \frac{1}{2}\kappa^2|\psi|^4\right) dx, \qquad (3.1)$$

$$\mathcal{E}(\psi, \mathbf{A}; D) = \mathcal{E}_0(\psi, \mathbf{A}; D) + (\kappa H)^2 \int_\Omega |\operatorname{curl}(\mathbf{A} - \mathbf{F})|^2 \, dx. \qquad (3.2)$$

In Lemma A.1, we constructed a vector field $\mathbf{F}$ satisfying

$$\mathbf{F} \in H^1_{\text{div}}(\Omega) \quad \text{and} \quad \operatorname{curl} \mathbf{F} = B_0 \quad \text{in } \Omega. \qquad (3.3)$$

Proof of Proposition 3.1.
Step 1 (Introducing a lattice of squares). We introduce the small parameter

$$\ell = \kappa^{-1/2}. \qquad (3.4)$$

Consider the lattice $L_\ell := \ell\mathbb{Z} \times \ell\mathbb{Z}$. Let

$$\mathcal{I}_\ell^j = \{z \in L_\ell : \overline{Q_\ell(z)} \subset \Omega_j\} \text{ for } j \in \{1, 2\}, \text{ and } \mathcal{I}_\ell = \mathcal{I}_\ell^1 \cup \mathcal{I}_\ell^2, \qquad (3.5)$$

where $Q_\ell(z)$ denotes the square of center z and side-length ℓ. By Assumption 1.1, the number

$$N = \text{Card } \mathcal{I}_\ell$$

satisfies

$$|\Omega|\ell^{-2} - \mathcal{O}(\ell^{-1}) \leq N \leq |\Omega|\ell^{-2} \quad (\ell \to 0_+). \qquad (3.6)$$

Step 2 (Defining a trial state). For all $z \in \mathcal{I}_\ell$, let $\varphi_z \in C^2(\overline{Q_\ell(z)})$ be the function introduced in Lemma A.2 and

$$b_z = \frac{H}{\kappa}|B_0(z)| \quad , \quad R_z = \ell\sqrt{\kappa H |B_0(z)|}, \quad \sigma_z = \frac{B_0(z)}{|B_0(z)|}. \qquad (3.7)$$

The function φ_z satisfies

$$\mathbf{F}(x) = \nabla\varphi_z(x) + B_0(z)\mathbf{A}_0(x - z), \qquad (x \in Q_\ell(z)). \qquad (3.8)$$

We define the function $v \in H_0^1(\Omega)$ as follows,

$$v(x) = \begin{cases} e^{i\kappa H\varphi_z} u_{b_z,R_z,\sigma_z}\left(\frac{R_z}{\ell}(x - z)\right) & \text{if } x \in Q_\ell(z), \\ 0 & \text{if } x \in \Omega\backslash\Omega_\ell, \end{cases}$$

where

$$\Omega_\ell = \left(\bigcup_{z \in \mathcal{I}_\ell} \overline{Q_\ell(z)}\right)^{\circ}, \qquad (3.9)$$

and $u_{b_z,R_z,\sigma_z} \in H_0^1(Q_R)$ is a minimizer of the functional in (2.1) (with $(b, R, \sigma) = (b_z, R_z, \sigma_z)$). In the sequel, we will omit the reference to (b_z, R_z, σ_z) in the notation u_{b_z,R_z,σ_z} and write simply

$$u_z = u_{b_z,R_z,\sigma_z}.$$

Step 3 (Energy of the trial state). We compute the energy of the configuration $(v, \mathbf{F})$. We have the obvious identities (see (3.1)–(3.3))

$$\begin{aligned}
\mathcal{E}(v, \mathbf{F}; \Omega) &= \int_\Omega \left(|(\nabla - i\kappa H\mathbf{F})v|^2 - \kappa^2|v|^2 + \frac{1}{2}\kappa^2|v|^4\right) dx \\
&= \sum_{z \in \mathcal{I}_\ell} \mathcal{E}_0(v, \mathbf{F}; Q_\ell(z)).
\end{aligned} \qquad (3.10)$$

Using (3.8), we write

$$\mathcal{E}_0(v, \mathbf{F}; Q_\ell(z)) = \mathcal{E}_0\big(e^{-i\kappa H \varphi_z}v, \sigma_z | B_0(z)| \mathbf{A}_0(x - z); Q_\ell(z)\big).$$

By doing the change of variable $y = \dfrac{R_z}{\ell}(x - z)$, we get

$$\mathcal{E}_0\big(e^{-i\kappa H \varphi_z}v, \sigma_z | B_0(z)| \mathbf{A}_0(x - z); Q_\ell(z)\big) = $$
$$\frac{\kappa}{H | B_0(z)|} \int_{Q_{R_z}} \Big(b_z |(\nabla_y - i\sigma_z \mathbf{A}_0)u_z|^2 - |u_z|^2 + \frac{1}{2}|u_z|^4 \Big) \, dy.$$

By using (2.4), we get

$$\mathcal{E}_0(v, \mathbf{F}; Q_\ell(z)) = \mathcal{E}_0\big(e^{-i\kappa H \varphi_z}v, \sigma_z | B_0(z)| \mathbf{A}_0(x - z); Q_\ell(z)\big) = \frac{1}{b_z} m_0(b_z, R_z).$$

Since $\ell = \kappa^{-1/2}$ and $H \geq c_1 \kappa$, $R_z \geq 1$ (see (3.7)). We use Theorem (2.1) to write

$$m_0(b_z, R_z) \leq g(b_z) R_z^2 + C R_z.$$

Consequently,

$$\mathcal{E}_0(v, \mathbf{F}; Q_\ell(z)) \leq \ell^2 \kappa^2 g\Big(\frac{H}{\kappa}| B_0(z)|\Big) + C\ell \kappa. \qquad (3.11)$$

We insert (3.11) into (3.10) to get

$$\mathcal{E}(v, \mathbf{F}; \Omega) = \sum_{z \in \mathcal{I}_\ell} \Big(\ell^2 \kappa^2 g\Big(\frac{H}{\kappa}| B_0(z)|\Big) + C\ell \kappa \Big)$$
$$\leq \kappa^2 \int_{\Omega_\ell} g\Big(\frac{H}{\kappa}| B_0(x)|\Big) \, dx + C\ell \kappa N,$$

where $N = \text{Card } \mathcal{I}_\ell$. Now, using (3.6) and the fact that $-1/2 \leq g(\cdot) \leq 0$, we get

$$\mathcal{E}(v, \mathbf{F}; \Omega) \leq \kappa^2 \int_\Omega g\Big(\frac{H}{\kappa}| B_0(x)|\Big) \, dx + \frac{1}{2}|\Omega \setminus \Omega_\ell|\kappa^2 + C\frac{\kappa}{\ell}.$$

To complete the proof of Proposition 3.1, we use $E_{\text{g.st}}(\kappa, H) \leq \mathcal{E}(v, \mathbf{F}; \Omega)$, $\ell = \kappa^{-1/2}$ and Assumption 1.1 which yields

$$|\Omega \setminus \Omega_\ell| = \mathcal{O}(\ell) \quad \text{as } \ell \to 0_+. \qquad (3.12)$$

$\square$

4   A Priori Estimates

In the derivation of a lower bound of the energy in (1.1), various error terms arise. These terms are controlled by the estimates that we present in this section.

Proposition 4.1. *If* $(\psi, \mathbf{A}) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$ *is a weak solution to* (1.3), *then*

$$\|\psi\|_{L^\infty} \leq 1.$$

A detailed proof of Proposition 4.1 can be found in [FH10, Proposition 10.3.1]. It only needs the assumption that $B_0 \in L^2(\Omega)$.

Proposition 4.1 is used in the next theorem to give *a priori* estimates on the solutions of the GL equations (1.3).

Theorem 4.2. *Let* $0 < c_1 < c_2$ *and* $\alpha \in (0, 1)$ *be constants. Suppose that the conditions in Assumption 1.1 hold. There exist two constants* $\kappa_0 > 0$ *and* $C > 0$ *such that, if* (1.4) *holds and* $(\psi, \mathbf{A}) \in H^1(\Omega) \times H^1_{\mathrm{div}}(\Omega)$ *is a solution of* (1.3), *then*

1. $\|(\nabla - i\kappa H\mathbf{A})\psi\|_{L^2(\Omega)} \leq C\kappa$;

2. $\|\operatorname{curl}(\mathbf{A} - \mathbf{F})\|_{L^2(\Omega)} \leq C/\kappa$;

3. $\mathbf{A} - \mathbf{F} \in H^2(\Omega)$ *and* $\|\mathbf{A} - \mathbf{F}\|_{H^2(\Omega)} \leq C/\kappa$;

4. $\mathbf{A} - \mathbf{F} \in C^{0,\alpha}(\overline{\Omega})$ *and* $\|\mathbf{A} - \mathbf{F}\|_{C^{0,\alpha}(\overline{\Omega})} \leq C/\kappa$.

Proof. The inequalities in items (1) and (2) of Theorem 4.2 follow from [FH10, Lemma 10.3.2].

Now we prove item (3) of this theorem. Let $a = \mathbf{A} - \mathbf{F} \in H^1_{\mathrm{div}}(\Omega)$. By (1.3), we know that $a \in H^1(\Omega)$ and $\operatorname{curl} a \in H^1_0(\Omega)$. Using Lemma B.1 and the second equation in (1.3), we get $a \in H^2(\Omega)$ and,

$$\|\mathbf{A} - \mathbf{F}\|_{H^2(\Omega)} \leq C\|\nabla(\operatorname{curl}(\mathbf{A} - \mathbf{F}))\|_{L^2(\Omega)} \leq \frac{C}{\kappa H}\|\overline{\psi}(\nabla - i\kappa H\mathbf{A})\psi\|_{L^2(\Omega)}.$$

Using the bound $H \geq c_1\kappa$, Proposition 4.1 and the estimate in item (1) of Theorem 4.2, we get the estimate in item (3) above.

Finally, the conclusion in item (4) in Theorem 4.2 is a consequence of the conclusion in item (3) and the Sobolev embedding of $H^2(\Omega)$ in $C^{0,\alpha}(\overline{\Omega})$. $\square$

Remark 4.3. In Theorem 4.2, the constant C in item (4) depends on α. Later in this paper, a fixed value of α is chosen. For this reason, we simply denote this constant by C instead of $C(\alpha)$.

Remark 4.4. In Theorem 4.2, Assumption 1.1 on the set Γ is used in the derivation of items (3) and (4). In fact, Assumption 1.1 ensures that the domains Ω_1 and Ω_2 satisfy the *cone condition*, which in turn allows us to use the Sobolev embedding theorems (see e.g. the proof of Lemma B.1).

5 Order Parameter Upper Bound

The aim of this section is to establish an upper bound for the L^4-norm of the order parameter, ψ, corresponding to a critical point, $(\psi, \mathbf{A})$, of the functional in (1.1). As a consequence, we will be able to complete the proof of Theorems 1.2 and 1.3 in the next section.

Recall that $Q_\ell(x_0)$ denotes the square of center x_0 and side length ℓ. In the statements of Lemma 5.1, Proposition 5.2 and Theorem 5.3, we will use the functional $\mathscr{E}_0$ in (3.1).

Lemma 5.1. *Let $\alpha \in (0, 1)$ and $0 < c_1 < c_2$ be constants. There exist positive constants C and κ_0 such that, if*

- (1.4) *holds;*

- $0 < \delta < 1$, $0 < \ell < 1$, $x_0 \in \Omega$;

- $\overline{Q_\ell(x_0)} \subset \Omega_j$ *for some* $j \in \{1, 2\}$;

- $(\psi, \mathbf{A}) \in H^1(\Omega, \mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$ *is a critical point of* (1.1);

- $b \in C^1(\overline{\Omega})$, $\|b\|_\infty \leq 1$;

then the following inequality holds

$$
\mathscr{E}_0(b\psi, \mathbf{A}; Q_\ell(x_0)) \geq (1 - \delta) \mathscr{E}_0(e^{-i\kappa H \eta} b\psi, \sigma_{x_0} |B_0(x_0)| \mathbf{A}_0(x - x_0); Q_\ell(x_0))
$$
$$
- C\big(\delta^{-1} \ell^{2\alpha+2} \kappa^2 + \delta \ell^2 \kappa^2\big),
$$

for a certain function $\eta \in H^2(\overline{Q_\ell(x_0)})$.

Proof. Let $\phi_{x_0}(x) = (A(x_0) - F(x_0)) \cdot x$. Using Theorem 4.2, we get for all $x \in Q_\ell(x_0)$

$$|A(x) - \nabla\phi_{x_0}(x) - F(x)| \leq \|(A - F)\|_{C^{0,\alpha}(\overline{\Omega}_j)}|x - x_0|^\alpha$$

$$\leq \frac{C}{\kappa}\ell^\alpha. \tag{5.1}$$

Define $\eta = \varphi_{x_0} + \phi_{x_0}$ where φ_{x_0} is the function introduced in Lemma A.2 and satisfying

$$F(x) = \nabla\varphi_{x_0}(x) + \sigma_{x_0}|B_0(x_0)|A_0(x - x_0), \quad \sigma_{x_0} = \frac{B_0(x_0)}{|B_0(x_0)|}, \quad x \in Q_\ell(x_0).$$

Let

$$u = e^{-i\kappa H\eta}b\psi. \tag{5.2}$$

Using the gauge invariance, the Cauchy-Schwarz inequality and (5.1), we write

$$\left|(\nabla - i\kappa H A)b\psi\right|^2 \geq (1 - \delta)\left|(\nabla - i\kappa H(\sigma_{x_0}|B_0(x_0)|A_0(x - x_0)))u\right|^2$$
$$- C\delta^{-1}\ell^{2\alpha}\kappa^2 b^2|\psi|^2. \tag{5.3}$$

Recalling the definition of u and using the estimates $\|\psi\|_\infty \leq 1$ and $\|b\|_\infty \leq 1$, we complete the proof. $\square$

Proposition 5.2. *Let $\gamma \in (0, 1)$. Under the assumptions in Lemma 5.1, it holds*

$$\mathcal{E}_0(b\psi, A; Q_\ell(x_0)) \geq \ell^2\kappa^2 g\left(\frac{H}{\kappa}|B_0(x_0)|\right) - C(\ell^{2\alpha+1}\kappa^2 + \ell\kappa + \ell^3\kappa^2), \tag{5.4}$$

and consequently,

$$\int_{Q_\ell(x_0)}|\psi|^4\,dx \leq -2\ell^2 g\left(\frac{H}{\kappa}|B_0(x_0)|\right) + C(\ell^{2\alpha+1} + \ell\kappa^{-1} + \ell^3 + \gamma^{-1}\kappa^{-2}). \tag{5.5}$$

Proof. Let

$$b = \frac{H}{\kappa}|B_0(x_0)|, \quad R = \ell\sqrt{\kappa H|B_0(x_0)|}, \quad \sigma_{x_0} = \frac{B_0(x_0)}{|B_0(x_0)|},$$

and define the rescaled function $v(x) = u((\ell/R)x + x_0)$ for all $x \in Q_R = (-R/2, R/2)^2$, where the function u is defined in (5.2). The change of variable $y = R/\ell(x - x_0)$ yields

$$\mathcal{E}_0(u, \sigma_{x_0}|B_0(x_0)|A_0(x - x_0); Q_\ell(x_0)) = \frac{1}{b}G_{b,Q_R}^{\sigma_{x_0}}(v).$$

Since $v \in H^1(Q_R)$, then by (2.3), (2.5) and Theorem 2.1,

$$
\begin{aligned}
\mathcal{E}_0\big(u, \sigma_{x_0} | B_0(x_0) | \mathbf{A}_0(x - x_0); Q_\ell(x_0)\big) &\geq \frac{1}{b} m(b, R) \\
&\geq \frac{1}{b}\big(g(b) R^2 - C R\big).
\end{aligned}
$$

Inserting this into the estimate in Lemma 5.1 and taking $\delta = \ell$ gives the proof of (5.4).

Now, we prove (5.5). Let $\hat{\ell} = (1 + \gamma)\ell$, and assume that ℓ is sufficiently small so that $\overline{Q_{\hat{\ell}}(x_0)} \subset \Omega_j$, for some $j \in \{1, 2\}$. Consider a smooth function f satisfying

$$
f = 1 \text{ in } Q_\ell(x_0), \quad f = 0 \text{ in } Q_{\hat{\ell}}(x_0)^{\complement}, \quad 0 \leq f \leq 1, \quad \text{and } |\nabla f| \leq C \gamma^{-1} \ell^{-1}. \tag{5.6}
$$

Using the first equation of (1.3), we multiply both sides by $f^2 \overline{\psi}$ and we integrate by parts over $Q_{\hat{\ell}}(x_0)$ to get

$$
\int_{Q_{\hat{\ell}}(x_0)} \big(|(\nabla - i\kappa H\mathbf{A}) f \psi|^2 - |\nabla f|^2 |\psi|^2\big)\, dx = \kappa^2 \int_{Q_{\hat{\ell}}(x_0)} \big(|\psi|^2 - |\psi|^4\big) f^2\, dx.
$$

Consequently,

$$
\mathcal{E}_0\big(f\psi, \mathbf{A}; Q_{\hat{\ell}}(x_0)\big) = \kappa^2 \int_{Q_{\hat{\ell}}(x_0)} f^2\Big(-1 + \frac{1}{2} f^2\Big)|\psi|^4\, dx + \int_{Q_{\hat{\ell}}(x_0)} |\nabla f|^2 |\psi|^2\, dx. \tag{5.7}
$$

By (5.6), we have $f = 1$ in $Q_\ell(x_0)$ and $-1 + 1/2 f^2 \leq -1/2$ in $Q_{\hat{\ell}}(x_0)$, thus

$$
\int_{Q_{\hat{\ell}}(x_0)} f^2\Big(-1 + \frac{1}{2} f^2\Big)|\psi|^4\, dx \leq -\frac{1}{2} \int_{Q_\ell(x_0)} |\psi|^4\, dx.
$$

We use the previous inequality, (5.7) and the estimate $\big|\operatorname{supp} |\nabla f|\big| \leq C \gamma \ell^2$ to obtain

$$
\mathcal{E}_0\big(f\psi, \mathbf{A}; Q_{\hat{\ell}}(x_0)\big) \leq -\frac{\kappa^2}{2} \int_{Q_\ell(x_0)} |\psi|^4\, dx + C \gamma^{-1}. \tag{5.8}
$$

We plug the lower bound in (5.4) into (5.8) for $h = f$ to complete the proof. $\qquad \square$

Theorem 5.3. *Let $\tau \in (3/2, 2)$ and $0 < c_1 < c_2$ be constants. Let $D \subset \Omega$ be an open set such that D and $\Omega \setminus \overline{D}$ have piecewise-smooth boundaries (with possibly a*

finite number of corners). There exist $\kappa_0 > 0$ and $C > 0$ such that if (1.4) *holds and* $(\psi, \mathbf{A}) \in H^1(\Omega, \mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$ *is a critical point of* (1.1)*, then*

$$\int_D |\psi|^4 \, dx \le -2 \int_D g\Big(\frac{H}{\kappa}|B_0(x)|\Big) \, dx + C\kappa^{-2+\tau}.$$

Proof. Let $\ell \in (0, 1)$, $\alpha \in (0, 1)$ and $\gamma \in (0, 1)$ be defined as follows

$$\alpha = \frac{1}{2(\tau - 1)}, \qquad \gamma = \kappa^{-2+\tau} \qquad \text{and } \ell = \kappa^{1-\tau}. \qquad (5.9)$$

In particular, we observe that ℓ is a function of κ such that $\ell \ll 1$ as $\kappa \to +\infty$. Consider the lattice $L_\ell = \ell\mathbb{Z} \times \ell\mathbb{Z}$ as in Proposition 3.1. Let

$$
\begin{aligned}
\mathcal{I}^1_\ell(D) &= \{z \in L_\ell : \overline{Q_\ell(z)} \subset D \cap \Omega_1\} \\
\mathcal{I}^2_\ell(D) &= \{z \in L_\ell : \overline{Q_\ell(z)} \subset D \cap \Omega_2\} \qquad (5.10) \\
\mathcal{I}_\ell(D) &= \mathcal{I}^1_\ell(D) \cup \mathcal{I}^2_\ell(D),
\end{aligned}
$$

$$N_1 = \mathrm{Card}\big(\mathcal{I}^1_\ell(D)\big), \ N_2 = \mathrm{Card}\big(\mathcal{I}^2_\ell(D)\big), \ N = N_1 + N_2 = \mathrm{Card}\big(\mathcal{I}_\ell(D)\big),$$

and

$$D_\ell = \Big(\bigcup_{z \in \mathcal{I}_\ell(D)} \overline{Q_\ell(z)} \Big)^\circ.$$

Under the assumption on D, it holds

$$|D|\ell^{-2} - \mathcal{O}(\ell^{-1}) \le N \le |D|\ell^{-2}, \qquad (5.11)$$

and

$$|D \backslash D_\ell| = \mathcal{O}(\ell) \quad (\ell \to 0_+). \qquad (5.12)$$

We write

$$
\begin{aligned}
\int_D |\psi|^4 \, dx &= \int_{D_\ell} |\psi|^4 \, dx + \int_{D \backslash \overline{D_\ell}} |\psi|^4 \, dx, \\
&= \sum_{z \in \mathcal{I}_\ell(D)} \int_{Q_\ell(z)} |\psi|^4 \, dx + \int_{D \backslash \overline{D_\ell}} |\psi|^4 \, dx.
\end{aligned}
$$

18

Using the previous equality together with Proposition 5.2, (5.11), (5.12), $|\psi| \leq 1$ and the fact that B_0 is a step function, we get

$$\int_D |\psi|^4 \, dx$$

$$\leq -2 \int_{D_\ell} g\left(\frac{H}{\kappa}|B_0(x)|\right) dx + CN\left(\ell^{2\alpha+1} + \ell\kappa^{-1} + \ell^3 + \gamma^{-1}\kappa^{-2}\right) + C\ell$$

$$\leq -2 \int_D g\left(\frac{H}{\kappa}|B_0(x)|\right) dx + C\left(\ell^{2\alpha-1} + \ell^{-1}\kappa^{-1} + \ell + \gamma^{-1}\kappa^{-2}\ell^{-2}\right).$$

In the last inequality, we used the fact that $g(\cdot) \leq 0$ and $D_\ell \subset D$. With the aforementioned choice of α, γ and ℓ, the proof is completed. $\qquad\square$

6 Proof of the main results: Energy and L^4-norm asymptotics

This section is devoted to the proof of Theorem 1.2 and Theorem 1.3.

Proof of Theorem 1.2. First, the upper bound in (1.6) follows from Theorem 5.3 by taking the particular case $D = \Omega$.

Next, we establish the lower bound in (1.7) in the case where $(\psi, \mathbf{A})$ is a minimizer of (1.1). To this end, we perform an integration by parts and use the boundary condition in (1.3) to obtain

$$\mathscr{E}_0(\psi, \mathbf{A}, \Omega) = -\frac{1}{2}\kappa^2 \int_\Omega |\psi|^4 \, dx. \tag{6.1}$$

Hence,

$$\mathrm{E}_{\mathrm{g.st}}(\kappa, H) \geq \mathscr{E}_0(\psi, \mathbf{A}, \Omega) = -\frac{1}{2}\kappa^2 \int_\Omega |\psi|^4 \, dx. \tag{6.2}$$

We use the upper bound of the ground-state energy in (3.1), and $\tau \in (3/2, 2)$ to get the desired result.

For the global estimates of the ground-state energy in (1.5), the upper bound is given in (3.1), while the lower bound can be deduced from (6.2) and Theorem 5.3 by taking $D = \Omega$.

It remains to establish the estimate of the magnetic energy: by (3.2), (6.1), Proposition 3.1 and Theorem 5.3, we have

$$\kappa^2 H^2 \int_\Omega |\operatorname{curl}\mathbf{A} - B_0|^2 \, dx \leq \mathrm{E}_{\mathrm{g.st}}(\kappa, H) - \mathscr{E}_0(\psi, \mathbf{A}; \Omega) \leq C\kappa^\tau.$$

$$\square$$

Proof of Theorem 1.3. The upper bound is given in Theorem 5.3. Next, we write

$$\int_D |\psi|^4 \, dx = \int_\Omega |\psi|^4 \, dx - \int_{\Omega \setminus \overline{D}} |\psi|^4 \, dx.$$

The assumption on the domain D allows us to apply the same argument in Theorem 5.3 to get a similar upper bound of the order parameter on the set $\Omega \setminus \overline{D}$ instead of D. This, together with the lower bound over Ω in Theorem 1.2, establish the lower bound in Theorem 1.3.

$\square$

7 Exponential decay and proof of Theorem 1.5

The inequality in Lemma 7.1 is well-known in the spectral theory of magnetic Schrödinger operators (e.g. [FH10, Lemma 1.4.1]).

Lemma 7.1. *Suppose that $D \subset \Omega$ is an open set, $\mathbf{a} \in C^2(D; \mathbb{R}^2)$ and* curl $\mathbf{a} = 1$. *For all $B \in \mathbb{R}$ and $\phi \in C_c^\infty(D)$, it holds*

$$\|(\nabla - iB\mathbf{a})\phi\|_{L^2(D)}^2 \geq |B| \, \|\phi\|_{L^2(D)}^2.$$

Lemma 7.1 is used to establish Lemma 7.2 below that will be a key ingredient in the proof of Theorem 1.5. Formally, one can view Lemma 7.2 as a slight generalization of Lemma 7.1 by taking $\kappa H = B$ and $\mathbf{A} = \mathbf{a}$. However, the difficulty is that curl $\mathbf{A} \neq 1$ and the vector field $\mathbf{A}$ depends on κ and H as we take $(\psi, \mathbf{A})_{\kappa, H}$ to be a critical point of (1.1). When $\mathbf{A}$ is smooth, independent of (κ, H) and curl $\mathbf{A} \neq 0$, the estimate in Lemma 7.2 is known to hold (e.g. [HM96]).

Under the assumptions in Theorem 1.5, there exist two constants $\kappa_0 > 0$ and $C > 0$ such that for all $\kappa \geq \kappa_0$ and $\phi \in C_c^\infty(\Omega_1)$,

Lemma 7.2.

$$\|(\nabla - i\kappa H\mathbf{A})\phi\|_{L^2(\Omega_1)}^2 \geq \kappa H\left(1 - C\kappa^{-\frac{1}{8}}\right)\|\phi\|_{L^2(\Omega_1)}^2.$$

Proof. Let $\phi \in C_c^\infty(\Omega_1)$. For all $j = (j_1, j_2) \in \mathbb{Z}^2$, let $S_j = (j_1 - 3/4, j_1 + 3/4) \times (j_2 - 3/4, j_2 + 3/4)$. We define the square $Q_{\ell, j}$ of side $3/2\ell$ as follows

$$Q_{\ell, j} = \{\ell x : x \in S_j\}.$$

Clearly, the squares $(Q_{\ell,j})$ finitely overlap. Consider the partition of unity (χ_j) in $\mathbb{R}^2$ satisfying

$$\sum_j |\chi_j|^2 = 1, \quad \sum_j |\nabla \chi_j|^2 \le C\ell^{-2}, \quad \text{and} \quad \operatorname{supp}\chi_j \subset Q_{\ell,j}.$$

We decompose the 'magnetic' kinetic energy term as follows

$$\int_{\Omega_1} |(\nabla - i\kappa H\mathbf{A})\phi|^2 \, dx = \sum_j \int_{\Omega_1} \left(|(\nabla - i\kappa H\mathbf{A})\chi_j\phi|^2 - |\phi\nabla\chi_j|^2 \right) dx$$

$$\ge \sum_j \int_{\Omega_1} \left(|(\nabla - i\kappa H\mathbf{A})\chi_j\phi|^2 \, dx \right)$$

$$- C\ell^{-2} \int_{\Omega_1} |\phi|^2 \, dx. \tag{7.1}$$

Let $x_j = \ell j$ be the center of the square $Q_{\ell,j}$. Define the real-valued function $\varphi_j(x) = (\mathbf{A}(x_j) - \mathbf{F}(x_j)) \cdot x$. By Theorem 4.2, for all $x \in \Omega_1 \cap Q_{\ell,j}$,

$$|\mathbf{A}(x) - \nabla\varphi_j(x) - \mathbf{F}(x)| \le C\ell^{3/4}\|(\mathbf{A} - \mathbf{F})\|_{C^{0,3/4}(\overline{\Omega}_1)} \le C\ell^{3/4}\kappa^{-1}.$$

We may write

$$|(\nabla - i\kappa H\mathbf{A})\chi_j\phi|^2 = \left| (\nabla - i\kappa H(\mathbf{A} - \nabla\varphi_j))e^{-i\kappa H\varphi_j}\chi_j\phi \right|^2$$

$$\ge \left(1 - \ell^{\frac{1}{2}}\right) |(\nabla - i\kappa H\mathbf{F})e^{-i\kappa H\varphi_j}\chi_j\phi|^2$$

$$- C\ell^{-1/2}\kappa^2 H^2 |\mathbf{A}(x) - \nabla\varphi_j(x) - \mathbf{F}(x)|^2|\chi_j\phi|^2$$

$$\ge (1 - \ell^{\frac{1}{2}})|(\nabla - i\kappa H\mathbf{F})e^{-i\kappa H\varphi_j}\chi_j\phi|^2 - C\ell H^2|\chi_j\phi|^2. \tag{7.2}$$

We integrate (7.2) on Ω_1 to obtain

$$\int_{\Omega_1} |(\nabla - i\kappa H\mathbf{A})\chi_j\phi|^2 \, dx \ge (1 - \ell^{\frac{1}{2}}) \int_{\Omega_1} |(\nabla - i\kappa H\mathbf{F})e^{-i\kappa H\varphi_j}\chi_j\phi|^2 \, dx$$

$$- C\ell H^2 \int_{\Omega_1} |\chi_j\phi|^2 \, dx. \tag{7.3}$$

Since $\operatorname{curl}\mathbf{F} = 1$ in Ω_1 and $\chi_j\phi \in C_c^\infty(\Omega_1)$, Lemma 7.1 applied in the domain Ω_1 yields

$$\int_{\Omega_1} |(\nabla - i\kappa H\mathbf{A})\chi_j\phi|^2 \, dx \ge (1 - \ell^{\frac{1}{2}})\kappa H \int_{\Omega_1} |\chi_j\phi|^2 \, dx - C\ell H^2 \int_{\Omega_1} |\chi_j\phi|^2 \, dx.$$

Returning back to (7.1), we obtain

$$\int_{\Omega_1} |(\nabla - i\kappa H\mathbf{A})\phi|^2 \, dx \geq \kappa H \left(1 - \ell^{\frac{1}{2}} - C\ell - C\ell^{-2}\kappa^{-2}\right) \int_{\Omega_1} |\phi|^2 \, dx.$$

The choice of $\ell = \kappa^{-\frac{1}{4}}$ yields

$$\int_{\Omega_1} |(\nabla - i\kappa H\mathbf{A})\phi|^2 \, dx \geq \kappa H \left(1 - C\kappa^{-\frac{1}{8}}\right) \int_{\Omega_1} |\phi|^2 \, dx.$$

$\square$

Proof of Theorem 1.5. Define the distance function t on Ω_1 as follows:

$$t(x) = \text{dist}(x, \partial\Omega_1).$$

Let $\tilde{\chi} \in C^\infty(\mathbb{R})$ be a function satisfying

$$\tilde{\chi} = 0 \ \text{ on } \left(-\infty, \frac{1}{2}\right], \quad \tilde{\chi} = 1 \ \text{ on } [1, +\infty) \ \text{ and } \ |\nabla\tilde{\chi}| \leq m,$$

where m is a universal constant.
Define the two functions χ and f on Ω_1 as follows:

$$\chi(x) = \tilde{\chi}\left(\sqrt{\kappa H}t(x)\right),$$

$$f(x) = \chi(x) \exp\left(\varepsilon\sqrt{\kappa H}\,t(x)\right),$$

where ε is a positive number whose value will be fixed later.
Using the first equation of (1.3), we multiply both sides by $f^2\overline{\psi}$ and integrate by parts over Ω_1, it results

$$\int_{\Omega_1} \left(|(\nabla - i\kappa H\mathbf{A})f\psi|^2 - |\nabla f|^2|\psi|^2\right) dx = \kappa^2 \int_{\Omega_1} \left(|\psi|^2 - |\psi|^4\right)f^2 \, dx$$

$$\leq \kappa^2 \int_{\Omega_1} |\psi|^2 f^2 \, dx. \qquad (7.4)$$

We combine the conclusions in (7.4) and Lemma 7.2 to get

$$\int_{\Omega_1} |\nabla f|^2|\psi|^2 \, dx \geq \left(\kappa H\left(1 - C\kappa^{-\frac{1}{8}}\right) - \kappa^2\right)\|f\psi\|^2_{L^2(\Omega_1)}$$

$$\geq \left(\lambda - C(1 + \lambda)\kappa^{-\frac{1}{8}}\right)\kappa^2\|f\psi\|^2_{L^2(\Omega_1)}. \qquad (7.5)$$

Now, we estimate the term involving ∇f on the right side of (7.5) as follows

$$\int_{\Omega_1} |\nabla f|^2 |\psi|^2 \, dx \le \varepsilon^2 \kappa H \|f\psi\|^2_{L^2(\Omega_1)} + C\kappa H \int_{\Omega_1 \cap \{\sqrt{\kappa H} t(x) \le 1\}} |\psi|^2 \, dx$$

Inserting (7.6) into (7.5) and dividing by κ^2 yields

$$\left(\lambda - C(1+\lambda)\kappa^{-\frac{1}{8}} - \varepsilon^2\right)\|f\psi\|^2_{L^2(\Omega_1)} \le C \int_{\Omega_1 \cap \{\sqrt{\kappa H} t(x) \le 1\}} |\psi|^2 \, dx.$$

We choose the constant ε such that $0 < \varepsilon < \sqrt{\lambda}$. That way, we get for κ sufficiently large,

$$\int_{\Omega_1 \cap \{\sqrt{\kappa H} t(x) \ge 1\}} |f\psi|^2 \, dx \le \widetilde{C} \int_{\Omega_1 \cap \{\sqrt{\kappa H} t(x) \le 1\}} |\psi|^2 \, dx, \qquad (7.6)$$

where $\widetilde{C}$ is a constant. Inserting (7.5) and (7.6) into (7.4) completes the proof of Theorem 1.5. $\qquad\square$

We conclude this paper by Theorem 7.3 below, whose proof is similar to that of Theorem 1.5.

Theorem 7.3 (Exponential decay in Ω_2). *Let $\lambda, c_2 > 0$ be two constants such that $1/|a| + \lambda < c_2$. There exist constants $C, \varepsilon, \kappa_0 > 0$ such that, if*

$$\kappa \ge \kappa_0, \quad \left(\frac{1}{|a|} + \lambda\right)\kappa \le H \le c_2\kappa, \quad (\psi, \mathbf{A})_{\kappa, H} \text{ is a solution of (1.3)},$$

then

$$\int_{\Omega_2 \cap \{\text{dist}(x, \partial\Omega_2) \ge \frac{1}{\sqrt{\kappa H}}\}} \left(|\psi|^2 + \frac{1}{\kappa H}|(\nabla - i\kappa H\mathbf{A})\psi|^2\right)$$

$$\exp\left(2\varepsilon\sqrt{\kappa H}\,\text{dist}(x, \partial\Omega_2)\right)dx \le C \int_{\Omega_2 \cap \{\text{dist}(x, \partial\Omega_2) \le \frac{1}{\sqrt{\kappa H}}\}} |\psi|^2 \, dx.$$

Acknowledgements

A Gauge transformation

Lemma A.1. *Suppose that Ω satisfies the conditions in Assumption 1.1. Let $B_0 \in L^2(\Omega)$. There exists a unique vector field $\mathbf{F} \in H^1_{\mathrm{div}}(\Omega)$ such that*

$$\mathrm{curl}\,\mathbf{F} = B_0.$$

Furthermore, F is in $C^\infty(\Omega_j)$, $j = 1, 2$.

Proof. Let $f \in H^2(\Omega) \cap H^1_0(\Omega)$ be the unique solution of $-\Delta f = B_0$ in Ω (e.g. [FH10]). The vector field $\mathbf{F} = (\partial_{x_2} f, -\partial_{x_1} f) \in H^1_{\mathrm{div}}(\Omega)$ and satisfies $\mathrm{curl}\,\mathbf{F} = B_0$.

  Since B_0 is constant in Ω_j for $j \in \{1, 2\}$, f the solution of $-\Delta f = B_0$ becomes in $C^\infty(\Omega_j)$. This yields that $\mathbf{F}$ is in $C^\infty(\Omega_j)$, $j \in \{1, 2\}$.

  The uniqueness of $\mathbf{F}$ is a consequence of the estimate (e.g. [FH10, Prop. D.2.1] or [Tem01, Appendix 1])

$$\forall\, a \in H^1_{\mathrm{div}}(\Omega)\,, \quad \|a\|_{H^1(\Omega)} \le C_* \|\,\mathrm{curl}\,a\|_{L^2(\Omega)}, \tag{A.1}$$

where $C_* > 0$ is a universal constant. $\qed$

Lemma A.2. *Let $j \in \{1, 2\}$, $\ell \in (0, 1)$, $x_0 \in \Omega$ and $Q_\ell(x_0) \subset \Omega_j$. There exists a function $\varphi_{x_0} \in C^2(Q_\ell(x_0))$ such that the magnetic potential $\mathbf{F}$ defined in Lemma A.1 satisfies*

$$\mathbf{F}(x) = B_0(x_0)\mathbf{A}_0(x - x_0) + \nabla\varphi_{x_0}(x), \qquad (x \in Q_\ell(x_0))$$

where B_0 is the function defined in 1.1 and $\mathbf{A}_0$ is the magnetic potential introduced in (2.2).

Proof. By the definition of $\mathbf{F}$ and $\mathbf{A}_0$ we have for all $x \in Q_\ell(x_0)$,

$$\mathrm{curl}\,\mathbf{F}(x) = B_0(x)\,\mathrm{curl}\,\mathbf{A}_0(x) \quad \text{in } Q_\ell(x_0).$$

Since $Q_\ell(x_0)$ is simply connected and B_0 is constant in $Q_\ell(x_0)$, we get the existence of the function φ_{x_0}. $\qed$

B   curl-div **elliptic estimate**

Lemma B.1. *Suppose that Ω is simply connected and satisfies the conditions in Assumption 1.1. There exists a constant $C > 0$ such that, if $a \in H^1_{\mathrm{div}}(\Omega)$ and $\mathrm{curl}\, a \in H^1_0(\Omega)$, then $a \in H^2(\Omega)$ and the following inequality holds*

$$\|a\|_{H^2(\Omega)} \leq C\|\nabla(\mathrm{curl}\, a)\|_{L^2(\Omega)}.$$

Proof. Let $f \in H^1_0(\Omega) \cap H^2(\Omega)$ be the unique solution of the Dirichlet problem $-\Delta f = \mathrm{curl}\, a$ in Ω. By the inequality in (A.1), we get that $a = \nabla^\perp f = (\partial_{x_2} f, -\partial_{x_1} f)$.

Since $f = 0$ on $\partial\Omega$ and $\mathrm{curl}\, a \in H^1(\Omega)$, we have by the elliptic estimates

$$\|f\|_{H^2(\Omega)} \leq C\big(\|\mathrm{curl}\, a\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\big),$$

$f \in H^3(\Omega)$ and

$$\|f\|_{H^3(\Omega)} \leq C\big(\|\mathrm{curl}\, a\|_{H^1(\Omega)} + \|f\|_{H^2(\Omega)}\big)$$
$$\leq C\big(\|\mathrm{curl}\, a\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)}\big).$$

This proves that $a \in H^2(\Omega)$. Now, since $f = 0$ and $\mathrm{curl}\, a = 0$ on $\partial\Omega$, we get by the Poincaré inequality

$$\|f\|_{H^3(\Omega)} \leq C\big(\|\nabla(\mathrm{curl}\, a)\|_{L^2(\Omega)} + \|\nabla f\|_{L^2(\Omega)}\big).$$

Since $a = \nabla^\perp f$, $\|\nabla f\|_{L^2(\Omega)} = \|a\|_{L^2(\Omega)}$ and consequently

$$\|\nabla f\|_{L^2(\Omega)} \leq C_*\|\mathrm{curl}\, a\|_{L^2(\Omega)} \qquad\qquad \text{[By (A.1)]}$$
$$\leq C\|\nabla(\mathrm{curl}\, a)\|_{L^2(\Omega)} \qquad \text{[By the Poincaré inquality]}.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Bibliography

[Att15a]   K. Attar. Energy and vorticity of the Ginzburg–Landau model with variable magnetic field. *Asymptot. Anal.*, 93(1-2):75–114, 2015.

[Att15b]   K. Attar. The ground state energy of the two dimensional Ginzburg–Landau functional with variable magnetic field. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 32(2):325–345, 2015.

[BNF07]   V. Bonnaillie-Noël and S. Fournais. Superconductivity in domains with corners. *Rev. Math. Phys.*, 19(06):607–637, 2007.

[CDG95]   S.J. Chapman, Q. Du, and M.D. Gunzburger. A Ginzburg–Landau type model of superconducting/normal junctions including Josephson junctions. *European J. Appl. Math.*, 6(2):97–114, 1995.

[DHS14]   N. Dombrowski, P.D. Hislop, and E. Soccorsi. Edge currents and eigenvalue estimates for magnetic barrier Schrödinger operators. *Asymptot. Anal.*, 89(3-4):331–363, 2014.

[FH10]   S. Fournais and B. Helffer. *Spectral Methods in Surface Superconductivity*, volume 77. Springer Science & Business Media, 2010.

[FK09]   S. Fournais and A. Kachmar. On the transition to the normal phase for superconductors surrounded by normal conductors. *J. Differ. Equ.*, 247(6):1637–1672, 2009.

[FK11]   S. Fournais and A. Kachmar. Nucleation of bulk superconductivity close to critical magnetic field. *Adv. Math.*, 226(2):1213–1258, 2011.

[FK13]   S. Fournais and A. Kachmar. The ground state energy of the three dimensional Ginzburg–Landau functional. Part I: Bulk regime. *Comm. Partial Differential Equations*, 38(2):339–383, 2013.

[GP99]   T. Giorgi and D. Phillips. The breakdown of superconductivity due to strong fields for the Ginzburg–Landau model. *SIAM J. Math. Anal.*, 30:341–359, 1999.

[HK15]   B. Helffer and A. Kachmar. The Ginzburg–Landau functional with vanishing magnetic field. *Arch. Ration. Mech. Anal.*, 218(1):55–122, 2015.

[HM96]   B. Helffer and A. Mohamed. Semiclassical analysis for the ground state energy of a Schrödinger operator with magnetic wells. *J. Funct. Anal.*, 138(1):40–81, 1996.

[HPRS16]   P.D. Hislop, N. Popoff, N. Raymond, and M. Sundqvist. Band functions in the presence of magnetic steps. *Math. Models Methods Appl. Sci.*, 26(1):161–184, 2016.

[HS08]   P.D. Hislop and E. Soccorsi. Edge currents for quantum Hall systems I: One-edge, unbounded geometries. *Rev. Math. Phys.*, 20(01):71–115, 2008.

[HS15]   P.D. Hislop and E. Soccorsi. Edge states induced by Iwatsuka Hamiltonians with positive magnetic fields. *J. Math. Anal. Appl.*, 422(1):594–624, 2015.

[Kac06]   A. Kachmar. The ground state energy of a magnetic Schrödinger operator and the effect of the de Gennes parameter. *J. Math. Phys.*, 47:1–32, 2006.

[LP99]   K. Lu and X.B. Pan. Estimates of the upper critical field for the Ginzburg–Landau equations of superconductivity. *Physica D*, 127(1):73–104, 1999.

[LP00]   K. Lu and X.B. Pan. Surface nucleation of superconductivity in 3-dimension. *J. Differential Equations*, 168(2):386–452, 2000.

[PK02]   X.B. Pan and K.H. Kwek. Schrödinger operators with non-degenerately vanishing magnetic fields in bounded domains. *Trans. Amer. Math. Soc.*, 354(10):4201–4227, 2002.

[PM93]   F.M. Peeters and A. Matulis. Quantum structures created by nonhomogeneous magnetic fields. *Phys. Rev. B*, 48(20):15166, 1993.

[RP00]   J. Reijniers and F.M. Peeters. Snake orbits and related magnetic edge states. *J. Phys. Condens. Matter*, 12(47):9771, 2000.

[SS03]   E. Sandier and S. Serfaty. The decrease of bulk superconductivity close to the second critical field in the Ginzburg–Landau model. *SIAM J. Math. Anal.*, 34(4):939–956, 2003.

[SS07]   E. Sandier and S. Serfaty. *Vortices in the Magnetic Ginzburg–Landau Model.* Progress in Nonlinear Partial Differential Equations and their Applications, Birkhäuser-Boston, 2007.

[Tem01]   R. Temam. *Navie–Stokes Equations: Theory and Numerical Analysis*, volume 343. American Mathematical Soc., 2001.

Paper II

The distribution of superconductivity near a magnetic barrier

Wafaa Assaad,[*] Ayman Kachmar,[†] Mikael Persson-Sundqvist[‡]

Abstract

We consider the Ginzburg–Landau functional in a two-dimensional simply connected domain with smooth boundary, in the situation when the applied magnetic field is piecewise constant with a jump discontinuity along a smooth curve. In the regime of large Ginzburg–Landau parameter and strong magnetic field, we study the concentration of the minimizing configurations along this discontinuity, by computing the energy of the minimizers and their weak limit in the sense of distributions.

1 Introduction

1.1 Motivation

The Ginzburg–Landau theory, introduced in [LG50], is a phenomenological macroscopic model describing the response of a superconducting sample to an external magnetic field, when the sample is close to its critical temperature T_c. The phenomenological quantities associated with a superconductor are the order parameter ψ and the magnetic potential $\mathbf{A}$, where $|\psi|^2$ measures the density of the superconducting Cooper pairs and $\operatorname{curl}\mathbf{A}$ represents the induced magnetic field in the sample.

In this paper, the superconducting sample is an infinite cylindrical domain subjected to a magnetic field with a direction parallel to the axis of the cylinder.

---

[*]Lund University, Department of Mathematics, Lund, Sweden
[†]Lebanese University, Department of Mathematics, Hadat, Lebanon
[‡]Lund University, Department of Mathematics, Lund, Sweden

For this specific geometry, it is enough to consider the horizontal cross section of the sample, $\Omega \subset \mathbb{R}^2$. The phenomenological configuration $(\psi, \mathbf{A})$ is then defined on the domain Ω.

The study of the Ginzburg–Landau model in the case of a *uniform* or a *smooth* non-uniform applied magnetic field has been the focus of much attention in the literature. We refer to the two monographs [FH10, SS07] for the uniform magnetic field case. Smooth magnetic fields are the subject of the papers [Att15a, Att15b, HK15, LP99, PK02]. Given the current interest in magnetic steps for various physical systems, we focus on the case where the applied magnetic field is a step function, which is not covered in the aforementioned papers.

Nonhomogeneous magnetic fields have been the focus of great amount of research. Current fabrication techniques allow the creation of such magnetic fields [FLBP94, STH+94, GGD+97], which opens new paths in quantum physics and possible applications [RP98, JBY+97, MJR97]. Indeed, these magnetic fields appear in models involved in nanophysics such as in quantum transport in 2DEG (bidimensional electron gas) (see [PM93, RP00] and references therein) and in the Ginzburg–Landau model in superconductivity [SJST69]. More recently, piecewise constant magnetic fields are considered in the analysis of transport properties in graphene [GDMH+08, ORK+08]. Such magnetic fields are interesting because they induce snake states, carriers of edge currents flowing in the interface separating the distinct values of the magnetic field–*the magnetic barrier* (for instance see [HPRS16, HS15, DHS14, HS08, RP00, PM93]). While such edge currents have been discussed for linear problems in earlier works, the main contribution of this manuscript lies in establishing their existence in the context of the non-linear Ginzburg–Landau functional in superconductivity, by examining the presence of superconductivity along the magnetic barrier. Our configuration is illustrated in Figure 1.

In an earlier contribution [AK16], the authors explored the influence of a step magnetic field on the distribution of bulk superconductivity, which highlighted the regime where an edge current might occur near the magnetic barrier. In this contribution, we will demonstrate the existence of such a current by providing examples where superconductivity concentrates at the interface separating the distinct values of the magnetic field.

1.2   The functional and the mathematical set-up

We assume that the domain Ω is open in $\mathbb{R}^2$, bounded, and simply connected. The Ginzburg–Landau (GL) free energy is given by the functional

$$\mathscr{E}_{\kappa,H}(\psi, \mathbf{A}) = \int_\Omega \left(\left| (\nabla - i\kappa H\mathbf{A})\psi \right|^2 - \kappa^2 |\psi|^2 + \frac{\kappa^2}{2} |\psi|^4 \right) dx$$

$$+ \kappa^2 H^2 \int_\Omega \left| \operatorname{curl} \mathbf{A} - B_0 \right|^2 dx, \quad (1.1)$$

with $\psi \in H^1(\Omega; \mathbb{C})$ and $\mathbf{A} = (A_1, A_2) \in H^1(\Omega; \mathbb{R}^2)$. Here, $\kappa > 0$ is a large GL parameter, the function $B_0 : \Omega \to [-1, 1]$ is the profile of the applied magnetic field, and $H > 0$ is the intensity of this applied magnetic field.

The parameter κ depends on the temperature and the type of the material. It is a physical characteristic scale of the sample, it measures the size of vortex cores (which is proportional to κ^{-1}, in some typical situations dependent on the strength of the applied magnetic field). Vortex cores are narrow regions in the sample, which corresponds to κ being a large parameter. That is the main reason behind our analysis of the asymptotic regime $\kappa \to +\infty$, following many early contributions addressing this asymptotic regime (see e.g. [SS07]). We work under the following assumptions on the domain Ω and the magnetic field B_0, which are quite generic as revealed from the illustration in Figure 2.

Assumption 1.1.

1. $\Omega_1 \subset \Omega$ and $\Omega_2 \subset \Omega$ are two disjoint open sets.

2. Ω_1 and Ω_2 have a finite number of connected components.

3. $\partial\Omega_1$ and $\partial\Omega_2$ are piecewise smooth with (possibly) a finite number of corners.

4. $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ is the union of a finite number of disjoint simple smooth curves $\{\Gamma_k\}_{k \in \mathscr{K}}$; we will refer to Γ as the magnetic barrier.

5. $\Omega = (\Omega_1 \cup \Omega_2 \cup \Gamma)^\circ$ and $\partial\Omega$ is smooth.

6. $\Gamma \cap \partial\Omega$ is either empty or finite.

7. For any $k \in \mathscr{K}$, if Γ_k intersects $\partial\Omega$ then the intersection is at two distinct points. This intersection is transversal, i.e. $\mathrm{T}_{\partial\Omega} \times \mathrm{T}_{\Gamma_k} \neq 0$ at the intersection point, where $\mathrm{T}_{\partial\Omega}$ and T_{Γ_k} are respectively unit tangent vectors of $\partial\Omega$ and Γ_k.
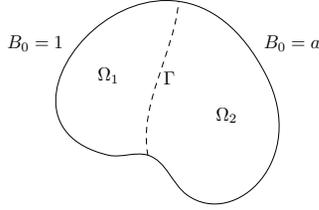
Figure 1: Schematic representation of the set Ω subjected to a step magnetic field B_0, with the magnetic barrier Γ.
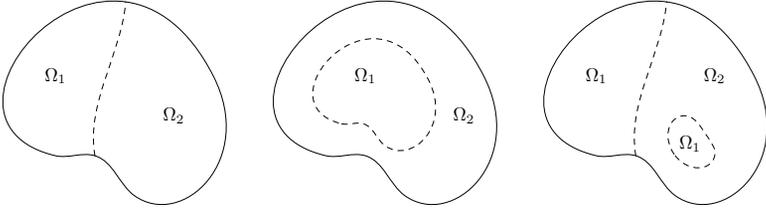
Figure 2: Schematic representations of the set Ω.

Assumption 1.2. $B_0 = \mathbb{1}_{\Omega_1} + a\mathbb{1}_{\Omega_2}$, *where* $a \in [-1, 1)\backslash\{0\}$ *is a given constant.*

The ground state of the superconductor describes its behaviour at equilibrium. It is obtained by minimizing the GL functional in (1.1) with respect to $(\psi, \mathbf{A})$. The corresponding energy is called the ground state energy, denoted by $E_{\text{g.st}}(\kappa, H)$, where

$$E_{\text{g.st}}(\kappa, H) = \inf\big\{\mathcal{E}_{\kappa,H}(\psi, \mathbf{A}) \; : \; (\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2)\big\}.$$

One may restrict the minimization of the GL functional to the space $H^1(\Omega; \mathbb{C}) \times H^1_{\text{div}}(\Omega)$ where

$$H^1_{\text{div}}(\Omega) = \big\{\mathbf{A} \in H^1(\Omega; \mathbb{R}^2) \; : \; \text{div}\, \mathbf{A} = 0 \text{ in } \Omega, \; \mathbf{A} \cdot \nu_{\partial\Omega} = 0 \text{ on } \partial\Omega\big\}.$$

Indeed, the functional in (1.1) enjoys the property of *gauge invariance*[1]. Consequently, the ground state energy can be written as follows (see [FH10, Appendix D])

$$E_{\text{g.st}}(\kappa, H) = \inf\big\{\mathcal{E}_{\kappa,H}(\psi, \mathbf{A}) \; : \; (\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\text{div}}(\Omega)\big\}. \qquad (1.2)$$

---

[1]It does not change under the transformation $(\psi, \mathbf{A}) \mapsto (e^{i\varphi\kappa H}\psi, \mathbf{A} + \nabla\varphi)$, for any (say smooth) function $\varphi : \mathbb{R}^2 \to \mathbb{R}$. The physically meaningful quantities are the gauge invariant ones, such as $|\psi|$ and curl $\mathbf{A}$.

This restriction allows us to make profit from some well-known regularity properties of vector fields in $H^1_{\mathrm{div}}(\Omega)$ (see [AK16, Appendix B]).

Critical points $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$ of $\mathscr{E}_{\kappa, H}$ are weak solutions of the following GL equations:

$$\begin{cases} \left(\nabla - i\kappa H\mathbf{A}\right)^2 \psi = \kappa^2 (|\psi|^2 - 1)\psi & \text{in } \Omega, \\ -\nabla^\perp \left(\operatorname{curl}\mathbf{A} - B_0\right) = \frac{1}{\kappa H}\operatorname{Im}\left(\overline{\psi}(\nabla - i\kappa H\mathbf{A})\psi\right) & \text{in } \Omega, \\ \nu \cdot (\nabla - i\kappa H\mathbf{A})\psi = 0 & \text{on } \partial\Omega, \\ \operatorname{curl}\mathbf{A} = B_0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Here,

$$(\nabla - i\kappa H\mathbf{A})^2 \psi = \Delta\psi - i\kappa H(\operatorname{div}\mathbf{A})\psi - 2i\kappa H\mathbf{A}\cdot\nabla\psi - \kappa^2 H^2|\mathbf{A}|^2\psi$$

and $\nabla^\perp = (\partial_{x_2}, -\partial_{x_1})$ is the Hodge gradient.

1.3   Some earlier results for uniform magnetic fields

The value of the ground state energy $\mathrm{E}_{\mathrm{g.st}}(\kappa, H)$ depends on κ and H in a non-trivial fashion. The physical explanation is that a superconductor undergoes phase transitions as the intensity of the applied magnetic field varies.

To illustrate the dependence on the intensity of the applied magnetic field, we assume that κ is large and $H = b\kappa$, for some fixed parameter $b > 0$. Such magnetic field strengths are considered in many papers (for instance see [AH07, LP99, Pan02, SS03]).

Assuming that the applied magnetic field is uniform, which corresponds to taking $B_0 = 1$ in (1.1), the following scenario takes place. If $b > \Theta_0^{-1}$, where $\Theta_0 \approx 0.59$ is a universal constant defined in (2.5) below, the only minimizers of the GL functional are the trivial states $(0, \mathbf{F})$, where $\operatorname{curl}\mathbf{F} = 1$ (see [GP99, LP99]). This corresponds in Physics to the destruction of superconductivity when the sample is submitted to a large external magnetic field, and occurs when the intensity H crosses a specific threshold value, the so-called *third critical field*, denoted by H_{C_3}.

Another well-known critical field is the *second critical field* H_{C_2}, which is much harder to define. When $H < H_{C_2}$, superconductivity is uniformly distributed in the interior of the sample (see [SS03]). This is the bulk superconductivity regime. When $H_{C_2} < H < H_{C_3}$, the surface superconductivity regime occurs: superconductivity disappears from the interior and is localized in a thin layer near

the boundary of the sample (see [AH07, HFPS11, Pan02, CR14]). The transition from surface to bulk superconductivity takes place when H varies around the critical value κ, which we informally take as the definition of H_{C_2} (see [FK11]).

One more critical field left is H_{C_1}. It marks the transition from the pure superconducting phase to the phase with vortices. We refer to [SS07] for its definition.

1.4 Expected behaviour under magnetic steps

Let us return back to the case where the magnetic field is a step function as in Assumption 1.2. At some stage, the expected behaviour of the superconductor in question deviates from the one submitted to a uniform magnetic field. Recently, this case was considered in [AK16] and the following was obtained. Suppose that $H = b\kappa$ and κ is large. If $b < 1/|a|$ then bulk superconductivity persists; if $b > 1/|a|$ then superconductivity disappears in the bulk of Ω_1 and Ω_2, and *may* nucleate in thin layers near $\Gamma \cup \partial\Omega$ (see Assumption 1.1 and Figure 1). The present contribution affirms the presence of superconductivity in the vicinity of Γ when b is greater than, but close to the value $1/|a|$, for some negative values of a. The precise statements are given in Theorems 1.7 and 1.11 below.

The aforementioned behaviour of the superconductor in presence of magnetic steps is consistent with the existing literature about the electron motion near the magnetic barrier at which the strength and/or the sign of the magnetic fields change (for instance see). Particularly, the case where $a \in [-1, 0)$ is called the *trapping magnetic step* (see [HPRS16]), where the discontinuous magnetic field may create supercurrents (*snake orbits*) flowing along the discontinuity edge. On the other hand, such supercurrents do not seem detectable in the case when $a \in (0, 1)$, which is called the *non-trapping magnetic step*. However, the approach was generally spectral where some properties of relevant linear models were analysed (see [HPRS16, HS15, Iwa85, RP00]), and no estimates for the non-linear GL energy in (1.1) were established in these cases.

The contribution of this paper together with [AK16] provide such estimates. Particularly in the case when $a \in [-1, 0)$ and $b > 1/|a|$, Theorems 1.7 and 1.11 below establish global and local asymptotic estimates for the ground state energy $E_{g.st}(\kappa, H)$, and the L^4-norm of the minimizing order parameter. These theorems assert the nucleation of superconductivity near the magnetic barrier Γ (and the surface $\partial\Omega$) when b crosses the threshold value $1/|a|$.

1.5   Main results

Our results are valid under the following additional assumption.

Assumption 1.3. *The parameter H depends on κ in the following manner*

$$H = b\kappa, \tag{1.4}$$

where b is a fixed parameter satisfying

$$b > \frac{1}{|a|}, \quad a \in [-1, 1)\setminus\{0\}.$$

Remark 1.4. Our study does not cover the potentially interesting case $a = 0$, which deserves to be studied independently in a future work. This case, referred to as *magnetic wall*, was considered in [RP00, HPRS16].

Remark 1.5. Even though the case $a \in (0, 1)$ is included in Assumption 1.3, it will not be central in our study (the reader may notice this in the majority of our theorems statements). The reason is that, our main concern is to analyse the interesting phenomenon happening when bulk superconductivity is only restricted to a narrow neighbourhood of the magnetic edge Γ, and this only occurs when the values of the two magnetic fields *interacting* near Γ are of *opposite signs*, that is when $a \in [-1, 0)$. This can be seen through the trivial cases in Section 3.2, and is consistent with the aforementioned literature findings (non-trapping magnetic steps). Moreover, the case $b < 1/|a|$ is treated previously in [AK16] and corresponds to the bulk regime.

The statements of the main theorems involve two non-decreasing continuous functions

$$e_a : \big[|a|^{-1}, +\infty\big) \to (-\infty, 0] \quad \text{and} \quad E_{\text{surf}} : [1, +\infty) \to (-\infty, 0],$$

respectively defined in (3.5) and (6.28) below. The energy E_{surf} has been studied in many papers (for instance see [CR14, FKP13, FK11, HFPS11, AH07, Pan02]). We will refer to E_{surf} as the *surface energy*. The function e_a is constructed in this paper, and we will refer to it as the *barrier energy*.

Remark 1.6. It is worthy of mention that $e_a(b)$ vanishes if and only if

- $a \in (0, 1)$; or

- $a \in [-1, 0)$ and $b \geq 1/\beta_a$, where β_a is defined in (2.11) below and satisfies $\beta_a \in (0, |a|)$ (see Theorem 2.6).

The surface energy $E_{\mathrm{surf}}(b)$ vanishes if and only if $b \geq \Theta_0^{-1}$, where Θ_0 is the constant defined in (2.5).

The main contribution of this paper is summarized in Theorems 1.7 and 1.11 below.

Theorem 1.7 (Global asymptotics). *For all $a \in [-1, 1)\backslash\{0\}$ and $b > 1/|a|$, the ground state energy $\mathrm{E}_{\mathrm{g.st}}(\kappa, H)$ in (1.2) satisfies, when $H = b\kappa$,*

$$\mathrm{E}_{\mathrm{g.st}}(\kappa, H) = E_a^{\mathrm{L}}(b)\kappa + o(\kappa) \qquad (\kappa \to +\infty), \tag{1.5}$$

where

$$E_a^{\mathrm{L}}(b) = b^{-1/2}\Big(|\Gamma|\,e_a(b) + |\partial\Omega_1 \cap \partial\Omega|\,E_{\mathrm{surf}}(b) + |\partial\Omega_2 \cap \partial\Omega|\,|a|^{-\frac{1}{2}}E_{\mathrm{surf}}\big(b|a|\big)\Big).$$

Furthermore, every minimizer $(\psi, \mathbf{A})_{\kappa, H} \in H^1(\Omega; \mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$ of the functional in (1.1) satisfies

$$\int_\Omega |\psi|^4\,dx = -2E_a^{\mathrm{L}}(b)\kappa^{-1} + o(\kappa^{-1}) \qquad (\kappa \to +\infty). \tag{1.6}$$

Remark 1.8. In the asymptotics displayed in Theorem 1.7, the term $|\Gamma|b^{-1/2}e_a(b)$ corresponds to the energy contribution of the magnetic barrier. The rest of the terms indicate the energy contributions of the surface of the sample. In light of Remark 1.6, the critical value $b = \beta_a^{-1}$ marks the transition between the superconducting and normal states along Γ.

Remark 1.9. The edge Γ creates vertices in the case where $\Gamma \cap \partial\Omega \neq \emptyset$ (see Figure 2) which may have non-trivial energy contributions hidden in the remainder term in (1.5). This case alters the breakdown of superconductivity too and shares some similarities with corner cases [BNF07, CG17, HK18, Ass19].

Remark 1.10. Theorem 1.7 does not cover the case when the intensity of the magnetic field satisfies $b = 1/|a|$. However, we expect that some additional bulk terms will contribute to the estimate of the energy in this case, by analogy with [FK11].

Our next result, Theorem 1.11 below, describes the local behaviour of the minimizing order parameter ψ, thereby enhancing the statement in Theorem 1.7. We define the following distribution in $\mathbb{R}^2$,

$$C_c^\infty(\mathbb{R}^2) \ni \varphi \mapsto \mathscr{T}^b(\varphi),$$

where

$$\mathscr{T}^b(\varphi) = -2b^{-\frac{1}{2}}\Big(e_a(b) \int_\Gamma \varphi \, d s_\Gamma + E_{\mathrm{surf}}(b) \int_{\partial\Omega_1 \cap \partial\Omega} \varphi \, d s$$

$$+ |a|^{-\frac{1}{2}} E_{\mathrm{surf}}\big(b|a|\big) \int_{\partial\Omega_2 \cap \partial\Omega} \varphi \, d s \Big). \quad (1.7)$$

Here $d s_\Gamma$ and $d s$ denote the arc-length measures on Γ and $\partial\Omega$ respectively.

Theorem 1.11 (Local asymptotics). *For all $a \in [-1, 1)\backslash\{0\}$ and $b > 1/|a|$, if $(\psi, \mathbf{A})_{\kappa, H} \in H^1(\Omega; \mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$ is a minimizer of the functional in (1.1) for $H = b\kappa$, then, as $\kappa \to +\infty$,*

$$\kappa \mathscr{T}^b_\kappa \rightharpoonup \mathscr{T}^b \text{ in } \mathscr{D}'(\mathbb{R}^2),$$

where $\mathscr{T}^b_\kappa$ is the distribution in $\mathbb{R}^2$ defined as follows

$$C^\infty_c(\mathbb{R}^2) \ni \varphi \mapsto \mathscr{T}^b_\kappa(\varphi) = \int_\Omega |\psi|^4 \varphi \, d x,$$

and the convergence of $\mathscr{T}^b_\kappa$ to $\mathscr{T}^b$ is understood in the following sense

$$\forall \, \varphi \in C^\infty_c(\mathbb{R}^2), \quad \lim_{\kappa \to +\infty} \kappa \mathscr{T}^b_\kappa(\varphi) = \mathscr{T}^b(\varphi). \quad (1.8)$$

Similarly as in [CR16b], we expect that the second correction term in the asymptotics in (1.8) will depend on the surface geometry of Γ and $\partial\Omega$, and will require a restrictive assumption on the way the support of the test function φ meets the edges Γ and $\partial\Omega$.

Discussion of the main results.   We will discuss the results in Theorems 1.7 and 1.11, in the interesting case where the magnetic barrier Γ intersects the boundary of Ω. Hence we will assume that $\partial\Omega_j \cap \partial\Omega \neq \emptyset$ for $j \in \{1, 2\}$. When this condition is violated, the discussion below can be adjusted easily.

The following observations mainly rely on Remark 1.6 and the order of the values $|a|\Theta_0$, Θ_0, β_a, and $|a|$.

- For $a = -1$, we have $\beta_a = \Theta_0 < |a|$ (see (2.26)). Consequently, in light of Remark 1.6:
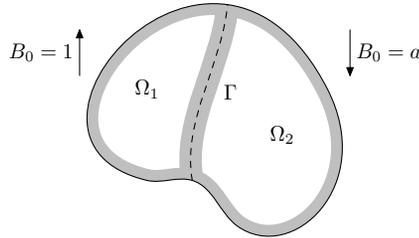
Figure 3: Superconductivity distribution in the set Ω subjected to a magnetic field B_0, in the regime where $a = -1$, $H = b\kappa$, and $|a|^{-1} < b < \Theta_0^{-1}$. The white regions are in a normal state, while the grey regions carry superconductivity.

- If $1 < b < \Theta_0^{-1}$, then the surface of the sample carries superconductivity and the entire bulk is in a normal state *except* for the region near the magnetic barrier (see Figure 3). Moreover, the energy contributions of the magnetic barrier and the surface of the sample are of the same order and described by the surface energy, since in this case $e_a(b) = E_{\mathrm{surf}}(b)$, see Remark 3.12. This behaviour is remarkably distinct from the case of a uniform applied magnetic field.

- If $b \geq \Theta_0^{-1}$, then all the aforementioned energy contributions vanish, $E_a^1(b) = 0$.

- For $a \in (-1, 0)$, comparing the values β_a, Θ_0 and $|a|$ is more subtle. In (2.18), (2.23) and Theorem 2.6 below, we show that

$$\forall\, a \in (-\Theta_0, 0), \qquad |a|\Theta_0 < \beta_a < |a| < \Theta_0. \qquad (1.9)$$

Moreover, numerical results about the variation of β_a with respect to a show that β_a is strictly decreasing for $a \in [-1, 0)$ (see Figure 5)[2]. Having $\beta_{-1} = \Theta_0$ (see (2.26)), this suggests that $\beta_a < \Theta_0$ for $a \in (-1, 0)$. However, such a result is not rigorously established yet.

With (1.9) in hand, Theorem 1.11 and Remark 1.6 indicate the following behaviour for $a \in (-\Theta_0, 0)$ and $b > |a|^{-1}$:

- The part of the sample's surface near $\partial\Omega_1 \cap \partial\Omega$ is not superconducting.

---

[2]The graph in Figure 5 was obtained after a numerical computation done by Virginie Bonnaillie-Noël.
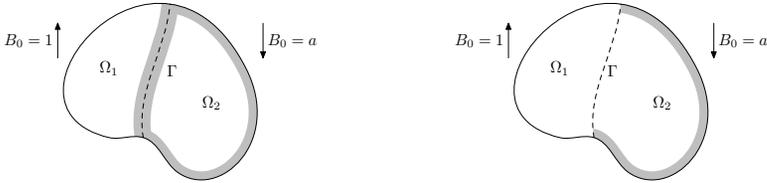
Figure 4: Superconductivity distribution in the set Ω subjected to a magnetic field B_0, in the regime where $a \in (-\Theta_0, 0)$, $H = b\kappa$, and respectively $|a|^{-1} < b < \beta_a^{-1}$ and $\beta_a^{-1} \leq b < |a|^{-1}\Theta_0^{-1}$. The white regions are in a normal state, while the grey regions carry superconductivity.
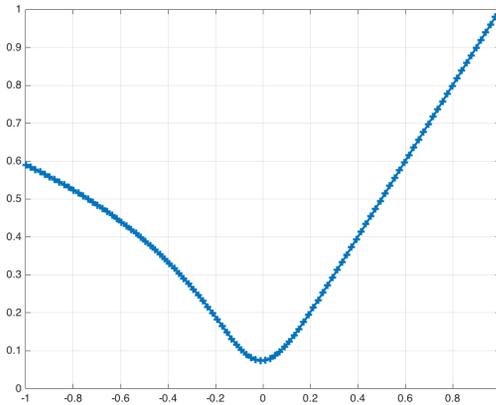
Figure 5: Variation of β_a with respect to a, for $a \in [-1, 1)\backslash\{0\}$.

– If $|a|^{-1} < b < \beta_a^{-1}$, then *surface superconductivity* is confined to the *part* of the surface near $\partial\Omega_2 \cap \partial\Omega$. At the same time, superconductivity is observed along the magnetic barrier Γ (see Figure 4). This behaviour is interesting for two reasons. Firstly, it demonstrates the existence of the *edge current* along the magnetic barrier, which is consistent with physics (see [HPRS16]). Secondly, it marks a distinct behaviour from the one known for uniform applied magnetic fields, in which case the whole surface carries superconductivity evenly (see for instance [HK17, FKP13, Pan02]).

– If $\beta_a^{-1} \leq b < |a|^{-1}\Theta_0^{-1}$, then superconductivity only survives along $\partial\Omega_2 \cap \partial\Omega$ (see Figure 4). Our results then display the strength of the applied magnetic field responsible for the breakdown of the edge

current along the barrier.

– If $b \geq |a|^{-1} \Theta_0^{-1}$, then all energy contributions in Theorem 1.7 disappear.

• For $a \in (0, 1)$, $\beta_a = a$ (see (2.19)). When $b > a^{-1}$, Theorem 1.11 reveals the absence of superconductivity along the magnetic barrier. As for the distribution of superconductivity along the surface of the sample, we distinguish between two regimes:

Regime 1, $a \in (0, \Theta_0]$. The part of the boundary, $\partial \Omega_1 \cap \partial \Omega$, does not carry superconductivity. It remains to inspect the energy contribution of $\partial \Omega_2 \cap \partial \Omega$. In that respect:

– If $a^{-1} < b < a^{-1} \Theta_0^{-1}$, then superconductivity exists along $\partial \Omega_2 \cap \partial \Omega$.

– If $b \geq a^{-1} \Theta_0^{-1}$, then superconductivity disappears along $\partial \Omega_2 \cap \partial \Omega$.

Regime 2, $a \in (\Theta_0, 1)$. We observe the following :

– If $a^{-1} < b < \Theta_0^{-1}$, then the entire surface of the sample is in a superconducting state, though the superconductivity distribution is not uniform.

– If $\Theta_0^{-1} \leq b < a^{-1} \Theta_0^{-1}$, then only $\partial \Omega_2 \cap \partial \Omega$ carries superconductivity.

– If $b \geq a^{-1} \Theta_0^{-1}$, then all the energy contributions in Theorem 1.7 vanish.

1.6 Notation

• The letter C denotes a positive constant whose value may change from one formula to another. Unless otherwise stated, the constant C depends on the value of a and the domain Ω, and is independent of κ and H.

• Let $a(\kappa)$ and $b(\kappa)$ be two positive functions. We write $a(\kappa) \approx b(\kappa)$, if there exist constants κ_0, C_1 and C_2 such that for all $\kappa \geq \kappa_0$, $C_1 a(\kappa) \leq b(\kappa) \leq C_2 a(\kappa)$.

- The quantity $o(1)$ indicates a function of κ, defined by universal quantities, the domain Ω, given functions, etc., and such that $|o(1)| \to 0$ as $\kappa \to +\infty$. Any expression $o(1)$ is independent of the minimizer $(\psi, \mathbf{A})$ of (1.1). Similarly, $\mathcal{O}(1)$ indicates a function of κ, absolutely bounded by a constant independent of the minimizers of (1.1).

- Let $n \in \mathbb{N}$, $p \in \mathbb{N}$, $N \in \mathbb{N}$, $\alpha \in (0,1)$, $K \subset \mathbb{R}^N$ be an open set. We use the following Hölder space

$$C^{n,\alpha}(\overline{K}) = \left\{ f \in C^n(\overline{K}) \mid \sup_{x \neq y \in K} \frac{|D^n f(x) - D^n f(y)|}{|x - y|^\alpha} < +\infty \right\}.$$

- Let $n \in \mathbb{N}$, $I \subset \mathbb{R}$ be an open interval. We use the space

$$B^n(I) = \left\{ u \in L^2(I) : x^i D^j u \in L^2(I), \ \forall i, j \in \mathbb{N} \text{ s.t. } i + j \leq n \right\}. \tag{1.10}$$

1.7 Heuristics of the proofs

In this section, we present our approach in an informal way, not organized according to the order of appearance of various effective models in the paper, but following a scheme highlighting some important links between these models.

We are mainly interested in examining the behaviour of the minimizer of the GL energy in (1.1) near the magnetic barrier Γ. Working under Assumption 1.3, one can use the (Agmon) decay estimates established in [AK16] (see Theorem 2.4) to neglect the bulk energy contribution and restrict the study near the edge Γ and the boundary $\partial\Omega$.

As the applied magnetic field behaves uniformly near $\partial\Omega \setminus \Gamma$, the study of surface superconductivity is the same as that in the case of uniform fields, frequently encountered in the literature. Therefore in Section 6.2, the reader is referred to the existing literature.

The rest of the paper mainly focuses on the study of superconductivity in a tubular neighbourhood of Γ. In Section 6, we decompose this neighbourhood into small cells, each of size $\mathcal{O}(\kappa^{-3/2})$, in order to establish the local asymptotics of the minimizer as well as the corresponding energy estimates as $\kappa \to +\infty$. This decomposition aims to reveal the existence of superconductivity in each of these small patches, in a certain regime of the applied magnetic field (i.e. for certain values of the parameter b, as in Assumption 1.3).

43

Using Frenet coordinates, cut-off functions, a suitable gauge transformation allowing to replace the induced magnetic field **A** by the applied magnetic field **F** (curl **F** = B_0, see Lemma 2.2), together with a rescaling argument (Sections 4–6), we may reduce the study of the GL energy in (1.1) into that of the 2D-effective energy $\mathcal{G}_{a,b,R}$ defined on the strip $S_R = (-R/2, R/2) \times (-\infty, +\infty)$, for $R > 1$ (Section 3)

$$\mathcal{G}_{a,b,R}(u) = \int_{S_R} \left(b \left| (\nabla - i\sigma \mathbf{A}_0) u \right|^2 - |u|^2 + \frac{1}{2} |u|^4 \right) dx,$$

with Dirichlet boundary conditions imposed on u, where $\sigma(x) = \mathbb{1}_{\mathbb{R}_+}(x_2) + a \mathbb{1}_{\mathbb{R}_-}(x_2)$ for $x = (x_1, x_2) \in \mathbb{R}^2$. Here, x_1 and x_2 are respectively the tangential and the normal coordinates with respect to the magnetic edge. We also consider the ground state energy

$$g_a(b, R) = \inf_u \mathcal{G}_{a,b,R}(u).$$

Hence, we launch an investigation of the new energy model, $\mathcal{G}_{a,b,R}$, with a step magnetic field. It is standard to begin by exploring the linear part of this energy, which leads us to the following linear magnetic Schrödinger operator defined in the plane (Section 2.4)

$$\mathcal{L}_a = -(\nabla - i\sigma \mathbf{A}_0)^2,$$

where $\mathbf{A}_0(x) = (-x_2, 0)$ for $x = (x_1, x_2) \in \mathbb{R}^2$. The ground state energy of this operator is denoted by β_a. One can easily see that the non-triviality of the energy $\mathcal{G}_{a,b,R}$ minimizer (that is when $g_a(b, R) \neq 0$) is equivalent to $1/|a| < b < 1/\beta_a$ (under Assumption 1.3). Therefore, to ensure the non-emptiness of the interval $(1/|a|, 1/\beta_a)$, thus the non-triviality of our study, we shall compare the values $|a|$ and β_a.

In order to get the aforementioned comparison (of $|a|$ and β_a), we use partial Fourier transform to perform a new reduction, this time of the 2D-operator $\mathcal{L}_a$ to a 1D-effective operator in $\mathbb{R}$, $h_a[\xi]$, parametrized by $\xi \in \mathbb{R}$ (Section 2.4):

$$h_a[\xi] = -\frac{d^2}{dt^2} + V_a(\xi, t),$$

with the effective potential

$$V_a(\xi, t) = \begin{cases} (\xi + at)^2 & (t < 0), \\ (\xi + t)^2 & (t > 0), \end{cases}$$

and with a lowest eigenvalue denoted by $\mu_a(\xi)$. The ground state energy β_a satisfies

$$\beta_a = \inf_{\xi \in \mathbb{R}} \mu_a(\xi).$$

Next, we provide information about this infimum by collecting some spectral properties of the operator $h_a[\xi]$. This 1D-operator has already been considered in the literature, and some spectral information was established experimentally and rigorously in earlier works (for instance see [HPRS16, HS15, DHS14, RP00, Iwa85]). However the approach in the aforementioned references was rather complicated, since all energy levels were examined. In addition, some of the spectral results we need in our study were not explicitly stated in these references. Therefore, for the sake of clarity and since we are only interested in the lowest eigenvalue, we opt to use a direct approach to provide such results (see Section 2.4). Moreover, our results slightly improve those of the aforementioned works (see Theorem 2.6). Our proofs call some spectral data of well-known effective models on the half-line (Section 2.3).

From Section 2.4, we collect the following useful properties:

- $\beta_a = a$, for $a \in (0, 1)$,

- $\beta_{-1} = \Theta_0$,

- $|a|\Theta_0 < \beta_a < |a|$, for $a \in [-1, 0)$.

Here, Θ_0 is the value in (2.5). Now, the comparison of β_a and a is in hand and a consequence of this is the following observation:

$$g_a(b, R) = 0, \quad \text{for } a \in (0, 1) \text{ and } b > \frac{1}{a}$$

while

$$g_a(b, R) < 0, \quad \text{for } a \in [-1, 0) \text{ and } \frac{1}{|a|} < b < \frac{1}{\beta_a}.$$

We highlight the contribution of Theorem 2.6 in obtaining the latter property. This gives us the desired information about the values of a and b for which our study is non-trivial. Subsequently, we neglect the case $a \in (0, 1)$ and proceed under the more restrictive assumption

$$a \in [-1, 0), \qquad \frac{1}{|a|} < b < \frac{1}{\beta_a}.$$

The main results about the reduced energy $\mathcal{G}_{a,b,R}$ are stated in Theorem 3.1. In particular, this theorem introduces the limiting energy $e_a(b)$ appearing in our main theorems (Theorems 1.7 and 1.11):

$$e_a(b) = \lim_{R \to +\infty} \frac{g_a(b, R)}{R}.$$

In addition, the bounds in the last item of this theorem are important to control the error terms arising while establishing the energy and minimizer estimates in Section 6. The proof of Theorem 3.1 occupies Section 3. It relies on the approach in [Pan02, FKP13] in the case of uniform fields, with some additional technical difficulties caused by the discontinuity of our magnetic field. For instance, we step carefully while establishing some regularity properties needed in proving the existence of $\mathcal{G}_{a,b,R}$ minimizer (see Lemmas B.3–B.6).

Finally, inspired by the recent work of Correggi–Rougerie [CR14] studying the surface superconductivity in the case of constant fields (more precisely by their energy lower bound proof), we interestingly prove that the 2D-limiting energy $e_a(b)$ is nothing but a one dimensional energy, $E_{a,b}^{1D}$, defined in Section 3.6. This reduction serves in providing a more explicit definition of the enregy $e_a(b)$ and suggests that the profile of the minimizing order parameter ψ near the edge is as follows (up to a gauge transformation):

$$\psi \approx f_0(b\kappa t)e^{i\bar{\xi}_0 b\kappa s} \tag{1.11}$$

where $(f_0, \bar{\xi}_0)$ is a minimizing couple of the energy $\mathscr{E}_{a,b,\xi}^{1D}$ defined in (3.16), s is the tangential distance along Γ and t is the normal distance to Γ. Such a profile suggests that the supercurrent along the edge Γ, $j = \text{Im}\big(\overline{\psi}(\nabla - i\kappa H\mathbf{A})\psi\big)$, behaves to leading order as $b\kappa\bar{\xi}_0 f_0(0)^2\vec{\tau}$, with $\vec{\tau}$ being a unit tangent vector along the edge Γ.

The rigorous derivation of (1.11) is not given in the present paper, but we expect that the analysis in this paper paves the way to a future investigation of the profile of ψ displayed in (1.11). In that respect, a special attention is required due to the non-homogeneity of the order parameter ψ as revealed in Theorem 1.11; indeed ψ seems to have different profiles along Γ and the parts of $\partial\Omega$.

One remarkable aspect of our proofs is that we have not used the *a priori* elliptic L^∞-estimate $\|(\nabla - i\kappa H\mathbf{A})\psi\|_\infty \leq C\kappa$. Such estimate is not known to hold in our case of discontinuous magnetic field B_0. Instead, we used the easy energy estimate $\|(\nabla - i\kappa H\mathbf{A})\psi\|_2 \leq C\kappa$ and the regularity of the curl-div system (see Theorem 2.3).

This also spares us the complex derivation of the L^∞-estimate (see [FH10, Chapter 11]).

1.8   Organization of the paper

Section 2 presents some preliminaries, particularly, a priori estimates, exponential decay results, and a linear 2D-operator with a step magnetic field. Theorem 2.6 is an improvement of a result in [HPRS16]. Section 3 introduces the 2D-reduced GL energy along with the barrier energy $e_a(\cdot)$. In Section 4, we present the Frenet coordinates defined in a tubular neighbourhood of the curve Γ. These coordinates are frequently used in the context of surface superconductivity [FH10, Appendix F]. In Section 5, we introduce a reference energy that describes the local behaviour of the full GL energy in (1.1). Section 6 is devoted for the analysis of the local behaviour of the minimizing order parameter near the edge Γ. Also, we recall well-known results about the local behaviour of the order parameter near the surface $\partial\Omega$. Finally, collecting all the estimates established in Section 6, we complete the proof of our main theorems (Theorems 1.7 and 1.11 above).

2   Preliminaries

2.1   A Priori Estimates

We present some celebrated estimates needed in the sequel to control the various errors arising while estimating the energy in (1.1).

Proposition 2.1. *If* $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2)$ *is a weak solution of* (1.3), *then*

$$\|\psi\|_{L^\infty(\Omega)} \leq 1.$$

The proof of Proposition 2.1 can be found in [FH10, Proposition 10.3.1].

Recall the magnetic field B_0 introduced in Assumption 1.2. In the next lemma, we will fix the gauge for the magnetic potential generating B_0 (see [AK16, Lemma A.1])

Lemma 2.2. *Suppose that the conditions in Assumptions 1.1 and 1.2 hold. There exists a unique vector field* $\mathbf{F} \in H^1_{\mathrm{div}}(\Omega)$ *such that*

$$\mathrm{curl}\ \mathbf{F} = B_0.$$

Furthermore, $\mathbf{F}$ *is in* $C^\infty(\Omega_i)$, $i = 1, 2$.

We collect below some useful estimates whose proofs are given in [AK16, Theorem 4.2].

Theorem 2.3. *Let $\alpha \in (0,1)$ be a constant. Suppose that the conditions in Assumptions 1.1 and 1.2 hold. There exists a constant $C > 0$ (dependent on b) such that if (1.4) is satisfied and $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$ is a solution of (1.3), then*

  1. $\|(\nabla - i\kappa H\mathbf{A})\psi\|_{L^2(\Omega)} \leq C\kappa.$

  2. $\|\mathrm{curl}(\mathbf{A} - \mathbf{F})\|_{L^2(\Omega)} \leq \dfrac{C}{\kappa}.$

  3. $\mathbf{A} - \mathbf{F} \in H^2(\Omega)$ *and* $\|\mathbf{A} - \mathbf{F}\|_{H^2(\Omega)} \leq \dfrac{C}{\kappa}.$

  4. $\mathbf{A} - \mathbf{F} \in C^{0,\alpha}(\overline{\Omega})$ *and* $\|\mathbf{A} - \mathbf{F}\|_{C^{0,\alpha}(\overline{\Omega})} \leq \dfrac{C}{\kappa}.$

2.2 Exponential decay of the order parameter

The following theorem displays a regime for the intensity of the applied magnetic field where the order parameter and the GL energy are exponentially small in the bulk of the domains Ω_1 and Ω_2.

Theorem 2.4. *Given $a \in [-1, 1) \setminus \{0\}$ and $b > 1/|a|$, there exist constants $\kappa_0 > 0$, $C > 0$, and $\alpha_0 > 0$ such that, if*

$$\kappa \geq \kappa_0, \ \kappa_0 \kappa^{-1} \leq \ell < 1, \ \text{and} \ (\psi, \mathbf{A}) \text{ is a solution of (1.3) for } H = b\kappa,$$

then

$$\int_{\Omega_j \cap \{\mathrm{dist}(x, \partial\Omega_j) \geq \ell\}} \left(|\psi|^2 + (\kappa H)^{-1} |(\nabla - i\kappa H\mathbf{A})\psi|^2 \right) dx \leq C\kappa^{-1} e^{-\alpha_0 \kappa \ell},$$

for $j \in \{1, 2\}$.

Remark 2.5. In the proof of Theorem 2.4 below, we see that $\alpha_0 \to 0$ when $b \to (1/|a|)_+$.

Proof of Theorem 2.4. The proof is a consequence of the Agmon-type estimates established in [AK16, Theorems 1.5 & 7.3]; indeed, for a fixed $b > 1/|a|$, there

exist $\kappa_0, C > 0$ such that, for $\kappa \geq \kappa_0$ and $H = b\kappa$,

$$\int_{\Omega_j \cap \{\text{dist}(x, \partial\Omega_j) \geq \frac{1}{\sqrt{\kappa H}}\}} \left(|\psi|^2 + \frac{1}{\kappa H} |(\nabla - i\kappa H A)\psi|^2 \right)$$
$$\exp\left(2\varepsilon \sqrt{\kappa H} \, \text{dist}(x, \partial\Omega_j) \right) dx \leq C \int_{\Omega_j \cap \{\text{dist}(x, \partial\Omega_j) \leq \frac{1}{\sqrt{\kappa H}}\}} |\psi|^2 \, dx, \quad (2.1)$$

for $j \in \{1, 2\}$ and $0 < \varepsilon < b - 1/|a|$. We modify the choice of κ_0 so that $\kappa_0 \geq 1/\sqrt{b}$. That way, for $\kappa \geq \kappa_0$ and $\kappa_0 \kappa^{-1} \leq \ell < 1$, we get $\ell \geq 1/\sqrt{\kappa H}$. Using (2.1), one can easily verify the claim of Theorem 2.4, with $\alpha_0 = \alpha_0(b) = 2\varepsilon\sqrt{b}$. $\qquad\square$

2.3 A Family of Sturm–Liouville operators on $L^2(\mathbb{R}_+)$

In this section, we will briefly present some spectral properties of the self-adjoint realization on $L^2(\mathbb{R}_+)$ of the Sturm–Liouville operator:

$$H[\gamma, \xi] = -\frac{d^2}{dt^2} + (t - \xi)^2, \qquad (2.2)$$

defined over the domain

$$\text{Dom}\left(H[\gamma, \xi]\right) = \{u \in B^2(\mathbb{R}_+) \; : \; u'(0) = \gamma u(0)\},$$

where $B^2(\mathbb{R}_+)$ is the space introduced in (1.10), and ξ and γ are two real parameters.

Denote by $\mu(\gamma, \xi)$ the lowest eigenvalue of the operator $H[\gamma, \xi]$

$$\mu(\gamma, \xi) = \inf \text{sp}\left(H[\gamma, \xi]\right). \qquad (2.3)$$

For all $\gamma \in \mathbb{R}$, we define

$$\Theta(\gamma) = \inf_{\xi \in \mathbb{R}} \mu(\gamma, \xi), \qquad (2.4)$$

The particular case where $\gamma = 0$ corresponds to the *Neumann* realization, and we use the following notation,

$$H^N[\xi] = H[0, \xi], \quad \mu^N(\xi) = \mu(0, \xi), \quad \Theta_0 = \Theta(0). \qquad (2.5)$$

For all $\gamma \in \mathbb{R}$, there exists a unique minimum $\xi(\gamma)$ for the function $\xi \mapsto \mu(\gamma, \xi)$. Furthermore (see [Kac06, Section 2.3])

$$\xi(\gamma) = \sqrt{\Theta(\gamma) + \gamma^2}. \qquad (2.6)$$

and

$$\forall \gamma \geq 0, \qquad 0 < \Theta(\gamma) < 1. \qquad (2.7)$$

2.4 An operator with a step magnetic field

Let $a \in [-1, 1) \setminus \{0\}$. We consider the magnetic potential $\mathbf{A}_0$ defined by

$$\mathbf{A}_0(x) = (-x_2, 0) \qquad \left(x = (x_1, x_2) \in \mathbb{R}^2\right), \tag{2.8}$$

which satisfies $\operatorname{curl} \mathbf{A}_0 = 1$. We define the step function σ as follows. For $x = (x_1, x_2) \in \mathbb{R}^2$,

$$\sigma(x) = \mathbb{1}_{\mathbb{R}_+}(x_2) + a \mathbb{1}_{\mathbb{R}_-}(x_2). \tag{2.9}$$

We introduce the self-adjoint *magnetic Hamiltonian*

$$\mathscr{L}_a = -(\nabla - i\sigma \mathbf{A}_0)^2, \tag{2.10}$$

with domain of definition

$$\operatorname{Dom}\left(\mathscr{L}_a\right) = \left\{u \in L^2(\mathbb{R}^2) \; : \; (\nabla - i\sigma \mathbf{A}_0)^j u \in L^2(\mathbb{R}^2), \text{ for } j \in \{1, 2\}\right\}.$$

The ground state energy of the operator $\mathscr{L}_a$ is denoted by

$$\beta_a = \inf \operatorname{sp}\left(\mathscr{L}_a\right). \tag{2.11}$$

Since the Hamiltonian $\mathscr{L}_a$ is invariant with respect to translations in the x_1-direction then, by using the partial Fourier transform with respect to the x_1-variable, we can reduce $\mathscr{L}_a$ to a family of Shrödinger operators on $L^2(\mathbb{R})$, $h_a[\xi]$, parametrized by $\xi \in \mathbb{R}$ and called *fiber operators* (see [HPRS16, HS15]). The operator $h_a[\xi]$ is defined by

$$h_a[\xi] = -\frac{d^2}{dt^2} + V_a(\xi, t), \tag{2.12}$$

with

$$V_a(\xi, t) = \begin{cases} (\xi + at)^2, & t < 0, \\ (\xi + t)^2, & t > 0. \end{cases} \tag{2.13}$$

The domain of $h_a[\xi]$ is given by

$$\operatorname{Dom}\left(h_a[\xi]\right) = \left\{u \in B^1(\mathbb{R}) : \left(-\frac{d^2}{dt^2} + V_a(\xi, t)\right)u \in L^2(\mathbb{R}), u'(0_+) = u'(0_-)\right\}.$$

The quadratic form associated to $h_a[\xi]$ is

$$q_a[\xi](u) = \int_{\mathbb{R}} \left(|u'(t)|^2 + V_a(\xi, t)|u(t)|^2\right) dt, \tag{2.14}$$

defined on the form domain

$$\mathrm{Dom}\left(q_a[\xi]\right) = B^1(\mathbb{R}). \tag{2.15}$$

The spectra of the operators $\mathscr{L}_a$ and $h_a[\xi]$ are linked together as follows (see [FH10, Section 4.3])

$$\mathrm{sp}(\mathscr{L}_a) = \overline{\bigcup_{\xi \in \mathbb{R}} \mathrm{sp}(h_a[\xi])}. \tag{2.16}$$

We introduce the lowest eigenvalue of the fiber operator $h_a[\xi]$,

$$\mu_a(\xi) = \inf \mathrm{sp}(h_a[\xi]) = \inf_{u \in B^1(\mathbb{R}), u \neq 0} \frac{q_a[\xi](u)}{\|u\|^2_{L^2(\mathbb{R})}}. \tag{2.17}$$

Consequently, for all $a \in [-1, 1)\backslash\{0\}$, we may express the ground state energy in (2.11) by

$$\beta_a = \inf_{\xi \in \mathbb{R}} \mu_a(\xi). \tag{2.18}$$

Below, we collect some properties of the eigenvalue $\mu_a(\xi)$.

The case $0 < a < 1$. This case is studied in [HS15, Iwa85]. The eigenvalue $\mu_a(\xi)$ is simple and is a decreasing function of ξ. The monotonicity of $\mu_a(\cdot)$ and its asymptotics in Proposition A.4 imply that

$$a < \mu_a(\xi) < 1 \qquad (\xi \in \mathbb{R}),$$

and that β_a, introduced in (2.11), satisfies

$$\beta_a = a. \tag{2.19}$$

The case $a = -1$. This case is studied in [HPRS16]. Using symmetry arguments, $\mu_{-1}(\xi)$ is simple and satisfies

$$\mu_{-1}(\xi) = \mu^N(-\xi), \tag{2.20}$$

where $\mu^N(\cdot)$ is introduced in (2.5). By (2.5)–(2.7),

$$0 < \min_{\xi \in \mathbb{R}} \mu_{-1}(\xi) = \mu_{-1}(\xi_{-1}) = \Theta_0 < 1, \tag{2.21}$$

where $\xi_{-1} = -\xi(0) = -\sqrt{\Theta_0}$, Θ_0 and $\xi(0)$ are respectively introduced in (2.5) and (2.7).

The case $-1 < a < 0$. See also [HPRS16] for the study of this case. The eigenvalue $\mu_a(\xi)$ is simple, and there exists $\xi_a < 0$ satisfying[3]

$$|a| \geq \mu_a(\xi_a) = \min_{\xi \in \mathbb{R}} \mu_a(\xi). \tag{2.22}$$

Moreover, we have (see Proposition A.6)

$$|a| \Theta_0 < \min_{\xi \in \mathbb{R}} \mu_a(\xi). \tag{2.23}$$

Combining the foregoing discussion in the case $a \in [-1, 0)$, we get that β_a, introduced in (2.11), satisfies

$$|a| \Theta_0 \leq \beta_a \leq |a|, \tag{2.24}$$

$$\beta_a = \mu_a(\xi_a) \text{ with } \xi_a < 0. \tag{2.25}$$

In particular,

$$\beta_{-1} = (\xi_{-1})^2 = \Theta_0. \tag{2.26}$$

In the next theorem, we will use a direct approach, different from the one in [HPRS16], to establish the existence of a global minimum ξ_a in the case when $a \in (-1, 0)$ and to prove that $\beta_a < |a|$. Theorem 2.6 slightly improves the estimates in [HPRS16], since it provides an upper bound of β_a stronger than $|a|$. This theorem is necessary to validate the hypothesis $1/|a| < 1/\beta_a$ in (3.7), under which we work in Section 3. Indeed, it guarantees the existence of a non-empty b-parameter region where the minimizer of the reduced GL energy $\mathcal{G}_{a,b,R}$, introduced in Section 3, is non-trivial, which is key in the study of this energy.

Theorem 2.6. *For all* $a \in (-1, 0)$, *there exists* $\xi < 0$ *such that* $\mu_a(\xi)$, *the lowest eigenvalue of the operator* $h_a[\xi]$, *satisfies*

$$\mu_a(\xi) < |a| \Theta \left(\sqrt{\frac{1}{2|a|(1 - |a|)}} \right) < |a|,$$

where $\Theta(\cdot)$ *is defined in (2.4). Consequently, the function* $\xi \mapsto \mu_a(\xi)$ *admits a global minimum satisfying*

$$\min_{\xi \in \mathbb{R}} \mu_a(\xi) < |a|.$$

---

[3] ξ_a was not explicitly proven to be negative in [HPRS16]. For the convenience of the reader, we show that $\xi_a < 0$ in Proposition A.7.

Proof. The proof is inspired by [Kac]. For all $\gamma \in \mathbb{R}$, let $\Theta(\gamma)$ and $\xi(\gamma)$ be the quantities introduced in (2.4) and (2.6) respectively $\big($such that $\Theta(\gamma) = \mu(\gamma, \xi(\gamma))\big)$. Denote by $\varphi_\gamma = \varphi_{\gamma, \xi(\gamma)}$ the positive L^2-normalized eigenfunction of the operator in (2.2) with eigenvalue $\Theta(\gamma)$. Define the function

$$u(t) = \begin{cases} \varphi_\gamma(0) \exp(-mt), & t \geq 0, \\ \varphi_\gamma(-\sqrt{|a|}t), & t < 0, \end{cases} \tag{2.27}$$

where γ and m are two positive constants to be fixed later. One can check that $u \in \mathrm{Dom}\big(q_a[\xi]\big)$, hence by the min-max principle, for all $\xi \in \mathbb{R}$,

$$\mu_a(\xi) \leq \frac{q_a[\xi](u)}{\|u\|_{L^2(\mathbb{R})}^2}. \tag{2.28}$$

Pick $\xi \in \mathbb{R}$. We will choose ξ precisely later. The quadratic form $q_a[\xi](u)$ defined in (2.14) can be decomposed as follows:

$$q_a[\xi](u) = q_a^{(1)}[\xi](u) + q_a^{(2)}[\xi](u),$$

where

$$q_a^{(1)}[\xi](u) = \int_0^{+\infty} \left(|u'(t)|^2 + |(t + \xi)u(t)|^2 \right) dt,$$

and

$$q_a^{(2)}[\xi](u) = \int_{-\infty}^0 \left(|u'(t)|^2 + |(at + \xi)u(t)|^2 \right) dt.$$

A simple computation gives

$$q_a^{(1)}[\xi](u) = \left(\frac{m}{2} + \frac{\xi^2}{2m} + \frac{\xi}{2m^2} + \frac{1}{4m^3} \right) |\varphi_\gamma(0)|^2. \tag{2.29}$$

On the other hand, for $t < 0$, we do the change of variable $y = -\sqrt{|a|}t$, which in turn yields

$$q_a^{(2)}[\xi](u) = \sqrt{|a|} \int_0^{+\infty} \left(|\varphi'_\gamma(y)|^2 + \left| \left(y + \frac{\xi}{\sqrt{|a|}} \right) \varphi_\gamma(y) \right|^2 \right) dy.$$

Now we select $\xi = -\sqrt{|a|}\xi(\gamma)$ (see (2.6)). That way we get

$$q_a^{(2)}[\xi](u) = \sqrt{|a|} \left(\Theta(\gamma) - \gamma |\varphi_\gamma(0)|^2 \right). \tag{2.30}$$

The definition of the function u in (2.27) yields

$$\int_{-\infty}^{+\infty} |u(t)|^2 \, dt = \frac{|\varphi_\gamma(0)|^2}{2m} + \frac{1}{\sqrt{|a|}}. \qquad (2.31)$$

Combining the results in (2.29)–(2.31) and using (2.7), we rewrite (2.28) as follows

$$\mu_a(\xi) \leq \frac{\sqrt{|a|}\,\Theta(\gamma)}{\frac{1}{\sqrt{|a|}} + \frac{|\varphi_\gamma(0)|^2}{2m}}$$

$$+ \frac{\left(\frac{m}{2} - \sqrt{|a|}\right)\gamma + \frac{|a|\left(\Theta(\gamma)+\gamma^2\right)}{2m} - \frac{\sqrt{|a|}\left(\Theta(\gamma)+\gamma^2\right)}{2m^2} + \frac{1}{4m^3}\right)|\varphi_\gamma(0)|^2}{\frac{1}{\sqrt{|a|}} + \frac{|\varphi_\gamma(0)|^2}{2m}}.$$

Since $0 < \Theta(\gamma) < 1$ for $\gamma > 0$,

$$\mu_a(\xi) \leq \frac{\sqrt{|a|}\,\Theta(\gamma) + \left(\frac{m}{2} - \sqrt{|a|}\right)\gamma + \frac{|a|(1+\gamma^2)}{2m} - \frac{\sqrt{|a|}\gamma}{2m^2} + \frac{1}{4m^3}\right)|\varphi_\gamma(0)|^2}{\frac{1}{\sqrt{|a|}} + \frac{|\varphi_\gamma(0)|^2}{2m}}.$$

Now we choose $\gamma = \sqrt{1/(2|a|(1-|a|))}$ and $m = \sqrt{|a|}\gamma$. Using again the fact that $\Theta(\gamma) < 1$, we obtain

$$\mu_a(\xi) \leq \frac{\sqrt{|a|}\,\Theta(\gamma)}{\frac{1}{\sqrt{|a|}} + \frac{|\varphi_\gamma(0)|^2}{2m}} < |a|\Theta(\gamma) < |a|. \qquad (2.32)$$

The existence of the global minimum is now standard (it is a consequence of Proposition A.4 in the appendix). $\qquad\square$

Remark 2.7. Collecting the foregoing results in (2.19)–(2.23) and Theorem 2.6, we deduce the following facts regarding the bottom of the spectrum of the operator $\mathscr{L}_a$ introduced in (2.10).

1. For all $a \in (0, 1)$, $\beta_a = a$.

2. For all $a \in [-1, 0)$, $|a|\Theta_0 \leq \beta_a < |a|$, and there exist $\xi_a < 0$ and a L^2-normalized function $\phi_a \in B^2(\mathbb{R})$ such that

$$h_a[\xi_a]\phi_a = \beta_a\phi_a \text{ in } \mathbb{R}, \qquad (2.33)$$

where $h_a[\cdot]$ is introduced in (2.12).

3   Reduced Ginzburg–Landau Energy

3.1   The functional and the main result

Assume that $a \in [-1, 1) \backslash \{0\}$ is fixed, σ is the step function defined in (2.9) and $\mathbf{A}_0$ is the magnetic potential defined in (2.8). For every $R > 1$, consider the strip

$$S_R = (-R/2, R/2) \times (-\infty, +\infty). \qquad (3.1)$$

We introduce the space

$$\mathscr{D}_R = \left\{ u \in L^2(S_R) \; : \; (\nabla - i\sigma\mathbf{A}_0)u \in L^2(S_R), \; u\left(x_1 = \pm\frac{R}{2}, x_2\right) = 0 \right\}. \qquad (3.2)$$

Note that the boundary condition in the domain $\mathscr{D}_R$ is meant in the trace sense. For $b > 0$, we define the following Ginzburg–Landau energy on $\mathscr{D}_R$ by

$$\mathscr{G}_{a,b,R}(u) = \int_{S_R} \left(b \left|(\nabla - i\sigma\mathbf{A}_0)u\right|^2 - |u|^2 + \frac{1}{2}|u|^4 \right) dx, \qquad (3.3)$$

along with the ground state energy

$$g_a(b, R) = \inf_{u \in \mathscr{D}_R} \mathscr{G}_{a,b,R}(u). \qquad (3.4)$$

Our objective is to prove

Theorem 3.1. *Assume that* $a \in [-1, 1) \backslash \{0\}$*,* $b \geq 1/|a|$*,* $R > 1$*,* $g_a(b, R)$ *is the ground state energy in* (3.4)*, and* β_a *is defined in* (2.11)*.*
  The following holds:

1. $g_a(b, R) \leq 0$.

2. *If* $a \in (0, 1)$*, then* $g_a(b, R) = 0$.

3. *If* $a \in [-1, 0)$*, then there exists a constant* $e_a(b) \leq 0$ *such that*

$$\lim_{R \to +\infty} \frac{g_a(b, R)}{R} = e_a(b). \qquad (3.5)$$

    Furthermore, $e_a(b) = 0$ *if and only if* $b \geq 1/\beta_a$.

4. *For all* $a \in [-1, 0)$*, the function* $[1/|a|, +\infty) \ni b \mapsto e_a(b)$ *is monotone non-decreasing and continuous.*

5. For all $a \in [-1, 0)$, there exists $C > 0$ such that

$$\forall R \geq 4, \quad e_a(b) \leq \frac{g_a(b, R)}{R} \leq e_a(b) + \frac{C}{R^{\frac{1}{3}}}. \tag{3.6}$$

The proof of Theorem 3.1, along with other properties of $e_a(b)$, will occupy the rest of this section.

3.2 The trivial case

We start by handling the trivial situation where the ground state energy vanishes:

Lemma 3.2. If $a \in [-1, 1) \backslash \{0\}$ and $b \geq 1/\beta_a$, then for all $R > 1$, $g_a(b, R) = 0$.

Remark 3.3.

1. Under the assumptions in Lemma 3.2, the function $u = 0 \in \mathcal{D}_R$ is the unique minimizer of the functional in (3.3).

2. When $a \in (0, 1)$, $\beta_a = a$ by Remark 2.7, hence Lemma 3.2 yields that $g_a(b, R) = 0$ for all $b \geq 1/a$ and $R > 1$.

Proof of Lemma 3.2. We have the obvious upper bound $g_a(b, R) \leq \mathcal{G}_{a,b,R}(0) = 0$. For the lower bound, pick an arbitrary function $u \in \mathcal{D}_R$ and extend it by zero on $\mathbb{R}^2$. Using the min-max principle, we get

$$\mathcal{G}_{a,b,R}(u) \geq b \beta_a \int_{S_R} |u|^2 \, dx + \int_{S_R} \left(-|u|^2 + \frac{1}{2}|u|^4 \right) dx \geq 0 \text{ since } b \geq \frac{1}{\beta_a}. \quad \square$$

3.3 Existence of minimizers

Now we handle the following case, which is complementary to that in Lemma 3.2

$$-1 \leq a < 0 \quad \text{and} \quad \frac{1}{|a|} \leq b < \frac{1}{\beta_a}, \tag{3.7}$$

where β_a is the lowest eigenvalue introduced in (2.11). Under the hypothesis in (3.7), we can prove the existence of a non-trivial minimizer of the functional in (3.3) along with decay estimates at infinity.

Proposition 3.4. *Assume that* (3.7) *holds and let* $R > 1$. *There exists a function* $\varphi_{a,b,R} \in \mathcal{D}_R$ *such that, for* R *large enough,* $\varphi_{a,b,R} \not\equiv 0$,

$$\mathcal{G}_{a,b,R}(\varphi_{a,b,R}) = g_a(b, R) \quad \text{and} \quad \|\varphi_{a,b,R}\|_{L^\infty(S_R)} \le 1. \qquad (3.8)$$

Here $\mathcal{G}_{a,b,R}$ *is the functional in* (3.3) *and* $g_a(b, R)$ *is the ground state energy in* (3.4).
   Furthermore, there exists a universal constant $C > 0$ *such that, for all* $R > 1$, *the function* $\varphi_{a,b,R}$ *satisfies*

$$\int_{S_R \cap \{|x_2| \ge 4\}} \frac{|x_2|}{\left(\ln |x_2|\right)^2} \left(\left|(\nabla - i\sigma\mathbf{A}_0)\varphi_{a,b,R}\right|^2 + |\varphi_{a,b,R}|^2\right) dx \le CbR, \qquad (3.9)$$

and

$$\int_{S_R} \left(b\left|(\nabla - i\sigma\mathbf{A}_0)\varphi_{a,b,R}\right|^2 + |\varphi_{a,b,R}|^2\right) dx \le CbR. \qquad (3.10)$$

   The proof of Proposition 3.4 relies on the approach in [FKP13, Theorem 3.6] and [Pan02]. It can be described in a heuristic manner as follows. The unboundedness of the set S_R makes the existence of the minimizer $\varphi_{a,b,R}$ in (3.8) non-trivial. To overcome this issue, we consider a reduced Ginzburg–Landau energy $\mathcal{G}_{a,b,R,m}$ defined on the bounded set $S_{R,m} = (-R/2, R/2) \times (-m, m)$, and we establish some decay estimates of its minimizer $\varphi_{a,b,R,m}$. Later, using a limiting argument on $\mathcal{G}_{a,b,R,m}$ and $\varphi_{a,b,R,m}$ for large values of m, we obtain the existence of the minimizer $\varphi_{a,b,R}$ together with the properties in Proposition 3.4. The details are given in Appendix B for the convenience of the reader.

3.4   The limit energy

In this section, we will prove the existence of the limit energy $e_a(b)$, defined as the limit of $g_a(b, R)/R$ as $R \to +\infty$. After that, we will study, when the parameter a is fixed, some properties of the function $b \mapsto e_a(b)$.
   In the sequel, we assume that a, b, R *are constants such that* $R \ge 1$ *and* (3.7) *holds.* The next lemma displays some simple, yet very important, property of the energy. This property is mainly needed in Theorem 3.1 to establish an upper bound of the limit energy $e_a(b)$.

Lemma 3.5. *Let* $n \in \mathbb{N}$. *Consider the ground state energy* $g_a(b, R)$ *defined in* (3.4), *then*

$$g_a(b, nR) \le n g_a(b, R).$$

Proof. Lemma 3.5 follows from the translation invariance of the integrand of $\mathcal{G}_{a,b,R}$ with respect to the variable x_1 and the Dirichlet boundary conditions, where $\mathcal{G}_{a,b,R}$ is defined in (3.3). $\qquad\square$

Our next result easily follows from the property of monotonicity with respect to the domain.

Lemma 3.6. *The function $R \mapsto g_a(b, R)$ defined in (3.4) is monotone non-increasing.*

The existence of the limit of $g_a(b, R)/R$ as $R \to +\infty$ will be derived from a well-known abstract result (see [FK13, Lemma 2.2]). To apply this abstract result, we need some estimates on the energy $g_a(b, R)$, that we give in Lemmas 3.7 and 3.8 below.

Lemma 3.7. *Let $g_a(b, R)$ be the ground state energy in (3.4). There exist positive constants C_1, C_2, and C_3 dependent only on a and b such that*

$$- C_1 R \leq \frac{g_a(b, R)}{1 - b\beta_a} \leq -C_2 R + \frac{C_3}{R}. \qquad (3.11)$$

Proof.
   Upper bound. Let $\theta \in C_c^\infty(\mathbb{R})$ be a smooth cut-off function satisfying

$$\operatorname{supp} \theta \subset \left(-\frac{1}{2}, \frac{1}{2} \right), \quad 0 \leq \theta \leq 1, \quad \theta = 1 \text{ in } \left[-\frac{1}{4}, \frac{1}{4} \right],$$

and let $\theta_R(x) = \theta(x/R)$. Recall the function $\phi_a \not\equiv 0$ defined in (2.33), we define in $\mathbb{R}^2$ the functions

$$\psi_a(x_1, x_2) = e^{i\xi_a x_1} \phi_a(x_2) \quad \text{and} \quad v(x_1, x_2) = \theta_R(x_1) \psi_a(x_1, x_2). \qquad (3.12)$$

The function ψ_a satisfies $-(\nabla - i\sigma A_0)^2 \psi_a = \beta_a \psi_a$. Multiplying this equation by $\overline{\psi_a}\theta_R^2$ and integrating by parts yield

$$\beta_a \int_{S_R} \theta_R^2(x_1)|\psi_a(x)|^2 \, dx = \int_{S_R} \theta_R^2(x_1)\big|(\nabla - i\sigma A_0)\psi_a(x)\big|^2 \, dx$$
$$+ 2 \int_{S_R} \theta_R(x_1)\theta_R'(x_1)\overline{\psi_a}(x)(\nabla - i\sigma A_0)\psi_a(x) \, dx.$$

Taking the real part of each side of the equation above, we get

$$\beta_a \int_{S_R} \theta_R^2(x_1) |\psi_a(x)|^2 \, dx = \int_{S_R} \theta_R^2(x_1) |(\nabla - i\sigma\mathbf{A}_0)\psi_a(x)|^2 \, dx$$

$$+ 2\mathrm{Re} \int_{S_R} \theta_R(x_1) \theta_R'(x_1) \overline{\psi_a}(x) (\nabla - i\sigma\mathbf{A}_0)\psi_a(x) \, dx$$

$$= \int_{S_R} |(\nabla - i\sigma\mathbf{A}_0)v|^2 \, dx - \int_{S_R} \theta_R'^2(x_1) |\psi_a(x)|^2 \, dx$$

$$= \int_{S_R} |(\nabla - i\sigma\mathbf{A}_0)v|^2 \, dx - \int_{S_R} \theta_R'^2(x_1) |\phi_a(x_2)|^2 \, dx.$$

Hence, using $\|\phi_a\|_{L^2(\mathbb{R})} = 1$ and the properties of θ_R in (3.12), we obtain

$$\int_{S_R} |(\nabla - i\sigma\mathbf{A}_0)v|^2 \, dx \le \beta_a \int_{S_R} \theta_R^2(x_1) |\psi_a(x)|^2 \, dx + \frac{C}{R}.$$

Consequently, for $t = \sqrt{(1 - b\beta_a)/\nu_a}$ and $\nu_a = \int_{\mathbb{R}} |\phi_a(x_2)|^4 \, dx_2$, we get

$$g_a(b, R) \le \mathcal{G}_{a,b,R}(tv)$$

$$\le t^2 R \left((b\beta_a - 1) + \frac{t^2}{2} \int_{\mathbb{R}} |\phi_a(x_2)|^4 \, dx_2 \right) + Cb\frac{t^2}{R}$$

$$\le (1 - b\beta_a) \left(-C_2 R + \frac{C_3}{R} \right),$$

where $C_2 = (1/2)t^2$ and $C_3 = Cb/\nu_a$.

Lower bound. Let $\varphi = \varphi_{a,b,R}$ be the minimizer in Proposition 3.4. It follows from the min-max principle that

$$g_a(b, R) = \mathcal{G}_{a,b,R}(\varphi) \ge (b\beta_a - 1) \int_{S_R} |\varphi|^2 \, dx.$$

By (3.10), $\int_{S_R} |\varphi|^2 \, dx \le CbR$, where $C > 0$ is some constant. Hence, choosing $C_1 = C/\beta_a$ establishes the desired lower bound. $\qquad\square$

Lemma 3.8. *There exists a universal constant C such that, for all $n \in \mathbb{N}$ and $\alpha \in (0, 1)$, the ground state energy $g_a(b, R)$ defined in (3.4) satisfies, for $R > 1$,*

$$\frac{g_a(b, n^2 R)}{n^2 R} \ge \frac{g_a(b, (1 + \alpha)^2 R)}{(1 + \alpha)^2 R} - Cb^2 \left(\alpha + \frac{1}{\alpha^2 R} \right). \tag{3.13}$$

Proof. Let $n \geq 1$ be a natural number, $\alpha \in (0,1)$ and consider the family of strips

$$S_j = \left(-n^2 - 1 - \alpha + (2j-1)\left(1 + \frac{\alpha}{2}\right), -n^2 - 1 + (2j+1)\left(1 + \frac{\alpha}{2}\right)\right) \times \mathbb{R},$$

for $j \in \mathbb{Z}$. Notice that the width of S_j is $2(1 + \alpha)$, and the overlapping occurs only between two adjacent strips (S_j and S_{j-1}, for any j). There exists a universal constant $\tilde{C} > 0$ and a partition of unity $(\chi_j)_{j \in \mathbb{Z}}$ of $\mathbb{R}^2$ such that

$$\sum_j |\chi_j|^2 = 1, \quad \operatorname{supp} \chi_j \subset S_j, \quad 0 \leq \chi_j \leq 1, \quad |\nabla \chi_j| \leq \frac{\tilde{C}}{\alpha},$$

and

$$\chi_j = 1 \text{ in } \left(-n^2 - 1 + (2j-1)\left(1 + \frac{\alpha}{2}\right), -n^2 - 1 - \alpha + (2j+1)\left(1 + \frac{\alpha}{2}\right)\right).$$

Since the overlapping is between a finite number of strips, one may further write

$$\sum_j |\chi_j|^2 = 1, \quad 0 \leq \chi_j \leq 1, \quad \sum_j |\nabla \chi_j|^2 \leq \frac{C}{\alpha^2}, \quad \operatorname{supp} \chi_j \subset S_j,$$

where C is some universal constant. Define

$$\chi_{R,j}(x) = \chi_j(2x/R),$$

$(\chi_{R,j})$ is then a new partition of unity satisfying

$$\sum_j |\chi_{R,j}|^2 = 1, \ 0 \leq \chi_{R,j} \leq 1, \ \sum_j |\nabla \chi_{R,j}|^2 \leq \frac{C}{\alpha^2 R^2}, \ \operatorname{supp} \chi_{R,j} \subset S_{R,j}, \quad (3.14)$$

where $S_{R,j} = \{xR/2 \ : \ x \in S_j\}$. The family of strips $(S_{R,j})_{j \in \{1,2,\dots,n^2\}}$ yields a covering of $S_{n^2 R} = \left(-n^2 R/2, n^2 R/2\right) \times \mathbb{R}$ by n^2 strips, each of width $(1 + \alpha)R$. Let $\varphi_{a,b,n^2 R} \in \mathcal{D}_{n^2 R}$ be the minimizer in Proposition 3.4. We decompose the energy associated to $\varphi_{a,b,n^2 R}$ as follows

$$g_a(b, n^2 R) \geq \sum_{j=1}^{n^2} \left(\mathcal{G}_{a,b,n^2 R}(\chi_{R,j}\varphi_{a,b,n^2 R}) - b\left\|(\nabla \chi_{R,j})\varphi_{a,b,n^2 R}\right\|_{L^2\left(S_{n^2 R}\right)}^2\right)$$

$$\geq \sum_{j=1}^{n^2} \mathcal{G}_{a,b,n^2 R}(\chi_{R,j}\varphi_{a,b,n^2 R}) - C\frac{b^2 n^2}{\alpha^2 R}.$$

The first inequality above follows from the celebrated *IMS localization formula* (see [CFKS09, Theorem 3.2]), while the second comes from (3.10) and the properties of $(\chi_{R,j})$ in (3.14). Notice that $\chi_{R,j}\varphi_{a,b,n^2R}$ is supported in an infinite strip of width $(1+\alpha)R$. By energy translation invariance along the x_1-direction, we have

$$\mathscr{G}_{a,b,n^2R}(\chi_{R,j}\varphi_{a,b,n^2R}) \geq g_a(b,(1+\alpha)R).$$

As a consequence,

$$g_a(b,n^2R) \geq n^2 g_a(b,(1+\alpha)R) - C\frac{b^2 n^2}{\alpha^2 R}.$$

For $R \geq 1$, dividing both sides by n^2R and using the monotonicity of $R \mapsto g_a(b,R)$, we get

$$\begin{aligned}
\frac{g_a(b,n^2R)}{n^2R} &\geq \frac{g_a\big(b,(1+\alpha)R\big)}{R} - \frac{Cb^2}{\alpha^2 R^2} \\
&\geq \frac{g_a\big(b,(1+\alpha)^2R\big)}{(1+\alpha)^2R} - Cb^2\Big(\alpha + \frac{1}{\alpha^2 R^2}\Big) \\
&\geq \frac{g_a\big(b,(1+\alpha)^2R\big)}{(1+\alpha)^2R} - Cb^2\Big(\alpha + \frac{1}{\alpha^2 R}\Big). \qquad \square
\end{aligned}$$

3.5 Proof of Theorem 3.1

Here we will verify all the statements appearing in Theorem 3.1. Noticing that $\mathscr{G}_{a,b,R}(0) = 0$, we get Item (1). The second item is already proven in Lemma 3.2.

Defining $e_a(b) = 0$ for $b \geq 1/\beta_a$, the items (3) and (5) hold trivially since $g_a(b,R) = 0$ in this case. We handle now the case where $1/|a| \leq b < 1/\beta_a$. Define in $\mathbb{R}$ the two functions $d_{a,b}(l) = g_a(b,l^2)$ and $f_{a,b}(l) = d_{a,b}(l)/l^2$. Using Lemmas 3.6–3.8, we see that the functions $d_{a,b}(l)$ and $f_{a,b}(l)$ satisfy the following properties:

- $d_{a,b}(\cdot)$ is non-positive, monotone non-increasing, and $f_{a,b}(\cdot)$ is bounded.

- For $l \geq 1$, $f_{a,b}(nl) \geq f_{a,b}((1+\alpha)l) - C\big(\alpha + \frac{1}{\alpha^2 l^2}\big)$, where $C > 0$ is a constant dependent on b and independent from l, n and α.

Then, by [FK13, Lemma 2.2], the following limit exists

$$\lim_{R \to +\infty} \frac{g_a(b,R)}{R} = \lim_{l \to +\infty} f_{a,b}(l) = e_a(b),$$

and for $R \geq 4$

$$\frac{g_a(b, R)}{R} \leq e_a(b) + \frac{2C}{R^{\frac{1}{3}}}.$$

Moreover, for every integer $n \geq 1$, Lemma 3.5 asserts that,

$$g_a(b, nR) \leq n \, g_a(b, R).$$

Dividing both sides by nR and taking $n \to +\infty$ yields $e_a(b) \leq g_a(b, R)/R$.

By Lemma 3.7, $e_a(b) < 0$; that the function $e_a(\cdot)$ is monotone non-decreasing follows from the monotonicity of the function $b \mapsto g_a(b, R)$; the continuity of the function $e_a(\cdot)$ follows from the estimates in (3.10) and the bounds in (3.6) (see [FKP13, Theorem 3.13]).

3.6 An effective one-dimensional energy

Assume that $a \in [-1, 1) \setminus \{0\}$ and $b > 0$. For all $\xi \in \mathbb{R}$, consider the functional $\mathscr{E}_{a,b,\xi}^{1D}$ defined over the space $B^1(\mathbb{R})$

$$\mathscr{E}_{a,b,\xi}^{1D}(f) = \int_{\mathbb{R}} \left(b|f'(t)|^2 + bV_a(\xi, t)|f(t)|^2 - |f(t)|^2 + \frac{1}{2}|f(t)|^4 \right) dt, \tag{3.15}$$

where $V_a(\xi, t)$ is introduced in (2.13). Let

$$E_{a,b}^{1D}(\xi) = \inf_{f \in B^1(\mathbb{R})} \mathscr{E}_{a,b,\xi}^{1D}(f). \tag{3.16}$$

We would like to find a relationship between the 2D-energy in (3.4) and the 1D-energy in (3.16) for some specific value of ξ. The existing results on the Ginzburg–Landau functional with a uniform magnetic field suggest that we should select ξ so as to minimize the function $\xi \mapsto E_{a,b}^{1D}(\xi)$, see [AH07, CR14, Pan02].

In light of Remark 3.3, we will assume that a and b satisfy

$$a \in [-1, 0) \qquad \text{and} \qquad b \geq \frac{1}{|a|}. \tag{3.17}$$

We can list some elementary properties of the functional $\mathscr{E}_{a,b,\xi}^{1D}$ in (3.15):

Proposition 3.9. *Let $a \in [-1, 0)$ and $b \geq 1/|a|$.*

1. The functional $\mathscr{E}^{1D}_{a,b,\xi}$ has a non-trivial minimizer in $B^1(\mathbb{R})$ if and only if $1/|a| \leq b < 1/\mu_a(\xi)$. Furthermore, one can find a positive minimizer $f_{a,b,\xi}$, dependent on a and b, such that any minimizer has the form $c f_{a,b,\xi}$ where $c \in \mathbb{C}$ and $|c| = 1$.

2. Any minimizer f of $\mathscr{E}^{1D}_{a,b,\xi}$ satisfies $\|f\|_\infty \leq 1$ and the equation:

$$-f''(t) + V_a(\xi,t)f(t) = \frac{1}{b}(1 - |f(t)|^2)f(t), \quad \text{for } t \in \mathbb{R}.$$

3. For $1/|a| < b < 1/\beta_a$, there exists $\tilde{\xi}_0$, dependent on a and b, such that

$$E^{1D}_{a,b}(\tilde{\xi}_0) = \inf_{\xi \in \mathbb{R}} E^{1D}_{a,b}(\xi).$$

4. (Feynman–Hellmann)

$$\int_{-\infty}^{0} (at + \tilde{\xi}_0)|f_{\tilde{\xi}_0}(t)|^2 \, dt + \int_{0}^{+\infty} (t + \tilde{\xi}_0)|f_{\tilde{\xi}_0}(t)|^2 \, dt = 0.$$

5. Any minimizer f of $\mathscr{E}^{1D}_{a,b,\xi}$ satisfies

$$E^{1D}_{a,b}(\xi) = -\frac{1}{2} \int_{\mathbb{R}} f^4(x_2) \, dx_2.$$

Remark 3.10. Guided by the numerical computations of [HPRS16, Section 1.3], we expect that:

- the global minimum β_a, defined in (2.18), is attained at a *unique* point ξ_a;

- ξ_a is the unique critical point of the function $\xi \mapsto \mu_a(\xi)$.

However, such results have not been analytically proven yet.

The proof of Proposition 3.9 may be derived as done in [FH10, Section 14.2] devoted to the analysis of the following 1D-functional

$$\mathscr{E}^{1D}_{b,\xi}(f) = \int_{0}^{+\infty} \left(b|f'(t)|^2 + b(t + \xi)^2|f(t)|^2 - |f(t)|^2 + \frac{1}{2}|f(t)|^4 \right) dt,$$

defined over the space $B^1(\mathbb{R}_+)$. We introduce the energies

$$E^{1D}_b(\xi) = \inf_{f \in B^1(\mathbb{R}_+)} \mathscr{E}^{1D}_{b,\xi}(f),$$

63

and

$$E_b^{1D} = \inf_{\xi \in \mathbb{R}} E_b^{1D}(\xi). \tag{3.18}$$

The ground state energy in (3.18) plays a crucial role in the study of surface superconductivity under the presence of a uniform magnetic field (see e.g. [AH07, FH10, HFPS11, CR14]). Let $E_{g.st}^{unif}(\kappa, H)$ be the ground state energy of the functional in (1.1) for $B_0 = 1$. Assuming that $H = b\kappa$ and $1 < b < \Theta_0^{-1}$, Θ_0 is the constant in (2.5), then as $\kappa \to +\infty$,

$$E_{g.st}^{unif}(\kappa, H) = |\partial\Omega|\kappa b^{-\frac{1}{2}} E_b^{1D} + \mathcal{O}(1), \tag{3.19}$$

where the remainder term $\mathcal{O}(1)$ depends on the geometry and is explicitly computed in [CR16a, CR16b, CDR17].

That has been conjectured by Pan [Pan02], then proven by Almog–Helffer and Helffer–Fournais–Persson [AH07, HFPS11] under a restrictive assumption on b, using a spectral approach. In the whole regime $b \in (1, \Theta_0^{-1})$, the upper bound part in (3.19) easily holds (see [FH10, Section 14.4.2]), while the matching lower bound is more difficult to obtain and has been finally proven by Correggi–Rougerie [CR14]. The proof of Correggi–Rougerie, based on the non-negativity of a certain *cost function*, was markedly different from the spectral approach of [AH07, HFPS11].

Going back to our step magnetic field problem and the one dimensional energy in (3.15), we prove the following theorem.

Theorem 3.11. *Assume that* $-1 \le a < 0$ *and* $1/|a| < b < 1/\beta_a$, *where* β_a *is defined in (2.11). Then, the energy* $e_a(b)$ *introduced in (3.5) satisfies*

$$e_a(b) = E_{a,b}^{1D},$$

where

$$E_{a,b}^{1D} = \inf_{\xi \in \mathbb{R}} E_{a,b}^{1D}(\xi), \tag{3.20}$$

and $E_{a,b}^{1D}(\cdot)$ *is defined in (3.16).*

Remark 3.12. By a symmetry argument, Theorem 3.11 trivially holds in the case $a = -1$, namely

$$e_{-1}(b) = E_{-1,b}^{1D} = E_b^{1D}.$$

To prove Theorem 3.11, we will adopt the method of [CR14], which relies on remarkable identities, including an energy splitting [LM99], along with the

non-positivity of a certain *potential function* and the non-negativity of another *cost function*.

We propose the *potential* and *cost* functions of our problem. These are defined as follows,

$$F_0(t) = \begin{cases} 2 \displaystyle\int_t^0 (a\eta + \tilde{\xi}_0) f_0^2(\eta) \, d\eta, & t \leq 0, \\[4mm] 2 \displaystyle\int_0^t (\eta + \tilde{\xi}_0) f_0^2(\eta) \, d\eta, & t > 0, \end{cases} \tag{3.21}$$

and

$$K_0(t) = f_0^2(t) + F_0(t), \qquad \text{for } t \in \mathbb{R}, \tag{3.22}$$

where $\tilde{\xi}_0$ and $f_0 = f_{a,b,\tilde{\xi}_0}$ are introduced in Proposition 3.9. We recall the set S_R in (3.1) and the energy $\mathscr{G}_{a,b,R}$ in (3.3) defined over the space $\mathscr{D}_R$ in (3.2). Let $u \in C_0^\infty(S_R)$ (note that this space is dense in $\mathscr{D}_R$ with respect to the norm $\|(\nabla - i\sigma\mathbf{A}_0)u\|_{L^2(S_R)} + \|u\|_{L^2(S_R)}$). Since $f_0 > 0$ on $\mathbb{R}$ (see Proposition 3.9), we may introduce the function v via the relation

$$u(x_1, x_2) = e^{i\tilde{\xi}_0 x_1} f_0(x_2) v(x_1, x_2), \quad (x_1, x_2) \in \mathbb{R}^2. \tag{3.23}$$

Lemma 3.13. *It holds*
$$\mathscr{G}_{a,b,R}(u) = R E_{a,b}^{\mathrm{1D}} + \mathscr{E}_0(v),$$

where

$$\mathscr{E}_0(v) = \int_{S_R} b f_0^2(x_2) \Big(|\partial_{x_2} v|^2 + |\partial_{x_1} v|^2 + 2(\sigma x_2 + \tilde{\xi}_0)(iv, \partial_{x_1} v) $$
$$+ \frac{1}{2b} f_0^2(x_2) \big(1 - |v|^2\big)^2 \Big) \, dx, \quad (3.24)$$

$$(iv, \partial_{x_1} v) = \frac{i}{2}\big(v \partial_{x_1} \bar{v} - \bar{v} \partial_{x_1} v\big), \tag{3.25}$$

and σ is defined in (2.9).

Proof. Note that

$$\mathscr{G}_{a,b,R}(u) = \int_{S_R} \Big(b|\partial_{x_2} u|^2 + b|(\partial_{x_1} + i\sigma x_2)u|^2 - |u|^2 + \frac{1}{2}|u|^4 \Big) \, dx. \tag{3.26}$$

We will compute each term of $\mathcal{G}_{a,b,R}(u)$ apart. Starting with

$$\int_{S_R} |\partial_{x_2} u|^2 \, dx = \int_{S_R} \left(|\partial_{x_2} f_0|^2 |v|^2 + f_0^2 |\partial_{x_2} v|^2 \right) dx + \int_{S_R} f_0 \partial_{x_2} f_0 \partial_{x_2} |v|^2 \, dx. \tag{3.27}$$

An integration by parts yields

$$\int_{S_R} f_0 \partial_{x_2} f_0 \partial_{x_2} |v|^2 \, dx = -\int_{S_R} |\partial_{x_2} f_0|^2 |v|^2 \, dx$$

$$- \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{-\infty}^{0} f_0 \partial_{x_2}^2 f_0 |v|^2 \, dx - \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{0}^{+\infty} f_0 \partial_{x_2}^2 f_0 |v|^2 \, dx, \tag{3.28}$$

since the functions f_0 and f_0' vanish at $\pm\infty$. Plugging (3.28) in (3.27) and using the second item of Proposition 3.9, we find

$$\int_{S_R} |\partial_{x_2} u|^2 \, dx = \int_{S_R} \left(f_0^2 |\partial_{x_2} v|^2 + f_0 |v|^2 \left(-(\sigma x_2 + \tilde{\xi}_0)^2 f_0 + \frac{1}{b} f_0 (1 - f_0^2) \right) \right) dx. \tag{3.29}$$

Next, we compute the second term of $\mathcal{G}_{a,b,R}(u)$

$$\int_{S_R} \left| (\partial_{x_1} + i\sigma x_2) u \right|^2 \, dx = \int_{S_R} f_0^2 \Big(|\partial_{x_1} v|^2 + (\sigma x_2 + \tilde{\xi}_0)^2 |v|^2$$

$$+ 2(\sigma x_2 + \tilde{\xi}_0)(iv, \partial_{x_1} v) \Big) dx. \tag{3.30}$$

Moreover, by Proposition 3.9 we have

$$E_{a,b}^{1D} = \mathcal{E}_{a,b,\tilde{\xi}_0}^{1D}(f_0) = -\frac{1}{2} \int_{\mathbb{R}} f_0^4(x_2) \, dx_2. \tag{3.31}$$

We put (3.29)–(3.31) in (3.26) to complete the proof. □

Lemma 3.14. *Let F_0 and K_0 be the functions defined respectively in (3.21) and (3.22). If $F_0 \leq 0$ and $F_0(\pm\infty) := \lim_{t \to \pm\infty} F_0(t) = 0$, then*

$$\mathcal{G}_{a,b,R}(u) \geq R E_{a,b}^{1D} + \mathcal{E}_1(v),$$

where

$$\mathcal{E}_1(v) = \int_{S_R} b K_0(x_2) \left(|\partial_{x_2} v|^2 + |\partial_{x_1} v|^2 \right) dx + \frac{1}{2} \int_{S_R} f_0^4(x_2)(1 - |v|^2)^2 \, dx.$$

Proof. Note that

$$F_0'(t) = \begin{cases} -2(at + \tilde{\xi}_0)f_0^2(t), & t < 0, \\ 2(t + \tilde{\xi}_0)f_0^2(t), & t > 0. \end{cases}$$

Since $F_0(0) = 0$ and $F_0(\pm\infty) = 0$, we can handle the next term through an integration by parts:

$$2\int_{S_R} f_0^2(x_2)(\sigma x_2 + \tilde{\xi}_0)(iv, \partial_{x_1}v)\,dx$$

$$= -\int_{-\frac{R}{2}}^{\frac{R}{2}}\int_{-\infty}^{0}(iv, \partial_{x_1}v)\partial_{x_2}F_0\,dx + \int_{-\frac{R}{2}}^{\frac{R}{2}}\int_{0}^{+\infty}(iv, \partial_{x_1}v)\partial_{x_2}F_0\,dx$$

$$= \int_{-\frac{R}{2}}^{\frac{R}{2}}\int_{-\infty}^{0}F_0\partial_{x_2}(iv, \partial_{x_1}v)\,dx - \int_{-\frac{R}{2}}^{\frac{R}{2}}\int_{0}^{+\infty}F_0\partial_{x_2}(iv, \partial_{x_1}v)\,dx. \qquad (3.32)$$

Now we handle the integral involving the term in (3.25). An integration by parts yields

$$\int_{-\frac{R}{2}}^{\frac{R}{2}}\partial_{x_2}(iv, \partial_{x_1}v)\,dx_1 = i\int_{-\frac{R}{2}}^{\frac{R}{2}}\left(\partial_{x_2}v\partial_{x_1}\bar{v} - \partial_{x_2}\bar{v}\partial_{x_1}v\right)dx_1$$

$$+ \frac{i}{2}\left[v\partial_{x_2}\bar{v} - \bar{v}\partial_{x_2}v\right]_{-\frac{R}{2}}^{\frac{R}{2}} \qquad (3.33)$$

$$= i\int_{-\frac{R}{2}}^{\frac{R}{2}}\left(\partial_{x_2}v\partial_{x_1}\bar{v} - \partial_{x_2}\bar{v}\partial_{x_1}v\right)dx_1,$$

since $u = 0$ (and consequently v) for $x_1 = \pm R/2$.

We plug (3.33) into (3.32) and we use Cauchy's inequality to get

$$2\int_{S_R} f_0^2(x_2)(\sigma x_2 + \tilde{\xi}_0)(iv, \partial_{x_1}v)\,dx \geq -2\int_{S_R}|F_0||\partial_{x_1}v||\partial_{x_2}v|\,dx$$

$$\geq \int_{S_R}F_0\left(|\partial_{x_1}v|^2 + |\partial_{x_2}v|^2\right)dx,$$

since $F_0 \leq 0$. This completes the proof in light of Lemma 3.13 and the definition of the function K_0. □

Now, looking at the expression of $\mathcal{E}_1(v)$ in Lemma 3.14, we obtain

Lemma 3.15. *If $K_0 \geq 0$ then*

$$\mathcal{G}_{a,b,R}(u) \geq RE_{a,b}^{1D}.$$

Thus, if $F_0 \leq 0$, $F_0(\pm\infty) = 0$ and $K_0 \geq 0$, then we get the lower bound

$$\frac{\mathcal{G}_{a,b,R}(u)}{R} \geq E_{a,b}^{1D}. \tag{3.34}$$

Our next task is to verify these conditions. We have the following Feynman–Hellmann equation (see Proposition 3.9):

$$\int_{-\infty}^{0} (at + \tilde{\xi}_0) f_0^2(t) \, dt + \int_{0}^{+\infty} (t + \tilde{\xi}_0) f_0^2(t) \, dt = 0, \tag{3.35}$$

which can be expressed as follows

$$F_0(-\infty) + F_0(+\infty) = 0. \tag{3.36}$$

Regarding the function K_0, we get immediately from (3.22),

$$K_0(\pm\infty) = F_0(\pm\infty). \tag{3.37}$$

If we manage to prove that $F_0(\pm\infty) = 0$, then by the same argument in [CR14, Lemma 3.2 & Proposition 3.4], we may prove that $F_0 \leq 0$ and $K_0 \geq 0$.

Such an information is known in the particular case $a = -1$, thanks to symmetry considerations and [CR14]; indeed

$$\int_{-\infty}^{0} (at + \tilde{\xi}_0) f_0^2(t) \, dt = \int_{0}^{+\infty} (t + \tilde{\xi}_0) f_0^2(t) \, dt = 0. \tag{3.38}$$

In the asymmetric case when $a \in (-1, 0)$, one needs to work a little bit more for obtaining (3.38).

The next lemma will be useful for establishing that $F_0(\pm\infty) = 0$.

Lemma 3.16 (Alternative expression of F_0). *It holds*

$$F_0(t) = \begin{cases} \frac{1}{|a|}\left(-f_0'^2(t) + (at + \tilde{\xi}_0)^2 f_0^2(t) - \frac{1}{b}f_0^2(t) + \frac{1}{2b}f_0^4(t)\right), & t \leq 0, \\[2mm] -f_0'^2(t) + (t + \tilde{\xi}_0)^2 f_0^2(t) - \frac{1}{b}f_0^2(t) + \frac{1}{2b}f_0^4(t), & t > 0. \end{cases}$$

Proof. For $t \leq 0$ and $a < 0$, we have

$$F_0(t) = 2 \int_{t}^{0} (a\eta + \tilde{\xi}_0) f_0^2(\eta) \, d\eta$$

$$= \frac{1}{|a|} \int_{0}^{t} \partial_\eta (a\eta + \tilde{\xi}_0)^2 f_0^2(\eta) \, d\eta$$

$$= \frac{1}{|a|} \left(-f_0'^2(t) + (at + \tilde{\xi}_0)^2 f_0^2(t) - \frac{1}{b}f_0^2(t) + \frac{1}{2b}f_0^4(t)\right) + R_1$$

where

$$R_1 = \frac{1}{|a|}\left(f_0'^2(0) - \tilde{\xi}_0^2 f_0^2(0) + \frac{1}{b}f_0^2(0) - \frac{1}{2b}f_0^4(0)\right).$$

Similarly, one proves for $t > 0$ that

$$F_0(t) = -f_0'^2(t) + (t + \tilde{\xi}_0)^2 f_0^2(t) - \frac{1}{b}f_0^2(t) + \frac{1}{2b}f_0^4(t) + R_2,$$

where

$$R_2 = f_0'^2(0) - \tilde{\xi}_0^2 f_0^2(0) + \frac{1}{b}f_0^2(0) - \frac{1}{2b}f_0^4(0).$$

Now, we use the Feynman–Hellmann equation in (3.36) and the vanishing of f_0 and f_0' at ∞ to get

$$R_1 + R_2 = F_0(-\infty) + F_0(+\infty) = 0.$$

Since $R_1 = 1/|a|R_2$, we conclude that $R_1 = R_2 = 0$. $\qquad\square$

Lemma 3.17. *Let F_0 be the potential function defined in* (3.21). *It holds*

$$F_0(t) \le 0 \text{ for all } t \in \mathbb{R}, \quad \text{and } F_0(\pm\infty) = 0.$$

Proof. From the definition of F_0, we have $F_0(0) = 0$. In addition, the alternative expression of F_0 in Lemma 3.16 and the decay and vanishing of f_0 and f_0' at ∞ imply that

$$F_0(-\infty) = \lim_{t \to -\infty} \frac{1}{|a|}\left(-f_0'^2(t) + (at + \tilde{\xi}_0)^2 f_0^2(t) - \frac{1}{b}f_0^2(t) + \frac{1}{2b}f_0^4(t)\right) = 0,$$

and similarly that $F_0(+\infty) = 0$. Next, we will study the variations of F_0. Recall the derivative of F_0

$$F_0'(t) = \begin{cases} -2(at + \tilde{\xi}_0)f_0^2(t), & t < 0, \\ 2(t + \tilde{\xi}_0)f_0^2(t), & t > 0. \end{cases}$$

We know that $f_0 > 0$ in $\mathbb{R}$. Hence, assuming that $\tilde{\xi}_0 \ge 0$ yields that $F_0'(t) > 0$ for all $t > 0$, which contradicts the fact that $F_0(0) = F_0(+\infty) = 0$. This proves that $\tilde{\xi}_0 < 0$. Consequently, we find that $F_0' < 0$ in a right-neighbourhood of 0, and $F_0' > 0$ in a left-neighbourhood of 0. Since $F_0(0) = 0$, we find that $F_0 \le 0$ in a neighbourhood of 0.

On the other hand, $F_0'(t) = 0$ iff $t = -\tilde{\xi}_0 > 0$ or $t = -\tilde{\xi}_0/a < 0$. Having the additional properties $F_0(0) = 0$ and $F_0(\pm\infty) = 0$, we get that $F_0 \le 0$ in $\mathbb{R}$. $\qquad\square$

Remark 3.18. Along the proof of Lemma 3.17, we proved that any $\tilde{\xi}_0$ minimizing $E_{a,b}^{1D}(\cdot)$ satisfies $\tilde{\xi}_0 < 0$.

Now, we are ready to prove the non-negativity of the cost function K_0.

Lemma 3.19. *Let K_0 be the cost function defined in* (3.22). *It holds*

$$K_0(t) \geq 0 \text{ for all } t \in \mathbb{R}.$$

Proof. Lemma 3.17 and (3.37) simply imply that $K_0(\pm\infty) = 0$. Consequently if K_0 becomes negative at some point t, this definitely means the existence of a global minimum at some point t_0 in $\mathbb{R}^*$, since $K_0(0) > 0$. We have then $K_0(t_0) < 0$ and $K_0'(t_0) = 0$, where

$$K_0'(t) = \begin{cases} -2(at + \tilde{\xi}_0)f_0^2(t) + 2f_0(t)f_0'(t), & t < 0, \\ 2(t + \tilde{\xi}_0)f_0^2(t) + 2f_0(t)f_0'(t), & t > 0. \end{cases}$$

Since $K_0'(t_0) = 0$ and $f_0(t_0) > 0$, we get that

$$f_0'(t_0) = \begin{cases} (at_0 + \tilde{\xi}_0)f_0(t_0), & \text{if } t_0 < 0, \\ -(t_0 + \tilde{\xi}_0)f_0(t_0), & \text{if } t_0 > 0. \end{cases} \tag{3.39}$$

On the other hand, we may use the alternative expression of F_0 in Lemma 3.16 to write the function K_0 in the following form

$$K_0(t) = \begin{cases} \left(1 - \frac{1}{|a|b}\right)f_0^2(t) - \frac{1}{|a|}f_0'^2(t) + \frac{1}{|a|}(at + \tilde{\xi}_0)^2 f_0^2(t) + \frac{1}{2|a|b}f_0^4(t), & t \leq 0, \\ \left(1 - \frac{1}{b}\right)f_0^2(t) - f_0'^2(t) + (t + \tilde{\xi}_0)^2 f_0^2(t) + \frac{1}{2b}f_0^4(t), & t > 0. \end{cases} \tag{3.40}$$

Plug (3.39) into (3.40) to get

$$K_0(t_0) = \begin{cases} \left(1 - \frac{1}{|a|b}\right)f_0^2(t_0) + \frac{1}{2|a|b}f_0^4(t_0), & t_0 < 0, \\ \left(1 - \frac{1}{b}\right)f_0^2(t_0) + \frac{1}{2b}f_0^4(t_0), & t_0 > 0. \end{cases}$$

Since $a \in [-1, 0)$, $b > 1/|a|$ and $f_0 > 0$ everywhere in $\mathbb{R}$, then obviously $K_0(t_0) > 0$ which is the desired contradiction. $\qquad \square$

Collecting the aforementioned lemmas, we can now prove Theorem 3.11.

Proof of Theorem 3.11. The upper bound $e_a(b) \leq E_{a,b}^{1D}$ follows by using the trial function

$$u(x_1, x_2) = \chi_R(x_1) f_0(x_2) e^{i\tilde{\xi}_0 x_1},$$

and passing to the limit $R \to +\infty$. Here, $\tilde{\xi}_0$ and $f_0 = f_{a,b,\tilde{\xi}_0}$ are introduced in Proposition 3.9, and χ_R is a smooth cut-off function supported in S_R and satisfying $\chi_R(x_1) \in (0, 1)$ for $x_1 \in (-R/2, R/2)$, and $\chi_R = 1$ in $(-R/2 + 1, R/2 - 1)$.

The lower bound $e_a(b) \geq E_{a,b}^{1D}$ is a consequence of (3.34) after passing to the limit $R \to +\infty$. $\qquad\square$

4 The Frenet Coordinates

In this section, we assume that the set Γ consists of a single simple smooth curve that may intersect the boundary of Ω transversely in two points. In the general case, Γ consists of a finite number of such (disjoint) curves. By working on each component separately, we reduce to the simple case above.

To study the energy contribution along Γ, we will use the *Frenet coordinates* which are valid in a tubular neighbourhood of Γ. For more details regarding these coordinates, see e.g. [FH10, Appendix F]. We will list the basic properties of these coordinates here.

Let $(-|\Gamma|/2, |\Gamma|/2] \ni s \longmapsto M(s) \in \Gamma$ (respectively $[-|\Gamma|/2, |\Gamma|/2] \ni s \longmapsto M(s) \in \Gamma$) be the arc-length parametrization of Γ, when $\Gamma \cap \partial\Omega = \emptyset$ (respectively when $\Gamma \cap \partial\Omega \neq \emptyset$). The vector

$$T(s) = M'(s) \tag{4.1}$$

is the unit tangent vector to Γ at the point $M(s)$. Let $\nu(s)$ be the unit normal of Γ at the point $M(s)$ pointed toward Ω_1. The orientation of the parametrization M is displayed as follows

$$\det\big(T(s), \nu(s)\big) = 1.$$

The curvature k_r of Γ is defined by

$$T'(s) = k_r(s)\nu(s).$$

For $t_0 > 0$, we define

$$S(t_0) = \begin{cases} \left(-\frac{|\Gamma|}{2}, \frac{|\Gamma|}{2} \right] \times (-t_0, t_0), & \text{if } \Gamma \cap \partial\Omega = \emptyset, \\ \left(-\frac{|\Gamma|}{2}, \frac{|\Gamma|}{2} \right) \times (-t_0, t_0), & \text{if } \Gamma \cap \partial\Omega \neq \emptyset, \end{cases} \tag{4.2}$$

and the transformation

$$\Phi : \; S(t_0) \ni (s,t) \longmapsto M(s) + t\nu(s) \in \mathbb{R}^2. \tag{4.3}$$

For a sufficiently small t_0, Φ is a diffeomorphism from $S(t_0)$ to $\Gamma(t_0)$, where

$$\Gamma(t_0) := \mathrm{Im}\,\Phi. \tag{4.4}$$

The Jacobian of Φ is

$$a(s,t) = \det(D\Phi) = 1 - t\,k_r(s).$$

The inverse of Φ, Φ^{-1}, defines a system of coordinates for the tubular neighbourhood $\Gamma(t_0)$ of Γ,

$$\Phi^{-1}(x) = \Big(s(x), t(x) \Big).$$

To each function $u \in H_0^1\big(\Gamma(t_0)\big)$, we associate the function $\tilde{u} \in H^1\big(S(t_0)\big)$ as follows

$$\tilde{u}(s,t) = u\big(\Phi(s,t)\big). \tag{4.5}$$

We also associate to any vector field $A = (A_1, A_2) \in H_{\mathrm{loc}}^1(\mathbb{R}^2, \mathbb{R}^2)$, the vector field

$$\tilde{A} = (\tilde{A}_1, \tilde{A}_2) \in H^1\big(S(t_0)\big),$$

where

$$\tilde{A}_1(s,t) = a(s,t)A\big(\Phi(s,t)\big) \cdot T(s) \quad \text{and} \quad \tilde{A}_2(s,t) = A\big(\Phi(s,t)\big) \cdot \nu(s). \tag{4.6}$$

Then we have the following change of variable formulae:

$$\int_{\Gamma(t_0)} \big| (\nabla - iA)u \big|^2 \, dx$$

$$= \int_{-\frac{|\Gamma|}{2}}^{\frac{|\Gamma|}{2}} \int_{-t_0}^{t_0} \Big(a^{-2} \big| (\partial_s - i\tilde{A}_1)\tilde{u} \big|^2 + \big| (\partial_t - i\tilde{A}_2)\tilde{u} \big|^2 \Big) \, a \, ds \, dt$$

and

$$\int_{\Gamma(t_0)} |u(x)|^2 \, dx = \int_{-\frac{|\Gamma|}{2}}^{\frac{|\Gamma|}{2}} \int_{-t_0}^{t_0} |\tilde{u}|^2 \, a \, ds \, dt. \tag{4.7}$$

We define

$$\tilde{B}(s,t) = B\big(\Phi(s,t)\big), \qquad \text{for all } (s,t) \in S(t_0).$$

Note that

$$\operatorname{curl} \tilde{A}(s,t) = \partial_s \tilde{A}_2(s,t) - \partial_t \tilde{A}_1(s,t) = \big(1 - t\,k_r(s)\big)\tilde{B}(s,t). \tag{4.8}$$

The following lemma presents a special gauge transformation, that will allow us to express a given vector field in a canonical manner.

Lemma 4.1. *We assume that $a \in [-1,1)\backslash\{0\}$ and $[s_0, s_1] \subset (-|\Gamma|/2, |\Gamma|/2)$ such that $\Phi\big((s_0, s_1) \times (-t_0, t_0)\big) \subset \Omega$. If A is a vector field in $H^1(\Omega, \mathbb{R}^2)$ with $\operatorname{curl} A = \mathbb{1}_{\Omega_1} + a\mathbb{1}_{\Omega_2}$, then there exists a function $\omega_{s_0, s_1} \in H^2\big((s_0, s_1) \times (-t_0, t_0)\big)$ such that the vector field $\tilde{A}_{new} := \tilde{A} - \nabla_{(s,t)}\omega_{s_0, s_1}$ satisfies on $(s_0, s_1) \times (-t_0, t_0)$*

$$\big(\tilde{A}_{new}\big)_1 (s,t) = \begin{cases} -\big(t - \frac{t^2}{2}k_r(s)\big), & \text{if } t > 0; \\ -a\big(t - \frac{t^2}{2}k_r(s)\big), & \text{if } t < 0 \end{cases}; \qquad \big(\tilde{A}_{new}\big)_2 (s,t) = 0. \tag{4.9}$$

Proof. For $(s,t) \in (s_0, s_1) \times (-t_0, t_0)$, let $\omega_{s_0, s_1}(s,t) = \int_0^t \tilde{A}_2(s,t') \, dt' + \int_{s_0}^s \tilde{A}_1(s',0) \, ds'$. Obviously, $\big(\tilde{A}_{new}\big)_2 (s,t) = 0$ and by (4.8)

$$\big(\tilde{A}_{new}\big)_1 (s,t) = \int_0^t \big(\partial_t \tilde{A}_1(s,t') - \partial_s \tilde{A}_2(s,t')\big) \, dt'$$
$$= -\int_0^t (1 - t' k_r(s))\tilde{B}(s,t') \, dt',$$

which is the desired result since $\tilde{B}(s,t) = \begin{cases} 1, & \text{if } t > 0 \\ a, & \text{if } t < 0 \end{cases}$. $\qquad \square$

5 A Local Effective Energy

In this section, we will introduce a 'local version' of the Ginzburg–Landau functional in (1.1). For this local functional, we will be able to write precise estimates of the

ground state energy, which in turn will prove useful in estimating the ground state energy of the full functional in (1.1).

Select a positive number t_0 sufficiently small so that the Frenet coordinates of Section 4 are valid in the tubular neighbourhood $\Gamma(t_0)$ defined in (4.4). Let $0 < c_1 < c_2$ be fixed constants and ℓ be a parameter that is allowed to vary in such a manner that

$$c_1 \kappa^{-\frac{3}{4}} < \ell < c_2 \kappa^{-\frac{3}{4}}. \tag{5.1}$$

We will refer to (5.1) by writing $\ell \approx \kappa^{-3/4}$. Let $s_0 \in \left(-\frac{|\Gamma|}{2}, \frac{|\Gamma|}{2} \right)$. After performing a linear change of variable, we may assume that $s_0 = 0$ (for simplicity). For large values of κ, consider the neighbourhood of s_0

$$\mathcal{V}(\ell) = \left\{ (s, t) \in \Phi^{-1}(\Gamma(t_0)) \ : \ -\frac{\ell}{2} < s < \frac{\ell}{2}, \ -\ell < t < \ell \right\}. \tag{5.2}$$

Let $\tilde{F}$ be the magnetic potential defined in $\mathcal{V}(\ell)$ by

$$\tilde{F}(s, t) = \left(\tilde{F}_1(s, t), 0 \right) = \left(-\sigma\left(t - \frac{t^2}{2} k_r(s) \right), 0 \right), \tag{5.3}$$

where $\sigma = \sigma(s, t)$ was defined in (2.9). Consider the domain

$$\mathcal{D}_\ell = \left\{ u \in H_0^1(\mathcal{V}(\ell)) \cap L^\infty(\mathcal{V}(\ell)) \ : \ \|u\|_\infty \leq 1 \right\}. \tag{5.4}$$

For $u \in \mathcal{D}_\ell$, we define the (local) energy

$$G(u; \mathcal{V}(\ell)) = \int_{\mathcal{V}(\ell)} \left(a^{-2} |(\partial_s - i\kappa H \tilde{F}_1)u|^2 + |\partial_t u|^2 - \kappa^2 |u|^2 + \frac{\kappa^2}{2} |u|^4 \right) a \, ds \, dt, \tag{5.5}$$

where $a(s, t) = 1 - t k_r(s)$. Now we introduce the following ground state energy

$$G_0 = \inf_{u \in \mathcal{D}_\ell} G(u; \mathcal{V}(\ell)). \tag{5.6}$$

Using standard variational methods, one can prove the existence of a minimizer u_0 of G.

Our aim is to write matching upper and lower bounds for G_0, as $\kappa \to +\infty$, in the regime

$$H = b\kappa, \quad a \in [-1, 0) \quad \text{and} \quad b \geq \frac{1}{|a|}. \tag{5.7}$$

5.1 Lower bound of G_0

Lemma 5.1. *Under Assumption (5.7), there exist two constants $\kappa_0 > 1$ and $C > 0$ dependent only on a and b such that, if $\kappa \geq \kappa_0$ and ℓ is as in (5.1), then*

$$G_0 \geq b^{-\frac{1}{2}}\kappa\ell\, e_a(b) - C, \tag{5.8}$$

where G_0 and $e_a(b)$ are defined in (5.6) and (3.5) respectively.

Proof. Notice that $a(s,t)$ is bounded in the set $\mathscr{V}(\ell)$ as follows

$$1 - C\ell \leq a(s,t) \leq 1 + C\ell. \tag{5.9}$$

Consequently

$$G\big(u;\mathscr{V}(\ell)\big) \geq (1 - C\ell)\mathscr{J}(u) - C\kappa^2\ell\int_{\mathscr{V}(\ell)}|u|^2\,ds\,dt, \tag{5.10}$$

where

$$\mathscr{J}(u) = \int_{\mathscr{V}(\ell)}\left(\big|(\partial_s - i\kappa H\tilde{F}_1)u\big|^2 + |\partial_t u|^2 - \kappa^2|u|^2 + \frac{\kappa^2}{2}|u|^4\right)ds\,dt. \tag{5.11}$$

We apply Cauchy's inequality and the uniform bound of u to get

$$\mathscr{J}(u) \geq (1 - \kappa^{-\frac{1}{2}})\mathscr{T}(u) - C\left(\kappa^{\frac{3}{2}}\ell^2 + \kappa^{\frac{5}{2}}H^2\ell^6\right), \tag{5.12}$$

where

$$\mathscr{T}(u) = \int_{\mathscr{V}(\ell)}\left(\big|(\partial_s + i\sigma\kappa Ht)u\big|^2 + |\partial_t u|^2 - \kappa^2|u|^2 + \frac{\kappa^2}{2}|u|^4\right)ds\,dt.$$

We introduce the parameters $R = \sqrt{\kappa H}\ell$, $\gamma = \sqrt{\kappa H}s$, $\tau = \sqrt{\kappa H}t$, and define the re-scaled function

$$\check{u}(\gamma,\tau) = \begin{cases} u(s,t) & \text{if } (\gamma,\tau) \in (-R/2, R/2) \times (-R, R), \\ 0 & \text{otherwise.} \end{cases}$$

In the new scale, we may write

$$\mathscr{T}(u) = \int_{-R/2}^{R/2}\int_{-\infty}^{+\infty}\left(\big|(\partial_\gamma + i\sigma\tau)\check{u}\big|^2 + |\partial_\tau\check{u}|^2 - \frac{1}{b}|\check{u}|^2 + \frac{1}{2b}|\check{u}|^4\right)d\gamma\,d\tau$$

$$= \frac{1}{b}\mathscr{G}_{a,b,R}(\check{u}),$$

where $\mathcal{G}_{a,b,R}$ is the functional in (3.3), and $\breve{u} \in \mathcal{D}_R$ the domain in (3.2) (since $u \in \mathcal{D}_\ell$). Invoking Theorem 3.1, we conclude that

$$\mathcal{T}(u) \geq \frac{1}{b} R \, e_a(b). \tag{5.13}$$

We plug the estimates (5.12) and (5.13) in (5.10), then we use $e_a(b) \leq 0$ and the assumptions on κ and ℓ to complete the proof of Lemma 5.1. □

5.2 Upper bound of G_0

Lemma 5.2. *Under Assumption (5.7), there exist two constants $\kappa_0 > 1$ and $C > 0$ dependent only on a and b such that, if $\kappa \geq \kappa_0$ and ℓ is as in (5.1), then*

$$G_0 \leq b^{-\frac{1}{2}} \kappa \ell \, e_a(b) + C \kappa^{\frac{1}{8}}, \tag{5.14}$$

where G_0 and $e_a(b)$ are defined in (5.6) and (3.5) respectively.

Proof. For $R = \ell\sqrt{\kappa H}$, consider $\varphi = \varphi_{a,b,R}$ the minimizer of $\mathcal{G}_{a,b,R}$ defined in (3.8). We define the function u in $\mathcal{D}_\ell$ as follows

$$u(s,t) = \chi\left(\frac{t}{\ell}\right) \varphi\left(s\sqrt{\kappa H}, t\sqrt{\kappa H}\right), \tag{5.15}$$

where χ is a smooth cut-off function satisfying

$$0 \leq \chi \leq 1 \text{ in } \mathbb{R}, \quad \chi = 1 \text{ in } \left[-\frac{1}{2}, \frac{1}{2}\right] \quad \text{and} \quad \operatorname{supp}\chi \subset (-1,1).$$

Next, we define the following function (with the re-scaled variables)

$$v(\gamma, \tau) = u(s,t) \qquad \left((\gamma, \tau) \in (-R/2, R/2) \times (-R, R)\right),$$

with $\gamma = \sqrt{\kappa H}s$, $\tau = \sqrt{\kappa H}t$. Using (5.9) and (3.10), we get

$$G(u) \leq (1 + C\ell)\mathcal{J}(u) + C\kappa^2\ell \int_{\mathcal{V}(\ell)} |u|^2 \, ds \, dt \leq (1 + C\ell)\mathcal{K}(v) + C\kappa^2\ell^3, \tag{5.16}$$

where $\mathcal{J}(u)$ was defined in (5.11),

$$\mathcal{K}(v) = \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{-R}^{R} \left[\left| \left(\partial_\gamma + i\sigma\left(\tau - \varepsilon\frac{\tau^2}{2}k_r\left(\frac{\gamma}{\varepsilon}\right)\right)\right)v \right|^2 + |\partial_\tau v|^2 - \frac{1}{b}|v|^2 \right.$$
$$\left. + \frac{1}{2b}|v|^4 \right] d\gamma \, d\tau, \tag{5.17}$$

76

and $\varepsilon = 1/\sqrt{\kappa H}$.

Let $\chi_R(\tau) = \chi(\tau/R) = \chi(t/\ell)$. We will estimate now each term of $\mathcal{K}(v)$ apart, using mainly the estimates on the minimizer φ in (3.10) and the properties of the function χ_R. We start with the following two estimates that result from Cauchy–Schwarz inequality,

$$\int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{-R}^{R} |\partial_\tau v|^2 \, d\gamma \, d\tau \le (1 + \kappa^{-\frac{1}{4}}) \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{-\infty}^{+\infty} |\partial_\tau \varphi|^2 \, d\gamma \, d\tau + C \kappa^{-\frac{3}{4}} \ell^{-1},$$

and

$$\int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{-R}^{R} \left| \left(\partial_\gamma + i\sigma \left(\tau - \varepsilon \frac{\tau^2}{2} k_r \left(\frac{s}{\varepsilon} \right) \right) \right) v \right|^2 \, d\gamma \, d\tau$$

$$\le (1 + \kappa^{-\frac{1}{4}}) \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{-\infty}^{+\infty} |(\partial_\gamma + i\sigma\tau)\varphi|^2 \, d\gamma \, d\tau + C \kappa^{\frac{13}{4}} \ell^5. \quad (5.18)$$

Next, we may select R_0 sufficiently large so that, for all $R \ge R_0$,

$$|\tau| \ge \frac{R}{2} \implies \frac{|\tau|}{\ln^2 |\tau|} \ge R^{\frac{1}{2}}. \quad (5.19)$$

The decay of φ in (3.9), and (5.19) yield

$$\int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{-R}^{R} |v|^2 \, d\gamma \, d\tau = \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{-\infty}^{+\infty} |\varphi|^2 \, d\gamma \, d\tau$$

$$+ \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{-\infty}^{+\infty} (\chi_R^2(\tau) - 1)|\varphi|^2 \, d\gamma \, d\tau$$

$$\ge \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{-\infty}^{+\infty} |\varphi|^2 \, d\gamma \, d\tau - \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{|\tau| \ge R/2} |\varphi|^2 \, d\gamma \, d\tau$$

$$\ge \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{-\infty}^{+\infty} |\varphi|^2 \, d\gamma \, d\tau - C \kappa^{\frac{1}{2}} \ell^{\frac{1}{2}}.$$

Finally, we write the obvious inequality

$$\int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{-R}^{R} |v|^4 \, d\gamma \, d\tau \le \int_{-\frac{R}{2}}^{\frac{R}{2}} \int_{-\infty}^{+\infty} |\varphi|^4 \, d\gamma \, d\tau.$$

Gathering the foregoing estimates, we get

$$\mathcal{K}(v) \leq \frac{(1 + \kappa^{-\frac{1}{4}})}{b} \mathcal{G}_{a,b,R}(\varphi) + C\kappa^{\frac{1}{8}}. \tag{5.20}$$

Invoking Theorem 3.1, we implement (5.20) into (5.16) to get the desired upper bound. $\qquad \square$

6 Local Estimates

6.1 Superconductivity near the magnetic barrier

The aim of this section is to study the concentration of the minimizers $(\psi, \mathbf{A})$ of the functional in (1.1) near the set Γ that separates the values of the applied magnetic field (see Assumptions 1.1 and 1.2). This will be displayed by the local estimates of the Ginzburg–Landau energy and the L^4-norm of the Ginzburg–Landau parameter in Theorem 6.1.

We will introduce the necessary notations and assumptions. Starting with the local energy of the configuration $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$, in any open set $D \subset \Omega$ as follows

$$\mathcal{E}_0(\psi, \mathbf{A}; D) = \int_D \left(\left| (\nabla - i\kappa H\mathbf{A})\psi \right|^2 - \kappa^2 |\psi|^2 + \frac{1}{2}\kappa^2 |\psi|^4 \right) dx,$$

$$\mathcal{E}(\psi, \mathbf{A}; D) = \mathcal{E}_0(\psi, \mathbf{A}; D) + (\kappa H)^2 \int_\Omega |\operatorname{curl}\mathbf{A} - B_0|^2 \, dx. \tag{6.1}$$

Choose $t_0 > 0$ sufficiently small. For all $x \in \Gamma(t_0)$, define the point $p(x) \in \Gamma$ as follows

$$\operatorname{dist}(x, p(x)) = \operatorname{dist}(x, \Gamma).$$

Let $\ell \approx \kappa^{-3/4}$ satisfy (5.1) (for some fixed choice of the constants c_1 and c_2). For κ sufficiently large (hence ℓ sufficiently small), let $x_0 \in \Gamma \backslash \partial\Omega$ be chosen so that

$$\operatorname{dist}(x_0, \partial\Omega) > 2\ell. \tag{6.2}$$

Consider the following neighbourhood of x_0,

$$\mathcal{N}_{x_0}(\ell) = \left\{ x \in \mathbb{R}^2 \ : \ \operatorname{dist}_\Gamma(x_0, p(x)) < \frac{\ell}{2}, \ \operatorname{dist}_\Omega(x, p(x)) < \ell \right\}. \tag{6.3}$$

Thanks to (6.2), we have $\overline{\mathcal{N}_{x_0}(\ell)} \subset \Omega$. As a consequence of the assumption in (6.2), *all the estimates that we will write will hold uniformly with respect to the point x_0.*

We assume that $a \in [-1, 0)$ and $b > 0$ are fixed and satisfy

$$b > \frac{1}{|a|}. \tag{6.4}$$

When (6.4) holds, we are able to use the exponential decay of the Ginzburg–Landau parameter away from the set Γ and the surface $\partial \Omega$ (see Theorem 2.4).

Theorem 6.1. *Let $a \in [-1, 0)$ and $b > 1/|a|$. There exists $\kappa_0 > 0$ and a function $r : [\kappa_0, +\infty) \to (0, +\infty)$ such that $\lim_{\kappa \to +\infty} r(\kappa) = 0$ and the following is true. For $\kappa \geq \kappa_0$, $H = b\kappa$ and $\ell \approx \kappa^{-3/4}$ as in (5.1), for any $x_0 \in \Gamma$ satisfying (6.2), every minimizer $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$ of the functional in (1.1) satisfies*

$$\left| \mathscr{E}_0\left(\psi, \mathbf{A}; \mathscr{N}_{x_0}(\ell)\right) - b^{-\frac{1}{2}} \kappa \ell \, e_a(b) \right| \leq \kappa \ell \, r(\kappa), \tag{6.5}$$

and

$$\left| \frac{1}{\ell} \int_{\mathscr{N}_{x_0}(\ell)} |\psi|^4 \, dx + 2 b^{-\frac{1}{2}} \kappa^{-1} e_a(b) \right| \leq \kappa^{-1} r(\kappa), \tag{6.6}$$

where $\mathscr{N}_{x_0}(\cdot)$ is the set in (6.3), $\mathscr{E}_0$ is the local energy in (6.1), and $e_a(b)$ is the limiting energy in (3.5).

Furthermore, the function r is independent of the point $x_0 \in \Gamma$.

The proof of Theorem 6.1 follows by combining the results of Proposition 6.3 and Proposition 6.4 below, which are derived along the lines of [HK17, Section 4] in the study of local surface superconductivity.

Part of the proof of Theorem 6.1 is based on the following remark. After performing a translation, we may assume that the Frenet coordinates of x_0 are $(s = 0, t = 0)$ (see Section 4). Recall the local Ginzburg–Landau energy $\mathscr{E}_0$ introduced in (6.1). Let $\mathbf{F}$ be the vector field introduced in Lemma 2.2. We have the following relation

$$\mathscr{E}_0\left(u, \mathbf{F}; \mathscr{N}_{x_0}(\ell)\right) = G\left(\tilde{v}; \mathscr{V}(\ell)\right), \tag{6.7}$$

where G is defined in (5.5), $u \in H^1_0(\mathscr{N}_{x_0}(\ell))$, $\tilde{v} = e^{-i\kappa H \omega} \tilde{u}$, $\tilde{u}$ is the function associated to u by the transformation Φ^{-1} (see (4.5)), and $\omega = \omega_{-\ell, \ell}$ is the gauge function defined in Lemma 4.1.

Lower bound of the local energy

We start by establishing a lower bound for the local energy $\mathcal{E}_0\big(u, \mathbf{A}; \mathcal{N}_{x_0}(\ell)\big)$ for an arbitrary function $u \in H_0^1(\mathcal{N}_{x_0}(\ell))$ satisfying $|u| \leq 1$. *We will work under the assumptions made in this section*, notably, we assume that (6.4) holds, and $\ell \approx \kappa^{-3/4}$ (see (5.1)), and *in the regime where $H = b\kappa$*.

Proposition 6.2. *There exist two constants $\kappa_0 > 1$ and $C > 0$ such that, for $\kappa \geq \kappa_0$ and for all $x_0 \in \Gamma$ satisfying (6.2), the following is true. If*

- $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H_{\mathrm{div}}^1(\Omega)$ *is a solution of* (1.3).

- $u \in H_0^1(\mathcal{N}_{x_0}(\ell))$ *satisfies* $|u| \leq 1$.

then

$$\mathcal{E}_0\big(u, \mathbf{A}; \mathcal{N}_{x_0}(\ell)\big) \geq b^{-\frac{1}{2}} \kappa \ell \, e_a(b) - C,$$

where $\mathcal{N}_{x_0}(\cdot)$ is the neighbourhood in (6.3), $\mathcal{E}_0$ is the functional in (6.1), and $e_a(b)$ is the limiting energy in (3.5).

Proof. Let $\alpha \in (0, 1)$ and $\mathbf{F}$ be the vector field introduced in Lemma 2.2. We define the function $\phi_{x_0}(x) = \big(\mathbf{A}(x_0) - \mathbf{F}(x_0)\big) \cdot x$. As a consequence of the fourth item in Theorem 2.3, we get the following useful approximation of the vector potential $\mathbf{A}$

$$|\mathbf{A}(x) - \nabla \phi_{x_0}(x) - \mathbf{F}(x)| \leq \frac{C}{\kappa} \ell^\alpha, \qquad \text{for } x \in \mathcal{N}_{x_0}(\ell). \tag{6.8}$$

We choose $\alpha = 2/3$ in (6.8). Let $h = e^{-i\kappa H \phi_{x_0}} u$. Using (6.8), Cauchy's inequality, and the uniform bound $|h| \leq 1$, we may write

$$\mathcal{E}_0(u, \mathbf{A}; \mathcal{N}_{x_0}(\ell)) \geq (1 - \kappa^{-\frac{1}{2}}) \mathcal{E}_0(h, \mathbf{F}; \mathcal{N}_{x_0}(\ell)) - C\Big(\kappa^{\frac{3}{2}} \ell^2 + \kappa^{\frac{5}{2}} \ell^{\frac{10}{3}}\Big).$$

Now, define the function $\tilde{v} = e^{-i\kappa H \omega} \tilde{h}$ on $\Phi^{-1}(\mathcal{N}_{x_0}(\ell))$, where $\tilde{h}$ is the function associated to h by the transformation Φ^{-1} (see (4.5)), and $\omega = \omega_{s_0, s_1}$ is the function introduced in Lemma 4.1 with $s_0 = -\ell$ and $s_1 = \ell$. We may use the relation in (6.7) to write

$$\mathcal{E}_0(u, \mathbf{A}; \mathcal{N}_{x_0}(\ell)) \geq (1 - \kappa^{-\frac{1}{2}}) G(\tilde{v}; \mathcal{V}(\ell)) - C\Big(\kappa^{\frac{3}{2}} \ell^2 + \kappa^{\frac{5}{2}} \ell^{\frac{10}{3}}\Big).$$

Finally, the lower bound in Lemma 5.1, together with the inequality $e_a(b) \leq 0$, yield the claim of the inequality. $\qquad\square$

Sharp upper bound on the L^4- norm

We will derive a lower bound of the local energy $\mathscr{E}_0\big(\psi, \mathbf{A}; \mathscr{N}_{x_0}(\ell)\big)$ and an upper bound of the L^4-norm of ψ, valid for any critical point $(\psi, \mathbf{A})$ of the functional in (1.1). Again, we remind the reader that *we assume that* (6.4) *holds,* $\ell \approx \kappa^{-3/4}$ (see (5.1)) *and* $H = b\kappa$.

Proposition 6.3. *There exist two constants $\kappa_0 > 1$ and $C > 0$ such that, for all $x_0 \in \Gamma$ satisfying* (6.2), *the following is true. If $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$ is a critical point of the functional in* (1.1) *for $\kappa \geq \kappa_0$, then*

$$\mathscr{E}_0\big(\psi, \mathbf{A}; \mathscr{N}_{x_0}(\ell)\big) \geq b^{-\frac{1}{2}}\kappa\ell\, e_a(b) - C\kappa^{\frac{3}{16}}, \tag{6.9}$$

and

$$\frac{1}{\ell}\int_{\mathscr{N}_{x_0}(\ell)} |\psi|^4\, dx \leq -2b^{-\frac{1}{2}}\kappa^{-1} e_a(b) + C\kappa^{-\frac{17}{16}}. \tag{6.10}$$

Here $\mathscr{N}_{x_0}(\cdot)$, $\mathscr{E}_0$, and $e_a(b)$ are respectively defined in (6.3), (6.1), *and* (3.5).

Proof. In the sequel, $\gamma = \kappa^{-3/16}$ and κ is sufficiently large so that $\gamma \in (0,1)$. We denote by $\hat{\ell} = (1 + \gamma)\ell$.

Consider a smooth function f satisfying

$$\begin{aligned}
&f = 1 \text{ in } \mathscr{N}_{x_0}(\ell), \quad f = 0 \text{ in } \mathscr{N}_{x_0}\big(\hat{\ell}\big)^{\complement}, \\
&0 \leq f \leq 1, \quad |\nabla f| \leq C\gamma^{-1}\ell^{-1} \text{ and } |\Delta f| \leq C\gamma^{-2}\ell^{-2} \text{ in } \Omega.
\end{aligned} \tag{6.11}$$

Proof of (6.9)*.* We use the following simple identity (see [KN16, p. 2871])

$$\begin{aligned}
\int_{\mathscr{N}_{x_0}(\hat{\ell})} \big|(\nabla - i\kappa H\mathbf{A})f\psi\big|^2\, dx &= \int_{\mathscr{N}_{x_0}(\hat{\ell})} \big|f(\nabla - i\kappa H\mathbf{A})\psi\big|^2\, dx \\
&\quad - \int_{\mathscr{N}_{x_0}(\hat{\ell})} f\Delta f |\psi|^2\, dx.
\end{aligned} \tag{6.12}$$

Having in hand (6.12), $|\psi| \leq 1$ and $\big|\mathrm{supp}(\Delta f)\big| \leq C\gamma\ell^2$, we can write

$$\int_{\mathscr{N}_{x_0}(\hat{\ell})} \big|(\nabla - i\kappa H\mathbf{A})f\psi\big|^2\, dx \leq \int_{\mathscr{N}_{x_0}(\hat{\ell})} \big|f(\nabla - i\kappa H\mathbf{A})\psi\big|^2\, dx + C\gamma^{-1}.$$

On the other hand, we write

$$\int_{\mathcal{N}_{x_0}(\hat{\ell})} f^2 |\psi|^2 \, dx = \int_{\mathcal{N}_{x_0}(\hat{\ell})} |\psi|^2 \, dx - \int_{\mathcal{N}_{x_0}(\hat{\ell}) \cap \{\text{dist}(x,\Gamma) \leq \gamma\ell\}} (1 - f^2)|\psi|^2 \, dx$$

$$- \int_{\mathcal{N}_{x_0}(\hat{\ell}) \cap \{\text{dist}(x,\Gamma) > \gamma\ell\}} (1 - f^2)|\psi|^2 \, dx. \quad (6.13)$$

Recall that $\gamma = \kappa^{-3/16}$, then $\gamma\ell \gg \kappa^{-1}$ which, together with (6.4), allow us to use the exponential decay of $|\psi|^2$ in $\mathcal{N}_{x_0}(\hat{\ell}) \cap \{\text{dist}(x, \Gamma) > \gamma\ell\}$ (see Theorem 2.4). Consequently, the integral over $\mathcal{N}_{x_0}(\hat{\ell}) \cap \{\text{dist}(x, \Gamma) > \gamma\ell\}$ in (6.13) is exponentially small when $\kappa \to +\infty$; in addition, we have

$$\left| \text{supp}(1 - f^2) \cap \mathcal{N}_{x_0}(\hat{\ell}) \cap \{\text{dist}(x, \Gamma) \leq \gamma\ell\} \right| = \mathcal{O}(\gamma^2\ell^2),$$

this yields

$$\int_{\mathcal{N}_{x_0}(\hat{\ell})} f^2 |\psi|^2 \, dx \geq \int_{\mathcal{N}_{x_0}(\hat{\ell})} |\psi|^2 \, dx - C\gamma^2\ell^2.$$

Hence,

$$\mathcal{E}_0\big(f\psi, \mathbf{A}; \mathcal{N}_{x_0}(\hat{\ell})\big) \leq \mathcal{E}_0\big(\psi, \mathbf{A}; \mathcal{N}_{x_0}(\hat{\ell})\big) + C\kappa^{\frac{3}{16}}. \quad (6.14)$$

The fact that $f\psi \in H_0^1\big(\mathcal{N}_{x_0}(\hat{\ell})\big)$ and $|f\psi| \leq 1$ allows us to use the lower bound result established in Proposition 6.2, for $u = f\psi$. This yields together with (6.14)

$$\mathcal{E}_0\big(\psi, \mathbf{A}; \mathcal{N}_{x_0}(\hat{\ell})\big) \geq b^{-\frac{1}{2}} \kappa \hat{\ell} e_a(b) - C\kappa^{\frac{3}{16}}. \quad (6.15)$$

This completes the proof of (6.9), but with $\hat{\ell}$ appearing instead of ℓ. However, this is not harmful, as we could start the argument with $\check{\ell} = (1 + \gamma)^{-1}\ell$ in place of ℓ and then modify $\hat{\ell}$ accordingly; in this case we would get $\hat{\ell} = (1 + \gamma)\check{\ell} = \ell$ as required.

Proof of (6.10). In light of (1.3), we get using integration by parts (see [FK11, (6.2)])

$$\int_{\mathcal{N}_{x_0}(\hat{\ell})} \left(\left| (\nabla - i\kappa H\mathbf{A}) f\psi \right|^2 - |\nabla f|^2 |\psi|^2 \right) dx = \kappa^2 \int_{\mathcal{N}_{x_0}(\hat{\ell})} \left(|\psi|^2 - |\psi|^4 \right) f^2 \, dx.$$

Consequently,

$$\mathcal{E}_0\big(f\psi, \mathbf{A}; \mathcal{N}_{x_0}(\hat{\ell})\big) = \kappa^2 \int_{\mathcal{N}_{x_0}(\hat{\ell})} f^2 \left(-1 + \frac{1}{2}f^2 \right) |\psi|^4 \, dx + \int_{\mathcal{N}_{x_0}(\hat{\ell})} |\nabla f|^2 |\psi|^2 \, dx.$$

$$(6.16)$$

Since $f = 1$ in $\mathcal{N}_{x_0}(\ell)$ and $-1 + 1/2f^2 \leq -1/2$ in $\mathcal{N}_{x_0}(\hat{\ell})$, we get

$$\int_{\mathcal{N}_{x_0}(\hat{\ell})} f^2\left(-1 + \frac{1}{2}f^2\right)|\psi|^4 \, dx \leq -\frac{1}{2}\int_{\mathcal{N}_{x_0}(\ell)} |\psi|^4 \, dx.$$

We use the previous inequality, (6.16) and the estimate $\big|\operatorname{supp}|\nabla f|\big| \leq C\gamma\ell^2$ to obtain

$$\mathcal{E}_0\big(f\psi, \mathbf{A}; \mathcal{N}_{x_0}(\hat{\ell})\big) \leq -\frac{\kappa^2}{2}\int_{\mathcal{N}_{x_0}(\ell)} |\psi|^4 \, dx + C\kappa^{\frac{3}{16}}. \tag{6.17}$$

Finally we plug the lower bound in Proposition 6.2 into (6.17). $\qquad\square$

Sharp lower bound on the L^4-norm

Complementary to Proposition 6.3, we will prove Proposition 6.4 below, whose conclusion holds for minimizing configurations only. We continue working under *the assumption that* (6.4) *holds,* $\ell \approx \kappa^{-3/4}$ (see (5.1)) *and* $H = b\kappa$.

Proposition 6.4. *There exist two constants* $\kappa_0 > 1$ *and* $C > 0$ *such that, for all* $x_0 \in \Gamma$ *satisfying* (6.2), *the following is true. If* $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\operatorname{div}}(\Omega)$ *is a* minimizer *of the functional in* (1.1) *for* $\kappa \geq \kappa_0$, *then*

$$\mathcal{E}_0\big(\psi, \mathbf{A}; \mathcal{N}_{x_0}(\ell)\big) \leq b^{-\frac{1}{2}}\kappa\ell\, e_a(b) + C\kappa^{\frac{3}{16}}, \tag{6.18}$$

and

$$\frac{1}{\ell}\int_{\mathcal{N}_{x_0}(\ell)} |\psi|^4 \, dx \geq -2b^{-\frac{1}{2}}\kappa^{-1}e_a(b) - C\kappa^{-\frac{17}{16}}. \tag{6.19}$$

Here $\mathcal{N}_{x_0}(\cdot)$, $\mathcal{E}_0$, *and* $e_a(b)$ *are respectively defined in* (6.3), (6.1), *and* (3.5).

Proof. The proof is divided into five steps.

Step 1 (Construction of a test function and decomposition of the energy). The construction of the test function is inspired from that by Sandier and Serfaty, in their study of bulk superconductivity in [SS03]. For $\gamma = \kappa^{-3/16}$ and $\hat{\ell} = (1 + \gamma)\ell$, we define the function

$$u(x) = \mathbb{1}_{\mathcal{N}_{x_0}(\hat{\ell})}(x)e^{i\kappa H\phi_{x_0}(x)}v_0(x) + \eta(x)\psi(x), \tag{6.20}$$

where $v_0(x) = \left(e^{i\kappa H\omega}u_0\right) \circ \Phi^{-1}(x)$ for $x \in \mathcal{N}_{x_0}(\hat{\ell})$, ϕ_{x_0} is the gauge function introduced in (6.8), $\omega = \omega_{s_0,s_1}$ is the function introduced in Lemma 4.1 for $s_0 = -\hat{\ell}$

and $s_1 = \hat{\ell}$, Φ is the coordinate transformation in (4.3), u_0 is a minimizer of the functional $G(\cdot, \mathcal{V}(\hat{\ell}))$ defined in (5.5), and η is a smooth function satisfying

$$\eta = 0 \text{ in } \mathcal{N}_{x_0}(\hat{\ell}), \quad \eta = 1 \text{ in } \mathcal{N}_{x_0}\big((1 + 2\gamma)\ell\big)^{\complement},$$
$$0 \le \eta \le 1, \quad |\nabla \eta| \le C\gamma^{-1}\ell^{-1}, \quad \text{and } |\Delta \eta| \le C\gamma^{-2}\ell^{-2} \text{ in } \Omega. \tag{6.21}$$

Recall the energies defined in (1.1) and (6.1). Let us write the obvious decomposition

$$\mathcal{E}_0(\cdot, \mathbf{A}; \Omega) = \mathcal{E}_0\big(\cdot, \mathbf{A}; \mathcal{N}_{x_0}(\hat{\ell})\big) + \mathcal{E}_0\big(\cdot, \mathbf{A}; \mathcal{N}_{x_0}(\hat{\ell})^{\complement}\big).$$

Adding the magnetic energy term $\kappa^2 H^2 \|\operatorname{curl} \mathbf{A} - B_0\|^2_{L^2(\Omega)}$ on both sides, we obtain the following identity,

$$\mathcal{E}_{\kappa,H}(\cdot, \mathbf{A}) = \mathcal{E}_0\big(\cdot, \mathbf{A}; \mathcal{N}_{x_0}(\hat{\ell})\big) + \mathcal{E}\big(\cdot, \mathbf{A}; \mathcal{N}_{x_0}(\hat{\ell})^{\complement}\big),$$

since the same magnetic energy term is present in both energies $\mathcal{E}_{\kappa,H}(\cdot, \mathbf{A})$ and $\mathcal{E}\big(\cdot, \mathbf{A}; \mathcal{N}_{x_0}(\hat{\ell})^{\complement}\big)$. We denote by

$$\mathcal{E}_1(\cdot, \mathbf{A}) = \mathcal{E}_0\big(\cdot, \mathbf{A}; \mathcal{N}_{x_0}(\hat{\ell})\big), \quad \mathcal{E}_2(\cdot, \mathbf{A}) = \mathcal{E}_0\big(\cdot, \mathbf{A}; \mathcal{N}_{x_0}(\tilde{\ell}) \backslash \mathcal{N}_{x_0}(\hat{\ell})\big),$$
$$\text{and } \mathcal{E}_3(\cdot, \mathbf{A}) = \mathcal{E}\big(\cdot, \mathbf{A}; \mathcal{N}_{x_0}(\tilde{\ell})^{\complement}\big), \tag{6.22}$$

where $\tilde{\ell} = (1 + 2\gamma)\ell$. Hence, we get the following decomposition of the functional in (1.1),

$$\mathcal{E}_{\kappa,H}(\cdot, \mathbf{A}) = \mathcal{E}_1(\cdot, \mathbf{A}) + \mathcal{E}_2(\cdot, \mathbf{A}) + \mathcal{E}_3(\cdot, \mathbf{A}).$$

Step 2 (Estimating $\mathcal{E}_1(u, \mathbf{A})$). Using (6.8) for $\alpha = 2/3$, $|v_0| \le 1$ and the Cauchy-Schwarz inequality, we write

$$\mathcal{E}_1(u, \mathbf{A}) \le (1 + \kappa^{-\frac{1}{2}})\mathcal{E}_0\big(v_0, \mathbf{F}; \mathcal{N}_{x_0}(\hat{\ell})\big) + C. \tag{6.23}$$

But by (6.7), we have $\mathcal{E}_0\big(v_0, \mathbf{F}; \mathcal{N}_{x_0}(\hat{\ell})\big) = G\big(u_0, \mathcal{V}(\hat{\ell})\big)$. Hence, Lemma 5.2 and (6.23) imply

$$\mathcal{E}_1(u, \mathbf{A}) \le b^{-\frac{1}{2}}\kappa\hat{\ell}\, e_a(b) + C\kappa^{\frac{1}{8}}. \tag{6.24}$$

Step 3 (Estimating $\mathcal{E}_2(u, \mathbf{A})$). Notice that $u = \eta\psi$ with $0 \le \eta \le 1$ in $\mathcal{N}_{x_0}(\tilde{\ell}) \backslash \mathcal{N}_{x_0}(\hat{\ell})$. Then, we do a straightforward computation, similar to the one done in the proof of (6.14), replacing f by η and $\mathcal{N}_{x_0}(\hat{\ell})$ by $\mathcal{N}_{x_0}(\tilde{\ell}) \backslash \mathcal{N}_{x_0}(\hat{\ell})$. This gives the following relation between $\mathcal{E}_2(u, \mathbf{A})$ and $\mathcal{E}_2(\psi, \mathbf{A})$

$$\mathcal{E}_2(u, \mathbf{A}) \le \mathcal{E}_2(\psi, \mathbf{A}) + C\kappa^{\frac{3}{16}}. \tag{6.25}$$

Step 4 (Estimating $\mathcal{E}_1(\psi, \mathbf{A})$). Since $(\psi, \mathbf{A})$ is a minimizer of the functional $\mathcal{E}_{\kappa,H}$ defined in (1.1), we write $\mathcal{E}_{\kappa,H}(\psi, \mathbf{A}) \leq \mathcal{E}_{\kappa,H}(u, \mathbf{A})$. Notice that $\mathcal{E}_3(u, \mathbf{A}) = \mathcal{E}_3(\psi, \mathbf{A})$, then

$$\mathcal{E}_1(\psi, \mathbf{A}) + \mathcal{E}_2(\psi, \mathbf{A}) \leq \mathcal{E}_1(u, \mathbf{A}) + \mathcal{E}_2(u, \mathbf{A}).$$

We plug (6.24) and (6.25) into the previous inequality to get

$$\mathcal{E}_1(\psi, \mathbf{A}) \leq b^{-\frac{1}{2}} \kappa \hat{\ell} e_a(b) + C \kappa^{\frac{3}{16}}. \tag{6.26}$$

Recalling that $\mathcal{E}_1(\psi, \mathbf{A}) = \mathcal{E}_1\big(\psi, \mathbf{A}; \mathcal{N}_{x_0}(\hat{\ell})\big)$, we see that (6.26) is nothing but (6.18) with $\hat{\ell}$ appearing instead of ℓ.

Step 5 (Lower bound of the L^4-norm of ψ). Consider the function f defined in (6.11). We use the properties of this function, mainly that $f = 1$ in $\mathcal{N}_{x_0}(\ell)$ and $0 \leq f \leq 1$ in Ω, to obtain

$$\int_{\mathcal{N}_{x_0}(\hat{\ell})} f^2\Big(-1 + \frac{1}{2}f^2\Big)|\psi|^4\,dx \geq -\frac{1}{2}\int_{\mathcal{N}_{x_0}(\ell)} |\psi|^4\,dx - \int_{\mathcal{N}_{x_0}(\hat{\ell})\setminus\mathcal{N}_{x_0}(\ell)} |\psi|^4\,dx.$$

Following an argument similar to the one for (6.13), we divide the set $\mathcal{N}_{x_0}(\hat{\ell})\setminus\mathcal{N}_{x_0}(\ell)$ into $\big(\mathcal{N}_{x_0}(\hat{\ell})\setminus\mathcal{N}_{x_0}(\ell)\big) \cap \{\text{dist}(x, \Gamma) \leq \gamma\ell\}$ and $\big(\mathcal{N}_{x_0}(\hat{\ell})\setminus\mathcal{N}_{x_0}(\ell)\big) \cap \{\text{dist}(x, \Gamma) > \gamma\ell\}$, and we use this time the exponential decay of $|\psi|^4$, deduced from Theorem 2.4, to get

$$\int_{\mathcal{N}_{x_0}(\hat{\ell})} f^2\Big(-1 + \frac{1}{2}f^2\Big)|\psi|^4\,dx \geq -\frac{1}{2}\int_{\mathcal{N}_{x_0}(\ell)} |\psi|^4\,dx - C\kappa^{-\frac{15}{8}}. \tag{6.27}$$

Inserting (6.27) into (6.16) gives

$$\mathcal{E}_1(f\psi, \mathbf{A}) \geq -\frac{\kappa^2}{2}\int_{\mathcal{N}_{x_0}(\ell)} |\psi|^4\,dx - C\kappa^{\frac{1}{8}}.$$

The previous inequality together with (6.14) and (6.26) establish the lower bound in (6.19). □

Proof of Theorem 6.1

Gather results in Propositions 6.3 and 6.4.

6.2 Surface Superconductivity

In this section, we are concerned in the local behaviour of the sample near the boundary of Ω, under the assumption

$$b > \frac{1}{|a|}, \qquad a \in [-1, 0).$$

The analysis of superconductivity near $\partial\Omega$ in our case of a step magnetic field (B_0 satisfying 1.2) is essentially the same as that in the uniform field case, since B_0 is constant in each of Ω_1 and Ω_2. Thereby, the results presented in this section are well-known in the literature since the celebrated work of Saint–James and de Gennes [SJG63]. We refer to [CG17, CR16a, CR16b, CR14, FKP13, FK11, HFPS11, AH07, FH05, Pan02, LP99] for rigorous results in general 2D and 3D samples subjected to a constant magnetic field, and to [NSG⁺09] for recent experimental results. Particularly, local surface estimates were recently established in [HK17], when $B_0 \in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$. We will adapt these results to our discontinuous magnetic field (see Theorem 6.5 below).

The statement of Theorem 6.5 involves the surface energy E_{surf}, that we introduce in the next section.

The surface energy function

Let $b \geq 1$ and Θ_0 be the value defined in (2.5). When $b \in (1, \Theta_0^{-1})$, the surface energy has been described by the 1D-energy, E_b^{1D}, introduced in (3.18) (see [CR14, CR16a, CR16b, CDR17, AH07, HFPS11] and references therein).

This same energy was introduced earlier in the literature via a 2D-reduced Ginzburg–Landau functional defined in what follows. Let $R > 1$. We consider

$$\mathcal{W}(U_R) \ni \phi \mapsto \mathscr{E}_{b,R}(\phi) = \int_{U_R} \left(b \left| (\nabla - i\mathbf{A}_0)\phi \right|^2 - |\phi|^2 + \frac{1}{2}|\phi|^4 \right) d\gamma \, d\tau,$$

where $(\gamma, \tau) \in \mathbb{R}^2$, $\mathbf{A}_0(\gamma, \tau) = (-\tau, 0)$, $U_R = (-R/2, R/2) \times (0, +\infty)$, and

$$\mathcal{W}(U_R) = \left\{ u \in L^2(U_R) \;:\; (\nabla - i\mathbf{A}_0)u \in L^2(U_R), \, u(\pm R, \cdot) = 0 \right\}.$$

The boundary condition in the domain $\mathcal{W}(U_R)$ is meant in the trace sense. Let $d(b, R)$ be the ground state energy defined by

$$d(b, R) = \inf_{\phi \in \mathcal{W}(U_R)} \mathscr{E}_{b,R}(\phi).$$

Pan proved in [Pan02] the existence of a non-decreasing continuous function $E_{\mathrm{surf}} : [1, \Theta_0^{-1}] \to (-\infty, 0]$ such that

$$E_{\mathrm{surf}}(b) = \lim_{R \to +\infty} \frac{d(b, R)}{R}, \tag{6.28}$$

Later, it was proven that (see e.g. [CR14])

$$E_{\mathrm{surf}}(b) = E_b^{\mathrm{1D}}, \text{ for } b \in (1, \Theta_0^{-1}).$$

One important property of the function $E_{\mathrm{surf}}(\cdot)$ is (see [FH05])

$$E_{\mathrm{surf}}(\Theta_0^{-1}) = 0 \text{ and } E_{\mathrm{surf}}(b) < 0, \text{ for all } b \in \left[1, \Theta_0^{-1}\right). \tag{6.29}$$

This property allows us to extend the function $E_{\mathrm{surf}}(\cdot)$ continuously to $[1, +\infty)$, by setting it to zero on $[\Theta_0^{-1}, +\infty)$. This extension of the surface energy is still denoted by E_{surf} for simplicity.

Local surface superconductivity

Let $t_0 > 0$ and $j \in \{1, 2\}$. We define the following set

$$\Omega_j(t_0) = \left\{ x \in \Omega_j : \operatorname{dist}\left(x, \partial\Omega_j \cap \partial\Omega\right) < t_0 \right\}. \tag{6.30}$$

Assume that t_0 is sufficiently small, then for any $x \in \Omega_j(t_0)$, there exists a unique point $p(x) \in \partial\Omega_j \cap \partial\Omega$ satisfying

$$\operatorname{dist}\left(x, \partial\Omega_j \cap \partial\Omega\right) = \operatorname{dist}\left(x, p(x)\right).$$

Let $\ell \approx \kappa^{-3/4}$ be the parameter in (5.1). Assume that κ is sufficiently large and choose $x_0 \in \partial\Omega_j \cap \partial\Omega$ satisfying

$$\operatorname{dist}(x_0, \Gamma) > 2\ell. \tag{6.31}$$

We introduce the following neighbourhood of x_0

$$\mathcal{N}_{x_0}^j(\ell) = \left\{ x \in \Omega_j : \operatorname{dist}_{\partial\Omega}\left(x_0, p(x)\right) < \frac{\ell}{2}, \operatorname{dist}_\Omega\left(x, p(x)\right) < \ell \right\}. \tag{6.32}$$

The assumption on x_0 in (6.31) guarantees that $\overline{\mathcal{N}_{x_0}^j(\ell)} \subset \Omega_j$. Consequently, the estimates in Theorem 6.5 below hold uniformly with respect to the point x_0.

Recall the magnetic field B_0 defined in Assumption 1.2 ($B_0 = \mathbb{1}_{\Omega_1} + a\mathbb{1}_{\Omega_2}$).

Theorem 6.5. *Let $a \in [-1, 0)$ and $b > 1/|a|$. There exists $\kappa_0 > 0$ and a function $\check{r} : [\kappa_0, +\infty) \to (0, +\infty)$ such that $\lim_{\kappa \to +\infty} \check{r}(\kappa) = 0$ and the following is true. For $\kappa \geq \kappa_0$, $H = b\kappa$, ℓ as in (5.1), $j \in \{1, 2\}$, $x_0 \in \partial\Omega_j \cap \partial\Omega$ satisfying (6.31), and every minimizer $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$ of the functional in (1.1), we have*

$$\left| \mathscr{E}_0\left(\psi, \mathbf{A}; \mathscr{N}^j_{x_0}(\ell)\right) - b^{-\frac{1}{2}} |B_0(x_0)|^{-\frac{1}{2}} \kappa\ell\, E_{\mathrm{surf}}\left(b\,|B_0(x_0)|\right) \right| \leq \kappa\ell\check{r}(\kappa),$$

and

$$\left| \frac{1}{\ell} \int_{\mathscr{N}^j_{x_0}(\ell)} |\psi|^4\, dx + 2b^{-\frac{1}{2}} |B_0(x_0)|^{-\frac{1}{2}} \kappa^{-1} E_{\mathrm{surf}}\left(b\,|B_0(x_0)|\right) \right| \leq \kappa^{-1}\check{r}(\kappa),$$

where $\mathscr{N}^j_{x_0}(\cdot)$ is the set in (6.32), and $\mathscr{E}_0$ is the local energy in (6.1).

   Furthermore, the function $\check{r}$ is independent of the point x_0.

The estimates in Theorem 6.5 are established in [HK17], when the function B_0 is smooth. Since B_0 is constant in the neighbourhood $\mathscr{N}^j_{x_0}(\ell)$, the proof in [HK17] still holds in our case.

6.3   Proof of Main Results

In this section, we work under the conditions of Theorems 1.7 and 1.11. We will gather the results of the two previous sections to establish the two aforementioned theorems.

Proof of Theorem 1.11

We will decompose the sample Ω into the sets $\Gamma^*(\ell)$, $\Omega^*_1(\ell)$, $\Omega^*_2(\ell)$, $\Omega_{\mathrm{bulk}}(\ell)$ and $T(\ell)$ introduced below in this section (see Figure 6), and we will analyse the behaviour of the minimizer in each of these sets. *We assume ℓ to be the parameter in (5.1) which satisfies $\ell \approx \kappa^{-3/4}$.*

In a neighbourhood of the magnetic barrier.   We start by introducing the set $\Gamma^* = \Gamma^*(\ell)$ which covers almost all of the set Γ. Recall the assumption that Γ consists of a finite collection of simple disjoint smooth curves that may intersect $\partial\Omega$ *transversely*. For the simplicity of the exposition, we will focus on the particular case of a single curve intersecting $\partial\Omega$ at two points. The construction below may be adjusted to cover the general case by considering every single component of
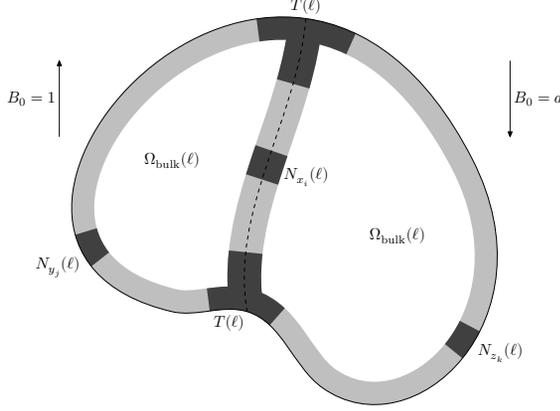
Figure 6: Schematic representation of the sets $\Omega_{\mathrm{bulk}}(\ell)$, $T(\ell)$, $\mathcal{N}_{x_i}(\ell)$, $\mathcal{N}_{y_j}(\ell)$ and $\mathcal{N}_{z_k}(\ell)$, where $\ell \approx \kappa^{-3/4}$. In the regime where $a \in [-1, 0)$, $H = b\kappa$ and $1/|a| < b$, the white region is in the normal state, while the other regions may carry superconductivity.

Γ separately. We may select two constants $\ell_0 \in (0, 1)$ and $c > 2$, and for all $\ell \in (0, \ell_0)$, a collection of pairwise distinct points $(x_i)_{i=1}^N \subset \Gamma$ such that,

$$(x_i)_{i=1}^N \subset \{x \in \Gamma \,:\, \mathrm{dist}(x, \partial\Omega) > c\ell\},$$

$$\forall\, i \in \{1, ..., N-1\}, \quad \mathrm{dist}_\Gamma(x_i, x_{i+1}) = \ell,$$

and

$$\{x \in \Omega \,:\, \mathrm{dist}(x, \Gamma) < \ell,\ \mathrm{dist}(x, \partial\Omega) > c\ell\} \subset \Gamma^*(\ell) := \left(\bigcup_{i=1}^N \overline{\mathcal{N}_{x_i}(\ell)}\right)^\circ, \quad (6.33)$$

where $\mathcal{N}_{x_i}(\ell)$ is the set introduced in (6.3). The family $\left(\mathcal{N}_{x_i}(\ell)\right)_{1 \leq i \leq N}$ consists of pairwise disjoint sets. The number N depends on ℓ as follows

$$|\Gamma|\ell^{-1} - \mathcal{O}(1) \leq N \leq |\Gamma|\ell^{-1}, \quad (\ell \to 0). \qquad (6.34)$$

In a neighbourhood of the boundary. Now, we define the two sets $\Omega_1^* = \Omega_1^*(\ell)$ and $\Omega_2^* = \Omega_2^*(\ell)$ which cover almost all of the set $\partial\Omega$. In a similar fashion to the definition of $\Gamma^*(\ell)$, we fix $\ell_0 \in (0, 1)$ and $c > 2$ and we select two collections of points

$$(y_j)_{j=1}^{N_1} \subset \{x \in \partial\Omega_1 \,:\, \mathrm{dist}(x, \Gamma) > c\ell\}$$

$$\text{and } (z_k)_{k=1}^{N_2} \subset \{x \in \partial\Omega_2 \,:\, \mathrm{dist}(x, \Gamma) > c\ell\},$$

such that

$$\text{dist}_{\partial\Omega_1}(y_j, y_{j+1}) = \ell \quad \text{and} \quad \text{dist}_{\partial\Omega_2}(z_k, z_{k+1}) = \ell,$$

for $1 \leq j \leq N_1 - 1$ and $1 \leq k \leq N_2 - 1$. Furthermore,

$$\{x \in \Omega \; : \; \text{dist}(x, \partial\Omega_1) < \ell, \; \text{dist}(x, \Gamma) > c\ell\} \subset \Omega_1^*(\ell) := \left(\bigcup_{j=1}^{N_1} \overline{\mathcal{N}_{y_j}(\ell)} \right)^{\circ},$$

$$(6.35)$$

$$\{x \in \Omega \; : \; \text{dist}(x, \partial\Omega_2) < \ell, \; \text{dist}(x, \Gamma) > c\ell\} \subset \Omega_2^*(\ell) := \left(\bigcup_{k=1}^{N_2} \overline{\mathcal{N}_{z_k}(\ell)} \right)^{\circ},$$

$$(6.36)$$

where $\mathcal{N}_{y_j}^1(\ell)$ and $\mathcal{N}_{z_k}^2(\ell)$ are defined in (6.32). The numbers N_1 and N_2 depend on ℓ as follows

$$|\partial\Omega_1|\ell^{-1} - \mathcal{O}(1) \leq N_1 \leq |\partial\Omega_1|\ell^{-1} \text{ and } |\partial\Omega_2|\ell^{-1} - \mathcal{O}(1) \leq N_2 \leq |\partial\Omega_2|\ell^{-1},$$

as ℓ tends to 0.

The bulk set.   Next, we introduce the set $\Omega_{\text{bulk}} = \Omega_{\text{bulk}}(\ell)$ representing the bulk of the sample:

$$\Omega_{\text{bulk}}(\ell) = \left\{ x \in \Omega \; : \; \text{dist}(x, \partial\Omega_1 \cup \partial\Omega_2) > \ell \right\}. \qquad (6.37)$$

In a neighbourhood of the Γ-zone.   We finally introduce the remaining set in the decomposition of Ω, the neighbourhood $T = T(\ell)$ of $\Gamma \cap \partial\Omega$

$$T(\ell) := \Omega \setminus \overline{\left(\bigcup_{j=1}^{2} \Omega_j^*(\ell) \cup \Gamma^*(\ell) \cup \Omega_{\text{bulk}}(\ell) \right)}.$$

The definition of the sets Γ^*, Ω_1^*, Ω_2^* and Ω_{bulk} in (6.33), (6.35), (6.36) and (6.37) ensures that $|T| = \mathcal{O}(\ell^2)$ as $\ell \to 0$.

Behaviour of the minimizer.   Now, we are ready to prove the convergence of $|\psi|^4$ in the sense of distributions, claimed in Theorem 1.11.

Let $\varphi \in C_c^\infty(\mathbb{R}^2)$. We have

$$\kappa \mathcal{T}_\kappa^b(\varphi) = \kappa \int_{\Omega_{\text{bulk}}} |\psi|^4 \varphi \, dx + \kappa \int_T |\psi|^4 \varphi \, dx + \kappa \int_{\Gamma^*} |\psi|^4 \varphi \, dx$$

$$+ \kappa \int_{\Omega_1^*} |\psi|^4 \varphi \, dx + \kappa \int_{\Omega_2^*} |\psi|^4 \varphi \, dx. \quad (6.38)$$

We will estimate each of these right hand side integrals. Starting with

$$\left| \kappa \int_{\Omega_{\text{bulk}}} |\psi|^4 \varphi \, dx \right| \leq \kappa \|\varphi\|_{L^\infty(\Omega)} \int_{\Omega_{\text{bulk}}} |\psi|^4 \, dx = o(1), \quad (6.39)$$

by the exponential decay of ψ in Theorem 2.4.

Secondly, since $\|\psi\|_{L^\infty(\Omega)} \leq 1$ (see Proposition 2.1), $|T| = \mathcal{O}(\ell^2)$ as $\ell \to 0$ and by (5.1), we get

$$\left| \kappa \int_T |\psi|^4 \varphi \, dx \right| \leq C\kappa\ell^2 = o(1), \quad (6.40)$$

C is a constant independent of κ.

Next, we have (see (6.33))

$$\kappa \int_{\Gamma^*} |\psi|^4 \varphi \, dx = \kappa \sum_{i=1}^N \int_{\mathcal{N}_{x_i}(\ell)} |\psi|^4 \varphi \, dx. \quad (6.41)$$

For $i \in \{1, ..., N\}$, let p_i and q_i be two points of Γ such that

$$\varphi(p_i) = \max_{x \in \mathcal{N}_{x_i}(\ell) \cap \Gamma} \varphi(x) \quad \text{and} \quad \varphi(q_i) = \min_{x \in \mathcal{N}_{x_i}(\ell) \cap \Gamma} \varphi(x).$$

We may write

$$\kappa \sum_{i=1}^N \int_{\mathcal{N}_{x_i}(\ell)} |\psi|^4 \varphi \, dx = \kappa \sum_{i=1}^N \int_{\mathcal{N}_{x_i}(\ell)} |\psi(x)|^4 \varphi(p_i) \, dx$$

$$+ \kappa \sum_{i=1}^N \int_{\mathcal{N}_{x_i}(\ell)} |\psi(x)|^4 \big(\varphi(x) - \varphi(p_i)\big) \, dx. \quad (6.42)$$

We estimate $|\varphi(x) - \varphi(p_i)|$ in $\mathcal{N}_{x_i}(\ell)$ by the mean value theorem. Using the size of $\mathcal{N}_{x_i}(\ell)$ and the bound $\|\psi\|_{L^\infty(\Omega)} \leq 1$, we get

$$\left| \int_{\mathcal{N}_{x_i}(\ell)} |\psi(x)|^4 \big(\varphi(x) - \varphi(p_i)\big) \, dx \right| \leq C \int_{\mathcal{N}_{x_i}(\ell)} |\psi(x)|^4 |x - p_i| \, dx \leq C\ell^3,$$

for some C independent of κ. Hence, by (5.1) and (6.34)

$$\kappa \sum_{i=1}^{N} \left| \int_{\mathcal{N}_{x_i}(\ell)} |\psi(x)|^4 \big(\varphi(x) - \varphi(p_i) \big) \, dx \right| \leq CN\kappa\ell^3 = o(1). \qquad (6.43)$$

On the other hand, using the uniform bounds in (6.10) and (6.19), we get

$$\left| \kappa \sum_{i=1}^{N} \int_{\mathcal{N}_{x_i}(\ell)} |\psi(x)|^4 \varphi(p_i) \, dx + 2b^{-\frac{1}{2}} e_a(b) \sum_{i=1}^{N} \ell \varphi(p_i) \right| \leq C\kappa^{-\frac{1}{16}} \sum_{i=1}^{N} \ell |\varphi(p_i)|,$$

where C is a constant independent of κ. We use further that $\sum_{i=1}^{N} \ell |\varphi(p_i)| \leq \|\varphi\|_\infty N\ell$ and $N\ell = \mathcal{O}(1)$ by (6.34). We get that

$$\left| \kappa \sum_{i=1}^{N} \int_{\mathcal{N}_{x_i}(\ell)} |\psi(x)|^4 \varphi(p_i) \, dx + 2b^{-\frac{1}{2}} e_a(b) \sum_{i=1}^{N} \ell \varphi(p_i) \right| \leq \tilde{C}\kappa^{-1/16}, \qquad (6.44)$$

where $\tilde{C}$ is a new constant independent of κ. Combining (6.41)–(6.44) yields

$$\kappa \int_{\Gamma^*} |\psi|^4 \varphi \, dx \geq -2b^{-\frac{1}{2}} e_a(b) \sum_{i=1}^{N} \ell \varphi(p_i) + o(1)$$

$$\geq -2b^{-\frac{1}{2}} e_a(b) \int_{\Gamma^* \cap \Gamma} \varphi \, ds_\Gamma + o(1), \qquad (6.45)$$

since our choice of the points (p_i) is such that the term $\sum_{i=1}^{N} \ell \varphi(p_i)$ is an upper Riemann sum of the function $\varphi(x)$ on the set $\Gamma^* \cap \Gamma$. Similarly, using $\sum_{i=1}^{N} \ell \varphi(q_i)$ the lower Riemann sum of the function $\varphi(x)$ on the set $\Gamma^* \cap \Gamma$, we get

$$\kappa \int_{\Gamma^*} |\psi|^4 \varphi \, dx \leq -2b^{-\frac{1}{2}} e_a(b) \sum_{i=1}^{N} \ell \varphi(q_i) + o(1)$$

$$\leq -2b^{-\frac{1}{2}} e_a(b) \int_{\Gamma^* \cap \Gamma} \varphi \, ds_\Gamma + o(1). \qquad (6.46)$$

We combine (6.45) and (6.46) to obtain

$$\kappa \int_{\Gamma^*} |\psi|^4 \varphi \, dx = -2b^{-\frac{1}{2}} e_a(b) \int_{\Gamma^* \cap \Gamma} \varphi \, ds_\Gamma + o(1).$$

But by (6.34)

$$\left| \int_{\Gamma \backslash (\Gamma^* \cap \Gamma)} \varphi \, ds_\Gamma \right| \le \|\varphi\|_{L^\infty(\Omega)} |\Gamma \backslash (\Gamma^* \cap \Gamma)| \le C\ell = o(1).$$

Hence,

$$\kappa \int_{\Gamma^*} |\psi|^4 \varphi \, dx = -2b^{-\frac{1}{2}} e_a(b) \int_{\Gamma} \varphi \, ds_\Gamma + o(1). \qquad (6.47)$$

One can proceed similarly to prove that

$$\kappa \int_{\Omega_1^*} |\psi|^4 \varphi \, dx = -2b^{-\frac{1}{2}} E_{\text{surf}}(b) \int_{\partial\Omega_1 \cap \partial\Omega} \varphi \, ds + o(1)$$

$$\text{and } \kappa \int_{\Omega_2^*} |\psi|^4 \varphi \, dx = -2|a|^{-\frac{1}{2}} b^{-\frac{1}{2}} E_{\text{surf}}(b|a|) \int_{\partial\Omega_2 \cap \partial\Omega} \varphi \, ds + o(1). \quad (6.48)$$

Gathering pieces in (6.39), (6.40), (6.47) and (6.48), we establish Theorem 1.11.

Proof of Theorem 1.7

We apply Theorem 1.11 for $\varphi \in C_c^\infty(\mathbb{R}^2)$ such that $\varphi = 1$ in a neighbourhood of Ω to get (1.6).

Multiplying both sides of the first equation in (1.3) by $\bar\psi$ then integrating by parts give

$$\mathrm{E}_{\text{g.st}}(\kappa, H) = \mathscr{E}(\psi, \mathbf{A}; \Omega) \ge \mathscr{E}_0(\psi, \mathbf{A}; \Omega) = -\frac{1}{2}\kappa^2 \int_\Omega |\psi|^4 \, dx, \qquad (6.49)$$

where $\mathscr{E}(\psi, \mathbf{A}; \cdot)$ and $\mathscr{E}_0(\psi, \mathbf{A}; \cdot)$ are the energies in (6.1). Using (6.49) and (1.6), we get the lower bound of $\mathrm{E}_{\text{g.st}}(\kappa, H)$ in (1.5).

The upper bound of $\mathrm{E}_{\text{g.st}}(\kappa, H)$ can be derived by the help of a suitable trial configuration. We are still considering the parameter ℓ as in (5.1). Let $\mathbf{F}$ be the magnetic potential introduced in Lemma 2.2. We define the function $h_\Gamma \in H^1(\Omega; \mathbb{C}) \cap H_0^1(\Gamma^*(\ell))$

$$h_\Gamma(x) = \sum_{i=1}^{N} \mathbb{1}_{\mathscr{N}_{x_i}(\ell)}(x) v_i(x),$$

where $\Gamma^*(\ell)$ and $\mathscr{N}_{x_i}(\ell)$ are respectively the sets in (6.33) and (6.3), $v_i(x) = \left(e^{i\kappa H \omega} u_i \right) \circ \Phi^{-1}(x)$, $\omega = \omega_{-\ell,\ell}$ is the gauge function in Lemma 4.1,

Φ is the coordinate transformation in (4.3), u_i is defined by $u_i(s, t) = u_0(s - s_i, t)$ for $(s_i, t_i) = \Phi^{-1}(x_i)$, and u_0 is the minimizer of $G(\cdot, \mathcal{V}(\ell))$ defined in (5.5). From the definition of v_i, we derive the following (see (6.7))

$$\mathcal{E}_0\big(v_i, \mathbf{F}; \mathcal{N}_{x_i}(\ell)\big) = G\big(u_0, \mathcal{V}(\ell)\big).$$

The previous identity together with Lemma 5.2, (6.34) and ($\ell \approx \kappa^{-3/4}$) give

$$\mathcal{E}_0\big(b_\Gamma, \mathbf{F}; \Omega\big) = \sum_{i=1}^N \mathcal{E}_0\big(v_i, \mathbf{F}; \mathcal{N}_{x_i}(\ell)\big) \le |\Gamma| b^{-1/2} \kappa e_a(b) + o(\kappa), \quad (\kappa \to +\infty).$$
$$(6.50)$$

Similarly, for $j \in \{1, 2\}$, using the results of Theorem 6.5, one may define a function $b_j \in H^1(\Omega; \mathbb{C}) \cap H_0^1\big(\Omega_j^*(\ell)\big)$ satisfying

$$\mathcal{E}_0\big(b_1, \mathbf{F}; \Omega_1^*(\ell)\big) \le |\partial\Omega_1 \cap \partial\Omega| b^{-\frac{1}{2}} \kappa E_{\mathrm{surf}}(b) + o(\kappa),$$
$$\mathcal{E}_0\big(b_2, \mathbf{F}; \Omega_2^*(\ell)\big) \le |\partial\Omega_2 \cap \partial\Omega| b^{-\frac{1}{2}} |a|^{-\frac{1}{2}} \kappa E_{\mathrm{surf}}\big(b|a|\big) + o(\kappa), \qquad (6.51)$$

where $\Omega_j^*(\ell)$ is defined in (6.35) and (6.36). Now, we define the trial function

$$b(x) = \mathbb{1}_{\Gamma^*(\ell)}(x) b_\Gamma(x) + \mathbb{1}_{\Omega_1^*(\ell)}(x) b_1(x) + \mathbb{1}_{\Omega_2^*(\ell)}(x) b_2(x),$$

Noticing that $\mathrm{E}_{\mathrm{g.st}}(\kappa, H) \le \mathcal{E}\big(b, \mathbf{F}; \Omega\big) = \mathcal{E}_0\big(b, \mathbf{F}; \Omega\big)$ (see (6.1)), we gather the results in (6.50) and (6.51) to derive the upper bound in (1.5).

Acknowledgements

A Some Spectral Properties of Fiber Operators

A.1 Harmonic oscillators on the semi-axis

Let $\xi \in \mathbb{R}$. Besides the Robin and Neumann realizations of the harmonic oscillator, we introduce the Dirichlet realization of

$$H^D[\xi] = -\frac{d^2}{dt^2} + (t - \xi)^2, \tag{A.1}$$

with domain

$$\text{Dom}\left(H^D[\xi]\right) = \{u \in B^2(\mathbb{R}_+) \; : \; u(0) = 0\},$$

and lowest eigenvalue

$$\mu^D(\xi) = \inf \text{sp}\left(H^D[\xi]\right). \tag{A.2}$$

The perturbation theory [Kat66] ensures that the functions

$$\xi \mapsto \mu^D(\xi), \qquad \xi \mapsto \mu^N(\xi), \qquad \text{and} \qquad \xi \mapsto \mu(\gamma, \xi)$$

are analytic, where $\mu(\gamma, \xi)$ and $\mu^N(\xi)$ are respectively defined in (2.3) and (2.5). We list the following well-known spectral properties (for instance see [DH93, RS72, Kac06]):

Proposition A.1. *The function $\xi \mapsto \mu^D(\xi)$ introduced in* (A.2) *satisfies*

$$\lim_{\xi \to -\infty} \mu^D(\xi) = +\infty \quad \text{and} \quad \lim_{\xi \to +\infty} \mu^D(\xi) = 1.$$

For all $\gamma \in \mathbb{R}$, the function $\xi \mapsto \mu(\gamma, \xi)$ introduced in (2.3) *satisfies*

$$\lim_{\xi \to -\infty} \mu(\gamma, \xi) = +\infty \quad \text{and} \quad \lim_{\xi \to +\infty} \mu(\gamma, \xi) = 1.$$

A.2 Spectral properties of the operator $h_a[\xi]$

Let $a \in [-1, 1) \backslash \{0\}$ and $\xi \in \mathbb{R}$. Recall the operator $h_a[\xi]$ introduced in (2.13) and its associated quadratic form $q_a[\xi]$ defined in (2.14). The embedding of the domain of $q_a[\xi]$ is compact in $L^2(\mathbb{R})$, hence the spectrum of $h_a[\xi]$ is an increasing sequence of eigenvalues converging to $+\infty$. The lowest eigenvalue is denoted by $\mu_a(\xi)$.

The result in the following proposition may be derived similarly as done in [FH10, Section 3.2.1]:

Proposition A.2. *The lowest eigenvalue $\mu_a(\xi)$ of $h_a[\xi]$ is simple. Furthermore, there exists a positive eigenfunction $g_{a,\xi}$ normalized with respect to the norm $\| \cdot \|_{L^2(\mathbb{R})}$. $g_{a,\xi}$ is the unique function satisfying such properties.*

The functions $\xi \mapsto \mu_a(\xi)$ and $\xi \mapsto g_{a,\xi}$ are in C^∞ by the perturbation theory (see [FH10, Theorem C.2.2]).

The bounds in Lemma A.3 are useful for establishing Proposition A.4, which is crucial in our study of the eigenvalue $\mu_a(\xi)$ (see Section 2.4).

Lemma A.3. *Let $a \in [-1, 1)\backslash\{0\}$ and $\xi \in \mathbb{R}$. It holds*

- *If $a \in (0, 1)$, then*

$$\min\left(\mu^N(-\xi), \, a\mu^N\left(\frac{\xi}{\sqrt{a}}\right)\right) \leq \mu_a(\xi) \leq \min\left(\mu^D(-\xi), \, a\mu^D\left(\frac{\xi}{\sqrt{a}}\right)\right).$$

- *If $a \in [-1, 0)$, then*

$$\min\left(\mu^N(-\xi), \, |a|\mu^N\left(-\frac{\xi}{\sqrt{|a|}}\right)\right) \leq \mu_a(\xi)$$

$$\leq \min\left(\mu^D(-\xi), \, |a|\mu^D\left(-\frac{\xi}{\sqrt{|a|}}\right)\right). \quad \text{(A.3)}$$

Proof. We will prove the lemma in the case $a \in (-1, 0)$. The proof follows similarly in the case $a \in (0, 1)$.

We start by establishing the upper bound in (A.3). Let $\xi \in \mathbb{R}$. Consider $u = u^D_{-\xi}$ the normalized eigenfunction of the operator $H^D[-\xi]$ defined in (A.1), corresponding to the lowest eigenvalue $\mu^D(-\xi)$. Then

$$\mu^D(-\xi) = \int_0^{+\infty} \left(|u'(t)|^2 + (t + \xi)^2|u(t)|^2\right) dt.$$

Noticing that $u \in H^1_0(\mathbb{R}_+)$, we extend it by zero on $\mathbb{R}_-$ (the extension is still denoted by u for simplicity). Hence, we have $q_a[\xi](u) = \mu^D(-\xi)$, where $q_a[\xi]$ is the quadratic form in (2.14). Using the min-max principle, we get

$$\mu_a[\xi] \leq \frac{q_a[\xi](u)}{\|u\|^2_{L^2(\mathbb{R})}} = \mu^D(-\xi).$$

Similarly, using $v = v^D_{-\xi/\sqrt{|a|}}$ the normalized eigenfunction of $H^D[-\xi/\sqrt{|a|}\,]$ corresponding to the lowest eigenvalue $\mu^D(-\xi/\sqrt{|a|})$, we can prove that

$$\mu^D\left(\frac{-\xi}{\sqrt{|a|}}\right) = \int_0^{+\infty} \left(|v'(t)|^2 + \left(t + \frac{\xi}{\sqrt{|a|}}\right)^2|v(t)|^2\right) dt \geq \frac{1}{|a|}\mu_a(\xi),$$

by the min-max principle, after employing the change of variable $x = -t/\sqrt{|a|}$ and extending the resulting function by 0 on $\mathbb{R}_+$.

Next, we establish the lower bound in (A.3). We consider $g = g_{a,\xi}$ the normalized eigenfunction of the operator $h_a[\xi]$ corresponding to the lowest eigenvalue $\mu_a(\xi)$ (see Proposition A.2). We have

$$\mu_a(\xi) = \int_{-\infty}^{0} \left(|g'(t)|^2 + (at+\xi)^2 |g(t)|^2 \right) dt + \int_{0}^{+\infty} \left(|g'(t)|^2 + (t+\xi)^2 |g(t)|^2 \right) dt. \tag{A.4}$$

Using the min-max principle, we write a lower bound for each integral appearing in (A.4). Indeed,

$$\int_{0}^{+\infty} \left(|g'(t)|^2 + (t+\xi)^2 |g(t)|^2 \right) dt \geq \mu^N(-\xi) \int_{0}^{+\infty} |g(t)|^2 \, dt, \tag{A.5}$$

and

$$\int_{-\infty}^{0} \left(|g'(t)|^2 + (at+\xi)^2 |g(t)|^2 \right) dt \geq |a| \mu^N \left(\frac{-\xi}{\sqrt{|a|}} \right) \int_{-\infty}^{0} |g(t)|^2 \, dt. \tag{A.6}$$

Note that, for obtaining (A.6), we performed the change of variable $x = -\sqrt{|a|}\, t$ which yielded

$$\int_{-\infty}^{0} \left(|g'(t)|^2 + (at+\xi)^2 |g(t)|^2 \right) dt = \sqrt{|a|} \int_{0}^{+\infty} \left(|w'(x)|^2 \right.$$
$$\left. + \left(x + \frac{\xi}{\sqrt{|a|}} \right)^2 |w(x)|^2 \right) dx,$$

and

$$\int_{-\infty}^{0} |g(t)|^2 \, dt = \frac{1}{\sqrt{|a|}} \int_{0}^{+\infty} |w(x)|^2 \, dx,$$

where $w(x) = g(-x/\sqrt{|a|})$.

Combining (A.4), (A.5) and (A.6), and using the normalization of g, we obtain the desired lower bound. $\qquad\square$

Proposition A.4. *Let $a \in [-1,1) \backslash \{0\}$. We have*

- *For $a \in (0,1)$,*

$$\lim_{\xi \to -\infty} \mu_a(\xi) = 1 \quad \text{and} \quad \lim_{\xi \to +\infty} \mu_a(\xi) = a.$$

- *For $a \in [-1, 0)$,*

$$\lim_{\xi \to -\infty} \mu_a(\xi) = |a| \quad \text{and} \quad \lim_{\xi \to +\infty} \mu_a(\xi) = +\infty.$$

Proof. It is sufficient to apply Proposition A.1 and Lemma A.3. □

Proposition A.5. *([HS15])*
    For any $a \in [-1, 1) \backslash \{0\}$ and $\xi \in \mathbb{R}$ we have

$$\partial_\xi \mu_a(\xi) = \left(1 - \frac{1}{a}\right) \left(g'_{a,\xi}(0)^2 + (\mu_a(\xi) - \xi^2) g_{a,\xi}(0)^2\right), \tag{A.7}$$

where $g_{a,\xi}$ is the eigenfunction in Proposition A.2.

Proof. (Feynman–Hellmann). For simplicity, we write μ, g and h respectively for $\mu_a(\xi)$, $g_{a,\xi}$, and $h_a[\xi]$. Differentiating with respect to ξ and integrating by parts in

$$(h - \mu)g = 0. \tag{A.8}$$

we get

$$\langle (\partial_\xi h - \partial_\xi \mu) g, g \rangle + \langle (h - \mu) \partial_\xi g, g \rangle = 0.$$

Hence using

$$\langle (h - \mu) \partial_\xi g, g \rangle = \langle \partial_\xi g, (h - \mu) g \rangle = 0,$$

and recalling that g is normalized, we obtain

$$\partial_\xi \mu = \langle \partial_\xi h g, g \rangle = 2 \int_{-\infty}^0 (\xi + at) g^2(t) \, dt + 2 \int_0^{+\infty} (\xi + t) g^2(t) \, dt. \tag{A.9}$$

Integrating by parts the right hand side of (A.9), and using (A.8) establish the result. □

Proposition A.6. *Let $a \in (-1, 0)$ and $\beta_a = \min_{\xi \in \mathbb{R}} \mu_a(\xi)$. We have*

$$|a| \Theta_0 < \beta_a,$$

where Θ_0 is the value in (2.5).

Proof. Let ξ_a be such that $\beta_a = \mu_a(\xi_a)$ (see [HPRS16]). We use the lower bound proof of Lemma A.3, with $g = g_{a,\xi_a}$ the positive normalized eigenfunction of the operator $h_a[\xi_a]$ corresponding to $\mu_a(\xi_a)$ (see Proposition A.2). We get

$$\mu_a(\xi_a) \geq |a| \Theta_0 \int_{-\infty}^{0} g^2(t)\, dt + \Theta_0 \int_{0}^{+\infty} g^2(t)\, dt. \qquad \text{(A.10)}$$

Since g is normalized and positive, and $|a|\Theta_0 < \Theta_0$ for $a \in (-1,0)$, the proof is completed. $\qquad\square$

Proposition A.7. *Let* $a \in (-1,0)$. *If* $\xi_a \in \mathbb{R}$ *satisfies* $\mu_a(\xi_a) = \min_{\xi \in \mathbb{R}} \mu_a(\xi)$, *then* $\xi_a < 0$.

Proof. Suppose that $\xi_a \geq 0$. Let g_{a,ξ_a} be the positive normalized eigenfunction of the operator $h_a[\xi_a]$ corresponding to the lowest eigenvalue $\mu_a(\xi_a)$ (Proposition A.2).

- If $\xi_a > 0$, then since $a < 0$, one sees that $q_a[0](g_{a,\xi_a}) < q_a[\xi_a](g_{a,\xi_a})$, where $q_a[\cdot]$ is the form in (2.14); consequently, the min-max principle gives $\mu_a(0) < \mu_a(\xi_a)$. This contradicts the definition of ξ_a.

- If $\xi_a = 0$, then by Proposition A.5,

$$0 = \partial_\xi \mu_a(\xi_a) = \left(1 - \frac{1}{a}\right) \left(g'_{a,0}(0)^2 + \mu_a(0) g_{a,0}(0)^2\right)^2 > 0,$$

since $a \in (-1,0)$, $g > 0$ and, by Proposition A.6, $\mu_a(0) > |a|\Theta_0 > 0$. $\qquad\square$

B   Decay estimates for the 2D-effective model

The aim of this appendix is to prove Proposition 3.4. Recall that we work under (3.7), namely,

$$-1 \leq a < 0 \quad \text{and} \quad \frac{1}{|a|} \leq b < \frac{1}{\beta_a},$$

where β_a is the lowest eigenvalue introduced in (2.11).

For every $m \in \mathbb{N}$ and $R > 1$, we introduce the set $S_{R,m} = (-R/2, R/2) \times (-m, m)$ and the functional

$$\mathscr{G}_{a,b,R,m}(u) = \int_{S_{R,m}} \left(b\left|(\nabla - i\sigma A_0)u\right|^2 - |u|^2 + \frac{1}{2}|u|^4\right) dx \qquad \text{(B.1)}$$

defined over the space

$$\mathscr{D}_{R,m} = \left\{ u \in L^2(S_{R,m}) \; : \; (\nabla - i\sigma A_0)u \in L^2(S_{R,m}), \right.$$
$$\left. u\left(x_1 = \pm\frac{R}{2}, \cdot\right) = u\left(\cdot, x_2 = \pm m\right) = 0 \right\}. \quad \text{(B.2)}$$

Here σ was defined in (2.9). Now we define the ground state energy

$$g_a(b, R, m) = \inf_{u \in \mathscr{D}_{R,m}} \mathscr{G}_{a,b,R,m}(u). \quad \text{(B.3)}$$

Lemma B.1. *There exists a universal constant $C > 0$, and for all $R > 1$, $m \geq 1$, there exists a function $\varphi_{a,b,R,m} \in \mathscr{D}_{R,m}$ satisfying,*

$$\|\varphi_{a,b,R,m}\|_{L^\infty(S_{R,m})} \leq 1, \quad \text{(B.4)}$$

$$\int_{S_{R,m} \cap \{|x_2| \geq 4\}} \frac{|x_2|}{\left(\ln|x_2|\right)^2} \left(\left|(\nabla - i\sigma A_0)\varphi_{a,b,R,m}\right|^2 + |\varphi_{a,b,R,m}|^2 \right) dx \leq CbR, \quad \text{(B.5)}$$

$$\int_{S_{R,m} \cap \{|x_2| \geq 4\}} \frac{|x_2|^3}{\left(\ln|x_2|\right)^2} |\varphi_{a,b,R,m}|^4 \, dx \leq Cb^2R, \quad \text{(B.6)}$$

and

$$\mathscr{G}_{a,b,R,m}(\varphi_{a,b,R,m}) = g_a(b, R, m). \quad \text{(B.7)}$$

Here $\mathscr{G}_{a,b,R,m}$ is the functional introduced in (B.1) and $g_a(b, R, m)$ is the ground state energy introduced in (B.3).

Proof. The boundedness and the regularity of the domain $S_{R,m}$ guarantee the existence of a minimizer $\varphi_m := \varphi_{a,b,R,m}$ of $\mathscr{G}_{a,b,R,m}$ in $\mathscr{D}_{R,m}$, satisfying

$$-b(\nabla - i\sigma A_0)^2 \varphi_m = (1 - |\varphi_m|^2)\varphi_m \qquad \text{in } S_{R,m}, \quad \text{(B.8)}$$

see e.g. [FH10, Chapter 11]. Furthermore, Proposition 10.3.1 in [FH10] ensures that

$$\|\varphi_m\|_{L^\infty(S_{R,m})} \leq 1.$$

Next, select $\chi \in C^\infty(\mathbb{R})$ such that $\chi(x_2) = 0$ if $|x_2| \leq 1$, and $\chi(x_2) = |x_2|^2/\ln|x_2|$ if $|x_2| \geq 4$. Consequently, the function χ satisfies

$$0 < |\chi'(x_2)| < \frac{3\sqrt{|x_2|}}{2\ln|x_2|} \qquad \text{for all } |x_2| \geq 4.$$

Multiply (B.8) by $\chi^2 \overline{\varphi_m}$ and integrate by parts,

$$\int_{S_{R,m}} \left(b \left| (\nabla - i\sigma A_0) \chi \varphi_m \right|^2 - \chi^2 |\varphi_m|^2 + \chi^2 |\varphi_m|^4 \right) dx = b \int_{S_{R,m}} \chi'^2 |\varphi_m|^2 \, dx. \quad \text{(B.9)}$$

Since the function $x \mapsto \chi(x_2) \varphi_m(x)$ is supported in $S_{R,m} \cap \{|x_2| \geq 1\}$ where $\mathrm{curl}(\sigma A_0) = \sigma$, we can apply the spectral inequality in [FH10, Lemma 1.4.1] to get, under the assumption $1/|a| \leq b < 1/\beta_a$,

$$b \int_{S_{R,m}} \left| (\nabla - i\sigma A_0) \chi \varphi_m \right|^2 dx \geq b \int_{S_{R,m}} |\sigma| \chi^2 |\varphi_m|^2 \, dx \geq \int_{S_{R,m}} \chi^2 |\varphi_m|^2 \, dx. \quad \text{(B.10)}$$

It follows from (B.9) and (B.10)

$$\begin{aligned}
\int_{S_{R,m}} \chi^2(x_2) |\varphi_m|^4 \, dx &\leq b \int_{S_{R,m}} \chi'^2(x_2) |\varphi_m|^2 \, dx \\
&\leq b \int_{S_{R,m} \cap \{|x_2| \geq 4\}} \chi'^2(x_2) |\varphi_m|^2 \, dx \\
&\quad + b \int_{S_{R,m} \cap \{|x_2| < 4\}} \chi'^2(x_2) |\varphi_m|^2 \, dx \\
&\leq Cb \int_{S_{R,m} \cap \{|x_2| \geq 4\}} \frac{|x_2|}{\left(\ln |x_2| \right)^2} |\varphi_m|^2 \, dx + CbR. \quad \text{(B.11)}
\end{aligned}$$

Using Hölder's inequality,

$$\begin{aligned}
\int_{S_{R,m} \cap \{|x_2| \geq 4\}} & \frac{|x_2|}{\left(\ln |x_2| \right)^2} |\varphi_m|^2 \, dx \\
&\leq \left(\int_{S_{R,m} \cap \{|x_2| \geq 4\}} \frac{1}{|x_2| (\ln |x_2|)^2} \, dx \right)^{\frac{1}{2}} \left(\int_{S_{R,m} \cap \{|x_2| \geq 4\}} \frac{|x_2|^3}{(\ln |x_2|)^2} |\varphi_m|^4 \, dx \right)^{\frac{1}{2}} \\
&\leq CR^{\frac{1}{2}} \left(\int_{S_{R,m} \cap \{|x_2| \geq 4\}} \frac{|x_2|^3}{|(\ln |x_2|)^2} |\varphi_m|^4 \, dx \right)^{\frac{1}{2}}. \quad \text{(B.12)}
\end{aligned}$$

Now, using Cauchy–Schwarz inequality together with (B.11) and (B.12), we obtain

$$\int_{S_{R,m} \cap \{|x_2| \geq 4\}} \frac{|x_2|^3}{(\ln |x_2|)^2} |\varphi_m|^4 \, dx \leq \int_{S_{R,m}} \chi^2(x_2) |\varphi_m|^4 \, dx$$

$$\leq C R^{\frac{1}{2}} b \Big(\int_{S_{R,m} \cap \{|x_2| \geq 4\}} \frac{|x_2|^3}{|(\ln |x_2|)^2} |\varphi_m|^4 \, dx \Big)^{\frac{1}{2}}$$

$$+ C b R$$

$$\leq C b^2 R + C b R.$$

$$(B.13)$$

Consequently, under the assumption $1 \leq 1/|a| \leq b < 1/\beta_a$, we get (B.6). Inserting (B.6) into (B.12), we get

$$\int_{S_{R,m} \cap \{|x_2| \geq 4\}} \frac{|x_2|}{\big(\ln |x_2| \big)^2} |\varphi_m|^2 \, dx \leq C b R. \tag{B.14}$$

We still need to establish

$$\int_{S_{R,m} \cap \{|x_2| \geq 4\}} \frac{|x_2|}{\big(\ln |x_2| \big)^2} \big|(\nabla - i \sigma \mathbf{A}_0) \varphi_m\big|^2 \, dx \leq C b R. \tag{B.15}$$

To that end, we select $\eta \in C^\infty(\mathbb{R})$ such that $\eta(x_2) = 0$ if $|x_2| \leq 1$, and $\eta(x_2) = \sqrt{|x_2|}/\ln |x_2|$ if $|x_2| \geq 4$. Multiplying the equation in (B.8) by $\eta^2 \overline{\varphi_m}$ and integrating over $S_{R,m}$, we get

$$b \int_{S_{R,m} \cap \{|x_2| \geq 4\}} \big|(\nabla - i \sigma \mathbf{A}_0) \eta(x_2) \varphi_m\big|^2 \, dx$$

$$= \int_{S_{R,m} \cap \{|x_2| \geq 4\}} \Big(\eta^2(x_2) |\varphi_m|^2 - \eta^2(x_2) |\varphi_m|^4 + b \eta'^2(x_2) |\varphi_m|^2 \Big) \, dx. \tag{B.16}$$

It is easy to check by a straightforward computation, and using Cauchy's inequality, that

$$\int_S \Big(\eta^2(x_2) \big|(\nabla - i \sigma \mathbf{A}_0) \varphi_m\big|^2 \Big) \, dx$$

$$\leq \int_S \Big(\big|(\nabla - i \sigma \mathbf{A}_0) \eta(x_2) \varphi_m\big|^2 - \eta'^2(x_2) |\varphi_m|^2 \Big) \, dx$$

$$+ 2 \big| Re \, \langle \varphi_m \eta'(x_2), \eta(x_2) (\nabla - i \sigma \mathbf{A}_0) \varphi_m \rangle \big|$$

$$\leq \int_S \Big(\big|(\nabla - i \sigma \mathbf{A}_0) \eta(x_2) \varphi_m\big|^2 + \frac{1}{2} \eta^2(x_2) \big|(\nabla - i \sigma \mathbf{A}_0) \varphi_m\big|^2 + \eta'^2(x_2) |\varphi_m|^2 \Big) \, dx.$$

where $S = S_{R,m} \cap \{|x_2| \geq 4\}$. Hence,

$$\int_{S_{R,m} \cap \{|x_2| \geq 4\}} \eta^2(x_2) |(\nabla - i\sigma A_0)\varphi_m|^2 \, dx$$
$$\leq 2 \int_{S_{R,m} \cap \{|x_2| \geq 4\}} \left(|(\nabla - i\sigma A_0)\eta(x_2)\varphi_m|^2 + \eta'^2(x_2)|\varphi_m|^2 \right) dx. \quad \text{(B.17)}$$

Combining (B.16) and (B.17), we get

$$b \int_{S_{R,m} \cap \{|x_2| \geq 4\}} \eta^2(x_2) |(\nabla - i\sigma A_0)\varphi_m|^2 \, dx$$
$$\leq 2 \int_{S_{R,m} \cap \{|x_2| \geq 4\}} \eta^2(x_2)|\varphi_m|^2 \, dx + 4b \int_{S_{R,m} \cap \{|x_2| \geq 4\}} \eta'^2(x_2)|\varphi_m|^2 \, dx. \quad \text{(B.18)}$$

The definition of η yields that, in $S_{R,m} \cap \{|x_2| \geq 4\}$, $\eta^2 = |x_2|/(\ln|x_2|)^2$, and $\eta'^2 \leq 4\eta^2$. Hence, (B.14) and (B.18) imply (B.15). $\qquad\square$

Corollary B.2. *There exists a universal constant $C > 0$ such that the minimizer $\varphi_{a,b,R,m}$ in Lemma B.1 satisfies, for all $R > 1$, $m \geq 1$,*

$$\int_{S_{R,m}} b \left| (\nabla - i\sigma A_0)\varphi_{a,b,R,m} \right|^2 + |\varphi_{a,b,R,m}|^2 \, dx \leq CbR. \quad \text{(B.19)}$$

Proof. For the sake of brevity, we will write φ_m for $\varphi_{a,b,R,m}$. Using (B.14) and the fact that $|x_2|/(\ln|x_2|)^2 \geq 1$ for $|x_2| \geq 4$, we get

$$\int_{S_{R,m} \cap \{|x_2| \geq 4\}} |\varphi_m|^2 \, dx \leq CbR.$$

On the other hand, using $\|\varphi_m\|_\infty \leq 1$ and $b > 1$ we get

$$\int_{S_{R,m} \cap \{|x_2| < 4\}} |\varphi_m|^2 \, dx \leq CbR.$$

Next, since φ_m satisfies

$$-b(\nabla - i\sigma A_0)^2 \varphi_m = (1 - |\varphi_m|^2)\varphi_m \qquad \text{in } S_{R,m},$$

a simple integration by parts over $S_{R,m}$ yields

$$\int_{S_{R,m}} b |(\nabla - i\sigma A_0)\varphi_m|^2 \, dx = \int_{S_{R,m}} |\varphi_m|^2 \, dx - \int_{S_{R,m}} |\varphi_m|^4 \, dx \leq CbR.$$

$\qquad\square$

Now, we will investigate the regularity of the minimizer $\varphi_{a,b,R,m}$ in Lemma B.1. We have to be careful at this point since the magnetic field is a step function and therefore has singularities. As a byproduct, we will extract a convergent subsequence of $(\varphi_{a,b,R,m})_{m \geq 1}$.

We will use the following terminology. Let $\Omega \subset \mathbb{R}^2$ be an open set. If $(u_m)_{m \geq 1}$ is a sequence in $H^k(\Omega)$, then by saying that (u_m) is bounded/convergent in $H^k_{\text{loc}}(\Omega)$, we mean that it is bounded/convergent in $H^k(K)$, for every $K \subset \Omega$ open and relatively compact. A similar terminology applies for boundedness or convergence in $C^{k,\alpha}_{\text{loc}}(\Omega)$: A sequence $(u_m)_{m \geq 1}$ is bounded/convergent in $C^{k,\alpha}_{\text{loc}}(\Omega)$ if it is bounded/convergent in $C^{k,\alpha}(\overline{K})$, for every $K \subset \Omega$ open and relatively compact.

Lemma B.3. *Assume that* (3.7) *holds. Let $R > 1$ and $\alpha \in (0, 1)$ be fixed. The sequence $(\varphi_{a,b,R,m})_{m \geq 1}$ defined by Lemma B.1 is bounded in $H^3_{\text{loc}}(S_R)$ and consequently in $C^{1,\alpha}_{\text{loc}}(S_R)$.*

Proof. For simplicity, we will write $\varphi_m = \varphi_{a,b,R,m}$. The proof is split into three steps.

Step 1. We first prove the boundedness of (φ_m) in $H^2_{\text{loc}}(S_R)$. Using (B.8) we may write

$$\Delta \varphi_m = \frac{1}{b}\left(|\varphi_m|^2 - 1\right)\varphi_m + 2i\,\sigma \mathbf{A}_0 \cdot \nabla \varphi_m + |\sigma|^2 |\mathbf{A}_0|^2 \varphi_m. \tag{B.20}$$

Let $K \subset S_R$ be open and relatively compact. Choose an open and bounded set $\widetilde{K}$ such that $\overline{K} \subset \widetilde{K} \subset S_R$. There exists $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$, $\widetilde{K} \subset S_{R,m}$ and by Cauchy's inequality,

$$\int_{\widetilde{K}} |\nabla \varphi_m|^2 \, dx \leq 2 \int_{\widetilde{K}} \left|(\nabla - i\sigma \mathbf{A}_0)\varphi_m\right|^2 dx + 2 \int_{\widetilde{K}} |\sigma|^2 |\mathbf{A}_0|^2 |\varphi_m|^2 \, dx.$$

Using $|\varphi_m| \leq 1$, the decay estimate in (B.19) and the boundedness of σ and $\mathbf{A}_0$ in $\widetilde{K}$, we get a constant $C = C(\widetilde{K}, R)$ such that

$$\int_{\widetilde{K}} |\nabla \varphi_m|^2 \, dx \leq C, \quad \text{and} \quad \int_{\widetilde{K}} |\Delta \varphi_m|^2 \, dx \leq C,$$

in light of (B.20). By the interior elliptic estimates (see for instance [FH10, Section E.4.1]), we get that $\varphi_m \in H^2(K)$ and

$$\|\varphi_m\|_{H^2(K)} \leq C\left(\|\Delta \varphi_m\|_{L^2(\widetilde{K})} + \|\varphi_m\|_{L^2(\widetilde{K})}\right) \leq \widetilde{C}, \tag{B.21}$$

where $\widetilde{C}$ is a constant independent from m. This proves that $(\varphi_m)_{m \geq 1}$ is bounded in $H^2_{\mathrm{loc}}(S_R)$.

Step 2. Here we will improve the result in Step 1 and prove that $(\varphi_m)_{m \geq 1}$ is bounded in $H^3_{\mathrm{loc}}(S_R)$. It is enough to prove that the sequence $(\nabla \varphi_m)_{m \geq 1}$ is bounded in $H^2_{\mathrm{loc}}(S_R)$.

Let $\varsigma_m = \partial_{x_2} \varphi_m$. We will prove that $(\Delta \varsigma_m)_{m \geq 1}$ is bounded in $L^2_{\mathrm{loc}}(S_R)$. Recall that, for all $x = (x_1, x_2) \in \mathbb{R}^2$,

$$A_0(x) = (-x_2, 0) \quad \text{and} \quad \sigma(x) = \mathbb{1}_{\mathbb{R}_+}(x_2) + a\mathbb{1}_{\mathbb{R}_-}(x_2),$$

hence,

$$\left(\sigma A_0\right)(x) = \left(-x_2 \mathbb{1}_{\mathbb{R}_+}(x_2) - ax_2 \mathbb{1}_{\mathbb{R}_-}(x_2), 0\right), \tag{B.22}$$

$$\left(\sigma^2 |A_0|^2\right)(x) = x_2^2 \mathbb{1}_{\mathbb{R}_+}(x_2) + a^2 x_2^2 \mathbb{1}_{\mathbb{R}_-}(x_2). \tag{B.23}$$

Obviously, the functions in (B.22) and (B.23) admit respectively the following weak partial derivatives

$$\partial_{x_2}\left(\sigma A_0\right)(x) = \left(-\mathbb{1}_{\mathbb{R}_+}(x_2) - a\mathbb{1}_{\mathbb{R}_-}(x_2), 0\right) = \left(-\sigma(x), 0\right) \tag{B.24}$$

$$\partial_{x_2}\left(\sigma^2 |A_0|^2\right)(x) = 2x_2 \mathbb{1}_{\mathbb{R}_+}(x_2) + 2a^2 x_2 \mathbb{1}_{\mathbb{R}_-}(x_2) = 2x_2 \sigma^2(x). \tag{B.25}$$

A straightforward computation using (B.20), (B.24) and (B.25) yields

$$\begin{aligned}
\Delta \varsigma_m &= \partial_{x_2} \Delta \varphi_m \\
&= \frac{1}{b} \varphi_m^2 \partial_{x_2} \overline{\varphi_m} + \frac{1}{b} |\varphi_m|^2 \partial_{x_2} \varphi_m - 2i\sigma x_2 \partial_{x_2} \partial_{x_1} \varphi_m \\
&\quad - 2i\sigma \partial_{x_1} \varphi_m + \sigma^2 x_2^2 \partial_{x_2} \varphi_m + 2\sigma^2 x_2 \varphi_m,
\end{aligned}$$

in the sense of weak derivatives. By Step 1, the sequence (φ_m) is bounded in $H^2_{\mathrm{loc}}(S_R)$. Consequently, since $|\varphi_m| \leq 1$, it is clear that $(\Delta \varsigma_m)_{m \geq 1}$ is bounded in $L^2_{\mathrm{loc}}(S_R)$. By the interior elliptic estimates, we get that $(\varsigma_m = \partial_{x_2} \varphi_m)_{m \geq 1}$ is bounded in $H^2_{\mathrm{loc}}(S_R)$.

In a similar fashion, we prove that $(\partial_{x_1} \varphi_m)_{m \geq 1}$ is bounded in $H^2_{\mathrm{loc}}(S_R)$.

Step 3. Finally, for every relatively compact open set $K \subset \Omega$, the space $H^3(K)$ is embedded in $C^{1,\alpha}(\overline{K})$. Consequently, (φ_m) is bounded in $C^{1,\alpha}_{\mathrm{loc}}(S_R)$. □

Lemma B.4. *Let* $\alpha \in (0,1)$. *Assume that* $R > 1$ *and that* (3.7) *holds. Let* $(\varphi_{a,b,R,m})_{m \geq 1}$ *be the sequence defined in Lemma B.1. There exist a function* $\varphi_{a,b,R} \in H^3_{\mathrm{loc}}(S_R)$ *and a subsequence, denoted by* $(\varphi_{a,b,R,m})_{m \geq 1}$, *such that*

$$\varphi_{a,b,R,m} \to \varphi_{a,b,R} \text{ in } H^2_{\mathrm{loc}}(S_R) \text{ and } \varphi_{a,b,R,m} \to \varphi_{a,b,R} \text{ in } C^{0,\alpha}_{\mathrm{loc}}(S_R).$$

Furthermore, $\varphi_{a,b,R} \in C^{1,\alpha}_{\mathrm{loc}}(S_R)$.

Proof. We continue writing φ_m for $\varphi_{a,b,R,m}$. Let $K \subset S_R$ be open and relatively compact. By Lemma B.3, $(\varphi_m)_{m \geq 1}$ is bounded in $H^3(K)$, hence it has a weakly convergent subsequence by the Banach–Alaoglu theorem. By the compact embedding of $H^3(K)$ in $H^2(K)$, and of $H^2(K)$ in $C^{0,\alpha}(\overline{K})$, we may extract a subsequence, that we denote by (φ_m), such that it is strongly convergent in $H^2(K)$ and $C^{0,\alpha}(\overline{K})$. The subsequence in Lemma B.4 and its limit are then constructed via the standard Cantor's diagonal process. □

Lemma B.5. *Let* $R > 1$ *and* $\varphi_{a,b,R}$ *be the function defined by Lemma B.4. The following statements hold:*

$$\varphi_{a,b,R} \in \mathscr{D}_R, \qquad |\varphi_{a,b,R}| \leq 1 \text{ in } S_R, \tag{B.26}$$

$$- b(\nabla - i\sigma \mathbf{A}_0)^2 \varphi_{a,b,R} = (1 - |\varphi_{a,b,R}|^2)\varphi_{a,b,R} \qquad \text{in } S_R, \tag{B.27}$$

$$\int_{S_R \cap \{|x_2| \geq 4\}} \frac{|x_2|}{(\ln |x_2|)^2} \left(|(\nabla - i\sigma \mathbf{A}_0)\varphi_{a,b,R}|^2 + |\varphi_{a,b,R}|^2 \right) dx \leq CbR, \tag{B.28}$$

$$\int_{S_R \cap \{|x_2| \geq 4\}} \frac{|x_2|^3}{(\ln |x_2|)^2} |\varphi_{a,b,R}|^4 \, dx \leq Cb^2 R, \tag{B.29}$$

$$\int_{S_R} \left(b|(\nabla - i\sigma \mathbf{A}_0)\varphi_{a,b,R}|^2 + |\varphi_{a,b,R}|^2 \right) dx \leq CbR, \tag{B.30}$$

where $C > 0$ *is a universal constant, and* $\mathscr{D}_R$ *is the space introduced in* (3.2).

Proof. Let $(\varphi_{a,b,R,m})$ be the subsequence in Lemma B.4. Again, we will use (φ_m) and φ for $(\varphi_{a,b,R,m})$ and $\varphi_{a,b,R}$ respectively.

By Lemma B.1, the inequality $|\varphi_m| \leq 1$ holds for all m. The inequality $|\varphi| \leq 1$ then follows from the uniform convergence of (φ_m) stated in Lemma B.4. By the convergence of (φ_m) in $H^2_{\text{loc}}(S_R)$ and $C^{0,\alpha}_{\text{loc}}(S_R)$, we get (B.27) from

$$-b(\nabla - i\sigma A_0)^2 \varphi_m = (1 - |\varphi_m|^2)\varphi_m.$$

Now we prove that $\varphi \in \mathscr{D}_R$. Pick an arbitrary integer $m_0 \geq 1$. For all $m \geq m_0$, $S_{R,m_0} \subset S_{R,m}$. Thus using the decay of φ_m in (B.19) we have

$$\int_{S_{R,m_0}} |\varphi_m|^2 \, dx \leq \int_{S_{R,m}} |\varphi_m|^2 \, dx \leq CbR.$$

The uniform convergence of (φ_m) to φ gives us

$$\int_{S_{R,m_0}} |\varphi|^2 \, dx = \lim_{m \to +\infty} \int_{S_{R,m_0}} |\varphi_m|^2 \, dx \leq CbR.$$

Taking $m_0 \to +\infty$, we write by the monotone convergence theorem,

$$\int_{S_R} |\varphi|^2 \, dx \leq CbR.$$

This proves that $\varphi \in L^2(S_R)$. Next we will prove that $(\nabla - i\sigma A_0)\varphi \in L^2(S_R)$. In light of the convergence of (φ_m) in $H^1_{\text{loc}}(S_R)$, we can refine the subsequence (φ_m) so that

$$(\nabla - i\sigma A_0)\varphi_m \to (\nabla - i\sigma A_0)\varphi \text{ a.e.}$$

Furthermore, by Lemma B.3, (φ_m) is bounded in $C^1_{\text{loc}}(S_R)$, hence in $C^1(S_{R,m_0})$, for all $m_0 \geq 1$. Using the dominated convergence theorem and the estimate in (B.19), we may write, for all $m_0 \geq 1$,

$$\int_{S_{R,m_0}} |(\nabla - i\sigma A_0)\varphi|^2 \, dx = \lim_{m \to +\infty} \int_{S_{R,m_0}} |(\nabla - i\sigma A_0)\varphi_m|^2 \, dx \leq CR.$$

Sending m_0 to $+\infty$ and using the monotone convergence theorem, we get

$$\int_{S_R} |(\nabla - i\sigma A_0)\varphi|^2 \, dx \leq CR.$$

Thus, we have proven that $\varphi, (\nabla - i\sigma A_0)\varphi \in L^2(S_R)$. It remains to prove that φ satisfies the boundary condition

$$\varphi\left(x_1 = \pm\frac{R}{2}, x_2\right) = 0, \qquad \text{for all } x_2 \in \mathbb{R}.$$

To see this, let $x_2 \in \mathbb{R}$. There exists m_0 such that $x_2 \in (-m_0, m_0)$. By the convergence of (φ_m) to φ in $C^{0,\alpha}(\overline{S_{R,m_0}})$, we get

$$\varphi\left(x_1 = \pm\frac{R}{2}, x_2\right) = \lim_{m\to+\infty} \varphi_m\left(x_1 = \pm\frac{R}{2}, x_2\right) = 0.$$

Finally, we may use similar limiting arguments to pass from the decay estimates of φ_m in (B.5) and (B.6) to the decay estimates of φ in (B.28) and (B.29). $\qquad\square$

Now, we are ready to establish the existence of a minimizer of the Ginzburg–Landau energy $\mathcal{G}(a, b, R)$ defined in the unbounded set S_R.

Lemma B.6. *Let $R > 1$. The function $\varphi_{a,b,R} \in \mathcal{D}_R$ defined in Lemma B.4 is a minimizer of $\mathcal{G}_{a,b,R}$, that is*

$$\mathcal{G}_{a,b,R}(\varphi_{a,b,R}) = g_a(b, R).$$

Here $\mathcal{G}_{a,b,R}$ is the functional introduced in (3.3) and $g_a(b, R)$ is the ground state energy defined in (3.4).

Proof. The proof is divided into three steps.

Step 1 (Convergence of the ground state energy). Let $g_a(b, R, m)$ and $g_a(b, R)$ be the energies defined in (B.3) and (3.4) respectively. In this step, we will prove that

$$\lim_{m\to+\infty} g_a(b, R, m) = g_a(b, R). \tag{B.31}$$

Let $u \in \mathcal{D}_{R,m}$. We can extend u by 0 to a function $\tilde{u} \in \mathcal{D}_R$. Consequently, we get $g_a(b, R, m) \geq g_a(b, R)$, for all $m \geq 1$. Thus, $\liminf_{m\to+\infty} g_a(b, R, m) \geq g_a(b, R)$. Next, we will prove that

$$\limsup_{m\to+\infty} g_a(b, R, m) \leq g_a(b, R). \tag{B.32}$$

Consider $(\varphi_n) \subset \mathcal{D}_R$ a minimizing sequence of $\mathcal{G}_{a,b,R}$, that is $g_a(b, R)$ is the limit of $\mathcal{G}_{a,b,R}(\varphi_n)$ as $n \to +\infty$. Let $\vartheta \in C_c^\infty(\mathbb{R})$ be a cut-off function satisfying

$$0 \leq \vartheta \leq 1 \text{ in } \mathbb{R}, \quad \operatorname{supp}\vartheta \subset (-1, 1), \quad \vartheta = 1 \text{ in } \left[-\frac{1}{2}, \frac{1}{2}\right].$$

Consider the re-scaled function $\vartheta_m(x_2) = \vartheta(x_2/m)$. The function $\vartheta_m(x_2)\varphi_n(x)$ restricted to $S_{R,m}$ belongs to $\mathcal{D}_{R,m}$ and consequently

$$g_a(b, R, m) \leq \mathcal{G}_{a,b,R}(\vartheta_m\varphi_n). \tag{B.33}$$

By Cauchy's inequality, for all $\varepsilon \in (0,1)$

$$\left|(\nabla - i\sigma\mathbf{A}_0)\vartheta_m\varphi_n\right|^2 \le (1+\varepsilon)\left|\vartheta_m(\nabla - i\sigma\mathbf{A}_0)\varphi_n\right|^2 + 2\varepsilon^{-1}\left|\nabla\vartheta_m\right|^2\left|\varphi_n\right|^2.$$

Thus, using the definition of the ground state energy $g_a(b, R, m)$ and the functional $\mathcal{G}_{a,b,R}$ in (B.3) and (3.3) respectively, we obtain

$$g_a(b, R, m) \le (1+\varepsilon)\mathcal{G}_{a,b,R}(\varphi_n) + \frac{2b\varepsilon^{-1}}{m^2}\|\vartheta'\|^2_{L^\infty(\mathbb{R})}\int_{S_R}|\varphi_n|^2\,dx$$

$$+ \int_{S_R}(1 - \vartheta_m^2 + \varepsilon)|\varphi_n|^2\,dx. \quad \text{(B.34)}$$

Introducing $\limsup\limits_{m\to+\infty}$ on both sides of (B.34), and using the dominated convergence theorem, we get

$$\limsup_{m\to+\infty} g_a(b, R, m) \le (1+\varepsilon)\mathcal{G}_{a,b,R}(\varphi_n) + \varepsilon\int_{S_R}|\varphi_n|^2\,dx.$$

Taking the successive limits as $\varepsilon \to 0_+$ then $n \to +\infty$, we get (B.32).

Step 2 (The L^4-norm of the limit function). Let $(\varphi_m = \varphi_{a,b,R,m})$ be the sequence in Lemma B.4 which converges to the function $\varphi = \varphi_{a,b,R}$. We would like to verify that the limit function φ is a minimizer of the functional $\mathcal{G}_{a,b,R}$. To that end, we will prove first that

$$\lim_{m\to+\infty}\int_{S_{R,m}}|\varphi_m|^4\,dx = \int_{S_R}|\varphi|^4\,dx. \quad \text{(B.35)}$$

We begin by proving that

$$\liminf_{m\to+\infty}\int_{S_{R,m}}|\varphi_m|^4\,dx \ge \int_{S_R}|\varphi|^4\,dx. \quad \text{(B.36)}$$

Pick a fixed integer $m_0 \ge 1$. Since $S_{R,m} \supset S_{R,m_0}$ for all $m \ge m_0$, the following inequality holds

$$\int_{S_{R,m}}|\varphi_m|^4\,dx \ge \int_{S_{R,m_0}}|\varphi_m|^4\,dx. \quad \text{(B.37)}$$

In addition, having in hand the uniform convergence of φ_m to φ on the compact set S_{R,m_0}, we get as $m \to \infty$

$$\int_{S_{R,m_0}} |\varphi_m|^4 \, dx \to \int_{S_{R,m_0}} |\varphi|^4 \, dx. \tag{B.38}$$

We introduce $\liminf_{m \to +\infty}$ on both sides of (B.37), and we use (B.38) to get

$$\liminf_{m \to +\infty} \int_{S_{R,m}} |\varphi_m|^4 \, dx \geq \int_{S_{R,m_0}} |\varphi|^4 \, dx.$$

This is true for every integer $m_0 \geq 1$. Consequently (B.36) simply follows by applying the monotone convergence theorem.

Next, we prove that

$$\limsup_{m \to +\infty} \int_{S_{R,m}} |\varphi_m|^4 \, dx \leq \int_{S_R} |\varphi|^4 \, dx. \tag{B.39}$$

Let C be the universal constant in (B.6), $\varepsilon > 0$ be fixed, and $R > 1$ be arbitrary. We select an integer $m_0 \geq 1$ such that

$$\frac{Cb^2 R}{m_0} < \varepsilon. \tag{B.40}$$

In light of (B.38), there exists $m_1 \geq m_0$ such that

$$\forall \, m \geq m_1, \quad \left| \int_{S_{R,m_0}} |\varphi_m|^4 \, dx - \int_{S_{R,m_0}} |\varphi|^4 \, dx \right| \leq \varepsilon.$$

Noticing that $\int_{S_{R,m_0}} |\varphi|^4 \, dx \leq \int_{S_R} |\varphi|^4 \, dx$, we may write for all $m \geq m_1$

$$\int_{S_{R,m_0}} |\varphi_m|^4 \, dx \leq \int_{S_R} |\varphi|^4 \, dx + \varepsilon. \tag{B.41}$$

On the other hand, for $|x_2| \geq m_0 \geq 1$ we have $m_0 \leq \dfrac{|x_2|^3}{\left(\ln |x_2| \right)^2}$. Thus, the estimate in (B.6) yields for all $m \geq m_0$,

$$\int_{S_{R,m} \cap \{|x_2| \geq m_0\}} |\varphi_m|^4 \, dx \leq \underbrace{\frac{Cb^2 R}{m_0}}_{\text{by (B.40)}} < \varepsilon. \tag{B.42}$$

Combining (B.41) and (B.42), we get for all $m \geq m_1 \geq m_0$

$$
\int_{S_{R,m}} |\varphi_m|^4 \, dx = \int_{S_{R,m_0}} |\varphi_m|^4 \, dx + \int_{S_{R,m} \cap \{|x_2| \geq m_0\}} |\varphi_m|^4 \, dx
$$
$$
\leq \int_{S_R} |\varphi|^4 \, dx + 2\varepsilon.
$$

Taking the successive limits as $m \to +\infty$ then $\varepsilon \to 0_+$, we get (B.39).

Step 3 (The limit function is a minimizer). The convergence in (B.35) is crucial in establishing that φ is a minimizer of $\mathcal{G}_{a,b,R}$. In light of Eq. (B.8), an integration by parts yields, for all $m \geq 1$,

$$
g_a(b, R, m) = -\frac{1}{2} \int_{S_{R,m}} |\varphi_m|^4 \, dx.
$$

We take $m \to +\infty$, and we use the results in (B.31) and (B.35). We get

$$
g_a(b, R) = -\frac{1}{2} \int_{S_R} |\varphi|^4 \, dx. \tag{B.43}
$$

By Lemma B.5, $\varphi \in \mathscr{D}_R$ and satisfies (B.27), so after integrating by parts, we get

$$
\mathcal{G}_{a,b,R}(\varphi) = -\frac{1}{2} \int_{S_R} |\varphi|^4 \, dx. \tag{B.44}
$$

Comparing (B.43) and (B.44) yields that $\mathcal{G}_{a,b,R}(\varphi) = g_a(b, R)$. $\qquad\square$

Proof of Proposition 3.4. All the properties stated in Proposition 3.4 (except the non-triviality of the minimizer) are simply a convenient collection in one place of already proven facts in Lemmas B.5 and B.6. With these properties in hand, the non-triviality of $\varphi_{a,b,R}$ follows from Lemma 3.7. $\qquad\square$

Bibliography

[AH07]   Y. Almog and B. Helffer. The distribution of surface superconductivity along the boundary: On a conjecture of X.B. Pan. *SIAM J. Math. Anal.*, 38(6):1715–1732, 2007.

[AK16]   W. Assaad and A. Kachmar. The influence of magnetic steps on bulk superconductivity. *Discrete Contin. Dyn. Syst. Ser. A*, 36:6623–6643, 2016.

[Ass19]   Wafaa Assaad. The breakdown of superconductivity in the presence of magnetic steps. 2019.

[Att15a] K. Attar. Energy and vorticity of the Ginzburg–Landau model with variable magnetic field. *Asymptot. Anal.*, 93(1-2):75–114, 2015.

[Att15b] K. Attar. The ground state energy of the two dimensional Ginzburg–Landau functional with variable magnetic field. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 32(2):325–345, 2015.

[BNF07]   V. Bonnaillie-Noël and S. Fournais. Superconductivity in domains with corners. *Rev. Math. Phys.*, 19(06):607–637, 2007.

[CDR17]   M. Correggi, B. Devanarayanan, and N. Rougerie. Universal and shape dependent features of surface superconductivity. *Eur. Phys. J. B*, 90(11):231, 2017.

[CFKS09] H.L. Cycon, R.G. Froese, W. Kirsch, and B. Simon. *Schrödinger Operators: With Application to Quantum Mechanics and Global Geometry*. Springer, 2009.

[CG17]   M. Correggi and E. L. Giacomelli. Surface superconductivity in presence of corners. *Rev. Math. Phys.*, 29(2):1750005, 2017.

[CR14]   M. Correggi and N. Rougerie. On the Ginzburg–Landau functional in the surface superconductivity regime. *Commun. Math. Phys.*, 332(3):1297–1343, 2014.

[CR16a]   M. Correggi and N. Rougerie. Boundary behavior of the Ginzburg–Landau order parameter in the surface superconductivity regime. *Arch. Ration. Mech. Anal.*, 219(1):553–606, 2016.

[CR16b]   M. Correggi and N. Rougerie. Effects of boundary curvature on surface superconductivity. *Lett. Math. Phys.*, 106(4):445–467, 2016.

[DH93]   M. Dauge and B. Helffer. Eigenvalues Variation I.: Neumann problem for Sturm–Liouville operators. *J. Differential Equations*, 104(2):243–262, 1993.

[DHS14]   N. Dombrowski, P.D. Hislop, and E. Soccorsi. Edge currents and eigenvalue estimates for magnetic barrier Schrödinger operators. *Asymptot. Anal.*, 89(3-4):331–363, 2014.

[FH05]   S. Fournais and B. Helffer. Energy asymptotics for type II superconductors. *Calc. Var. Partial Differ. Equ.*, 24(3):341–376, 2005.

[FH10]   S. Fournais and B. Helffer. *Spectral Methods in Surface Superconductivity*, volume 77. Springer Science & Business Media, 2010.

[FK11]   S. Fournais and A. Kachmar. Nucleation of bulk superconductivity close to critical magnetic field. *Adv. Math.*, 226(2):1213–1258, 2011.

[FK13]   S. Fournais and A. Kachmar. The ground state energy of the three dimensional Ginzburg–Landau functional. Part I: Bulk regime. *Comm. Partial Differential Equations*, 38(2):339–383, 2013.

[FKP13]   S. Fournais, A. Kachmar, and M. Persson. The ground state energy of the three dimensional Ginzburg–Landau functional. Part II: Surface regime. *J. Math. Pures Appl.*, 99(3):343–374, 2013.

[FLBP94]   C.L. Foden, M.L. Leadbeater, J.H. Burroughes, and M. Pepper. Quantum magnetic confinement in a curved two-dimensional electron gas. *J. Phys.: Condens. Matter*, 6(10):L127, 1994.

[GDMH⁺08]   T.K. Ghosh, A. De Martino, W. Häusler, L. Dell'Anna, and R. Egger. Conductance quantization and snake states in graphene magnetic waveguides. *Phys. Rev. B*, 77(8):081404, 2008.

[GGD⁺97]   A.K. Geim, I.V. Grigorieva, S.V Dubonos, J.G.S Lok, J.C Maan, A.E. Filippov, and Peeters F.M. *Nature (London)*, 390:259, 1997.

[GP99]   T. Giorgi and D. Phillips. The breakdown of superconductivity due to strong fields for the Ginzburg–Landau model. *SIAM J. Math. Anal.*, 30:341–359, 1999.

[HFPS11]   B. Helffer, S. Fournais, and M. Persson-Sundqvist. Superconductivity between H_{C_2} and H_{C_3}. *J. Spectr. Theory*, 1(3):273–298, 2011.

[HK15]   B. Helffer and A. Kachmar. The Ginzburg–Landau functional with vanishing magnetic field. *Arch. Ration. Mech. Anal.*, 218(1):55–122, 2015.

[HK17]   B. Helffer and A. Kachmar. Decay of superconductivity away from the magnetic zero set. *Calc. Var. Partial Differ. Equ.*, 56(5):130, 2017.

[HK18]   B. Helffer and A. Kachmar. The density of superconductivity in domains with corners. *Lett. Math. Phys.*, 108(9):2169–2187, 2018.

[HPRS16]   P.D. Hislop, N. Popoff, N. Raymond, and M. Sundqvist. Band functions in the presence of magnetic steps. *Math. Models Methods Appl. Sci.*, 26(1):161–184, 2016.

[HS08]   P.D. Hislop and E. Soccorsi. Edge currents for quantum Hall systems I: One-edge, unbounded geometries. *Rev. Math. Phys.*, 20(01):71–115, 2008.

[HS15]   P.D. Hislop and E. Soccorsi. Edge states induced by Iwatsuka Hamiltonians with positive magnetic fields. *J. Math. Anal. Appl.*, 422(1):594–624, 2015.

[Iwa85]   A. Iwatsuka. Examples of absolutely continuous Schrödinger operators in magnetic fields. *Publ. Res. I. Math. Sci.*, 21(2):385–401, 1985.

[JBY+97]   M. Johnson, B.R. Bennett, M.J. Yang, M.M. Miller, and B.V. Shanabrook. Hybrid Hall effect device. *Appl. Phys. Lett.*, 71(7):974–976, 1997.

[Kac]   A. Kachmar. On the perfect superconducting solution for a generalized Ginzburg–Landau equation.

[Kac06]   A. Kachmar. The ground state energy of a magnetic Schrödinger operator and the effect of the de Gennes parameter. *J. Math. Phys.*, 47:1–32, 2006.

[Kat66]   T. Kato. *Perturbation Theory for Linear Operators*. Springer-Verlag, New York, 1966.

[KN16]   A. Kachmar and M. Nasrallah. The distribution of 3D superconductivity near the second critical field. *Nonlinearity*, 29(9):2856–2887, 2016.

[LG50]   L.D. Landau and V.L. Ginzburg. On the theory of superconductivity. *J. Exp. Theor. Phys.*, 20:1064, 1950.

[LM99]   L. Lassoued and P. Mironescu. Ginzburg-landau type energy with discontinuous constraint. *J. Anal. Math.*, 77(1):1–26, 1999.

[LP99]   K. Lu and X.B. Pan. Estimates of the upper critical field for the Ginzburg–Landau equations of superconductivity. *Physica D*, 127(1):73–104, 1999.

[MJR97]   F.G. Monzon, M. Johnson, and M.L. Roukes. Strong Hall voltage modulation in hybrid ferromagnet/semiconductor microstructures. *Appl. Phys. Lett.*, 71(21):3087–3089, 1997.

[NSG+09]   Y.X. Ning, C.L. Song, Z.L. Guan, X.C. Ma, X. Chen, J.F. Jia, and Q.K. Xue. Observation of surface superconductivity and direct vortex imaging of a Pb thin island with a scanning tunneling microscope. *EPL*, 85(2):27004, 2009.

[ORK+08]   L. Oroszlany, P. Rakyta, A. Kormanyos, C.J. Lambert, and J. Cserti. Theory of snake states in graphene. *Phys. Rev. B*, 77(8):081403, 2008.

[Pan02]   X.B. Pan. Surface superconductivity in applied magnetic fields above H_{C_2}. *Comm. Math. Phys.*, 228(2):327–370, 2002.

[PK02]   X.B. Pan and K.H. Kwek. Schrödinger operators with non-degenerately vanishing magnetic fields in bounded domains. *Trans. Amer. Math. Soc.*, 354(10):4201–4227, 2002.

[PM93]   F.M. Peeters and A. Matulis. Quantum structures created by nonhomogeneous magnetic fields. *Phys. Rev. B*, 48(20):15166, 1993.

[RP98]   J. Reijniers and F.M. Peeters. Hybrid ferromagnetic/semiconductor Hall effect device. *Appl. Phys. Lett.*, 73(3):357–359, 1998.

[RP00]   J. Reijniers and F.M. Peeters. Snake orbits and related magnetic edge states. *J. Phys. Condens. Matter*, 12(47):9771, 2000.

[RS72]   M. Reed and B. Simon. *Methods of Modern Mathematical Physics I: Functional Analysis*. Academic Press, New York, 1972.

[SJG63]   D. Saint-James and P.G. de Gennes. Onset of superconductivity in decreasing fields. *Phys. Lett.*, 7(5):306–308, 1963.

[SJST69]   D. Saint-James, G. Sarma, and E. J. Thomas. Type II superconductivity. 1969.

[SS03]   E. Sandier and S. Serfaty. The decrease of bulk superconductivity close to the second critical field in the Ginzburg–Landau model. *SIAM J. Math. Anal.*, 34(4):939–956, 2003.

[SS07]   E. Sandier and S. Serfaty. *Vortices in the Magnetic Ginzburg–Landau Model.* Progress in Nonlinear Partial Differential Equations and their Applications, Birkhäuser-Boston, 2007.

[STH⁺94]   A. Smith, R. Taboryski, L.T. Hansen, C.B. Sørensen, P. Hedegård, and P.E. Lindelof. *Phys. Rev. B*, 50(19):14726, 1994.

Paper III

The breakdown of superconductivity in the presence of magnetic steps

Wafaa Assaad

Abstract

Many earlier works were devoted to the study of the breakdown of superconductivity in type-II superconducting bounded planar domains, submitted to smooth magnetic fields. In the present contribution, we consider a new situation where the applied magnetic field is piecewise-constant, and the discontinuity jump occurs along a smooth curve meeting the boundary transversely. To handle this situation, we perform a detailed spectral analysis of a new effective model. Consequently, we establish the monotonicity of the transition from a superconducting to a normal state. Moreover, we determine the location of superconductivity in the sample just before it disappears completely. Interestingly, the study shows similarities with the case of corner domains subjected to constant fields.

1 Introduction

The breakdown of superconductivity in type-II superconductors submitted to a sufficiently strong magnetic field is a celebrated phenomenon in physics [SJG63, LP00, HM01, HP03]. A theorem of Giorgi and Phillips [GP99] asserts that a superconducting sample with Ginzburg–Landau parameter κ, submitted to a constant magnetic field of strength H, permanently passes to the normal state when H exceeds some critical value. An important question in the literature has been to establish that the transition from the superconducting to the normal state is *monotone*, i.e. to prove that the sample is superconducting for all H less than the aforementioned critical value.

Such a monotonicity has been established in several geometric situations both in 2- and 3-dimensional settings in the case where the Ginzburg–Landau parameter

is big, and for large classes of smooth magnetic fields [FH06, FH07, FH09, Ray09, FP11, DR13]. In particular, the analysis of 2-dimensional domains with smooth boundary, submitted to uniform fields shows that the problem is related to a purely linear eigenvalue problem [FH06, FH07]. The case of corner domains was treated in [Bon05, BND06, BNF07].

However, a monotone transition is not guaranteed in general, and an *oscillatory* behavior occurs in certain geometric settings. One famous example is the Little–Parks effect for 2D annuli [LP62, Erd97, FPS15], where the topology of the sample causes the lack of monotonicity. Other examples of this oscillation effect were provided in [FPS15], in a case of a disc-shaped sample placed in a non-uniform magnetic field.

In the present paper, we focus on the case of a smooth domain placed in a *discontinuous* magnetic field. More precisely, we consider a long cylindrical superconducting domain with smooth cross-section, submitted to a magnetic field with direction parallel to the axis of the cylinder and whose profile is a step function. Such a case was not treated in the aforementioned literature. We mainly aim at answering the following questions:

- *Question 1.* How does the discontinuity of the magnetic field affect the monotonicity of the transition from the superconducting to the normal state?

- *Question 2.* Where is superconductivity localized right before it completely disappears from the sample?

As shown later in this article, the answers to these questions generate an interesting comparison between the case that we handle and another known case of corner domains submitted to constant magnetic fields (see Section 1.3).

1.1 The functional and the assumptions

Consider an open, bounded, and simply connected set Ω of $\mathbb{R}^2$. Assume that Ω is the horizontal cross-section of a long wire subjected to a magnetic field, whose profile is the function $B_0 \colon \Omega \to [-1, 1]$ and whose intensity is $H > 0$. The

Ginzburg–Landau (GL) free energy is given by the functional

$$\mathcal{E}_{\kappa,H}(\psi, \mathbf{A}) = \int_{\Omega} \left(\left| (\nabla - i\kappa H\mathbf{A})\psi \right|^2 - \kappa^2 |\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 \right) dx$$
$$+ \kappa^2 H^2 \int_{\Omega} \left| \operatorname{curl} \mathbf{A} - B_0 \right|^2 dx, \quad (1.1)$$

with $\psi \in H^1(\Omega; \mathbb{C})$ and $\mathbf{A} \in H^1(\Omega; \mathbb{R}^2)$. In physics, $\kappa > 0$ is a characteristic scale of the sample called the GL parameter, ψ is the order parameter with $|\psi|^2$ being a measure of the density of Cooper pairs, and $\mathbf{A}$ is the vector potential whose curl represents the induced magnetic field in the sample.

We carry out our analysis in the asymptotic regime $\kappa \to +\infty$, which corresponds in physics to extreme type-II superconductors. We work under the following assumptions on the domain Ω and the magnetic field B_0 (see Figure 1):

Assumption 1.1.

1. Ω_1 and Ω_2 are two disjoint open sets.

2. Ω_1 and Ω_2 have a finite number of connected components.

3. $\partial\Omega_1$ and $\partial\Omega_2$ are piecewise-smooth with a finite number of corners.

4. $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ is the union of a finite number of disjoint simple smooth curves $\{\Gamma_k\}_{k \in \mathcal{K}}$; we will refer to Γ as the magnetic barrier.

5. $\Omega = (\Omega_1 \cup \Omega_2 \cup \Gamma)^{\circ}$ and $\partial\Omega$ is smooth.

6. For any $k \in \mathcal{K}$, Γ_k intersects $\partial\Omega$ at two distinct points. This intersection is transversal, i.e. $T_{\partial\Omega} \times T_{\Gamma_k} \neq 0$ at the intersection point, where $T_{\partial\Omega}$ and T_{Γ_k} are respectively unit tangent vectors of $\partial\Omega$ and Γ_k.

7. $B_0 = \mathbb{1}_{\Omega_1} + a\mathbb{1}_{\Omega_2}$, where $a \in [-1, 1)\backslash\{0\}$ is a given constant.
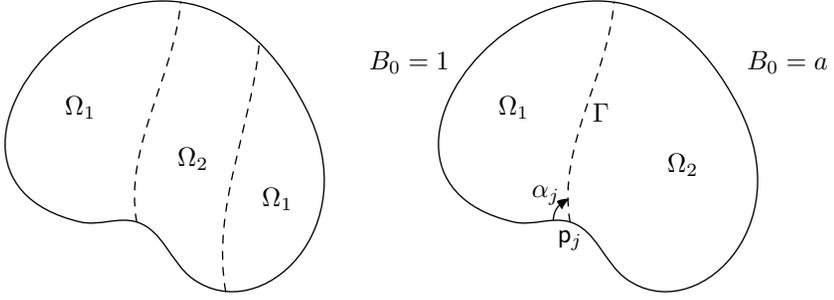
Figure 1: Schematic representation of the set Ω subjected to the piecewise-constant magnetic field B_0, with the magnetic barrier Γ.

Notation 1.2. *Since* $\Gamma \cap \partial\Omega$ *is finite, we denote by*

$$\Gamma \cap \partial\Omega = \{\mathsf{p}_j \,:\, j \in \{1, ..., n\}\},$$

where $n = \mathrm{Card}(\Gamma \cap \partial\Omega)$. *For all* $j \in \{1, ..., n\}$, *let* $\alpha_j \in (0, \pi)$ *be the angle between* Γ *and* $\partial\Omega$ *at the intersection point* p_j *(measured towards* Ω_1).

Since the functional in (1.1) is gauge invariant[1], one may restrict its minimization with respect to $(\psi, \mathbf{A})$ (originally done in $H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2)$) to the space $H^1(\Omega; \mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$, where

$$H^1_{\mathrm{div}}(\Omega) = \{\mathbf{A} \in H^1(\Omega; \mathbb{R}^2) \,:\, \mathrm{div}\,\mathbf{A} = 0 \text{ in } \Omega, \ \mathbf{A} \cdot \nu = 0 \text{ on } \partial\Omega\} \qquad (1.2)$$

and ν is a unit normal vector of $\partial\Omega$. This restriction is beneficial due to the nice regularity properties of the space $H^1_{\mathrm{div}}(\Omega)$ (see [AK16, Appendix B]). Hence, we introduce the following ground-state energy:

$$\mathrm{E}_{\mathrm{g.st}}(\kappa, H) = \inf\{\mathscr{E}_{\kappa, H}(\psi, \mathbf{A}) \,:\, (\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)\}. \qquad (1.3)$$

Critical points $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$ of $\mathscr{E}_{\kappa, H}$ are weak solutions of the following GL equations:

$$\begin{cases} (\nabla - i\kappa H\mathbf{A})^2 \psi = \kappa^2(|\psi|^2 - 1)\psi & \text{in } \Omega, \\ -\nabla^\perp(\mathrm{curl}\,\mathbf{A} - B_0) = \frac{1}{\kappa H}\,\mathrm{Im}\left(\overline{\psi}(\nabla - i\kappa H\mathbf{A})\psi\right) & \text{in } \Omega, \\ \nu \cdot (\nabla - i\kappa H\mathbf{A})\psi = 0 & \text{on } \partial\Omega, \\ \mathrm{curl}\,\mathbf{A} = B_0 & \text{on } \partial\Omega. \end{cases} \qquad (1.4)$$

---

[1]The physically meaningful quantities $|\psi|^2$, $\mathrm{curl}\,\mathbf{A}$ and $|(\nabla - i\kappa H\mathbf{A})\psi|^2$ are gauge invariant in the sense that they do not change under the transformation $(\psi, \mathbf{A}) \mapsto (e^{i\kappa H\varphi}\psi, \mathbf{A} + \nabla\varphi)$ for any $\varphi \in H^2(\Omega; \mathbb{R})$.

Here, $\nabla^{\perp} = (\partial_{x_2}, -\partial_{x_1})$.

1.2 Critical fields

Let $\mathbf{F} \in H^1_{\mathrm{div}}(\Omega)$ be the unique vector potential generating the *step* magnetic field B_0 (see (5.1)). For large κ, a result à la Giorgi–Phillips (Section 5) asserts that for sufficiently strong magnetic fields, H, the only solution of (1.4) is the normal state $(0, \mathbf{F})$. We want to prove the existence of a *unique* field where the transition to the normal state happens. To be consistent with the literature, we call this field *the third critical field* and denote it by $H_{C_3}(\kappa)$.

As mentioned, such a uniqueness result has been proved in many generic situations [FH06, FH07, FH09, Ray09, FP11, DR13]. In their analysis of constant magnetic fields, Fournais and Helffer [FH06, FH10] introduced several natural critical fields, called *global* and *local* fields: a monotone transition requires the global fields to coincide. To prove the equality of these fields (for large κ), Fournais and Helffer linked these global fields to local fields involving spectral data of a linear problem.

We adapt the definitions of the critical fields in [FH10] to our situation of a step magnetic field. For large κ, we consider the global fields:

$$\overline{H}_{C_3}(\kappa) = \inf\{H > 0 \,:\, \text{for all } H' > H, (0, \mathbf{F}) \text{ is the only minimizer of } \mathcal{E}_{\kappa,H'}\},$$
(1.5)

$$\underline{H}_{C_3}(\kappa) = \inf\{H > 0 \,:\, (0, \mathbf{F}) \text{ is the only minimizer of } \mathcal{E}_{\kappa,H}\}. \tag{1.6}$$

The latter field was first introduced by Lu and Pan [LP99]. We consider also the local fields:

$$\overline{H}^{\mathrm{loc}}_{C_3}(\kappa) = \inf\{H > 0 \,:\, \text{for all } H' > H, \ \lambda(\kappa H') \geq \kappa^2\}, \tag{1.7}$$

$$\underline{H}^{\mathrm{loc}}_{C_3}(\kappa) = \inf\{H > 0 \,:\, \lambda(\kappa H) \geq \kappa^2\}, \tag{1.8}$$

where $\lambda(\kappa H)$ stands for the ground-state energy of a Schrödinger operator with a step magnetic field, defined in Section 4.1. The equality between $\overline{H}^{\mathrm{loc}}_{C_3}(\kappa)$ and $\underline{H}^{\mathrm{loc}}_{C_3}(\kappa)$—and consequently between $\overline{H}_{C_3}(\kappa)$ and $\underline{H}_{C_3}(\kappa)$—depends on whether the function $b \mapsto \lambda(b)$ is monotone increasing for large b, a property that has been called 'strong diamagnetism'. In the settings of this paper, we prove this property in Section 6.

1.3 Main results

We present now our main results: Theorem 1.5 answers Question 1 in the introduction by establishing the existence and the uniqueness of the third critical field, for large κ, and providing asymptotics of this field. Question 2 is answered in Theorem 1.6, where we establish certain Agmon-type estimates that make precise the zone of nucleation of superconductivity before disappearing from the sample, and show that the size of this zone is of order κ^{-2}.

These results involve the following spectral quantities:

- $\Theta_0 \approx 0.59$ is the so-called de Gennes constant, introduced in Section 2.1 as the ground-state energy of the Neumann realization of the operator P_{1,U_π} in the half-space.

- $\mu(\alpha, a)$ is the ground-state energy of the Neumann realization of a Schrödinger operator with a step magnetic field in $\mathbb{R}^2_+$, introduced in Section 3.

The main theorems, namely Theorems 1.5 and 1.6, are established under the following additional assumption:

Assumption 1.3. *Suppose that Assumption 1.1 holds. For $j \in \{1, ..., n\}$, let α_j be the angle in Notation 1.2. We assume that $\mu(\alpha_j, a) < |a|\Theta_0$.*

We will discuss the conditions in this assumption later in the paper (see Section 1.4).

Remark 1.4. In Section 3.3, we provide particular examples of pairs (α_j, a) for which this assumption is satisfied.

Theorem 1.5. *There exists $\kappa_0 > 0$ such that if $\kappa \geq \kappa_0$ and $\lambda(\cdot)$ is as in (4.4), then the equation*

$$\lambda(\kappa H) = \kappa^2$$

admits a unique solution $H = H_{C_3}(\kappa)$ which can be estimated as follows:

$$H_{C_3}(\kappa) = \frac{\kappa}{\min\limits_{j \in \{1, ..., n\}} \mu(\alpha_j, a)} + \mathcal{O}(\kappa^{\frac{1}{2}}), \quad \text{as } \kappa \to +\infty. \tag{1.9}$$

Furthermore for $\kappa \geq \kappa_0$, the critical fields defined in (1.5) and (1.6) coincide and satisfy

$$\overline{H}_{C_3}(\kappa) = \underline{H}_{C_3}(\kappa) = H_{C_3}(\kappa).$$

It is worth comparing the asymptotics of the third critical field in Theorem 1.5 with these established in the literature, for smooth domains or corner domains submitted to uniform magnetic fields. In bounded planar domains with smooth boundary, the third critical field has the following asymptotics as κ tends to $+\infty$ [LP99, HM01, HP03, FH06, FH07]:

$$H_{C_3}^{\mathrm{unif}}(\kappa) = \frac{\kappa}{B\,\Theta_0} + o(\kappa),$$

when the applied field has a constant (positive) value B. In corner domains, a richer physics is produced for stronger applied magnetic fields, since the corners allow superconductivity to survive longer in the regime $\kappa/(B\,\Theta_0) \le H < H_{C_3}^{\mathrm{cor}}(\kappa)$, where B is the constant field and $H_{C_3}^{\mathrm{cor}}(\kappa)$ is the third critical field in the corner situation. More precisely, the following asymptotics were established in certain geometric settings [Bon05, BND06, BNF07]:

$$H_{C_3}^{\mathrm{cor}}(\kappa) = \frac{\kappa}{B\,\Lambda} + o(\kappa), \tag{1.10}$$

where Λ is the ground-state energy of the infinite sector operator with opening angle α, introduced in Section 2.1, and α is the angle corresponding to the corners with the smallest such a ground-state energy. The result has been established under the assumption that α fulfils $\Lambda < \Theta_0$, which is known to be true for the opening angles $\alpha \in (0, \alpha_0)$, $\alpha_0 \approx 0.595\pi$ (see Section 2.1).

Theorem 1.5 shows a similarity between the situation in the present paper and that in the corner domains submitted to uniform fields. In the former situation, the magnetic field, having a jump discontinuity along a curve that *cuts the boundary*, has enlarged the scope of the field's strengths where superconductivity still survive in the sample, exactly as the corners do in the latter situation. Indeed, we see that $H_{C_3}(\kappa)$ is of the same order but *strictly larger* than $H_{C_3}^{\mathrm{unif}}(\kappa)$, where $H_{C_3}^{\mathrm{unif}}(\kappa)$ corresponds to the constant field $B = |a|$.

Our next result makes even clearer the similarity between the two aforementioned situations. It is known that the corners attract the Cooper pairs (see e.g. [BNF07, HK18]). Indeed, under certain geometric/spectral conditions [BNF07, Assumption 1.3], some asymptotics of the global energy established in [BNF07] suggest the existence of intermediate phases, between the surface phase and the normal phase, in which superconductivity can be confined to the corners satisfying particular spectral conditions–the energetically favourable corners. Moreover, [BNF07] asserts the nucleation of superconductivity at least at a corner of the domain having the smallest opening angle, before its breakdown.

Recently, the results of [BNF07] have been sharpened in [HK18] where some asymptotics of the *local* energy affirm the confinement of superconductivity to the energetically favourable corners.

In our case, the Cooper pairs can be attracted by the intersection points of the magnetic barrier Γ and the boundary $\partial\Omega$. Indeed, working under the spectral conditions in Assumption 1.3, Theorem 1.6 suggests the following: when $\kappa/(|a|\Theta_0) \leq H < H_{C_3}(\kappa)$, superconductivity can successively nucleate near the intersection points of Γ and $\partial\Omega$, $\{p_j\}_j$, according to the ordering of their spectral parameters $\{\mu(\alpha_j, a)\}_j$. Furthermore, this theorem asserts that superconductivity is eventually localized near at least one of the points p_k admitting the smallest parameter $\mu(\alpha_k, a)$, before vanishing in the entire sample.

Theorem 1.6. *Take $\mu > 0$ satisfying*

$$\min_{j \in \{1,\dots,n\}} \mu(\alpha_j, a) \leq \mu < |a|\Theta_0.$$

We define

$$S = \left\{ p_j \in \Gamma \cap \partial\Omega \ : \ \mu(\alpha_j, a) \leq \mu \right\}.$$

There exist positive constants R_0, κ_0, C and β such that for all $\kappa \geq \kappa_0$, if

$$H \geq \frac{\kappa}{\mu},$$

and $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$ is a solution of (1.4)*, then*

$$\int_\Omega e^{\beta\sqrt{\kappa H}\,\mathrm{dist}(x,S)} \left(|\psi|^2 + \frac{1}{\kappa H}|(\nabla - i\kappa H\mathbf{A})\psi|^2 \right) dx$$

$$\leq C \int_{\{\sqrt{\kappa H}\,\mathrm{dist}(x,S) < R_0\}} |\psi|^2 \, dx. \quad (1.11)$$

This paper is an integral part of the stream of research that was started in [AK16, AKPS19]. Throughout these papers, we present tools for studying the distribution of superconductivity in a smooth domain submitted to a step magnetic field satisfying Assumption 1.1 (the SDSF case), when κ is large, considering various regimes of the intensity of this magnetic field. We particularly aim at detecting any behavior of the sample that is distinct from the well-known behavior of a smooth domain submitted to a uniform magnetic field (the SDUF case) or a corner domain submitted to a uniform field (the CDUF case). Such a distinction is not exhibited in the intensity-regime of [AK16]. However, in the intensity-regime of [AKPS19], the sample's

behavior in the SDSF case is remarkable. It can be dramatically different from the behavior in both the SDUF and CDUF cases. The present paper records another interesting magnetic conduct. In the intensity-regime of this paper, the SDSF case shows analogy to the CDUF case. This analogy is noteworthy, especially when contrasted to the discrepancy between these two cases, observed in [AKPS19].

Below, we summarize our results under the following three intensity-regime scenarios:

- In the intensity-regime $H < \kappa/|a|$: [AK16] establishes the existence of superconductivity in the whole bulk of Ω, and the results of our SDSF case are similar to those of the SDUF and CDUF cases (see e.g. [SS03]).

- In the intensity-regime $\kappa/|a| \leq H \leq \kappa/(|a|\Theta_0)$: [AK16] shows that superconductivity is lost in the bulk of Ω_1 and Ω_2. In [AKPS19], we affirm the nucleation of superconductivity near $\partial\Omega \cup \Gamma$. This nucleation can be *global* (along the entire $\partial\Omega \cup \Gamma$) or *partial* (along certain parts of $\partial\Omega \cup \Gamma$), according to the values of H and $a \in [-1, 1)\backslash\{0\}$ (see [AKPS19, Section 1.5]). This differs from what occurs in a smooth/corners domain, submitted to the uniform magnetic field[2] $B = |a|$ and considered in the same intensity-regime. Indeed, in the latter case, if the boundary is smooth then superconductivity is *exclusively* and *uniformly* localized along this boundary [Pan02, AH07, HFPS11, CR14]. Recently, [CG17] proved that this uniform distribution is not affected (to leading order) by the presence of corners.

- In the intensity-regime $H > \kappa/(|a|\Theta_0)$: the discussion is done under Assumption 1.3. Here, the distribution of superconductivity is dictated by the existence of intersection of the discontinuity curve Γ and the boundary of the sample. Before its breakdown, superconductivity is shown to be confined to the points of $\partial\Omega \cap \Gamma$. As explained in the discussion after Theorems 1.5 and 1.6, the sample's behavior differs in some aspects from that in the SDUF case but shows similarities with that in the CDUF case, when the uniform field is $B = |a|$.

---

[2]We choose the value $|a|$ for the uniform magnetic field just to facilitate the comparison between our SDSF case and the SDUF/CDUF case. Choosing a different value for this field will not qualitatively affect the comparison.

Based on the above observations, the combined results of our three papers highlight the peculiarity of the discontinuous case that we handle: according to the intensity-regime, the SDSF case may resemble to (or deviate from) one or both of the SDUF and CDUF cases. Particularly, the two schematic phase-diagrams in Figure 2 graphically illustrate the comparison between the SDSF case, with the step magnetic field B_0, and the CDUF case, with the uniform field $B = |a|$. These diagrams show the distribution of superconductivity in the sample according to the intensity of the applied magnetic field. In each case, we plot some critical lines in the (κ, H)-plane (for large κ) representing the following:

$$H_{C_2}(\kappa) = \frac{\kappa}{|a|}, \quad H_C^{\mathrm{int}}(\kappa) = \frac{\kappa}{|a|\Theta_0}, \quad H_{C_3}^{\mathrm{step}}(\kappa) = H_{C_3}(\kappa) \text{ in } (1.9),$$

$$\text{and } H_{C_3}^{\mathrm{cor}}(\kappa) \text{ as in } (1.10).$$

In the SDSF diagram, the configurations of the sample between $H_{C_2}(\kappa)$ and $H_C^{\mathrm{int}}(\kappa)$ illustrate different instances of the sample's behavior, occurring according to the values of H and a (see [AKPS19, Section 1.5]).
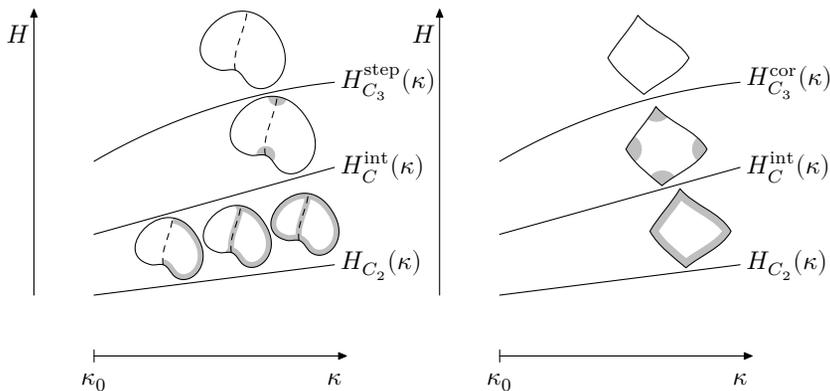
Figure 2: Schematic phase-diagrams: the SDSF case to the left and the CDUF case to the right. Only the grey regions carry superconductivity.

1.4 Heuristic considerations and outline of the approach

The discussion in this section is quite informal and is done under the assumptions stated in the introduction (mainly Assumptions 1.1 and 1.3, and that κ is large). It aims at presenting the workflow in a simple way. Recall that the two principal results are Theorems 1.5 and 1.6.

A sort of Giorgi–Phillips result established in Section 5 asserts that our sample stops superconducting when submitted to large magnetic fields. We aim at proving the following: with increasing values of the applied field, there is a one-way phase-transition between superconducting and normal states; once a superconducting sample passes to the normal state it remains in this state. This goal can be achieved by proving that the global critical fields, $\overline{H}_{C_3}(\kappa)$ and $\underline{H}_{C_3}(\kappa)$, defined in Section 1.2, coincide.

As it is usually the case in the study of breakdown of superconductivity, the equality of the global fields is not directly established. Instead, the analysis is more manageable when these fields are linked to local ones, $\overline{H}_{C_3}^{\mathrm{loc}}(\kappa)$ and $\underline{H}_{C_3}^{\mathrm{loc}}(\kappa)$, also introduced in Section 1.2. These local fields involve the ground-state energy $\lambda(b)$ of the linear Schrödinger operator

$$\mathscr{P}_{b,\mathbf{F}} = -(\nabla - i\,b\mathbf{F})^2$$

defined on Ω with magnetic Neumann boundary conditions (Section 4.1). Here b is a positive parameter, and $\mathbf{F} \in H_{\mathrm{div}}^1(\Omega)$ is the vector potential satisfying curl $\mathbf{F} = \mathbb{1}_{\Omega_1} + a\mathbb{1}_{\Omega_2}$ ($a \in [-1,1)\backslash\{0\}$). The spirit behind linking the four aforementioned critical fields is that, close to the phase of transition from superconducting to normal state, the problem can be viewed as linear. Indeed, when $\psi \approx 0$ and $\mathbf{A} \approx \mathbf{F}$, the first equation in (1.4) can be approximated by

$$-(\nabla - i\,b\mathbf{F})^2\psi = \kappa^2\psi \quad \text{in } \Omega.$$

This approximation of the problem by a linear one is the implicit reason behind establishing that $\overline{H}_{C_3}(\kappa) = \overline{H}_{C_3}^{\mathrm{loc}}(\kappa)$ and $\underline{H}_{C_3}(\kappa) \geq \underline{H}_{C_3}^{\mathrm{loc}}(\kappa)$ (Section 8). Since $\overline{H}_{C_3}(\kappa) \geq \underline{H}_{C_3}(\kappa)$, the equality of the global fields is now equivalent to that of the local ones, which in its turn can be concluded from the fact that the function $b \mapsto \lambda(b)$ is strictly increasing for large values of b. This monotonicity result is proved in Section 6 (see Proposition 6.3), but its main ingredients are prepared in Section 4.

In the aforementioned sections, we generally follow the highways in [HM01, Bon03, Bon05, FH07, FH10] where similar problems are handled in the case of smooth magnetic fields. However, the particularity of the step magnetic field case that we handle causes deviations at several stages of the analysis. Indeed, our discontinuous situation involves particular models while reducing the problem to other effective ones. Furthermore, a careful analysis and additional techniques are

required when working in an environment with a low level of regularity compared to the smooth field case. Some examples showing such a particularity will be presented while continuing this discussion below.

The asymptotic bounds of the ground-state energy $\lambda(b)$ in Theorem 4.2 are key-elements in the monotonicity argument. In the lower bound proof (Section 4.2), a partition of unity allows the local examination of the energy in four main regions of Ω: the interior of Ω away from $\partial\Omega$, the neighbourhood of $\partial\Omega$ away from Γ, the neighbourhood of Γ away from $\partial\Omega$, and the vicinity of the intersection points, p_j, of Γ and $\partial\Omega$.

The study in the first two regions is the same as that in the uniform field case, since the field curl $\mathbf{F}$ is constant in each of the sets Ω_1 and Ω_2. Hence, the results are borrowed from the existing literature (e.g. [FH10]). In these two regions, the energy admits lower bounds of order $|a|b$ and $|a|\Theta_0 b$ respectively.

By suitable change of variables (Sections 4.2 and B.1), gauge transformations and rescaling arguments, we link the study in the two remaining regions to the effective operators with step magnetic fields, $\mathscr{L}_a$ and $\mathscr{H}_{\alpha,a}$, defined on $\mathbb{R}^2$ and $\mathbb{R}^2_+$ respectively. The operator $\mathscr{L}_a$ is introduced in Section 2.2. It has been studied earlier in [HPRS16, AKPS19] (and the references therein), and the following bounds of the corresponding ground-state energy, β_a, were established: $|a|\Theta_0 \leq \beta_a \leq |a|$. The analysis of the operator $\mathscr{H}_{\alpha,a}$ in Section 3 is new. A further comment about this operator is given later in the current section. At the moment we are mainly interested in the upper bound, $\mu(\alpha, a) \leq |a|\Theta_0$, of the ground-state energy of $\mathscr{H}_{\alpha,a}$ (see Remark 3.2). Consequently, we get the following spectral ordering

$$\mu(\alpha, a) \leq |a|\Theta_0 \leq \beta_a \leq |a|,$$

which yields a lower bound of $\lambda(b)$ with leading order $\min_{j\in\{1,\dots,n\}} \mu(\alpha_j, a)b$.

We note that the fulfilment of Assumption 1.3 is not required while establishing the lower bound result. It is while deriving a matching upper bound of the energy that this assumption is useful (see Section 4.2). Indeed, under Assumption 1.3 the energies $\{\mu(\alpha_j, a)\}_j$ are eigenvalues (Remark 3.2). In particular, the minimal energy $\min_j \mu(\alpha_j, a)$ is an eigenvalue. This validates the construction of the trial function involving an eigenfunction corresponding to this minimal energy, in the proof of Proposition 4.7. In the rest of the paper, we work under Assumption 1.3 each time the argument requires the upper bound of $\lambda(b)$.

In addition to the bounds in Section 4.2, certain linear Agmon estimates established in Theorem 6.1 are used to get the monotonicity result in Proposition 6.3.

The proof of this proposition is an adaptation of that in [FH07, Theorem 1.1] to our step field situation. It employs the leading order term of $\lambda(b)$, sparing us the complexity of using higher order expansions of this energy as in e.g. [FH06, BNF07]. However, the discontinuity of our field as well as the way the magnetic field meets the boundary impose more complicated techniques on the argument (see the discussion below Proposition 6.3). Moreover, the proof contains a perturbation argument using the independence of the linear operator domain from the parameter b (see (4.3)). Whereas establishing such an independence is standard in the case of smooth fields, our case requires a particular argument given in Appendix C.

Consequently, we conclude that the value of the equal global and local fields—the third critical field $H_{C_3}(\kappa)$—is the unique solution of the equation $\lambda(\kappa H) = \kappa^2$ (Proposition 6.5). Asymptotic estimates of this field are given in Proposition 6.7. The aforementioned results (in Sections 6 and 8) constitute the proof of Theorem 1.5.

The second main result of this work, namely Theorem 1.6, is established in Section 7. The proof is given under Assumption 1.3 which implies the exclusive nucleation of superconductivity near the points of $\Gamma \cap \partial\Omega$ corresponding to the minimal energy $\min_j \mu(\alpha_j, a)$, right before its breakdown. Lemma 7.1 is essential in the proof. It mainly relies on the local energy estimates in Proposition 4.6, together with a simple, yet important, link between the fields $\mathbf{A}$ and $\mathbf{F}$, done in small patches of the sample (see (7.1)).

The discussion done so far shows the main contribution of the operator $\mathscr{H}_{\alpha,a}$, defined in Section 3, to our problem. We conclude this outline with a brief spectral description of this operator. $\mathscr{H}_{\alpha,a}$ is defined on the half-plane with magnetic Neumann boundary condition, and depends on the two parameters $\alpha \in (0, \pi)$ and $a \in [-1, 1)\backslash\{0\}$. It is reminiscent of the operator $\mathscr{L}_a$ defined on the plane (Section 2.2), since each of them involves a step magnetic field:

$$\operatorname{curl} \mathbf{A}_{\alpha,a}(x) = \mathbb{1}_{D_a^1}(x) + a\mathbb{1}_{D_a^2}(x), \qquad x \in \mathbb{R}_+^2 \quad \text{(for } \mathscr{H}_{\alpha,a}\text{)},$$

$$\operatorname{curl} \sigma\mathbf{A}_0(x) = \mathbb{1}_{\mathbb{R}_+}(x_1) + a\mathbb{1}_{\mathbb{R}_-}(x_1), \qquad x \in \mathbb{R}^2 \quad \text{(for } \mathscr{L}_a\text{)}.$$

However, due to the dependence of $\mathscr{H}_{\alpha,a}$ on the angle α, the study of this operator combines spectral properties of both the operator $\mathscr{L}_a$ and the sector operator with a constant field defined in Section 2.1. Actually, our analysis reveals more spectral similarities with the latter operator. Yet, as it will be shown in Section 3, the discontinuity of the magnetic field in our operator makes the study technically more challenging than that of the sector operator.

By using Persson's lemma in Appendix A, we show that the bottom of the essential spectrum of $\mathcal{H}_{\alpha,a}$ is $|a|\Theta_0$. This implies that the ground-state energy satisfies

$$\mu(\alpha, a) \leq |a|\Theta_0. \tag{1.12}$$

As mentioned earlier in this section, we are interested in the pairs (α, a) for which the inequality in (1.12) is strict and consequently the energy $\mu(\alpha, a)$ is an eigenvalue. The existence of such pairs validates Assumption 1.3 under which this work is done. Let us call here such pairs *admissible pairs*. The pair $(\pi/2, -1)$ is admissible and is directly derived by a symmetry argument (Proposition 3.8).

Certainly, the continuity of $\mu(\alpha, a)$ with respect to the parameters α and a, once verified, would provide more admissible pairs living in a neighbourhood of $(\pi/2, -1)$ (more generally near any already found admissible pair). However, such a regularity result is hard to establish in our case. In fact, a continuity result of $\mu(\alpha, a)$ with respect to a is reached after a lengthy proof in Section 3.2. Still, we did not succeed to prove the continuity with respect to α. The way the operator depends on α prevents making profit of the techniques used in earlier works (e.g. [Bon03, Section 5.3]) in similar situations while studying the sector operator (see discussion in Section 3.2).

The continuity of the energy with respect to a extends the admissibility result at $(\pi/2, -1)$ to other pairs (α, a) for which $\alpha = \pi/2$ (Proposition 3.8). A more complicated (rigorous) computation is done in Proposition 3.9 seeking more admissible pairs, in particular pairs with $\alpha \neq \pi/2$. The proof of this proposition is inspired by the approach in [ELPO18]. It starts with some techniques that facilitate the adoption of such an approach. Then it uses a variational argument with convenient test functions to establish a sufficient condition for a pair (α, a) to be admissible. After this proposition, an illustration using Mathematica is given to show a region of admissible pairs in the vicinity of $(\pi/2, -1)$ (see discussion below the proposition, and Figure 3).

At this point, it is worth comparing our results to those in [ELPO18], in order to highlight the challenges created by the step magnetic field. The argument in [ELPO18] shows the existence of bound states for any sector operator with opening angle α, such that $\alpha \in (0, \alpha_0)$ and $\alpha_0 \approx 0.595\pi$. Similar methods adapted to our situation yield the existence of bound states of the operator $\mathcal{H}_{\alpha,a}$ for distinct values of α. Yet, these values are still near $\pi/2$ (Figure 3). Also note that the corresponding values of a are negative (near -1), and no positive values of a are provided by these methods.

The spectral study of the operator $\mathscr{H}_{\alpha,a}$, that occupies Section 3 (and Appendix A), is an essential contribution of the present article.

1.5 Notation

- The letter C denotes a positive constant whose value may change from one formula to another.

- Let $\beta \in (0, 1)$. We use the following Hölder space:

$$\mathscr{C}^{0,\beta}(\overline{\Omega}) = \left\{ f \in \mathscr{C}(\overline{\Omega}) \mid \sup_{x \neq y \in \Omega} \frac{|f(x) - f(y)|}{|x - y|^{\beta}} < +\infty \right\}.$$

1.6 Organization of the paper

The rest of the paper is divided into seven sections. In Section 2, we summarize some useful properties of certain known 2D model operators. The operator $\mathscr{H}_{\alpha,a}$ is analysed in Section 3. The spectral data of the model operators are used in Section 4 while studying the linear eigenvalue problem. The breakdown of superconductivity under strong magnetic fields is proved in Section 5. In Section 6, we establish the eigenvalue monotonicity result when κ is large. Consequently, we deduce the equality of the local critical fields and provide certain asymptotics of them as κ tends to $+\infty$. The non-linear Agmon estimates in Theorem 1.6 are established in Section 7. Finally, in Section 8 we show the equality of the global and local critical fields for large κ and conclude the result in Theorem 1.5. The appendices gather technical estimates that we use here and there.

2 Some model operators

We present self-adjoint realizations of some Schrödinger operators with magnetic fields in open sets of $\mathbb{R}^2$. A spectral study of these operators can be found in the literature (for instance see [Jad01, Bon03, FH10, AKPS19]).

2.1 Operators with a constant magnetic field

Let U be an open and simply connected domain of $\mathbb{R}^2$. Let $b > 0$, and $\mathbf{A}_0$ be the constant magnetic potential defined by

$$\mathbf{A}_0(x) = (0, x_1) \qquad \left(x = (x_1, x_2) \in \mathbb{R}^2\right). \tag{2.1}$$

If $U = \mathbb{R}^2$, we consider the self-adjoint operator

$$P_{b,\mathbb{R}^2} = -(\nabla - i b \mathbf{A}_0)^2,$$

defined on the domain

$$\mathrm{Dom}\, P_{b,\mathbb{R}^2} = \left\{ u \in L^2(\mathbb{R}^2) \ : \ (\nabla - i b \mathbf{A}_0)^j u \in L^2(\mathbb{R}^2), \text{ for } j \in \{1,2\} \right\}.$$

If $U \subsetneq \mathbb{R}^2$, we assume that ∂U is piecewise-smooth with possibly a finite number of corners. In this case we consider the Neumann realization of the self-adjoint operator

$$P_{b,U} = -(\nabla - i b \mathbf{A}_0)^2,$$

defined on the domain

$$\mathrm{Dom}\, P_{b,U} = \big\{ u \in L^2(U) \ : \ (\nabla - i b \mathbf{A}_0)^j u \in L^2(U),$$
$$\text{for } j \in \{1,2\}, (\nabla - i b \mathbf{A}_0) \cdot \nu|_{\partial U} = 0 \big\},$$

where ν is a unit normal vector of ∂U (when it exists). Let

$$Q_{b,U}(u) = \int_U \left| (\nabla - i b \mathbf{A}_0) u \right|^2 dx$$

be the associated quadratic form defined on

$$\mathrm{Dom}\, Q_{b,U} = \left\{ u \in L^2(U) \ : \ (\nabla - i b \mathbf{A}_0) u \in L^2(U) \right\}.$$

We denote the bottom of the spectrum of $P_{b,U}$ by $\lambda_U(b)$.

The case where U is an angular sector in the plane corresponds to an important *sector* operator. For $0 < \alpha \le \pi$, we define the domain U_α in polar coordinates

$$U_\alpha = \left\{ r(\cos\theta, \sin\theta) \in \mathbb{R}^2 \ : \ r \in (0, \infty),\ 0 < \theta < \alpha \right\}.$$

Using a simple scaling argument, one can prove the following relation between the spectra of the operators P_{b,U_α} and P_{1,U_α}:

$$\text{sp}\, P_{b,U_\alpha} = b\,\text{sp}\, P_{1,U_\alpha}.$$

Therefore, we may restrict to the case $b = 1$ and define

$$\mu(\alpha) = \lambda_{U_\alpha}(1).$$

The special case of $\alpha = \pi$ (the half-plane) has been intensively studied. In this case we denote

$$\Theta_0 := \mu(\pi). \qquad (2.2)$$

Numerical computation shows that $\Theta_0 = 0.5901....$ We note that $\mu(\pi)$ is not an eigenvalue of P_{1,U_π}.

It was conjectured that $\mu(\alpha)$ is an eigenvalue satisfying $\mu(\alpha) < \Theta_0$, for all $\alpha \in (0, \pi)$ (see e.g. [Bon05, Remark 2.4]). This conjecture has been proved for $\alpha \in (0, \alpha_0)$ where $\alpha_0 \approx 0.595\pi$ [Jad01, Bon05, ELPO18]. The validity of the conjecture for all $\alpha \in (0, \pi)$ is still not settled, although numerical evidence suggests it (see [BNDMV07]). When $\mu(\alpha)$ is an eigenvalue, let u_α be a corresponding normalized eigenfunction.

2.2 An operator with a step magnetic field in the plane

Let $a \in [-1, 1)\backslash\{0\}$. For $x \in \mathbb{R}^2$, let σ be a step function defined as follows:

$$\sigma(x) = \mathbb{1}_{\mathbb{R}_+}(x_1) + a\mathbb{1}_{\mathbb{R}_-}(x_1).$$

We introduce the self-adjoint operator

$$\mathscr{L}_a = -(\nabla - i\sigma A_0)^2, \quad \text{with}$$
$$\text{Dom}\,\mathscr{L}_a = \{u \in L^2(\mathbb{R}^2) \,:\, (\nabla - i\sigma A_0)^j u \in L^2(\mathbb{R}^2), \text{ for } j \in \{1,2\}\},$$

and A_0 is the magnetic potential in (2.1). We denote the ground-state energy of $\mathscr{L}_a$ by

$$\beta_a = \inf \text{sp}\,(\mathscr{L}_a).$$

A spectral analysis of the operator $\mathscr{L}_a$ has been done in [HPRS16] and [AKPS19] (see also [Iwa85, HS15] and references therein), and β_a is found to satisfy:

- For $0 < a < 1$, $\beta_a = a$.

- For $a = -1$, $\beta_a = \Theta_0$.

- For $-1 < a < 0$, $|a|\Theta_0 < \beta_a < |a|$.

3 A new operator with a step magnetic field in the half-plane

In this section we introduce a Schrödinger operator with a step magnetic field in $\mathbb{R}_+^2$. To the best of our knowledge, the spectral analysis of this operator is considered for the first time in this contribution. The ground-state energy of this model operator is involved in the leading order of the third critical field $H_{C_3}(\kappa)$, for large values of κ (see Theorem 1.5), and it also appears when to determining the zone of concentration of superconductivity in the sample Ω, for large κ and for sufficiently strong magnetic fields (see Theorem 1.6).

Let $a \in [-1, 1) \backslash \{0\}$ and $\alpha \in (0, \pi)$. We define the sets D_α^1 and D_α^2 in polar coordinates as follows:

$$D_\alpha^1 = \{r(\cos\theta, \sin\theta) \in \mathbb{R}^2 \ : \ r \in (0, \infty), \ 0 < \theta < \alpha\},$$
$$D_\alpha^2 = \{r(\cos\theta, \sin\theta) \in \mathbb{R}^2 \ : \ r \in (0, \infty), \ \alpha < \theta < \pi\}. \tag{3.1}$$

Consider in $\mathbb{R}_+^2$ the Neumann realization of the operator

$$\mathcal{H}_{\alpha,a} = -\left(\nabla - i\mathbf{A}_{\alpha,a}\right)^2, \tag{3.2}$$

where $\mathbf{A}_{\alpha,a} = \left(0, A_{\alpha,a}\right)$ is the magnetic potential[3] such that:

$$\text{For } \alpha \in (0, \pi/2), \quad A_{\alpha,a}(x_1, x_2) = \begin{cases} x_1 + \frac{a-1}{\tan\alpha}x_2, & \text{if } (x_1, x_2) \in D_\alpha^1, \\ ax_1, & \text{if } (x_1, x_2) \in D_\alpha^2, \end{cases} \tag{3.3}$$

$$\text{for } \alpha \in (\pi/2, \pi), \quad A_{\alpha,a}(x_1, x_2) = \begin{cases} x_1, & \text{if } (x_1, x_2) \in D_\alpha^1, \\ ax_1 + \frac{1-a}{\tan\alpha}x_2, & \text{if } (x_1, x_2) \in D_\alpha^2, \end{cases} \tag{3.4}$$

$$\text{and} \quad A_{\frac{\pi}{2},a}(x_1, x_2) = \begin{cases} x_1, & \text{if } (x_1, x_2) \in D_{\pi/2}^1, \\ ax_1, & \text{if } (x_1, x_2) \in D_{\pi/2}^2. \end{cases} \tag{3.5}$$

The potential $\mathbf{A}_{\alpha,a}$ is in $H^1(\mathbb{R}_+^2, \mathbb{R}^2)$ and satisfies $\text{curl}\,\mathbf{A}_{\alpha,a} = \mathbb{1}_{D_\alpha^1} + a\mathbb{1}_{D_\alpha^2}$. The operator $\mathcal{H}_{\alpha,a}$ is defined on the domain

$$\text{Dom}\,\mathcal{H}_{\alpha,a} = \{u \in L^2(\mathbb{R}_+^2) \ : \ (\nabla - i\mathbf{A}_{\alpha,a})^j u \in L^2(\mathbb{R}_+^2),$$
$$\text{for } j \in \{1, 2\}, (\nabla - i\mathbf{A}_{\alpha,a}) \cdot (0, 1)|_{\partial(\mathbb{R}_+^2)} = 0\}. \tag{3.6}$$

---

[3]One may choose a simpler magnetic potential than $\mathbf{A}_{\alpha,a}$, but the choice in (3.3), (3.4) and (3.5) will prove useful in Section 4 (see Lemma 4.3).

The associated quadratic form, $q_{\alpha,a}$, is defined as

$$q_{\alpha,a}(u) = \int_{\mathbb{R}^2_+} \left| (\nabla - i\mathbf{A}_{\alpha,a})u \right|^2 dx, \quad \text{with} \tag{3.7}$$

$$\operatorname{Dom} q_{\alpha,a} = \left\{ u \in L^2(\mathbb{R}^2_+) \ : \ (\nabla - i\mathbf{A}_{\alpha,a})u \in L^2(\mathbb{R}^2_+) \right\}.$$

Let

$$\mu(\alpha, a) = \inf_{\substack{u \in \operatorname{Dom} q_{\alpha,a} \\ u \neq 0}} \frac{q_{\alpha,a}(u)}{\|u\|^2_{L^2(\mathbb{R}^2_+)}}, \tag{3.8}$$

be the bottom of the spectrum of $\mathcal{H}_{\alpha,a}$.

3.1 Bottom of the essential spectrum

Theorem 3.1. *Let $\alpha \in (0, \pi)$ and $a \in [-1, 1)\backslash\{0\}$. Then $\inf \operatorname{sp}_{\mathrm{ess}}(\mathcal{H}_{\alpha,a}) = |a|\Theta_0$.*

We refer to Appendix A for the proof of Theorem 3.1. Our proof is an adaptation of the corresponding proof for sector operators [Bon03, Section 3], which in turn is a generalization of Persson's lemma for unbounded domains in $\mathbb{R}^2$ and Neumann realizations, and is based on ideas in [Per60, Hel99, Agm14].

Remark 3.2. From Theorem 3.1, it follows that $\mu(\alpha, a) \leq |a|\Theta_0$ for all $\alpha \in (0, \pi)$ and $a \in [-1, 1)\backslash\{0\}$, and if $\mu(\alpha, a) < |a|\Theta_0$, then $\mu(\alpha, a)$ is an eigenvalue of $\mathcal{H}_{\alpha,a}$.

3.2 A continuity result

The operator $\mathcal{H}_{\alpha,a}$ depends on the parameters α and a. Some change of variable techniques have been previously used for other parameter dependent operators (see e.g. [Bon03, Section 5.3]) to link the problem to an operator with a fixed domain, independent of the parameters. This allows the use of the perturbation theory [FH10, Appendix C] to prove certain regularity properties of the ground-state energy. Unfortunately, such techniques may not be useful in our case. This causes difficulties in establishing some smoothness results with respect to α. The aim of this section is to prove the continuity of $\mu(\alpha, a)$ with respect to a.

Proposition 3.3. *Let $\alpha \in (0, \pi)$. The function $a \mapsto \mu(\alpha, a)$ is continuous for $a \in [-1, 1)\backslash\{0\}$.*

The proof of Proposition 3.3 mainly relies on establishing that $\mu(\alpha, a)$ is the limit of another ground-state energy, $\mu(\alpha, a, r)$, of an operator with associated form domain that is independent of α and a. Then, the continuity of $a \mapsto \mu(\alpha, a)$ is deduced from that of $a \mapsto \mu(\alpha, a, r)$. This will be made more precise in what follows. Let $B_r = B(0, r)$ be the ball of radius $r > 0$, and $B_r^+ = B_r \cap \mathbb{R}_+^2$. Define

$$\mathscr{D}_r = \left\{ u \in H^1(B_r^+) \ : \ u = 0 \text{ on } \partial B_r \cap \mathbb{R}_+^2 \right\}. \tag{3.9}$$

Let

$$\mu(\alpha, a, r) = \inf_{\substack{u \in \mathscr{D}_r \\ u \neq 0}} \frac{\left\| (\nabla - i\mathbf{A}_{\alpha,a})u \right\|_{L^2(B_r^+)}}{\left\| u \right\|_{L^2(B_r^+)}}. \tag{3.10}$$

Lemma 3.4. *The function $a \mapsto \mu(\alpha, a, r)$ is continuous.*

The proof of the lemma above is standard, but presented in Appendix A for the convenience of the reader.

Remark 3.5. Note that the form domain $\mathscr{D}_r$ is independent of the parameter a, and that for a fixed function $u \in \mathscr{D}_r$, $a \mapsto \left\| (\nabla - i\mathbf{A}_{\alpha,a})u \right\|_{L^2(B_r^+)}^2$ is holomorphic. Consequently, one can apply the perturbation theory to prove more regularity of $a \mapsto \mu(\alpha, a, r)$ (see [FH10, Appendix C]). However, we will be satisfied by the continuity result of Lemma 3.4 to establish Proposition 3.3.

Remark 3.6. Unfortunately, the perturbation theory might not be helpful in proving the smoothness of $\alpha \mapsto \mu(\alpha, a, r)$, despite of the independence of the domain $\mathscr{D}_r$ from α. Moreover, the continuity of $\alpha \mapsto \mu(\alpha, a, r)$ is not obvious; a technical difficulty comes from the possibility that

$$\liminf_{h \to 0} \sup_{\substack{u \in \mathscr{D}_r \\ \|u\|_{L^2(B_r^+)}=1}} \int_\alpha^{\alpha+h} |u|^2 \, dx > 0$$

which prevents us from comparing the eigenvalues $\mu(\alpha, a, r)$ and $\mu(\alpha + h, a, r)$ using the min-max principle.

The next lemma gives an energy lower bound for the functions in the domain of $q_{\alpha,a}$, supported away from the origin. The proof is also provided in Appendix A.

Consider the set

$$\mathscr{M}_r = \left\{ u \in \text{Dom} \, q_{\alpha,a} \ : \ u = 0 \text{ in } B_r^+ \right\}. \tag{3.11}$$

Lemma 3.7. *Let $\alpha \in (0, \pi)$ and $a \in [-1, 1) \backslash \{0\}$. There exists a constant $C > 0$, independent of a and dependent on α, such that for all $r > 0$ and any non-zero function $u \in \mathcal{M}_r$, it holds*

$$q_{\alpha,a}(u) \geq \left(|a| \Theta_0 - \frac{C}{r^2} \right) \|u\|^2_{L^2(\mathbb{R}^2_+)}.$$

Proof of Proposition 3.3. First, we prove that

$$\lim_{r \to +\infty} \mu(\alpha, a, r) = \mu(\alpha, a). \tag{3.12}$$

By a standard application of the min-max principle, we see that $r \mapsto \mu(\alpha, a, r)$ is decreasing in $\mathbb{R}_+$. Indeed, for $r > 0$, we may extend any $u \in \mathcal{D}_r$ by zero outside B_r (the extension is still denoted by u for simplicity), hence for $\rho > r$ and $u \in \mathcal{D}_r$ we view $u \in \mathcal{D}_\rho$. Consequently, $\lim_{r \to +\infty} \mu(\alpha, a, r)$ exists.

Since $D_r \subset \text{Dom } q_{\alpha,a}$, $\mu(\alpha, a) \leq \lim_{r \to +\infty} \mu(\alpha, a, r)$ is straightforward. It remains to establish $\mu(\alpha, a) \geq \lim_{r \to +\infty} \mu(\alpha, a, r)$. Let $u \in \text{Dom } q_{\alpha,a}$. Consider a smooth cut-off function f_r supported in B_r such that

$$0 \leq f_r \leq 1, \quad f_r = 1 \text{ in } B_{\frac{r}{2}}, \quad \text{and } |\nabla f_r| \leq \frac{C}{r}, \tag{3.13}$$

for some universal constant $C > 0$. We have

$$\|(\nabla - i\mathbf{A}_{\alpha,a}) f_r u\|^2_{L^2(\mathbb{R}^2_+)} = \|f_r (\nabla - i\mathbf{A}_{\alpha,a}) u\|^2_{L^2(\mathbb{R}^2_+)} + \|u|\nabla f_r|\|^2_{L^2(\mathbb{R}^2_+)}$$
$$+ 2 \operatorname{Re} \langle u \nabla f_r, f_r (\nabla - i\mathbf{A}_{\alpha,a}) u \rangle. \tag{3.14}$$

Then by (3.13) and (3.14), we bound $\|f_r (\nabla - i\mathbf{A}_{\alpha,a}) u\|^2_{L^2(\mathbb{R}^2_+)}$ from below by

$$\|(\nabla - i\mathbf{A}_{\alpha,a}) f_r u\|^2_{L^2(\mathbb{R}^2_+)} - \frac{C}{r^2} \|u\|^2_{L^2(\mathbb{R}^2_+)} - \frac{C}{r} \|u\|_{L^2(\mathbb{R}^2_+)} \|f_r (\nabla - i\mathbf{A}_{\alpha,a}) u\|_{L^2(\mathbb{R}^2_+)}$$

which, in turn, by the min-max principle can be bounded below by

$$\mu(\alpha, a, r) \|f_r u\|^2_{L^2(\mathbb{R}^2_+)} - \frac{C}{r^2} \|u\|^2_{L^2(\mathbb{R}^2_+)} - \frac{C}{r} \|u\|_{L^2(\mathbb{R}^2_+)} \|f_r (\nabla - i\mathbf{A}_{\alpha,a}) u\|_{L^2(\mathbb{R}^2_+)}.$$

Hence, having $q_{\alpha,a}(u) \geq \|f_r (\nabla - i\mathbf{A}_{\alpha,a}) u\|^2_{L^2(\mathbb{R}^2_+)}$, we get

$$\frac{q_{\alpha,a}(u)}{\|u\|^2_{L^2(\mathbb{R}^2_+)}} \geq \mu(\alpha, a, r) \frac{\|f_r u\|^2_{L^2(\mathbb{R}^2_+)}}{\|u\|^2_{L^2(\mathbb{R}^2_+)}} - \frac{C}{r^2} - \frac{C}{r} \frac{\|f_r (\nabla - i\mathbf{A}_{\alpha,a}) u\|_{L^2(\mathbb{R}^2_+)}}{\|u\|_{L^2(\mathbb{R}^2_+)}}.$$

Taking r to $+\infty$ and using the dominated convergence theorem, we obtain

$$\frac{q_{\alpha,a}(u)}{\|u\|^2_{L^2(\mathbb{R}^2_+)}} \geq \lim_{r\to+\infty} \mu(\alpha,a,r).$$

Since $u \in \mathrm{Dom}\, q_{\alpha,a}$ is arbitrary, we conclude that $\mu(\alpha,a) \geq \lim_{r\to+\infty} \mu(\alpha,a,r)$.

Next, we establish a useful lower bound of $\mu(\alpha,a)$. Let $r > 0$ and consider a partition of unity, (φ_r, χ_r), of $\mathbb{R}^2$ satisfying

$$\mathrm{supp}\, \varphi_r \subset B_r, \quad \mathrm{supp}\, \chi_r \subset (B_{\frac{r}{2}})^{\complement}, \quad |\nabla\varphi_r|^2 + |\nabla\chi_r|^2 \leq \frac{C}{r^2},$$

for some universal constant $C > 0$. Let $u \in \mathrm{Dom}\, q_{\alpha,a}$ such that $\|u\|_{L^2(\mathbb{R}^2_+)} = 1$. The IMS formula ([CFKS09, Theorem 3.2]) ensures that

$$q_{\alpha,a}(u) \geq q_{\alpha,a}(\varphi_r u) + q_{\alpha,a}(\chi_r u) - \frac{C}{r^2}. \tag{3.15}$$

Note that $\varphi_r u \in \mathscr{D}_r$ and $\chi_r u \in \mathscr{M}_{r/2}$, where $\mathscr{M}_{r/2}$ is defined in (3.11). Thus

$$q_{\alpha,a}(\varphi_r u) + q_{\alpha,a}(\chi_r u) \geq \mu(\alpha,a,r)\|\varphi_r u\|_{L^2(\mathbb{R}^2_+)} + |a|\,\Theta_0 \|\chi_r u\|_{L^2(\mathbb{R}^2_+)} - \frac{C}{r^2}, \tag{3.16}$$

for some C that is independent of a and r. In the above inequality, we used (3.10) and Lemma 3.7. Combining (3.15) and (3.16) gives

$$\mu(\alpha,a) \geq \min\left(\mu(\alpha,a,r), |a|\,\Theta_0\right) - \frac{C}{r^2}, \tag{3.17}$$

for C independent of a and r.

Finally, we establish the continuity of $a \mapsto \mu(\alpha,a)$. Let $r > 0$ and $h \in \mathbb{R}$ such that $a + h \in [-1, 1)\setminus\{0\}$. By (3.12) and the monotonicity of $r \mapsto \mu(\alpha,a,r)$ (see Step 1.), we have

$$\mu(\alpha, a + h) \leq \mu(\alpha, a + h, r).$$

Hence, using the continuity of $a \mapsto \mu(\alpha,a,r)$ (Lemma 3.4) gives

$$\limsup_{h\to 0} \mu(\alpha, a + h) \leq \mu(\alpha, a, r).$$

Let r tend to $+\infty$ and use (3.12) to get $\limsup_{h\to 0} \mu(\alpha, a + h) \leq \mu(\alpha,a)$. Next, by (3.17) we have

$$\mu(\alpha, a + h) \geq \min\left(\mu(\alpha, a + h, r), |a + h|\,\Theta_0\right) - \frac{C}{r^2}. \tag{3.18}$$

But Theorem 3.1 asserts that

$$|a + b|\Theta_0 \geq |a|\Theta_0 - |b|\Theta_0 \geq \mu(\alpha, a) - |b|\Theta_0. \tag{3.19}$$

We plug (3.19) in (3.18) and we insert $\liminf_{b \to 0}$ to obtain

$$\liminf_{b \to 0} \mu(\alpha, a + b) \geq \min\left(\mu(\alpha, a, r), \mu(\alpha, a)\right) - \frac{C}{r^2} \geq \mu(\alpha, a) - \frac{C}{r^2}.$$

In the above inequality, Lemma 3.4 and the monotonicity of $r \mapsto \mu(\alpha, a, r)$ are used again. Take r to $+\infty$ and use (3.12) to conclude that $\liminf_{b \to 0} \mu(\alpha, a + b) \geq \mu(\alpha, a)$. □

3.3 Bound states

In what follows, we provide particular values of α and a where $\mu(\alpha, a)$ is an eigenvalue (see Propositions 3.8 and 3.9), then we conclude with establishing some decay result of the corresponding eigenfunction(s) (see Theorem 3.10).

Proposition 3.8. *There exists $\gamma_0 \in (0, 1)$ such that, for all $a \in [-1, -1 + \gamma_0)$, the bottom of the spectrum of $\mathcal{H}_{\pi/2, a}$, $\mu(\pi/2, a)$, is an eigenvalue.*

Proof. Let $u := u_{\pi/2}$ be a normalized eigenfunction associated with the eigenvalue $\mu(\pi/2)$ introduced in Section 2.1. Consider a function $\hat{u}$ in $\mathbb{R} \times \mathbb{R}_+$ satisfying

$$\hat{u}(x_1, x_2) = \begin{cases} u(x_1, x_2) & \text{if } x_1 > 0, \\ u(-x_1, x_2) & \text{if } x_1 < 0. \end{cases}$$

For $a = -1$, a simple computation yields that $\hat{u} \in \mathrm{Dom}\, q_{\pi/2, -1}$ and satisfies

$$\frac{q_{\frac{\pi}{2}, -1}(\hat{u})}{\|\hat{u}\|_{L^2(\mathbb{R}_+^2)}^2} = \mu\left(\frac{\pi}{2}\right) < \Theta_0$$

(see Section 2.1). Hence, the min-max principle ensures that $\mu(\pi/2, -1) < \Theta_0$, which establishes that this ground-state energy is an eigenvalue (see Remark 3.2). The rest of the proof follows from the continuity of $a \mapsto \mu(\pi/2, a)$ at $a = -1$ (see Proposition 3.3). □

Inspired by the construction in [ELPO18, Proof of Theorem 1.1] in the study of corner domains, we establish a sufficient condition on the angle α and the number a under which $\mu(\alpha, a)$ is an eigenvalue.

Proposition 3.9. *For $\alpha \in (0, \pi)$ and $a \in [-1, 1) \backslash \{0\}$, consider the function $P_{\alpha,a}$: $(0, +\infty) \to \mathbb{R}$ defined by*

$$P_{\alpha,a}(x) = Ax^2 - \frac{\pi}{2}|a|\Theta_0 x + \frac{\pi}{2},$$

with

$$A = \frac{1}{64}\operatorname{csch}(\pi)\Big((1-a)\pi\big((-1+a)\cosh(\pi-2\alpha) + 4\cosh(\pi-\alpha) - 4a\cosh(\alpha)\big)$$
$$+ 8\big(\alpha + a^2(\pi-\alpha)\big)\sinh(\pi) - (3 - 2a + 3a^2)\pi\cosh(\pi) + 4a\pi\Big).$$

If there exists $x = x(\alpha, a) > 0$ such that $P_{\alpha,a}(x) < 0$, then $\mu(\alpha, a)$ is an eigenvalue of the operator $\mathcal{H}_{\alpha,a}$.

Proof. Fix $a \in [-1, 1)\backslash\{0\}$ and $\alpha \in (0, \pi)$. Recall the notation in the introduction of Section 3. There exists a function $\varphi \in H^1_{\mathrm{loc}}(\mathbb{R}^2_+)$ such that the vector potential $\mathbf{A}_{\alpha,a}$ satisfies on $\mathbb{R}^2_+$

$$\mathbf{A}_{\alpha,a} = \check{\sigma}\mathbf{A}_1 + \nabla\varphi,$$

where $\mathbf{A}_1(x) = 1/2(-x_2, x_1)$ (with $x = (x_1, x_2)$) and

$$\check{\sigma}(x) = \begin{cases} 1 & \text{if } x \in D^1_\alpha, \\ a & \text{if } x \in D^2_\alpha \end{cases}$$

[Lei83, Lemma 1.1]. An explicit definition of this function is the following:

$$\text{For } \alpha \in (0, \pi/2], \ \varphi(x) = \begin{cases} \frac{1}{2}x_1 x_2 + \frac{a-1}{2}\cot\alpha \, x_2^2 & \text{if } x \in D^1_\alpha, \\ \frac{a}{2}x_1 x_2 & \text{if } x \in D^2_\alpha, \end{cases}$$

$$\text{For } \alpha \in (\pi/2, \pi), \ \varphi(x) = \begin{cases} \frac{1}{2}x_1 x_2 & \text{if } x \in D^1_\alpha, \\ \frac{a}{2}x_1 x_2 + \frac{1-a}{2}\cot\alpha \, x_2^2 & \text{if } x \in D^2_\alpha. \end{cases}$$

Hence, considering the quadratic form

$$\check{q}(v) = \int_{\mathbb{R}^2_+} \big|(\nabla - i\check{\sigma}\mathbf{A}_1)v\big|^2 \, dx,$$

with domain

$$\operatorname{Dom}\check{q} = \big\{v \in L^2(\mathbb{R}^2_+) \ : \ (\nabla - i\check{\sigma}\mathbf{A}_1)v \in L^2(\mathbb{R}^2_+)\big\},$$

we get for all $v \in \mathrm{Dom}\, \check{q}$

$$\check{q}(v) = q_{\alpha,a}(e^{i\varphi}v). \tag{3.20}$$

The quadratic form $\check{q}$ is expressed in polar coordinates $(\rho, \theta) \in \check{D}^{pol} := (0, +\infty) \times (0, \pi)$ as follows

$$\check{q}^{pol}(v) = \int_0^\pi \int_0^{+\infty} \left(|\partial_\rho v|^2 + \frac{1}{\rho^2}\left|\left(\partial_\theta - i\check{\sigma}^{pol}\frac{\rho^2}{2}\right)v\right|^2 \right)\rho\, d\rho\, d\theta,$$

where $\check{\sigma}^{pol}(\rho, \theta) = \check{\sigma}(x_1, x_2)$ and

$$\mathrm{Dom}\, \check{q}^{pol} = \left\{ v \in L^2_\rho(\check{D}^{pol}) : \partial_\rho v \in L^2_\rho(\check{D}^{pol}), \frac{1}{\rho}\left(\partial_\theta - i\check{\sigma}^{pol}\frac{\rho^2}{2}\right)v \in L^2_\rho(\check{D}^{pol}) \right\}.$$

For any set $D \subset \mathbb{R}^2$, $L^2_\rho(D)$ denotes the weighted space of weight ρ. Just to easily follow the computation steps in [ELPO18], we consider further the quadratic form $\tilde{q}^{pol}$, defined on $\tilde{D}^{pol} := (0, +\infty) \times (-\pi + \alpha, \alpha)$ by

$$\tilde{q}^{pol}(u) = \int_{-\pi+\alpha}^{\alpha} \int_0^{+\infty} \left(|\partial_\rho u|^2 + \frac{1}{\rho^2}\left|\left(\partial_\theta + i\tilde{\sigma}^{pol}\frac{\rho^2}{2}\right)u\right|^2 \right)\rho\, d\rho\, d\theta,$$

where

$$\mathrm{Dom}\, \tilde{q}^{pol} = \left\{ u \in L^2_\rho(\tilde{D}^{pol}) : \partial_\rho u \in L^2_\rho(\tilde{D}^{pol}), \frac{1}{\rho}\left(\partial_\theta + i\tilde{\sigma}^{pol}\frac{\rho^2}{2}\right)u \in L^2_\rho(\tilde{D}^{pol}) \right\},$$

and

$$\tilde{\sigma}^{pol}(\rho, \theta) = \begin{cases} a & \text{if } (\rho, \theta) \in (0, +\infty) \times (-\pi + \alpha, 0), \\ 1 & \text{if } (\rho, \theta) \in (0, +\infty) \times (0, \alpha). \end{cases} \tag{3.21}$$

Performing a suitable symmetry and rotation of domain, we get for all $u \in \mathrm{Dom}\, \tilde{q}^{pol}$

$$\tilde{q}^{pol}(u) = \check{q}^{pol}(v), \tag{3.22}$$

where $v(\rho, \theta) = u(\rho, -\theta + \alpha)$.

In light of the above discussion (more precisely using (3.20) and (3.22)), a sufficient condition for $\mu(\alpha, a)$ to be an eigenvalue is to find a test function $u_* \in \mathrm{Dom}\, \tilde{q}^{pol}$ satisfying

$$\tilde{q}^{pol}(u_*) < |a|\Theta_0 \|u_*\|^2_{L^2_\rho(\tilde{D}^{pol})}. \tag{3.23}$$

This follows from Remark 3.2 and the min-max principle. To this end, we consider the function

$$u_*(\rho, \theta) = e^{-\beta \frac{\rho^2}{2}} e^{-i\rho g(\theta)},$$

where $g: (-\pi + \alpha, \alpha) \to \mathbb{R}$ is a piecewise-differentiable function, $\beta > 0$, g and β to be suitably chosen later. We define the functional $\mathscr{I}$ on $\mathrm{Dom}\,\tilde{q}^{pol}$ by

$$u \mapsto \mathscr{I}[u] = \tilde{q}^{pol}(u) - |a|\Theta_0 \|u\|^2_{L^2_\rho(\tilde{D}^{pol})}.$$

Then establishing (3.23) is equivalent to showing that

$$\mathscr{I}[u_*] < 0. \tag{3.24}$$

An elementary computation yields

$$\begin{aligned}
\mathscr{I}[u_*] = &\int_0^{+\infty} \rho e^{-\beta \rho^2}\, d\rho \int_{-\pi+\alpha}^0 \left(g^2 + (\partial_\theta g)^2 - |a|\Theta_0\right) d\theta \\
&- \int_0^{+\infty} \rho^2 e^{-\beta \rho^2}\, d\rho \int_{-\pi+\alpha}^0 a\partial_\theta g\, d\theta \\
&+ \int_0^{+\infty} \rho e^{-\beta \rho^2}\, d\rho \int_0^\alpha \left(g^2 + (\partial_\theta g)^2 - |a|\Theta_0\right) d\theta \\
&- \int_0^{+\infty} \rho^2 e^{-\beta \rho^2}\, d\rho \int_0^\alpha \partial_\theta g\, d\theta \\
&+ \left(\pi\beta^2 + \frac{1}{4}\left(\alpha + a^2(\pi - \alpha)\right)\right) \int_0^{+\infty} \rho^3 e^{-\beta \rho^2}\, d\rho.
\end{aligned}$$

Let $\mathscr{E}_n = \int_0^{+\infty} \rho^n e^{-\beta \rho^2}\, d\rho$, for $n \geq 0$. We use the equalities $\mathscr{E}_1 = 1/(2\beta)$, $\mathscr{E}_2 = \sqrt{\pi}/(4\beta^{3/2})$, and $\mathscr{E}_3 = 1/(2\beta^2)$ [GR15, Equations 3.461] to conclude that

$$\mathscr{I}[u_*] = \frac{1}{2\beta} \int_{-\pi+\alpha}^0 \left(g^2 + (\partial_\theta g)^2\right) d\theta - a\frac{\sqrt{\pi}}{4\beta^{\frac{3}{2}}} g(\theta)\Big|_{-\pi+\alpha}^0$$
$$+ \frac{1}{2\beta} \int_0^\alpha \left(g^2 + (\partial_\theta g)^2\right) d\theta - \frac{\sqrt{\pi}}{4\beta^{\frac{3}{2}}} g(\theta)\Big|_0^\alpha + \frac{\pi}{2} - \frac{|a|\Theta_0\pi}{2\beta} + \frac{1}{8\beta^2}\left(\alpha + a^2(\pi - \alpha)\right). \tag{3.25}$$

We choose further

$$g(\theta) = \begin{cases} c_1 e^\theta + c_2 e^{-\theta} & \text{if } -\pi + \alpha < \theta \leq 0, \\ c_3 e^\theta + c_4 e^{-\theta} & \text{if } 0 < \theta < \alpha, \end{cases}$$

where c_1, c_2, c_3, c_4 are real coefficients satisfying $c_1 + c_2 = c_3 + c_4$. This condition on the coefficients is imposed to guarantee the continuity of the function g. The choice of g is motivated by a similar one in [ELPO18, Section 2.1], which was optimal within a certain class of test functions. We plug this g into (3.25) and get

$$
\mathscr{I}[u_*] = \frac{(2 - e^{-2\alpha} - e^{-2\pi + 2\alpha})}{2\beta} c_1^2 + \frac{(e^{2\pi - 2\alpha} - e^{-2\alpha})}{2\beta} c_2^2 + \frac{(e^{2\alpha} - e^{-2\alpha})}{2\beta} c_3^2
$$
$$
+ \frac{(1 - e^{-2\alpha})}{\beta} c_1 c_2 + \frac{(-1 + e^{-2\alpha})}{\beta} c_1 c_3 + \frac{(-1 + e^{-2\alpha})}{\beta} c_2 c_3
$$
$$
+ \frac{(1 - a - e^{-\alpha} + a e^{-\pi + \alpha})\sqrt{\pi}}{4\beta^{\frac{3}{2}}} c_1 + \frac{(1 - a - e^{-\alpha} + a e^{\pi - \alpha})\sqrt{\pi}}{4\beta^{\frac{3}{2}}} c_2
$$
$$
+ \frac{(e^{-\alpha} - e^{\alpha})\sqrt{\pi}}{4\beta^{\frac{3}{2}}} c_3 + \frac{4\pi\beta^2 - 4\pi\beta|a|\Theta_0 + a^2(\pi - \alpha) + \alpha}{8\beta^2}.
$$

$\mathscr{I}[u_*]$ is a quadratic expression in c_1, c_2 and c_3. Minimizing $\mathscr{I}[u_*]$ with respect to these coefficients yields a unique solution (c_1, c_2, c_3), where

$$
c_1 = \frac{e^{\pi - 2\alpha}\big((a - 1)e^\pi + (a - 1)e^{\pi + 2\alpha} + 2e^\alpha(e^\pi - a)\big)\sqrt{\pi}\big(-1 + \coth(\pi)\big)}{16\sqrt{\beta}}
$$
$$
c_2 = \frac{\big(a - 1 + (a - 1)e^{2\alpha} + 2(1 - a e^\pi)e^\alpha\big)\sqrt{\pi}\big(-1 + \coth(\pi)\big)}{16\sqrt{\beta}}
$$
$$
c_3 = \frac{e^{-\alpha}\big(-a + e^\pi + (a - 1)\cosh(\pi - \alpha)\big)\sqrt{\pi}\operatorname{csch}(\pi)}{8\sqrt{\beta}}.
$$

We compute the corresponding $\mathscr{I}[u_*]$, taking $x = 1/\beta$. We get $\mathscr{I}[u_*] = P_{\alpha, a}(x)$. This result together with (3.24) complete the proof. $\qquad\square$

Computation. Bonnaillie has established in [BN12] a lower bound, Θ_0^{low}, of Θ_0 equal to $0.590106125 - 10^{-9}$. For each $\alpha \in (0, \pi)$, $a \in [-1, 1)\setminus\{0\}$ and $x > 0$ we set $P_{\alpha, a, \Theta_0^{\text{low}}}(x) = Ax^2 - \pi/2|a|\Theta_0^{\text{low}}x + \pi/2$, for A in Proposition 3.9. then $P_{\alpha, a}(x) \leq P_{\alpha, a, \Theta_0^{\text{low}}}(x)$. Our *rigorous* computation shows that, for all $\alpha \in (0, \pi)$ and $a \in [-1, 1)\setminus\{0\}$, $P_{\alpha, a, \Theta_0^{\text{low}}}(x)$ admits a minimum with respect to x, attained at a positive value $x_0 = x_0(\alpha, a)$. Then, we use Mathematica to plot the region of the pairs (α, a) where $\min_{x>0} P_{\alpha, a, \Theta_0^{\text{low}}}(x) = P_{\alpha, a, \Theta_0^{\text{low}}}(x_0) < 0$. The shaded region
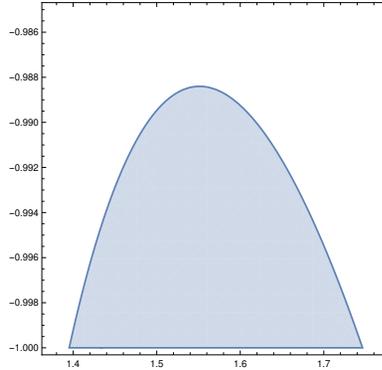
Figure 3: The horizontal axis represents the angles α and the vertical axis represents the values of a. For (α, a) in the shaded region, $\mu(\alpha, a)$ is an eigenvalue.

in Figure 3 represents these pairs. Consequently, the corresponding $P_{\alpha,a}(x_0)$ is negative and the corresponding $\mu(\alpha, a)$ is an eigenvalue.

In the case where $\mu(\alpha, a)$ is the lowest eigenvalue of the operator $\mathscr{H}_{\alpha,a}$, let $v_{\alpha,a}$ be a corresponding normalized eigenfunction. The following theorem reveals a decay of the eigenfunction $v_{\alpha,a}$, for large values of $|x|$. We omit the proof of Theorem 3.10, and we refer for details to the similar proof in [Bon03, Theorem 9.1].

Theorem 3.10. *Let $\alpha \in (0, \pi)$ and $a \in [-1, 1) \backslash \{0\}$. Consider the case where $\mu(\alpha, a)$ is the lowest eigenvalue of the operator $\mathscr{H}_{\alpha,a}$ introduced in (3.2), and let $v_{\alpha,a}$ be a corresponding normalized eigenfunction. For all δ such that $0 < \delta < |a|\Theta_0 - \mu(\alpha, a)$, there exists a constant $C_{\delta,\alpha}$ such that*

$$\|v_{\alpha,a} e^{\phi}\|_{L^2(\mathbb{R}^2_+)} + q_{\alpha,a}(v_{\alpha,a} e^{\phi}) \le C_{\delta,\alpha},$$

where $q_{\alpha,a}$ is the quadratic form in (3.7), and ϕ is a function defined in $\mathbb{R}^2_+$ as follows:

$$\phi(x) = \sqrt{|a|\Theta_0 - \mu(\alpha, a) - \delta} |x|, \qquad \text{for all } x \in \mathbb{R}^2_+.$$

4   The linear problem

4.1   The linear operator

The parameter dependent magnetic Shrödinger operator has been extensively studied in the literature in the case of regular/corner planar domains, submitted to *smooth* magnetic fields (see e.g. [HM96, BS98, LP99, LP00, HM01, Bon05, FH10]).

Let $b > 0$, $\mathbf{E} \in H^1(\Omega, \mathbb{R}^2)$. We consider the Neumann realization of the self-adjoint operator in the domain Ω (satisfying Assumption 1.1):

$$\mathscr{P}_{b,\mathbf{E}} = -(\nabla - i\,b\mathbf{E})^2 \quad \text{with}$$
$$\text{Dom}\,\mathscr{P}_{b,\mathbf{E}} = \big\{ u \in L^2(\Omega) \,:\, (\nabla - i\,b\mathbf{E})^j u \in L^2(\Omega),$$
$$j \in \{1, 2\}, \, (\nabla - i\,b\mathbf{E}) \cdot \nu|_{\partial\Omega} = 0 \big\}, \quad (4.1)$$

where ν is a unit normal vector of $\partial\Omega$. The associated quadratic form is

$$Q_{b,\mathbf{E}}(u) = \int_\Omega \big| (\nabla - i\,b\mathbf{E}) u \big|^2 \, dx \quad \text{with} \tag{4.2}$$
$$\text{Dom}\,Q_{b,\mathbf{E}} = \big\{ u \in L^2(\Omega) \,:\, (\nabla - i\,b\mathbf{E}) u \in L^2(\Omega) \big\}.$$

If $\mathbf{E} = \mathbf{F}$, where $\mathbf{F} \in H^1_{\text{div}}(\Omega)$ *is the magnetic potential in* (5.1) *satisfying* curl $\mathbf{F} = B_0 = \mathbb{1}_{\Omega_1} + a\mathbb{1}_{\Omega_2}$ *for a fixed* $a \in [-1, 1)\backslash\{0\}$, then the operator and the form domains are respectively

$$\text{Dom}\,\mathscr{P}_{b,\mathbf{F}} = \big\{ u \in H^2(\Omega) \,:\, \nabla u \cdot \nu|_{\partial\Omega} = 0 \big\} \quad \text{and} \quad \text{Dom}\,Q_{b,\mathbf{F}} = H^1(\Omega). \tag{4.3}$$

The bottom of the spectrum

$$\lambda(b) = \inf_{\substack{u \in \text{Dom}\,Q_{b,\mathbf{F}} \\ u \neq 0}} \frac{Q_{b,\mathbf{F}}(u)}{\|u\|^2_{L^2(\Omega)}}. \tag{4.4}$$

is an eigenvalue.

Remark 4.1. Compared to smooth magnetic fields cases, an extra argument is required to establish that the domains of $\mathscr{P}_{b,\mathbf{F}}$ and $Q_{b,\mathbf{F}}$ are independent of the parameter b, as in (4.3), in our case of a step magnetic field (curl $\mathbf{F} = B_0$). This argument is given in Appendix C. The domains' independence of b will be crucial while applying the perturbation theory in Proposition 6.3 later.

4.2 Bounds of the ground-state energy

Theorem 4.2. *Under Assumption 1.3, there exist* $b_0, C > 0$ *such that for all* $b \geq b_0$, *we have*

$$-C b^{\frac{3}{4}} \leq \lambda(b) - b \min_{j \in \{1,\dots,n\}} \mu(\alpha_j, a) \leq C b^{\frac{3}{5}}.$$

Note that the error in the upper bound can be improved to be $\mathcal{O}(b^{1-\rho})$, for any $\rho \in (0, 1/2)$ (see Remark 4.8).

The rest of this section is devoted to the proof of Theorem 4.2. More precisely, the lower and upper bounds in this theorem are established in Proposition 4.5 and 4.7 respectively. The same techniques in [HM01] and [Bon05] are used here. We introduce a partition of unity to localize our analysis to different zones in Ω, then we compare our linear operator to an operator with a constant magnetic field in $\mathbb{R}^2$ (if the zone is in $\Omega \setminus \Gamma$), an operator with a constant magnetic field in $\mathbb{R}^2_+$ (if the zone meets the boundary away from Γ), an operator with a step magnetic field in $\mathbb{R}^2$, introduced in Section 2.2 (if the zone meets Γ away from $\partial\Omega$), and finally an operator with a step magnetic field in $\mathbb{R}^2_+$, introduced in Section 3 (if the zone meets $\Gamma \cap \partial\Omega$).

Localization using a partition of unity

Let $0 < \rho < 1$. For $R_0 > 0$, we can find a partition of unity, χ_j, satisfying (when restricted to $\overline{\Omega}$):

$$\sum_j |\chi_j|^2 = 1, \quad \sum_j |\nabla\chi_j|^2 \leq C R_0^{-2} b^{2\rho} \text{ and } \operatorname{supp}(\chi_j) \subset B(z_j, R_0 b^{-\rho}) \text{ s.t.}$$

$$\begin{cases} \text{either} & \operatorname{supp}(\chi_j) \cap (\partial\Omega \cup \Gamma) = \emptyset, \\ \text{or} & z_j \in \partial\Omega \setminus \Gamma \text{ and } \operatorname{supp}(\chi_j) \cap \Gamma = \emptyset, \\ \text{or} & z_j \in \Gamma \setminus \partial\Omega \text{ and } \operatorname{supp}(\chi_j) \cap \partial\Omega = \emptyset, \\ \text{or} & z_j = \mathsf{p}_j, \end{cases} \quad (4.5)$$

where C is independent of R_0 and b. *The index j is chosen such that $z_j = \mathsf{p}_j$, for $j \in \{1, ..., n\}$, where $\mathsf{p}_j \in \Gamma \cap \partial\Omega$.* For $u \in \operatorname{Dom} Q_{b,\mathbf{F}}$, the IMS formula asserts that

$$Q_{b,\mathbf{F}}(u) = \underbrace{\sum Q_{b,\mathbf{F}}(\chi_j u)}_{\text{int}} + \underbrace{\sum Q_{b,\mathbf{F}}(\chi_j u)}_{\text{bnd}}$$
$$+ \underbrace{\sum Q_{b,\mathbf{F}}(\chi_j u)}_{\text{bar}} + \underbrace{\sum Q_{b,\mathbf{F}}(\chi_j u)}_{\text{T}} - \sum_j \left\| |\nabla\chi_j| u \right\|^2_{L^2(\Omega)}, \quad (4.6)$$

where

$$\begin{aligned} \text{int} &:= \{j \; : \; z_j \in \Omega \setminus \Gamma\}, & \text{bnd} &:= \{j \; : \; z_j \in \partial\Omega \setminus \Gamma\}, \\ \text{bar} &:= \{j \; : \; z_j \in \Gamma \setminus \partial\Omega\}, & \text{T} &:= \{j \; : \; j = 1, ..., n\}. \end{aligned}$$

We will optimize later the choice of ρ and R_0 for our various problems.

Change of variables

In order to study the energy contribution near $\Gamma \cap \partial\Omega$, we will carry out the computation in adapted coordinates in this zone. Recall that we are working under Assumption 1.1 (see also Notation 1.2). For $j \in \{1, ..., n\}$, there exist $r_j > 0$ and a local diffeomorphism $\Psi = \Psi_j$ of $\mathbb{R}^2$ satisfying the following (see Appendix B.2):

$$\Psi(\mathsf{p}_j) = (0,0), \qquad |J_\Psi|(\mathsf{p}_j) = |J_{\Psi^{-1}}|(0,0) = 1, \tag{4.7}$$

and there exists a neighbourhood $\mathcal{U}_j$ of $(0,0)$ such that

$$\Psi\big(B(\mathsf{p}_j, r_j) \cap \Omega_1\big) = \mathcal{U}_j \cap D_1^{\alpha_j}, \quad \Psi\big(B(\mathsf{p}_j, r_j) \cap \Omega_2\big) = \mathcal{U}_j \cap D_2^{\alpha_j},$$

and consequently,

$$\Psi\big(B(\mathsf{p}_j, r_j) \cap (\partial\Omega_1 \backslash \Gamma)\big) = \mathcal{U}_j \cap \mathbb{R}_+ \times \{0\},$$

$$\Psi\big(B(\mathsf{p}_j, r_j) \cap (\partial\Omega_2 \backslash \Gamma)\big) = \mathcal{U}_j \cap \mathbb{R}_- \times \{0\},$$

$$\Psi\big(B(\mathsf{p}_j, r_j) \cap \Gamma\big) = \mathcal{U}_j \cap (\hat{x}_2 = \hat{x}_1 \tan \alpha_j).$$

Here, $(\hat{x}_1, \hat{x}_2) := \Psi(x_1, x_2)$, and the sets $D_1^{\alpha_j}$ and $D_2^{\alpha_j}$ were defined in (3.1). We assume further that the radii r_j are sufficiently small so that $\big(B(\mathsf{p}_j, r_j)\big)_{j \in \{1,...,n\}}$ is a family of disjoint balls. The smoothness of Ψ, the fact that $\{1, ..., n\}$ is finite, the assumptions in (4.7) and a Taylor expansion prove the existence of $C > 0$, *independent of j*, such that the Jacobians J_Ψ and $J_{\Psi^{-1}}$ satisfy

$$\big||J_\Psi(x)| - 1\big| \le C\ell \qquad \text{and} \qquad \big||J_{\Psi^{-1}}(\hat{x})| - 1\big| \le C\ell, \tag{4.8}$$

for all $x \in B(\mathsf{p}_j, \ell) \subset B(\mathsf{p}_j, r_j)$ and $\hat{x} = \Psi(x)$. Let $\mathbf{E} = (E_1, E_2) \in H^1(\Omega; \mathbb{R}^2)$ be such that $\mathrm{curl}\, \mathbf{E} = B$, for $B \in L^2(\mathbb{R}^2)$, and let $u \in \mathrm{Dom}\, Q_{b,\mathbf{E}}$ (see (4.2)) such that $\mathrm{supp}\, u \subset B(\mathsf{p}_j, r_j)$. Consider the magnetic potential $\hat{\mathbf{E}} = (\hat{E}_1, \hat{E}_2) \in H^1\big(\Psi(B(\mathsf{p}_j, r_j)) \cap \mathbb{R}_+^2, \mathbb{R}^2\big)$ satisfying $\hat{E}_1\, d\hat{x}_1 + \hat{E}_2\, d\hat{x}_2 = E_1\, dx_1 + E_2\, dx_2$, and the function $\hat{u}$, defined in $\Psi(B(\mathsf{p}_j, r_j)) \cap \mathbb{R}_+^2$ by $\hat{u}(\hat{x}) = u\big(\Psi^{-1}(\hat{x})\big)$. Furthermore, let

$$\hat{B}(\hat{x}) = B\big(\Psi^{-1}(\hat{x})\big), \qquad \text{for all } \hat{x} \in \Psi(B(\mathsf{p}_j, r_j)) \cap \mathbb{R}_+^2.$$

One can check that

$$\operatorname{curl} \hat{\mathbf{E}} = \partial_{\hat{x}_1} \hat{E}_2 - \partial_{\hat{x}_2} \hat{E}_1 = \hat{B} J_{\Psi^{-1}}, \tag{4.9}$$

and

$$Q_{b,\mathbf{E}}(u) = \int_D \sum_{1 \le k,m \le 2} G_{k,m}(\hat{x}) (\partial_{\hat{x}_k} - i b \hat{E}_k) \hat{u}(\hat{x})$$

$$\overline{(\partial_{\hat{x}_m} - i b \hat{E}_m) \hat{u}(\hat{x})} \, |J_{\Psi^{-1}}(\hat{x})| \, d\hat{x}. \tag{4.10}$$

Here $D = \Psi(B(\mathsf{p}_j, r_j)) \cap \mathbb{R}_+^2$ and $G_{k,m}(\hat{x})$ are the elements of the matrix $G(\hat{x}) = (d\Psi)(d\Psi)^t \,_{|\Psi^{-1}(\hat{x})}$.

Note that $G(0,0)$ is the identity matrix. Then, for any $\ell < r_j$, one may apply Taylor's formula in $\Psi(B(\mathsf{p}_j, \ell))$ to prove that

$$|G_{k,m}(\hat{x}) - \delta_{k,m}| \le C\ell, \tag{4.11}$$

for some $C > 0$ independent of j. The following lemma presents a particular transformation, that will allow us to express a given vector field in a canonical manner.

Lemma 4.3. *Let $a \in [-1, 1) \setminus \{0\}$, and $B(0, \ell) \subset \Psi(B(\mathsf{p}_j, r_j))$ be a ball of radius ℓ. Consider the vector potential $\mathbf{F} \in H^1_{\mathrm{div}}(\Omega)$ satisfying $\operatorname{curl} \mathbf{F} = \mathbb{1}_{\Omega_1} + a \mathbb{1}_{\Omega_2}$. There exists a function $\varphi_\ell \in H^2(B(0, \ell) \cap \mathbb{R}_+^2)$ such that the vector potential $\hat{\mathbf{F}}_{\mathrm{g}} := \hat{\mathbf{F}} - \nabla_{\hat{x}_1, \hat{x}_2} \varphi_\ell$ defined in $B(0, \ell) \cap \mathbb{R}_+^2$, satisfies*

$$(\hat{F}_{\mathrm{g}})_1 = 0, \quad (\hat{F}_{\mathrm{g}})_2 = A_{\alpha,a} + f,$$

where $A_{\alpha,a}$ is the potential introduced in (3.2), f is a continuous function satisfying $|f(\hat{x}_1, \hat{x}_2)| \le C(\hat{x}_1^2 + |\hat{x}_1 \hat{x}_2|)$, for some $C > 0$ independent of j.

Proof. Define

$$\varphi_\ell(\hat{x}_1, \hat{x}_2) = \int_0^{\hat{x}_1} \hat{F}_1(\hat{x}_1', \hat{x}_2) \, d\hat{x}_1' + \int_0^{\hat{x}_2} \hat{F}_2(0, \hat{x}_2') \, d\hat{x}_2',$$

for $(\hat{x}_1, \hat{x}_2) \in B(0, \ell) \cap \mathbb{R}_+^2$. Obviously $(\hat{F}_{\mathrm{g}})_1 = 0$. Furthermore, a simple computation using (4.8) and (4.9) yields

$$(\hat{F}_{\mathrm{g}})_2(\hat{x}_1, \hat{x}_2) = \int_0^{\hat{x}_1} (1 + \mathcal{O}(\hat{x}_1)) \hat{B}(\hat{x}_1', \hat{x}_2) \, d\hat{x}_1'.$$

Recalling the definition of $\hat{B}$, we complete the proof. Note that the independence of the constant C from j follows from the fact that the points p_j are finite. $\square$

Lower bound of $\lambda(b)$

In this section we are working under Assumption 1.1 and we use Notation 1.2. We do not require Assumption 1.3 to be fulfilled. Let $u \in \operatorname{Dom} Q_{b,\mathbf{F}}$ (see (4.2)). We use the relation (4.6) to localize the estimates. The error term $\sum_j \big\| |\nabla \chi_j| u \big\|^2_{L^2(\Omega)}$ is estimated using (4.5)

$$\sum_j \big\| |\nabla \chi_j| u \big\|^2_{L^2(\Omega)} \le C R_0^{-2} b^{2\rho} \|u\|^2_{L^2(\Omega)}. \tag{4.12}$$

Recall that the magnetic potential $\mathbf{F} \in H^1_{\mathrm{div}}(\Omega)$ satisfies $\operatorname{curl} \mathbf{F} = B_0 = \mathbb{1}_{\Omega_1} + a\mathbb{1}_{\Omega_2}$.

Estimating $\sum_{\mathrm{int}} Q_{b,\mathbf{F}}(\chi_j u)$.

Let $j \in \mathrm{int}$. Notice that $\chi_j u \in H^1(\Omega)$, and $\chi_j u$ is supported in Ω, then a well-known spectral property (see [FH10, Lemma 1.4.1]) assures that

$$\sum_{\mathrm{int}} Q_{b,\mathbf{F}}(\chi_j u) \ge \sum_{\mathrm{int}} |B_0| b \int_\Omega |\chi_j|^2 |u|^2 \, dx \ge |a| b \sum_{\mathrm{int}} \|\chi_j u\|^2_{L^2(\Omega)}. \tag{4.13}$$

Estimating $\sum_{\mathrm{bnd}} Q_{b,\mathbf{F}}(\chi_j u)$.

Let $j \in \mathrm{bnd}$. Notice that $\operatorname{curl} \mathbf{F}$ is constant in $B(z_j, R_0 b^{-\rho})$. This allows us to use the local lower bound estimates in [FH10, Section 8.2.2], in the case of a smooth magnetic field. Using our notation, we present here the result in [FH10]: there exists a universal constant $C > 0$ (independent of j) such that when b is sufficiently large,

$$Q_{b,\mathbf{F}}(v) \ge \big(|a| \Theta_0 b - C r_1(R_0, b) \big) \|v\|^2_{L^2(\Omega)},$$

where v is any function such that $v \in \operatorname{Dom} Q_{b,\mathbf{F}}$ and $\operatorname{supp}(v) \subset B(z_j, R_0 b^{-\rho})$, and

$$r_1(R_0, b) = b^{\frac{1}{2}} + \eta b + R_0^4 \eta^{-1} b^{2-4\rho} + R_0^2 b^{1-\rho}, \tag{4.14}$$

for arbitrary $\eta \in (0, 1)$. Consequently for $v = \chi_j u$, we conclude that

$$\sum_{\mathrm{bnd}} Q_{b,\mathbf{F}}(\chi_j u) \ge \big(|a| \Theta_0 b - C r_1(R_0, b) \big) \sum_{\mathrm{bnd}} \|\chi_j u\|^2_{L^2(\Omega)}. \tag{4.15}$$

Estimating $\sum_{\mathrm{bar}} Q_{b,\mathbf{F}}(\chi_j u)$.

Here, we will use the local transformation Φ introduced in Section B.1. In particular, a key-ingredient is the change of gauge in Lemma B.1, that will link (locally) the form $Q_{b,\mathrm{F}}$ to the spectral value β_a defined in Section 2.2.

Let $j \in \mathrm{bar}$. Consider $v \in \mathrm{Dom}\, Q_{b,\mathrm{F}}$ such that $\mathrm{supp}\, v \subset B(z_j, R_0 b^{-\rho})$. After possibly performing a translation in the s variable, we may assume that $\Phi^{-1}(z_j) = (0,0)$. Assume that b is sufficiently large so that $B(z_j, R_0 b^{-\rho}) \subset \Gamma(t_0) \cap \Omega$ (see Appendix B.1). The transformation Φ associates to $\mathbf{F}$ the vector potential $\tilde{\mathbf{F}}$ in (B.3), and to v a function $\tilde{v} = v \circ \Phi$ defined in $\Phi^{-1}\big(B(z_j, R_0 b^{-\rho})\big)$. Using the estimates in (B.2), the change of variables formulae in (B.4), and the support of $\tilde{v}$, one may deduce the existence of $C > 0$, independent of j, such that

$$(1 - C R_0 b^{-\rho}) \int_{\Phi^{-1}\big(B(z_j, R_0 b^{-\rho})\big)} \big|(\nabla - i b \tilde{\mathbf{F}})\tilde{v}\big|^2 \, dx \le Q_{b,\mathrm{F}}(v)$$

$$\le (1 + C R_0 b^{-\rho}) \int_{\Phi^{-1}\big(B(z_j, R_0 b^{-\rho})\big)} \big|(\nabla - i b \tilde{\mathbf{F}})\tilde{v}\big|^2 \, dx, \quad (4.16)$$

Next, we will make profit of the gauge result in Lemma B.1. Thanks to (B.2) and the support of v, one may note the existence of $c_0 > 0$ such that

$$\Phi^{-1}\big(B(z_j, R_0 b^{-\rho})\big) \subset B(0, c_0 R_0 b^{-\rho}) \subset (-|\Gamma|/2, |\Gamma|/2) \times (-t_0, t_0),$$

for large b. We define

$$\tilde{v}_{\mathrm{g}}(s, t) = \tilde{v}(s, t) e^{-i b \omega(s,t)},$$

for $(s, t) \in \Phi^{-1}\big(B(z_j, R_0 b^{-\rho})\big)$, where $\omega = \omega_\ell$ is the function in Lemma B.1 and $\ell = c_0 R_0 b^{-\rho}$. One can easily check that

$$\int_{\Phi^{-1}\big(B(z_j, R_0 b^{-\rho})\big)} \big|(\nabla - i b \tilde{\mathbf{F}})\tilde{v}\big|^2 \, dx = \int_{\Phi^{-1}\big(B(z_j, R_0 b^{-\rho})\big)} \big|(\nabla - i b \tilde{\mathbf{F}}_{\mathrm{g}})\tilde{v}_{\mathrm{g}}\big|^2 \, dx. \quad (4.17)$$

Consequently, it suffices to estimate the right hand side of (4.17). We extend $\tilde{v}$ and $\tilde{v}_{\mathrm{g}}$ by zero in $\mathbb{R}^2$. Using Cauchy's inequality and the support of $\tilde{v}_{\mathrm{g}}$, we get for $\delta \in (0,1)$,

$$\int_{\Phi^{-1}\big(B(z_j, R_0 b^{-\rho})\big)} \big|(\nabla - i b \tilde{\mathbf{F}}_{\mathrm{g}})\tilde{v}_{\mathrm{g}}\big|^2 \, dx$$

$$\ge (1 - b^{-\delta}) \int_{\mathbb{R}^2} \Big(\big|(\partial_s + i b \sigma t)\tilde{v}_{\mathrm{g}}\big|^2 + |\partial_t \tilde{v}_{\mathrm{g}}|^2\Big) \, ds \, dt$$

$$- C R_0^4 b^{2 - 4\rho + \delta} \int_{\mathbb{R}^2} |\tilde{v}_{\mathrm{g}}|^2 \, ds \, dt, \quad (4.18)$$

where $\sigma = \sigma(s,t) = \mathbb{1}_{\mathbb{R}_+}(t) + a\mathbb{1}_{\mathbb{R}_-}(t)$. Performing a suitable change of gauge and a scaling, one can use the spectral properties of the operator $\mathscr{L}_a$, in Section 2.2, to conclude that

$$\int_{\mathbb{R}^2} \left(|(\partial_s + ib\sigma t)\tilde{v}_{\mathrm{g}}|^2 + |\partial_t\tilde{v}_{\mathrm{g}}|^2 \right) ds\, dt \geq \beta_a b \int_{\mathbb{R}^2} |\tilde{v}_{\mathrm{g}}|^2 \, ds\, dt. \qquad (4.19)$$

Implementing (4.19) in (4.18) yields

$$\int_{\Phi^{-1}\left(B(z_j, R_0 b^{-\rho})\right)} \left|(\nabla - ib\tilde{\mathbf{F}}_{\mathrm{g}})\tilde{v}_{\mathrm{g}}\right|^2 dx$$
$$\geq \left(\beta_a b - Cb^{1-\delta} - CR_0^4 b^{2-4\rho+\delta}\right) \int_{\mathbb{R}^2} |\tilde{v}_{\mathrm{g}}|^2 \, ds\, dt. \quad (4.20)$$

Now, we estimate the L^2-norm of $\tilde{v}_{\mathrm{g}}$. We have

$$\int_{\mathbb{R}^2} |\tilde{v}_{\mathrm{g}}|^2 \, ds\, dt = \int_{\mathbb{R}^2} |\tilde{v}|^2 \, ds\, dt = \int_{B(z_j, R_0 b^{-\rho})} |v|^2 J_{\Phi^{-1}} \, dx.$$

Hence by (B.2) and the support of v, there exists $C > 0$ independent of j such that

$$(1 - CR_0 b^{-\rho}) \int_\Omega |v|^2 \, dx \leq \int_{\mathbb{R}^2} |\tilde{v}_{\mathrm{g}}|^2 \, ds\, dt \leq (1 + CR_0 b^{-\rho}) \int_\Omega |v|^2 \, dx. \quad (4.21)$$

Plug (4.17), (4.20), and (4.21) into (4.16) to obtain

$$Q_{b,\mathbf{F}}(v) \geq \left(\beta_a b - Cr_2(R_0, b)\right) \int_\Omega |v|^2 \, dx,$$

where

$$r_2(R_0, b) = R_0 b^{1-\rho} + b^{1-\delta} + R_0^4 b^{2-4\rho+\delta}. \qquad (4.22)$$

We consider now the particular case where $v = \chi_j u$, and we conclude that

$$\sum_{\mathrm{bar}} Q_{b,\mathbf{F}}(\chi_j u) \geq \left(\beta_a b - Cr_2(R_0, b)\right) \sum_{\mathrm{bar}} \|\chi_j u\|_{L^2(\Omega)}^2. \qquad (4.23)$$

Estimating $\sum_{\mathrm{T}} Q_{b,\mathbf{F}}(\chi_j u)$.

The techniques we use below are quite similar to the ones used in estimating the $\sum_{\mathrm{bar}} Q_{b,\mathbf{F}}(\chi_j u)$. We will make profit of the local transformation Ψ introduced in

Section 4.2, and particularly of the change of gauge in Lemma 4.3, to locally link the form $Q_{b,\mathrm{F}}$ to $\mu(\cdot, a)$ defined in (3.8).

Let $j \in \mathrm{T}$. Consider $v \in \mathrm{Dom}\, Q_{b,\mathrm{F}}$ such that $\mathrm{supp}\, v \subset B(z_j, R_0 b^{-\rho})$. We use the change of variables introduced in Section 4.2, valid in a neighbourhood of z_j, to locally send the domain in Ω onto $\mathbb{R}_+^2$. b is assumed large enough so that $B(z_j, R_0 b^{-\rho}) \subset B(z_j, r_j)$. We associate to v the function $\hat{v} = v \circ \Psi^{-1}$, defined in $\Psi(B(z_j, R_0 b^{-\rho}))$. We may use the transformation formula in (4.10) and the properties in (4.8) and (4.11) to conclude that

$$(1 - CR_0 b^{-\rho}) \int_{\Psi(B(z_j, R_0 b^{-\rho})) \cap \mathbb{R}_+^2} \left| (\nabla - i b \hat{\mathbf{F}}) \hat{v} \right|^2 dx \leq Q_{b,\mathrm{F}}(v)$$

$$\leq (1 + CR_0 b^{-\rho}) \int_{\Psi(B(z_j, R_0 b^{-\rho})) \cap \mathbb{R}_+^2} \left| (\nabla - i b \hat{\mathbf{F}}) \hat{v} \right|^2 dx, \quad (4.24)$$

where $\hat{\mathbf{F}}$ is the transform of $\mathbf{F}$ by Ψ, and $C > 0$ is a constant independent of j.

In addition, due to the support of v and (4.8), we note the existence of $c_1 > 0$ such that $\Psi(B(z_j, R_0 b^{-\rho})) \subset B(0, c_1 R_0 b^{-\rho}) \subset \Psi(B(z_j, r_j))$, for large b. Consequently, the gauge transform in Lemma 4.3 allows us to write

$$\int_{\Psi(B(z_j, R_0 b^{-\rho})) \cap \mathbb{R}_+^2} \left| (\nabla - i b \hat{\mathbf{F}}) \hat{v} \right|^2 dx$$

$$= \int_{\Psi(B(z_j, R_0 b^{-\rho})) \cap \mathbb{R}_+^2} \left| (\nabla - i b \hat{\mathbf{F}}_{\mathrm{g}}) \hat{v}_{\mathrm{g}} \right|^2 d\hat{x}, \quad (4.25)$$

where $\hat{v}_{\mathrm{g}}(\hat{x}) = \hat{v}(\hat{x}) e^{-i b \varphi(\hat{x})}$, for $\hat{x} \in \Psi(B(z_j, R_0 b^{-\rho})) \cap \mathbb{R}_+^2$. Here $\varphi = \varphi_\ell$, for $\ell = c_1 R_0 b^{-\rho}$, is the gauge function in Lemma 4.3, and $\hat{\mathbf{F}}_{\mathrm{g}}$ is the magnetic potential in the aforementioned lemma. Let $\delta \in (0, 1)$. Recall the potential $\mathbf{A}_{\alpha, a}$ introduced in (3.2). Extending $\hat{v}$ and $\hat{v}_{\mathrm{g}}$ by zero in $\mathbb{R}_+^2$, the Cauchy's inequality applied in (4.25), and the support of the function $\hat{v}$ imply

$$\int_{\Psi(B(z_j, R_0 b^{-\rho})) \cap \mathbb{R}_+^2} \left| (\nabla - i b \hat{\mathbf{F}}) \hat{v} \right|^2 dx \geq (1 - b^{-\delta}) \int_{\mathbb{R}_+^2} \left| (\nabla - i b \mathbf{A}_{\alpha_j, a}) \hat{v}_{\mathrm{g}} \right|^2 d\hat{x}$$

$$- CR_0^4 b^{2 - 4\rho + \delta} \int_{\mathbb{R}_+^2} |\hat{v}_{\mathrm{g}}|^2 d\hat{x}, \quad (4.26)$$

where α_j is the corresponding angle to the point z_j, defined in Notation 1.2. Hence,

using a simple scaling argument we write

$$\int_{\Psi(B(z_j, R_0 b^{-\rho})) \cap \mathbb{R}^2_+} \left| (\nabla - i b \hat{\mathbf{F}}) \hat{v} \right|^2 dx$$

$$\geq \left(\mu(\alpha_j, a) b - C b^{1-\delta} - C R_0^4 b^{2-4\rho+\delta} \right) \int_{\mathbb{R}^2_+} |\hat{v}_g|^2 d\hat{x}, \quad (4.27)$$

where $\mu(\alpha_j, a)$ is the value in (3.8) corresponding to the angle α_j. But

$$\int_{\mathbb{R}^2_+} |\hat{v}_g|^2 d\hat{x} = \int_{B(z_j, R_0 b^{-\rho}) \cap \Omega} |v|^2 |J_\Psi| dx.$$

Thus, using (4.8) we get

$$(1 - C R_0 b^{-\rho}) \int_\Omega |v|^2 dx \leq \int_{\mathbb{R}^2_+} |\hat{v}_g|^2 d\hat{x} \leq (1 + C R_0 b^{-\rho}) \int_\Omega |v|^2 dx. \quad (4.28)$$

Plug (4.27) and (4.28) into (4.24) to obtain

$$Q_{b,\mathbf{F}}(v) \geq \left(\mu(\alpha_j, a) b - C r_3(R_0, b) \right) \|v\|^2_{L^2(\Omega)}, \quad (4.29)$$

where

$$r_3(R_0, b) = R_0 b^{1-\rho} + b^{1-\delta} + R_0^4 b^{2-4\rho+\delta}. \quad (4.30)$$

Taking the particular case $v = \chi_j u$, we infer from (4.29) that

$$\sum_T Q_{b,\mathbf{F}}(\chi_j u) \geq \left(\min_{j \in T} \mu(\alpha_j, a) b - C r_3(R_0, b) \right) \sum_T \|\chi_j u\|^2_{L^2(\Omega)}. \quad (4.31)$$

Let

$$r(R_0, b) = \max \left(r_1(R_0, b), r_2(R_0, b), r_3(R_0, b) \right), \quad (4.32)$$

where r_1, r_2, r_3 defined in (4.14), (4.22) and (4.30) respectively. The estimates in (4.12), (4.13), (4.15), (4.23), and (4.31) give the following lower bound of $Q_{b,\mathbf{F}}(u)$:

$$Q_{b,\mathbf{F}}(u) \geq |a| b \sum_{\text{int}} \|\chi_j u\|^2_{L^2(\Omega)} + |a| \Theta_0 b \sum_{\text{bnd}} \|\chi_j u\|^2_{L^2(\Omega)}$$

$$+ \beta_a b \sum_{\text{bar}} \|\chi_j u\|^2_{L^2(\Omega)} + \min_{j \in T} \mu(\alpha_j, a) b \sum_T \|\chi_j u\|^2_{L^2(\Omega)}$$

$$- C \left(r(R_0, b) + R_0^{-2} b^{2\rho} \right) \|u\|^2_{L^2(\Omega)}. \quad (4.33)$$

We may extract particular results from the discussion done above, which we present in the following two propositions:

Proposition 4.4. *There exists $C > 0$, and for all $R_0 > 1$ there exists $b_0 > 0$ such that for $b \geq b_0$ and $u \in \mathrm{Dom}\, Q_{b,\mathrm{F}}$, it holds*

$$Q_{b,\mathrm{F}}(u) \geq \int_\Omega \left(U_b(x) - C R_0^{-2} b^{2\rho} \right) |u(x)|^2 \, dx,$$

where $U_b(x)$ is given by

$$
\begin{cases}
|a|\, b & \mathrm{dist}(x, \partial\Omega \cup \Gamma) \geq R_0 b^{-\rho}, \\
\beta_a b - C r(R_0, b) & \mathrm{dist}(x, \partial\Omega) \geq R_0 b^{-\rho} \,\&\, \mathrm{dist}(x, \Gamma) < R_0 b^{-\rho}, \\
|a|\,\Theta_0 b - C r(R_0, b) & \mathrm{dist}(x, \partial\Omega) < R_0 b^{-\rho} \,\&\, x \notin \bigcup_{j=1}^{n} B(\mathsf{p}_j, R_0 b^{-\rho}), \\
\mu(\alpha_j, a) b - C r(R_0, b) & j \in \{1, ..., n\}\,,\ x \in B(\mathsf{p}_j, R_0 b^{-\rho}),
\end{cases}
$$

where $r(R_0, b)$ is the term in (4.32), $\mu(\alpha_j, a)$ and Θ_0 are introduced in (3.8) and (2.2) respectively.

Proof. Let $R_0 > 1$ and $b > 0$ be large. Define the following partition of Ω:

$$Z_1 = \left\{ x \in \Omega \,:\, \mathrm{dist}(x, \partial\Omega \cup \Gamma) \geq R_0 b^{-\rho} \right\}.$$

$$Z_2 = \left\{ x \in \Omega \,:\, \mathrm{dist}(x, \partial\Omega) \geq R_0 b^{-\rho},\ \mathrm{dist}(x, \Gamma) < R_0 b^{-\rho} \right\}.$$

$$Z_3 = \left\{ x \in \Omega \,:\, \mathrm{dist}(x, \partial\Omega) < R_0 b^{-\rho},\ x \notin \bigcup_{j \in \mathrm{T}} B(\mathsf{p}_j, R_0 b^{-\rho}) \right\}.$$

$$Z_4 = \bigcup_{j \in \mathrm{T}} B(\mathsf{p}_j, R_0 b^{-\rho}) \cap \Omega,$$

and consider the partition of unity in Section 4.2. Clearly, we have

$$\bigcup_{j \in \mathrm{T}} B(z_j, R_0 b^{-\rho}) \subset Z_4 = (Z_1 \cup Z_2 \cup Z_3)^{\complement}, \quad \bigcup_{j \in \partial\Omega \setminus \Gamma} B(z_j, R_0 b^{-\rho}) \subset (Z_1 \cup Z_2)^{\complement},$$

$$\text{and} \quad \bigcup_{j \in \Gamma \setminus \partial\Omega} B(z_j, R_0 b^{-\rho}) \subset (Z_1)^{\complement}.$$

Hence, using the lower bounds established in (4.13), (4.15), (4.23), and (4.31) and the ordering $\max_j \mu(\alpha_j, a) \leq |a|\,\Theta_0 \leq \beta_a \leq |a|$ (see Theorem 3.1 and Section 2.2), the IMS formula yields the proof. □

Note again that $\min_{j \in \{1, ..., n\}} \mu(\alpha_j, a) \leq |a|\,\Theta_0 \leq \beta_a \leq |a|$ (see Section 2.2 and Theorem 3.1). *We choose $R_0 = 1$, $\rho = 3/8$, $\delta = 1/4$ and $\eta = b^{-1/4}$* in (4.33). Consequently, the min-max principle implies the following:

Proposition 4.5. *Under Assumption 1.1, there exist $b_0, C > 0$ such that for all $b \geq b_0$,*

$$\lambda(b) \geq \min_{j \in \{1,...,n\}} \mu(\alpha_j, a) b - C b^{\frac{3}{4}},$$

where $\lambda(b)$ and $\mu(\alpha_j, a)$ are the values in (4.4) *and* (3.8) *respectively.*

The previous result is nothing but the lower bound in Theorem 4.2, established under the weaker Assumption 1.1.

In the non-linear Agmon estimates (see Theorem 1.6), we need the localization zone to have the right surface scale, namely $\{\text{dist}(x, S) < R_0 b^{-1/2}\}$ (for $b = \kappa H$). For this purpose, it is more convenient to choose the parameters in the above lower bound study as follows: $\rho = \delta = 1/2$, $\eta = b^{-1/2}$, *and R_0 large*, even though the lower bound estimate may appear weaker. With this choice of parameters, Proposition 4.4 becomes:

Proposition 4.6. *There exists $C > 0$, and for all $R_0 > 1$ there exists $b_0 > 0$ such that for $b \geq b_0$ and $u \in \text{Dom } Q_{b,F}$, it holds*

$$Q_{b,F}(u) \geq \int_\Omega \left(U_b^{(2)}(x) - C \frac{b}{R_0^2} \right) |u(x)|^2 \, dx,$$

where $U_b^{(2)}(x)$ is given by

$$\begin{cases} |a| b & \text{dist}(x, \partial\Omega \cup \Gamma) \geq R_0 b^{-\frac{1}{2}}, \\ \beta_a b - C R_0^4 b^{\frac{1}{2}} & \text{dist}(x, \partial\Omega) \geq R_0 b^{-\frac{1}{2}} \,\&\, \text{dist}(x, \Gamma) < R_0 b^{-\frac{1}{2}}, \\ |a| \Theta_0 b - C R_0^4 b^{\frac{1}{2}} & \text{dist}(x, \partial\Omega) < R_0 b^{-\frac{1}{2}} \,\&\, x \notin \bigcup_{j=1}^{n} B(\mathsf{p}_j, R_0 b^{-\frac{1}{2}}), \\ \mu(\alpha_j, a) b - C R_0^4 b^{\frac{1}{2}} & j \in \{1, ..., n\}, \; x \in B(\mathsf{p}_j, R_0 b^{-\frac{1}{2}}). \end{cases}$$

Here $\mu(\alpha_j, a)$ and Θ_0 are the values in (3.8) *and* (2.2) *respectively.*

Upper bound of $\lambda(b)$

In the next proposition, we establish the upper bound in Theorem 4.2.

Proposition 4.7. *Under Assumption 1.3, there exist $b_0, C > 0$ such that for all $b \geq b_0$,*

$$\lambda(b) \leq \min_{j \in \{1,...,n\}} \mu(\alpha_j, a) b + C b^{\frac{3}{5}},$$

where $\lambda(b)$ is the value in (4.4).

Proof. Let $k \in \{1, ..., n\}$ be such that $\mu(\alpha_k, a) = \min_{j \in \{1,...,n\}} \mu(\alpha_j, a)$, and let p_k be the corresponding intersection point of Γ and $\partial\Omega$ (see Notation 1.2). We will establish the desired upper bound by defining a suitable test function, localized in a neighbourhood of p_k. To this end, we consider a smooth cut-off function, χ, satisfying

$$0 \leq \chi \leq 1 \text{ in } \mathbb{R}^2, \quad \chi = 1 \text{ in } B(0, 1/2) \quad \text{and} \quad \operatorname{supp}\chi \subset B(0, 1).$$

Let $b > 0$ be sufficiently large such that

$$B(0, b^{-2/5}) \subset \Psi\big(B(\mathsf{p}_k, r_k)\big), \tag{4.34}$$

where r_k is the radius introduced in Section 4.2. We define the function $\hat{\chi}$ in $\mathbb{R}^2$ by $\hat{\chi}(\hat{x}) = \chi(b^{\frac{2}{5}}\hat{x})$. Consequently,

$$0 \leq \hat{\chi} \leq 1, \, \hat{\chi} = 1 \text{ in } B\big(0, 1/2b^{-2/5}\big), \operatorname{supp}\hat{\chi} \subset B(0, b^{-2/5}), \, |\nabla_{\hat{x}}\hat{\chi}| \leq Cb^{2/5}. \tag{4.35}$$

We define the following test function in Ω:

$$u(x) = \begin{cases} \hat{u} \circ \Psi(x) & \text{if } x \in \Psi^{-1}\big(B(0, b^{-2/5})\big) \cap \Omega, \\ 0 & \text{otherwise,} \end{cases} \tag{4.36}$$

where Ψ is the diffeomorphism in Section 4.2,

$$\hat{u}(\hat{x}) = \begin{cases} \hat{\chi}(\hat{x})u_0(\hat{x})e^{ib\varphi(\hat{x})} & \text{if } \hat{x} \in B(0, b^{-2/5}) \cap \mathbb{R}^2_+, \\ 0 & \text{otherwise,} \end{cases}$$

and $u_0(\hat{x}) = \sqrt{b}v_0(\sqrt{b}\hat{x})$, for all $\hat{x} \in \mathbb{R}^2_+$. Here v_0 is a normalized eigenfunction corresponding to $\mu(\alpha_k, a)$ (see Remark 3.2), and $\varphi = \varphi_\ell$ is the gauge function in Lemma 4.3, for $\ell = b^{-2/5}$ (b satisfies (4.34)). We will prove that

$$\frac{Q_{b,\mathbf{F}}(u)}{\|u\|^2_{L^2(\Omega)}} \leq b\mu(\alpha_k, a) + Cb^{\frac{3}{5}}. \tag{4.37}$$

Upper bound of $Q_{b,\mathbf{F}}(u)$. We establish the upper bound in several steps.

Step 1 (Change of variables). We use the properties of Ψ in Section 4.2 to get

$$Q_{b,\mathbf{F}}(u) \leq (1 + Cb^{-\frac{2}{5}}) \int_{B(0,b^{-2/5}) \cap \mathbb{R}^2_+} \big|(\nabla - ib\hat{\mathbf{F}})\hat{u}\big|^2 \, dx, \tag{4.38}$$

for some $C > 0$ (see (4.8), (4.10) and (4.11)).

Step 2 (Change of gauge). We use the change of gauge in Lemma 4.3 to write

$$\int_{B(0,b^{-2/5})\cap\mathbb{R}_+^2} \left|(\nabla - i\,b\hat{\mathbf{F}})\hat{u}\right|^2 dx = \int_{B(0,b^{-2/5})\cap\mathbb{R}_+^2} \left|(\nabla - i\,b\hat{\mathbf{F}}_{\mathrm{g}})\hat{\chi}u_0\right|^2 dx, \quad (4.39)$$

where $\hat{\mathbf{F}}_{\mathrm{g}}$ is the vector potential in Lemma 4.3.

Step 3 (Link to $\mu(\alpha_k, a)$). By Lemma 4.3, we have

$$\int_{B(0,b^{-2/5})\cap\mathbb{R}_+^2} \left|(\nabla - i\,b\hat{\mathbf{F}}_{\mathrm{g}})\hat{\chi}u_0\right|^2 dx$$

$$\leq \int_{B(0,b^{-2/5})\cap\mathbb{R}_+^2} \left(\left|\partial_{\hat{x}_1}(\hat{\chi}u_0)\right|^2 + \left|\left(\partial_{\hat{x}_2} - i\,b(A_{\alpha_k,a} + f)\right)\hat{\chi}u_0\right|^2 \right) d\hat{x}.$$

Recall that the function f satisfies $|f(\hat{x}_1, \hat{x}_2)| \leq C(\hat{x}_1^2 + |\hat{x}_1\hat{x}_2|)$, for some constant $C > 0$. Let $y = \sqrt{b}\hat{x}$, for $\hat{x} \in \mathbb{R}_+^2$. We define the function $\tilde{\chi}$ in $\mathbb{R}_+^2$ such that

$$\tilde{\chi}(y) = \chi(b^{-\frac{1}{10}}y) = \chi(b^{\frac{2}{5}}\hat{x}) = \hat{\chi}(\hat{x}).$$

Note that

$$0 \leq \tilde{\chi} \leq 1 \text{ in } \mathbb{R}^2, \quad \tilde{\chi} = 1 \text{ in } B(0, 1/2b^{1/10}),$$

$$\operatorname{supp}\tilde{\chi} \subset B(0, b^{1/10}), \quad |\nabla_y\tilde{\chi}| \leq C b^{-1/10}, \quad (4.40)$$

and $(\hat{\chi}u_0)(\hat{x}) = \sqrt{b}(v_0\tilde{\chi})(y)$. Hence, a simple computation yields that

$$\int_{B(0,b^{-2/5})\cap\mathbb{R}_+^2} \left|(\nabla - i\,b\hat{\mathbf{F}}_{\mathrm{g}})\hat{\chi}u_0\right|^2 dx = b \int_{\mathbb{R}_+^2} \left|\partial_{y_1}(\tilde{\chi}v_0)\right|^2 dy +$$

$$b \int_{\mathbb{R}_+^2} \left|\left(\partial_{y_2} - i\left(A_{\alpha_k,a}(y_1,y_2) + b^{-\frac{1}{2}}\mathcal{O}(y_1^2) + b^{-\frac{1}{2}}\mathcal{O}(y_1 y_2)\right)\right)\tilde{\chi}v_0\right|^2 dy. \quad (4.41)$$

Below, we estimate each term of the right hand side of (4.41) apart. We start by estimating the term $\int \left|\partial_{y_1}(\tilde{\chi}v_0)\right|^2 dy$. We use Cauchy's inequality and (4.40) to get

$$\int_{\mathbb{R}_+^2} \left|\partial_{y_1}(\tilde{\chi}v_0)\right|^2 dy \leq (1 + b^{-\frac{1}{2}}) \int_{\mathbb{R}_+^2} \left|\tilde{\chi}\partial_{y_1}v_0\right|^2 dy + C b^{\frac{1}{2}} \int_{\mathbb{R}_+^2} \left|v_0\partial_{y_1}\tilde{\chi}\right|^2 dy$$

$$\leq (1 + b^{-\frac{1}{2}}) \int_{\mathbb{R}_+^2} \left|\partial_{y_1}v_0\right|^2 dy$$

$$+ C b^{\frac{3}{10}} \int_{\left(B(0,b^{\frac{1}{10}})\setminus B(0,\frac{1}{2}b^{\frac{1}{10}})\right)\cap\mathbb{R}_+^2} \left|v_0\right|^2 dy. \quad (4.42)$$

To control the error term in (4.42), we use the following result derived from the decay of the eigenfunction v_0 established in Theorem 3.10 (taking $\delta = (|a|\Theta_0 - \mu(\alpha_k, a))/2$ in the aforementioned theorem):

$$\int_{\left(B(0, b^{\frac{1}{10}})\backslash B(0, \frac{1}{2}b^{\frac{1}{10}})\right)\cap\mathbb{R}_+^2} |v_0|^2\, dy \leq e^{-C_2 b^{\frac{1}{10}}} \int_{\mathbb{R}_+^2} e^{2\phi}|v_0|^2\, dy \leq C_1 e^{-C_2 b^{\frac{1}{10}}}. \quad (4.43)$$

Here $C_1 = C_{\delta, \alpha_k}$, $C_2 = \sqrt{(|a|\Theta_0 - \mu(\alpha_k, a))/2}$, and ϕ is the function introduced in Theorem 3.10. Plugging (4.43) in (4.42), we get for large values of b, and for some positive constants $\tilde{C}_1$ and $\tilde{C}_2$

$$\int_{\mathbb{R}_+^2} |\partial_{y_1}(\tilde{\chi}v_0)|^2\, dy \leq (1 + b^{-\frac{1}{2}}) \int_{\mathbb{R}_+^2} |\partial_{y_1}v_0|^2\, dy + \tilde{C}_1 e^{-\tilde{C}_2 b^{\frac{1}{10}}}. \quad (4.44)$$

Now we estimate the second term in the right hand side of (4.41):

$$\int_{\mathbb{R}_+^2} \left|\left(\partial_{y_2} - i\left(A_{\alpha_k, a} + b^{-\frac{1}{2}}\mathcal{O}(y_1^2) + b^{-\frac{1}{2}}\mathcal{O}(y_1 y_2)\right)\right)\tilde{\chi}v_0\right|^2\, dy$$

$$\leq (1 + b^{-\frac{1}{2}}) \int_{\mathbb{R}_+^2} \left|(\partial_{y_2} - iA_{\alpha_k, a})v_0\right|^2\, dy + Cb^{-\frac{1}{2}} \int_{\mathbb{R}_+^2} y_1^4 |v_0|^2\, dy$$

$$+ Cb^{-\frac{1}{2}} \int_{\mathbb{R}_+^2} y_1^2 y_2^2 |v_0|^2\, dy + Cb^{\frac{1}{2}} \int_{\mathbb{R}_+^2} |\partial_{y_2}\tilde{\chi}|^2 |v_0|^2\, dy. \quad (4.45)$$

In (4.45), we used Cauchy's inequality together with the properties of $\tilde{\chi}$ in (4.40). In a similar fashion of establishing (4.43), we use (4.40) together with the exponential decay in Theorem 3.10 to estimate

$$b^{\frac{1}{2}} \int_{\left(B(0, b^{\frac{1}{10}})\backslash B(0, \frac{1}{2}b^{\frac{1}{10}})\right)\cap\mathbb{R}_+^2} |\partial_{y_2}\tilde{\chi}|^2 |v_0|^2\, dy \leq \tilde{C}_1 e^{-\tilde{C}_2 b^{\frac{1}{10}}}. \quad (4.46)$$

Moreover, the aforementioned exponential decay shows that $y \mapsto y_1^2 v_0(y)$ and $y \mapsto y_1 y_2 v_0(y)$ are square integrable in $\mathbb{R}_+^2$, that is there exists $C > 0$ such that

$$\int_{\mathbb{R}_+^2} y_1^4 |v_0|^2\, dy \leq C \quad \text{and} \quad \int_{\mathbb{R}_+^2} y_1^2 y_2^2 |v_0|^2\, dy \leq C. \quad (4.47)$$

From (4.45)–(4.47), we get

$$\int_{\mathbb{R}_+^2} \left|\left(\partial_{y_2} - i\left(A_{\alpha_k, a} + b^{-\frac{1}{2}}\mathcal{O}(y_1^2) + b^{-\frac{1}{2}}\mathcal{O}(y_1 y_2)\right)\right)\tilde{\chi}v_0\right|^2\, dy$$

$$\leq (1 + b^{-\frac{1}{2}}) \int_{\mathbb{R}_+^2} \left|(\partial_{y_2} - iA_{\alpha_k, a})v_0\right|^2\, dy + Cb^{-\frac{1}{2}}. \quad (4.48)$$

Since v_0 is a normalized eigenfunction of the operator $\mathcal{H}_{\alpha_k, a}$ (in (3.2)), corresponding to $\mu(\alpha_k, a)$, we have

$$\int_{\mathbb{R}^2_+} \left(\left| \partial_{y_1} v_0 \right|^2 + \left| (\partial_{y_2} - iA_{\alpha_k, a}) v_0 \right|^2 \right) dy = q_{\alpha_k, a}(v_0) = \mu(\alpha_k, a). \qquad (4.49)$$

Gathering pieces in (4.41), (4.44), (4.48), and (4.49) implies

$$\int_{B(0, b^{-2/5}) \cap \mathbb{R}^2_+} \left| (\nabla - i b \hat{\mathbf{F}}_g) \hat{\chi} u_0 \right|^2 dx \leq (1 + b^{-\frac{3}{8}}) b q_{\alpha_k, a}(v_0) + C b^{\frac{1}{2}}$$

$$\leq b \mu(\alpha_k, a) + C b^{\frac{1}{2}}. \qquad (4.50)$$

Finally, the estimates established in (4.38), (4.39), and (4.50) yield

$$Q_{b, \mathbf{F}}(u) \leq (1 + C b^{-\frac{2}{5}}) \left(b \mu(\alpha_k, a) + C b^{\frac{1}{2}} \right) \leq b \mu(\alpha_k, a) + C b^{\frac{3}{5}}. \qquad (4.51)$$

Lower bound of $\|u\|^2_{L^2(\Omega)}$. The definition of u in (4.36) and the property in (4.8) yield

$$\int_{\Omega} |u|^2 dx \geq (1 - C b^{-\frac{2}{5}}) \int_{B(0, b^{-\frac{2}{5}}) \cap \mathbb{R}^2_+} |\hat{u}|^2 d\hat{x}$$

$$= (1 - C b^{-\frac{2}{5}}) \int_{B(0, b^{-\frac{2}{5}}) \cap \mathbb{R}^2_+} |\hat{\chi} u_0|^2 d\hat{x}$$

$$= (1 - C b^{-\frac{2}{5}}) \int_{B(0, b^{\frac{1}{10}}) \cap \mathbb{R}^2_+} |\tilde{\chi} v_0|^2 dy$$

$$\geq (1 - C b^{-\frac{2}{5}}) \int_{B(0, \frac{1}{2} b^{\frac{1}{10}}) \cap \mathbb{R}^2_+} |v_0|^2 dy$$

$$= (1 - C b^{-\frac{2}{5}}) \left(1 - \int_{B(0, \frac{1}{2} b^{\frac{1}{10}})^C \cap \mathbb{R}^2_+} |v_0|^2 dy \right). \qquad (4.52)$$

Similarly to (4.43), we have

$$\int_{B(0, \frac{1}{2} b^{\frac{1}{10}})^C \cap \mathbb{R}^2_+} |v_0|^2 dy \leq C_1 e^{-C_2 b^{\frac{1}{10}}}.$$

Hence,

$$\int_{\Omega} |u|^2 dx \geq 1 - C b^{-\frac{2}{5}}. \qquad (4.53)$$

We gather the results in (4.51) and (4.53) to establish the claim in (4.37). Consequently the min-max principle completes the proof of Proposition 4.7. $\qquad \square$

Remark 4.8. The error established in Proposition 4.7 is not optimal. More generally, for any $\rho \in (0, 1/2)$, one may set $B(0, b^{-\rho})$ to be the support of $\hat{\chi}$ in (4.35). Then, by adjusting the choice of the parameters in the upper bound proof, one can get

$$\lambda(b) \leq \min_{j \in \{1,...,n\}} \mu(\alpha_j, a) b + C b^{1-\rho},$$

for all $b \geq b_0$.

5 Breakdown of superconductivity

Below, we prove that when the magnetic field is sufficiently large, the only solution of (1.4) is the normal state $(0, \mathbf{F})$, where $\mathbf{F} \in H^1_{\text{div}}(\Omega)$ is the vector potential in (5.1) (see Theorem 5.2).

5.1 A priori estimates

We present certain known estimates needed in the sequel to control the errors arising in our various approximations.

Proposition 5.1. *If* $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2)$ *is a weak solution of* (1.4)*, then*

$$\|\psi\|_{L^\infty(\Omega)} \leq 1.$$

We omit the proof of Proposition 5.1, and refer to the similar proof in [FH10, Proposition 10.3.1].

Recall the magnetic field $B_0 = \mathbb{1}_{\Omega_1} + a \mathbb{1}_{\Omega_2}$ with $a \in [-1, 1) \setminus \{0\}$, introduced in Assumption 1.1. There exists a unique vector potential $\mathbf{F} \in H^1_{\text{div}}(\Omega)$ such that (see [AK16, Lemma A.1])

$$\text{curl } \mathbf{F} = B_0. \tag{5.1}$$

Theorem 5.2. *Let* $\beta \in (0, 1)$. *Suppose that the conditions in Assumption 1.1 hold. There exists* $C > 0$ *such that for all* $\kappa > 0$, *if* $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\text{div}}(\Omega)$ *is a solution of* (1.4)*, then*

1. $\|(\nabla - i\kappa H \mathbf{A})\psi\|_{L^2(\Omega)} \leq \kappa \|\psi\|_{L^2(\Omega)}$.

2. $\|\text{curl}(\mathbf{A} - \mathbf{F})\|_{L^2(\Omega)} \leq \dfrac{C}{H} \|\psi\|_{L^2(\Omega)}$.

3. $\mathbf{A} - \mathbf{F} \in H^2(\Omega)$ *and* $\|\mathbf{A} - \mathbf{F}\|_{H^2(\Omega)} \leq \dfrac{C}{H} \|\psi\|_{L^2(\Omega)}$.

4. $\mathbf{A} - \mathbf{F} \in \mathscr{C}^{0,\beta}(\overline{\Omega})$ *and* $\|\mathbf{A} - \mathbf{F}\|_{\mathscr{C}^{0,\beta}(\overline{\Omega})} \leq \dfrac{C}{H}\|\psi\|_{L^2(\Omega)}$.

The proof of the previous theorem is given in [FH10, Lemma 10.3.2] and [AK16, Theorem 4.2].

5.2 Trivial minimizers

We adapt a result of Giorgi–Phillips [GP99] to our case of the step magnetic field B_0. Let $\mathbf{F} \in H^1_{\mathrm{div}}(\Omega)$ be the magnetic potential in (5.1), satisfying curl $\mathbf{F} = B_0$. Observe that $(0, \mathbf{F})$ is a critical point of the functional in (1.1), i.e. it is a weak solution of (1.4). In Theorem 5.3 below, we show that this trivial solution is the *unique minimizer of the functional in* (1.1), for sufficiently large values of H.

Theorem 5.3. *Under Assumption 1.1, there exist positive constants κ_1 and C_1 such that if $\kappa \geq \kappa_1$,*
$$H > C_1\kappa,$$
then $(0, \mathbf{F})$ is the unique solution of (1.4) *in* $H^1(\Omega) \times H^1_{\mathrm{div}}(\Omega)$.

Proof. Let $\kappa > 0$ and $H > 0$. Assume that the corresponding GL system (1.4) admits a non-trivial solution $(\psi, \mathbf{A}) \in H^1(\Omega) \times H^1_{\mathrm{div}}(\Omega)$. We mean by *non-trivial* that
$$\|\psi\|_{L^2(\Omega)} > 0. \tag{5.2}$$
We compare $\|(\nabla - i\kappa H\mathbf{F})\psi\|_{L^2(\Omega)}$ and $\|(\nabla - i\kappa H\mathbf{A})\psi\|_{L^2(\Omega)}$ using Cauchy's inequality
$$\|(\nabla - i\kappa H\mathbf{F})\psi\|^2_{L^2(\Omega)} \leq 2\|(\nabla - i\kappa H\mathbf{A})\psi\|^2_{L^2(\Omega)} + 2(\kappa H)^2\|(\mathbf{A} - \mathbf{F})\psi\|^2_{L^2(\Omega)}. \tag{5.3}$$

The estimates in Theorem 5.2 ensure that
$$\|(\nabla - i\kappa H\mathbf{A})\psi\|^2_{L^2(\Omega)} + (\kappa H)^2\|\mathbf{A} - \mathbf{F}\|^2_{L^2(\Omega)} \leq C\kappa^2\|\psi\|^2_{L^2(\Omega)}. \tag{5.4}$$

This inequality, together with $|\psi| \leq 1$, allow us to control the right hand side of (5.3) and get
$$\|(\nabla - i\kappa H\mathbf{F})\psi\|^2_{L^2(\Omega)} \leq C\kappa^2\|\psi\|^2_{L^2(\Omega)}.$$
Since $(\psi, \mathbf{A})$ is non-trivial, we get
$$\lambda(\kappa H) \leq C\kappa^2, \tag{5.5}$$

where $\lambda(\kappa H)$ is the value in (4.4).

On the other hand, let κ_0 be such that $\kappa_0 \geq b_0$, where b_0 is the constant in Proposition 4.5. Applying this Proposition, we get the existence of $\tilde{C} > 0$ such that for all $\kappa \geq \kappa_0$ and $H \geq 1$,

$$\lambda(\kappa H) \geq \tilde{C} \min\left(|a|\Theta_0, \min_j \mu(\alpha_j, a)\right)\kappa H. \tag{5.6}$$

We combine (5.5) and (5.6) to obtain the following: for all $\kappa \geq \kappa_0$ and $H \geq 1$, if the corresponding GL system (1.4) admits a non-trivial solution, then

$$\tilde{C} \min\left(|a|\Theta_0, \min_j \mu(\alpha_j, a)\right)\kappa H \leq \lambda(\kappa H) \leq C\kappa^2,$$

which in this case implies that

$$H \leq C_1 \kappa,$$

for $C_1 = C / \left(\tilde{C} \min\left(|a|\Theta_0, \min_j \mu(\alpha_j, a)\right)\right)$. This result can be reformulated as follows: For all $\kappa \geq \kappa_0$, if $H > \max(C_1\kappa, 1)$ then $\mathscr{E}_{\kappa,H}$ admits only trivial minimizers. Take $\kappa_1 \geq \max(\kappa_0, 1/C_1)$ so that for all $\kappa \geq \kappa_1$, $C_1\kappa \geq 1$. We have then proved Theorem 5.3. $\qquad\square$

6   Monotonicity of $\lambda(b)$

We consider $\lambda(b)$—the lowest eigenvalue of the operator $\mathscr{P}_{b,\mathbf{F}}$ defined in Section 4.1. We will establish the so-called strong diamagnetic property ([FH07]); $b \mapsto \lambda(b)$ is strictly increasing for large values of b (Proposition 6.3). This property will enable us to prove the first statement of Theorem 1.5 (Proposition 6.5). Moreover, we will provide the asymptotics of $H_{C_3}(\kappa)$ stated in Theorem 1.5 (Proposition 6.7).

Information about the localization of a ground-state of $\mathscr{P}_{b,\mathbf{F}}$ is needed while establishing the monotonicity result in Proposition 6.3. Theorem 6.1 below provides such localization (Agmon) estimates. Our argument is quite similar to that in [Bon03, Section 15]. Still, we give the proof of this theorem for completeness.

Recall the set $\Gamma \cap \partial\Omega = \{\mathsf{p}_j : j \in \{1, ..., n\}\}$. In this section, we assume that Assumption 1.3 holds. We denote by

$$\mu^* = \min_{j \in \{1,...,n\}} \mu(\alpha_j, a). \tag{6.1}$$

Let S^* be the set of points p_k corresponding to the minimal energy $\mu(\alpha_k, a)$

$$S^* = \{\mathsf{p}_k \in \Gamma \cap \partial\Omega : \mu(\alpha_k, a) = \mu^*\}, \tag{6.2}$$

As shown in the next theorem, a ground-state is localized near the points of S^*.

Theorem 6.1. *Under Assumption 1.3, there exist positive constants b_0, C, and ζ such that if $b \geq b_0$ and ψ is a ground-state of the operator $\mathscr{P}_{b,\mathbf{F}}$ then*

$$\int_\Omega e^{2\zeta\sqrt{b}\,\mathrm{dist}(x,S^*)}\left(|\psi|^2 + b^{-1}\big|(\nabla - ib\mathbf{F})\psi\big|^2\right) dx \leq C\|\psi\|^2_{L^2(\Omega)}. \tag{6.3}$$

Consequently, for all $N > 0$,

$$\int_\Omega \mathrm{dist}(x, S^*)^N|\psi|^2\,dx = \mathcal{O}\big(b^{-\frac{N}{2}}\big).$$

Proof. Let $R_0 > 1$. We define the real Lipschitz function

$$g(x) = \zeta\max\big(\mathrm{dist}(x, S^*), R_0 b^{-\frac{1}{2}}\big), \qquad x \in \Omega, \tag{6.4}$$

where $\zeta > 0$ is to be chosen later. An integration by parts yields

$$\mathrm{Re}\,\big\langle \mathscr{P}_{b,\mathbf{F}}\psi, e^{2\sqrt{b}g}\psi\big\rangle = Q_{b,\mathbf{F}}\big(e^{\sqrt{b}g}\psi\big) - b\big\||\nabla g|e^{\sqrt{b}g}\psi\big\|^2_{L^2(\Omega)}, \tag{6.5}$$

where $Q_{b,\mathbf{F}}$ is the quadratic form in (4.2). Hence, using (6.5) and the definition of ψ, we get

$$\lambda(b)\|e^{\sqrt{b}g}\psi\|^2 = Q_{b,\mathbf{F}}\big(e^{\sqrt{b}g}\psi\big) - b\big\||\nabla g|e^{\sqrt{b}g}\psi\big\|^2. \tag{6.6}$$

By Propositions 4.6 and 4.7, we have

$$Q_{b,\mathbf{F}}\big(e^{\sqrt{b}g}\psi\big) \geq \int_\Omega \big(U_b^{(2)}(x) - CbR_0^{-2}\big)\big|e^{\sqrt{b}g(x)}\psi(x)\big|^2\,dx, \tag{6.7}$$

and

$$\lambda(b) \leq \mu^*b + o(b). \tag{6.8}$$

Implementing (6.7) and (6.8) in (6.6), dividing by b and using the properties of the function $U_b^{(2)}$ in Proposition 4.6 yield

$$\int_{\{t(x)\geq R_0 b^{-\frac{1}{2}}\}} \big(\mu^{**} - CR_0^4 b^{-\frac{1}{2}} - CR_0^{-2}\big)\big|e^{\sqrt{b}g}\psi\big|^2\,dx$$

$$+ \int_{\{t(x)\leq R_0 b^{-\frac{1}{2}}\}} \big(\mu^* - CR_0^4 b^{-\frac{1}{2}} - CR_0^{-2}\big)\big|e^{\sqrt{b}g}\psi\big|^2\,dx$$

$$\leq \big(\mu^* + o(1)\big)\|e^{\sqrt{b}g}\psi\|^2 + \big\||\nabla g|e^{\sqrt{b}g}\psi\big\|^2. \tag{6.9}$$

Here $t(x) = \text{dist}(x, S^*)$, and μ^{**} is the minimum of all the $\mu(\alpha_j, a)$ that are strictly greater that μ^* (if such a $\mu(\alpha_j, a)$ does not exist, we take $\mu^{**} = |a|\Theta_0$). By (6.4), we have $\text{supp}(\nabla g) \subset \{t(x) \geq R_0 b^{-\frac{1}{2}}\}$ and $|\nabla g| \leq \zeta$. Consequently,

$$\big\| |\nabla g| e^{\sqrt{b}g}\psi \big\|^2 \leq \zeta^2 \int_{\{t(x) \geq R_0 b^{-\frac{1}{2}}\}} e^{2\sqrt{b}g} |\psi|^2 \, dx. \tag{6.10}$$

Hence, (6.9) yields

$$\int_{\{t(x) \geq R_0 b^{-\frac{1}{2}}\}} \big(\mu^{**} - \mu^* - o(1) - CR_0^4 b^{-\frac{1}{2}} - CR_0^{-2} - \zeta^2\big) \big| e^{\sqrt{b}g}\psi \big|^2 \, dx$$

$$\leq \int_{\{t(x) \leq R_0 b^{-\frac{1}{2}}\}} \big(CR_0^4 b^{-\frac{1}{2}} + CR_0^{-2} + o(1)\big) \big| e^{\sqrt{b}g}\psi \big|^2 \, dx. \tag{6.11}$$

We may choose $\zeta < \sqrt{\mu^{**} - \mu^*}$. Then using (6.11) and the definition of g in (6.4), there exist large positive constants R_0 and b_0 such that for all $b \geq b_0$

$$\int_\Omega e^{2\zeta\sqrt{b}\,\text{dist}(x,\partial\Omega)} |\psi|^2 \, dx \leq \tilde{C}(R_0, \zeta)\|\psi\|^2_{L^2(\Omega)}. \tag{6.12}$$

One can deduce the other part of (6.3) by gathering the estimates in (6.6), (6.10), (6.12) and the upper bound in Proposition 4.7. ☐

Remark 6.2. In similar situations in the literature, when the applied magnetic field is uniform, certain normal Agmon estimates were established showing the decay of the ground-state away from the boundary. Such decays were usually used in the proofs of the monotonicity of the ground-state energy (see [FH07, Section 2]). In the present work, one can similarly establish such a normal decay of the ground-state away from the boundary of $\Omega_1 \cup \Omega_2$. However as it will be explained later in this section, the localization result in Theorem 6.1 is sufficient while deriving the monotonicity of the ground-state in our step magnetic field case. Therefore, we opt not to state the normal estimates here.

Having the domain of $\mathscr{P}_{b,F}$ independent of b (see (4.3)), the existence of the left and right derivatives of $\lambda(b)$ is guaranteed by the analytic perturbation theory (see [Kat66]):

$$\lambda'_\pm(b) = \lim_{\varepsilon \to 0^\pm} \frac{\lambda(b + \varepsilon) - \lambda(b)}{\varepsilon}.$$

Proposition 6.3. *Under Assumption 1.3, the limits of $\lambda'_-(b)$ and $\lambda'_+(b)$ as $b \to +\infty$ exist, and we have*

$$\lim_{b \to +\infty} \lambda'_+(b, a) = \lim_{b \to +\infty} \lambda'_-(b, a) = \mu^*,$$

where $\mu^* = \min_{j \in \{1,\dots,n\}} \mu(\alpha_j, a) > 0$.
 Consequently, $b \mapsto \lambda(b)$ is strictly increasing, for large b.

 The proof of Proposition 6.3 is inspired by that of [FH07, Theorem 1.1], although the two proofs slightly differ at the technical level in a way that we will describe below.

 The argument in [FH07] avoids the use of a complete expansion of the ground-state energy. Such expansions have been used in other works such as [FH06, BNF07], and are usually difficult to establish. Fournais and Helffer succeeded to prove the monotonicity of the ground-state by only using its leading order asymptotics. Their proof mainly rely on the control of a certain (error) term, $\|\hat{\mathbf{A}}\psi\|_{L^2(\Omega)}$, appearing in the differentiation of the energy, where ψ is a ground-state of the linear operator and $\hat{\mathbf{A}}$ is a vector potential that we introduce below. In [FH07], they use the fact that their vector potential, denoted by $\mathbf{F}$, generates a constant magnetic field (curl $\mathbf{F} = 1$). In the case where the sample is not a disc, this implies the existence of a part of the boundary (away from the points with maximal curvature) where ψ is negligible. The remaining part, Ω_0, of the boundary (containing the points with maximal curvature) is a simply connected domain. Hence, a gauge transform is used to construct from the potential $\mathbf{F}$ another potential $\hat{\mathbf{A}} \in H^1(\Omega, \mathbb{R}^2)$ such that $|\hat{\mathbf{A}}| \leq C\,\mathrm{dist}(x, \partial\Omega)$ in Ω_0. This upper bound of $|\hat{\mathbf{A}}|$ compensates the fact that ψ is *big* in Ω_0, and, together with the normal and boundary Agmon estimates, allow to control $\|\hat{\mathbf{A}}\psi\|_{L^2(\Omega)}$.

 We adopt a parallel strategy where we use the leading order asymptotics of $\lambda(b)$ established in Theorem 4.2. The intersection points, p_j, of the magnetic edge Γ and the boundary $\partial\Omega$ play the role of the points with maximum curvature in [FH07]. However, the discontinuity of our magnetic field makes us take into consideration the way Γ intersects $\partial\Omega$, while constructing the gauge vector potential $\mathbf{F}_g$ (playing the role of $\hat{\mathbf{A}}$ in [FH07]). This generates a more complicated definition of $\mathbf{F}_g$ related to the geometry of the problem (Lemma 6.4). This definition guarantees that $\mathbf{F}_g$ is in $H^1(\Omega, \mathbb{R}^2)$ and satisfies $|\mathbf{F}_g| \leq C\,\mathrm{dist}(x, \mathsf{p}_j)$ in the vicinity of any point p_j. Consequently, the localization estimates in Theorem 6.1 are sufficient to

control the (error) term $\|\mathbf{F}_g \psi\|_{L^2(\Omega)}$. Here, ψ is a ground-state corresponding to the energy $\lambda(b)$.

Now, we present the approach in details. It is convenient to work in the so-called Frenet coordinates. For $t_0 > 0$, we define

$$\check{\Phi} \; : \; \frac{|\partial\Omega|}{2\pi} \mathbb{S}^1 \times (0, t_0) \ni (s, t) \longmapsto \gamma(s) + t\nu(s) \in \mathbb{R}^2.$$

where $\left(|\partial\Omega|/2\pi\right)\mathbb{S}^1 \ni s \mapsto \gamma(s) \in \partial\Omega$ is the arc length parametrization of $\partial\Omega$, oriented counterclockwise and $\nu(s)$ is the inward unit normal vector of $\partial\Omega$ at the point $\gamma(s)$. We assume that t_0 is sufficiently small so that $\check{\Phi}$ is a diffeomorphism, and we denote its image by $\Omega(t_0)$.

Notice that $t = \text{dist}(\check{\Phi}(s, t), \partial\Omega)$. The Jacobian of $\check{\Phi}$ satisfies $J_{\check{\Phi}} = 1 - t\,k(s)$. Here, $k(s)$ is the curvature of $\partial\Omega$ at the point $\gamma(s)$, which is bounded according to the assumptions on the domain. For more details about Frenet coordinates, see [FH10, Appendix F].

For $j \in \{1, ..., n\}$, let s_j be the abscissa of the point $\mathsf{p}_j \in \Gamma \cap \partial\Omega$ in the Frenet coordinates, that is $(\check{\Phi})^{-1}(\mathsf{p}_j) = (s_j, 0)$. We denote by $l = \min_{p,m} |s_p - s_m|$. For any positive ε such that $\varepsilon < \min(t_0, l/2)$, we define the set:

$$\mathscr{N}(\mathsf{p}_j, \varepsilon) = \left\{ x = \check{\Phi}(s, t) \; : \; 0 < t < \varepsilon, |s - s_j| < \varepsilon \right\}.$$

When the above conditions on ε hold, we choose an $\varepsilon_0 > 0$ to get a family of pairwise disjoint sets $\left(\mathscr{N}(\mathsf{p}_j, 2\varepsilon_0) \right)_{j=1}^n$ of $\Omega(t_0)$.

Lemma 6.4. *Let $\Gamma \cap \partial\Omega = \left\{ \mathsf{p}_j \; : \; j \in \{1, ..., n\} \right\}$. There exist $C > 0$ and a function $\varphi \in H^2(\Omega)$ such that $\mathbf{F}_g = \mathbf{F} + \nabla\varphi$ satisfies for any $j \in \{1, ..., n\}$*

$$|\mathbf{F}_g(x)| \leq C\,\text{dist}(x, \mathsf{p}_j), \; x \in \mathscr{N}(\mathsf{p}_j, \varepsilon_0).$$

Proof. Let $\check{\mathbf{F}} = (\check{F}_1, \check{F}_2)$ be the vector potential defined so that

$$F_1 dx_1 + F_2 dx_2 = \check{F}_1 ds + \check{F}_2 dt.$$

We have

$$\text{curl}_{s,t}\,\check{\mathbf{F}} = \partial_s\check{F}_2 - \partial_t\check{F}_1 = \begin{cases} 1 - t\,k(s), & \text{if } \check{\Phi}(s, t) \in \Omega_1, \\ a(1 - t\,k(s)), & \text{if } \check{\Phi}(s, t) \in \Omega_2. \end{cases}$$

We fix a point $\mathsf{p}_j \in \Gamma \cap \partial\Omega$ and we work locally in the set $\mathcal{N}(\mathsf{p}_j, 2\varepsilon_0)$. After performing a translation, we assume that the Frenet coordinates of p_j are $(0,0)$, but for simplicity we still denote by $\check{\Phi}$ the obtained diffeomorphism $\check{\Phi}^j$. Furthermore, let $\mathcal{N}_m = \mathcal{N}(\mathsf{p}_j, 2\varepsilon_0) \cap \Omega_m$, $m = 1, 2$. To fix computation, we assume w.l.o.g that $\check{\Phi}^{-1}(\partial\mathcal{N}_1 \cap \partial\Omega)$ (respectively $\check{\Phi}^{-1}(\partial\mathcal{N}_2 \cap \partial\Omega)$ is a subset of $\{(s,t) \ : \ s \geq 0, t = 0\}$ (respectively $\{(s,t) \ : \ s \leq 0, t = 0\}$). The curve $\Gamma \cap \mathcal{N}(\mathsf{p}_j, 2\varepsilon_0)$ is transformed to the curve $\check{\Gamma}$ in the (s,t)-plane. Under Assumption 1.1 (particularly Item 6) and due to the nature of the diffeomorphism $\check{\Phi}$, the curve $\check{\Gamma}$ is not tangent to the s-axis. Hence for sufficiently small ε_0, one may distinguish between three cases:

Case 1. $\check{\Gamma} \subset \{(s,t) \ : \ s > 0\}$.

Case 2. $\check{\Gamma} \subset \{(s,t) \ : \ s < 0\}$.

Case 3. $\check{\Gamma} \subset \{(s,t) \ : \ s = 0\}$.

In each of the first two cases, we assume that ε_0 is small enough so that the curve $\check{\Gamma}$ corresponds to a strictly monotonous function $s \mapsto f(s)$. We consider the vector potential $\check{\mathbf{F}}_g^j = \left(0, \check{F}_g^j\right) \in H^1\left(\check{\Phi}^{-1}\left(\mathcal{N}(\mathsf{p}_j, 2\varepsilon_0)\right)\right)$, where $\check{F}_g^j$ is defined in $\check{\Phi}^{-1}\left(\mathcal{N}(\mathsf{p}_j, 2\varepsilon_0)\right)$, in each of the three cases above, as follows:

Case 1. $\check{F}_g^j(s,t)$ is given by

$$s + (a-1)f^{-1}(t) - at\int_0^{f^{-1}(t)} k(s')\,ds' - t\int_{f^{-1}(t)}^s k(s')\,ds',\ s > 0\ \&\ s \geq f^{-1}(t),$$

and

$$as - at\int_0^s k(s')\,ds', \quad \text{elsewhere.}$$

Case 2. $\check{F}_g^j(s,t)$ is given by

$$as + (1-a)f^{-1}(t) - t\int_0^{f^{-1}(t)} k(s')\,ds' - at\int_{f^{-1}(t)}^s k(s')\,ds',\ s < 0\ \&\ s \leq f^{-1}(t),$$

and

$$s - t\int_0^s k(s')\,ds', \quad \text{elsewhere.}$$

Case 3.

$$\check{F}_g^j(s, t) = \begin{cases} s - t \int_0^s k(s') \, ds', & s > 0, \\ as - at \int_0^s k(s') \, ds', & s < 0. \end{cases}$$

Note that $\check{\Phi}^{-1}\big(\mathcal{N}(\mathsf{p}_j, 2\varepsilon_0)\big)$ is simply connected and $\mathrm{curl}_{s,t}\,\check{\mathbf{F}} = \mathrm{curl}_{s,t}\,\check{\mathbf{F}}_g^j$ in each of the aforementioned cases. Consequently, there exists a function $\check{\varphi}^j \in H^2\big(\Phi^{-1}\big(\mathcal{N}(\mathsf{p}_j, 2\varepsilon_0)\big)\big)$ such that

$$\check{\mathbf{F}} + \nabla_{s,t}\check{\varphi}^j = \check{\mathbf{F}}_g^j.$$

Having $k(s)$ bounded and ε_0 small, and using the properties of the diffeomorphism Φ, one can see that $|\check{\mathbf{F}}_g^j| \leq C_1|s| \leq C \,\mathrm{dist}(x, \mathsf{p}_j)$ for some $C_1, C > 0$.

Now, we consider $\chi \in \mathscr{C}^\infty(\overline{\Omega})$ such that

$$\mathrm{supp}\,\chi \subset \bigcup_{j=1}^n \mathcal{N}(\mathsf{p}_j, 2\varepsilon_0), \quad 0 \leq \chi \leq 1 \quad \text{and} \quad \chi = 1 \text{ in } \bigcup_{j=1}^n \mathcal{N}(\mathsf{p}_j, \varepsilon_0).$$

Hence, defining $\varphi(x) = \check{\varphi}^j\big((\check{\Phi})^{-1}(x)\big)\chi(x)$ completes the proof. □

Proof of Proposition 6.3. Let $b \geq 0$ and M be the multiplicity of $\lambda(b)$. Recall that the domains of the corresponding operator and quadratic form are independent of b (see (4.3)). The perturbation theory asserts the existence of $\varepsilon > 0$, and analytic functions

$$(b - \varepsilon, b + \varepsilon) \ni \beta \mapsto \psi_m \in H^2(\Omega)\setminus\{0\},$$

$$(b - \varepsilon, b + \varepsilon) \ni \beta \mapsto E_m \in \mathbb{R},$$

for $m = 1, ..., M$, such that the functions $\{\psi_m(b)\}$ are linearly independent and normalized in $L^2(\Omega)$, and

$$\mathscr{P}_{\beta,\mathbf{F}}\psi_m(\beta) = E_m(\beta)\psi_m(\beta), \quad E_m(b) = \lambda(b).$$

For small ε, there exist m_+ and m_- in $\{1, ..., M\}$ such that

$$\text{for } \beta \in (b, b + \varepsilon), \qquad E_{m_+}(\beta) = \min_{\{1,...,M\}} E_m(\beta),$$

$$\text{for } \beta \in (b - \varepsilon, b), \qquad E_{m_-}(\beta) = \min_{\{1,...,M\}} E_m(\beta).$$

Let $\mathbf{F}_g$ be the field introduced in Lemma 6.4, and $\mathscr{P}_{b,\mathbf{F}_g}$, $Q_{b,\mathbf{F}_g}$ be the operator and the quadratic form defined in (4.1) and (4.2) respectively. The operators $\mathscr{P}_{b,\mathbf{F}_g}$ and

$\mathscr{P}_{b,\mathbf{F}}$ are unitarily equivalent. Indeed, $\mathscr{P}_{b,\mathbf{F}_g} = e^{ib\varphi}\mathscr{P}_{b,\mathbf{F}}e^{-ib\varphi}$, where φ is the gauge function in Lemma 6.4. Let $\psi_{g,m\pm}(b) = e^{ib\varphi}\psi_{m\pm}(b)$ be normalized eigenfunctions of $\mathscr{P}_{b,\mathbf{F}_g}$, associated with the lowest ground-state energy $\lambda(b)$. By the first order perturbation theory, the derivatives $\lambda'_\pm(b)$ can be written as

$$\lambda'_\pm(b) = \frac{d}{d\beta}Q_{\beta,\mathbf{F}_g}\big(\psi_{g,m\pm}(\beta)\big)\big|_{\beta=b}$$
$$= 2\operatorname{Im}\big\langle \mathbf{F}_g\psi_{g,m\pm}(b),(\nabla - ib\mathbf{F}_g)\psi_{g,m\pm}(b)\big\rangle.$$

This implies for any $B > 0$

$$\lambda'_+(b) = \frac{Q_{b+B,\mathbf{F}_g}\big(\psi_{g,m+}(b)\big) - Q_{b,\mathbf{F}_g}\big(\psi_{g,m+}(b)\big)}{B} - B\int_\Omega |\mathbf{F}_g\psi_{g,m+}(b)|^2\,dx,$$
$$\geq \frac{\lambda(b+B) - \lambda(b)}{B} - B\int_\Omega |\mathbf{F}_g|^2|\psi_{g,m+}(b)|^2\,dx.$$

We decompose the integral in the right hand side of the previous inequality into two, one over $\bigcup_{j=1}^n \mathcal{N}(\mathsf{p}_j,\varepsilon_0)$ and the other over its complement. By Theorem 6.1 and Lemma 6.4, the first integral is bounded from above by Cb^{-1} (assuming b large). The second integral is bounded by $C\|\mathbf{F}_g\|_\infty^2 b^{-1}$, due to the exponential decay in Theorem 6.1[4]. These bounds imply that $\int_\Omega |\mathbf{F}_g|^2|\psi_{g,m+}(b)|^2$ is bounded by Cb^{-1}. Hence, choosing $B = \eta b$ for any $\eta > 0$ and using Propositions 4.5 and 4.7, we get

$$\liminf_{b\to+\infty} \lambda'_+(b) \geq \min_{j\in\{1,\dots,n\}} \mu(\alpha_j,a) - C\eta.$$

Since η is arbitrary, then

$$\liminf_{b\to+\infty} \lambda'_+(b) \geq \min_{j\in\{1,\dots,n\}} \mu(\alpha_j,a). \tag{6.13}$$

For $B < 0$, we use a similar argument to get

$$\limsup_{b\to+\infty} \lambda'_-(b) \leq \min_{j\in\{1,\dots,n\}} \mu(\alpha_j,a). \tag{6.14}$$

By the perturbation theory $\lambda'_+(b) \leq \lambda'_-(b)$. This together with (6.13) and (6.14) complete the proof. $\qquad\square$

---

[4] The fact that $\mathbf{F}_g \in L^\infty(\Omega)$ can be deduced from the explicit definition of this field (in Lemma 6.4) together with the boundedness of the potential $\mathbf{F}$ established in C.1.

Proposition 6.5. *Under Assumption 1.3, there exists $\kappa_0 > 0$ such that for all $\kappa \geq \kappa_0$, the equation in H*

$$\lambda(\kappa H) = \kappa^2$$

has a unique solution, which we denote by $H_{C_3}(\kappa)$.

Proof. Proposition 6.3 and the perturbation theory ensure the existence of b_0 such that $b \mapsto \lambda(b)$ is a strictly increasing continuous function from $[b_0, +\infty)$ onto $[\lambda(b_0), +\infty)$. We may choose b_0 sufficiently large so that for any $0 < b < b_0$, $\lambda(b) < \lambda(b_0)$. Let $\kappa_0 = \sqrt{\lambda(b_0)}$, then for all $\kappa \geq \kappa_0$, the equation

$$\lambda(\kappa H) = \kappa^2$$

admits a unique solution $H_{C_3}(\kappa) = \lambda^{-1}(\kappa^2)/\kappa$, where $\lambda^{-1}(\cdot)$ is the inverse function of $\lambda(\cdot)$ defined on $[\lambda(b_0), +\infty)$. $\square$

Remark 6.6. For $\kappa > 0$, recall the local critical fields, $\overline{H}_{C_3}^{\mathrm{loc}}(\kappa)$ and $\underline{H}_{C_3}^{\mathrm{loc}}(\kappa)$, defined in (1.7) and (1.8) respectively. For sufficiently large values of κ, the equality of these two critical fields follows easily from the result established in Proposition 6.5.

Proposition 6.7. *Under Assumption 1.3, there exists $\kappa_0 > 0$ such that for all $\kappa \geq \kappa_0$, the unique solution, $H = H_{C_3}(\kappa)$, to the equation*

$$\lambda(\kappa H) = \kappa^2$$

satisfies the following. There exist positive constants η_1 and η_2 such that

$$-\eta_1 \kappa^{\frac{1}{2}} \leq H_{C_3}(\kappa) - \frac{\kappa}{\min\limits_{j \in \{1,\dots,n\}} \mu(\alpha_j, a)} \leq \eta_2 \kappa^{\frac{1}{2}}.$$

Proof. We assume that κ is sufficiently large so that the results of Proposition 6.5 hold. We will suitably define two fields $H_1 = H_1(\kappa)$ and $H_2 = H_2(\kappa)$ satisfying

$$\lambda(\kappa H_1) < \kappa^2 \qquad \text{and} \qquad \lambda(\kappa H_2) > \kappa^2,$$

then the desired result follows by using the continuity of $b \mapsto \lambda(b)$.

Set $H_1 = \kappa/\mu^* - \eta_1 \kappa^{\delta_1}$, where $\eta_1 > 0$ and $\delta_1 \in (0, 1)$ are two constants to be chosen soon. For any fixed choice of η_1 and δ_1, we assume that κ is sufficiently

large so that $H_1 > 1$. Hence, Theorem 4.2 asserts the existence of $\kappa_0 > 0$ and $C > 0$ such that for all $\kappa \geq \kappa_0$,

$$
\begin{aligned}
\lambda(\kappa H_1) &\leq \mu^* \kappa H_1 + C(\kappa H_1)^{\frac{3}{4}} \\
&\leq \kappa^2 - \eta_1 \mu^* \kappa^{1+\delta_1} + C\kappa^{\frac{3}{2}} \big((\mu^*)^{-1} - \eta_1 \kappa^{-1+\delta_1}\big)^{\frac{3}{4}} \\
&\leq \kappa^2 - \eta_1 \mu^* \kappa^{1+\delta_1} + C(\mu^*)^{-3/4} \kappa^{\frac{3}{2}}.
\end{aligned}
$$

Choose $\delta_1 = 1/2$ and $\eta_1 > C(\mu^*)^{-7/4}$ (so that $-\eta_1 \mu^* + C(\mu^*)^{-3/4} < 0$). This choice of parameters yields

$$
\lambda(\kappa H_1) < \kappa^2, \qquad \text{for all } \kappa \geq \kappa_0.
$$

Similarly, set $H_2 = \kappa/\mu^* + \eta_2 \kappa^{\delta_2}$, where $\eta_2 > 0$ and $\delta_2 \in (0,1)$ are constants to be chosen. By Theorem 4.2, there exists $\kappa_0 > 0$ and $C > 0$ such that for all $\kappa \geq \kappa_0$,

$$
\lambda(\kappa H_2) \geq \mu^* \kappa H_2 - C(\kappa H_2)^{\frac{3}{4}} \geq \kappa^2 + \eta_2 \mu^* \kappa^{1+\delta_2} - C(\mu^*)^{-\frac{3}{4}} \kappa^{\frac{3}{2}}.
$$

Choose $\delta_2 = 1/2$ and η_2 such that $\eta_2 > C(\mu^*)^{-7/4}$ to obtain

$$
\lambda(\kappa H_2) > \kappa^2, \qquad \text{for all } \kappa \geq \kappa_0.
$$

$\square$

7 Proof of Theorem 1.6

The aim of this section is to establish Theorem 1.6. This theorem displays how, with an increasing field, the order parameter (in (1.4)) and the corresponding GL energy successively decay away from the intersection points of Γ and $\partial\Omega$, $\{p_j\}_j$, according to the ordering of the eigenvalues $\{\mu(\alpha_j, a)\}_j$. Moreover, it asserts the eventual localization of the order parameter near the point(s) p_k with the smallest corresponding eigenvalue μ^*.

The following lower bound is crucial in establishing Theorem 1.6.

Lemma 7.1. *Suppose that Ω satisfies Assumption 1.3. Let* $T = \{1, ..., n\}$ *and $\mu > 0$ satisfy $\mu^* \leq \mu < |a|\Theta_0$. Define*

$$
\Sigma = \{j \in T : \mu(\alpha_j, a) \leq \mu\}, S = \Big\{p_j \in \Gamma \cap \partial\Omega, \ j \in \Sigma\Big\}, d = \min_{j \in T \setminus \Sigma} \mu(\alpha_j, a) - \mu
$$

(in the case $\Sigma = T$, we set $d = |a| \Theta_0 - \mu|$). There exists $C > 0$, and for all $R_0 > 1$ there exists $\tilde{\kappa}_0 > 0$ such that for $\kappa \geq \tilde{\kappa}_0$, if $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$ is a critical point of (1.1), H satisfies $H \geq \kappa/\mu$ and $Q_{\kappa H, \mathbf{A}}$ is the form in (4.2), then for $\varphi \in \mathrm{Dom}\, Q_{\kappa H, \mathbf{A}}$ such that $\mathrm{dist}(\mathrm{supp}\, \varphi, S) \geq R_0 (\kappa H)^{-1/2}$ we have

$$Q_{\kappa H, \mathbf{A}}(\varphi) \geq \kappa H \left(\mu + \frac{d}{2} - \frac{C}{R_0^2} \right) \|\varphi\|^2_{L^2(\Omega)}.$$

Proof. Let $\varphi \in \mathrm{Dom}\, Q_{\kappa H, \mathbf{A}}$ be such that $\mathrm{dist}(\mathrm{supp}\, \varphi, S) \geq R_0 (\kappa H)^{-1/2}$, $\mathbf{F}$ be the vector potential defined in (5.1), and $\beta \in (0, 1)$. We consider the family of cut-off functions $(\chi_j)_{j \in \mathscr{P}}$ introduced in Section 4.2 for $b = \kappa H$ and $\rho = 1/2$. For all $j \in \mathscr{P}$, we define on $\overline{\Omega}$ the function $\phi_j(x) = \left(\mathbf{A}(z_j) - \mathbf{F}(z_j) \right) \cdot x$. As a consequence of the last item in Theorem 5.2, we may approximate the vector potential $\mathbf{A}$ as follows:

$$|\mathbf{A}(x) - \nabla \phi_j(x) - \mathbf{F}(x)| \leq C \frac{R_0^{\beta} (\kappa H)^{-\frac{1}{2}\beta}}{H}, \quad \text{for all } x \in B(z_j, R_0 (\kappa H)^{-\frac{1}{2}}) \cap \overline{\Omega}. \tag{7.1}$$

We choose $\beta = 3/4$ and we define $h = e^{-i\kappa H \phi_j} \varphi$. Using (7.1) and $H \geq \kappa/\mu$, Cauchy's inequality yields

$$\|(\nabla - i\kappa H \mathbf{A})\chi_j \varphi\|^2_{L^2(\Omega)} \geq (1 - \kappa^{-\frac{1}{2}}) \|(\nabla - i\kappa H \mathbf{F})\chi_j h\|^2_{L^2(\Omega)}$$
$$- C R_0^{\frac{3}{2}} \kappa \|\chi_j \varphi\|^2_{L^2(\Omega)}. \tag{7.2}$$

Notice that $\mathrm{supp}\, h = \mathrm{supp}\, \varphi$. Hence (7.2), $H \geq \kappa/\mu$, the support of φ and Proposition 4.6 assert that

$$\|(\nabla - i\kappa H \mathbf{A})\chi_j \varphi\|^2_{L^2(\Omega)} \geq \kappa H \left(\min_{j \in T \setminus \Sigma} \mu(\alpha_j, a) - \frac{C}{R_0^2} - C R_0^4 \kappa^{-1} \right) \|\chi_j \varphi\|^2_{L^2(\Omega)}.$$

Hence, the IMS formula gives

$$\|(\nabla - i\kappa H \mathbf{A})\varphi\|^2_{L^2(\Omega)} \geq \kappa H \left(\min_{j \in T \setminus \Sigma} \mu(\alpha_j, a) - \frac{C}{R_0^2} - C R_0^4 \kappa^{-1} \right) \|\varphi\|^2_{L^2(\Omega)}. \tag{7.3}$$

Choose $\tilde{\kappa}_0$ sufficiently large so that $C R_0^4 \tilde{\kappa}_0^{-1} < d/2$, for $d = \min_{j \in T \setminus \Sigma} \mu(\alpha_j, a) - \mu$.

Consequently, (7.3) yields for all $\kappa \geq \tilde{\kappa}_0$,

$$\|(\nabla - i\kappa H\mathbf{A})\varphi\|^2_{L^2(\Omega)} \geq \kappa H\Big(\min_{j \in T \setminus \Sigma} \mu(\alpha_j, a) - \frac{d}{2} - \frac{C}{R_0^2} \Big)\|\varphi\|^2_{L^2(\Omega)}$$

$$\geq \kappa H\Big(\mu + \frac{d}{2} - \frac{C}{R_0^2}\Big)\|\varphi\|^2_{L^2(\Omega)}. \qquad \square$$

Proof of Theorem 1.6. Let $R_0 > 1$. Take κ_0 in Theorem 1.6 to be $\kappa_0 = \max(\tilde{\kappa}_0, \kappa_1)$, where $\tilde{\kappa}_0$ and κ_1 are the constants in Lemma 7.1 and Theorem 5.3 respectively. Assume that $\mu^{-1} \leq C_1$, where C_1 is the constant in Theorem 5.3, else Equation (1.11) is evidently true for any positive constants C and β, in light of Theorem 5.3. So, the case examined below is

$$\kappa \geq \kappa_0, \quad \text{and } \mu^{-1} \leq \frac{H}{\kappa} \leq C_1. \qquad (7.4)$$

Let S be the set appearing in Lemma 7.1, $t(x) = \text{dist}(x, S)$, and $\tilde{\chi} \in \mathscr{C}^\infty(\mathbb{R})$ be a function satisfying

$$\tilde{\chi} = 0 \text{ on } (-\infty, 1/2] \quad \text{and} \quad \tilde{\chi} = 1 \text{ on } [1, +\infty).$$

We define the two functions χ and f as follows:

$$\chi(x) = \tilde{\chi}\big(R_0^{-1}(\kappa H)^{\frac{1}{2}}t(x)\big) \text{ and } f(x) = \chi(x)\exp\big(\beta(\kappa H)^{\frac{1}{2}}t(x)\big), \qquad (7.5)$$

where β is a positive constant whose value will be fixed soon. Integrating in the first equation of (1.4), we get

$$\int_\Omega |\nabla f|^2 |\psi|^2\, dx \geq \int_\Omega \big|(\nabla - i\kappa H\mathbf{A})f\psi\big|^2\, dx - \kappa^2 \int_\Omega |\psi|^2 f^2\, dx. \qquad (7.6)$$

Notice that the conditions in Lemma 7.1 are satisfied for $\varphi = f\psi$, hence we may apply this lemma to obtain

$$\int_\Omega |\nabla f|^2 |\psi|^2\, dx \geq \Big(\kappa H\Big(\mu + \frac{d}{2} - \frac{C}{R_0^2}\Big) - \kappa^2\Big)\|f\psi\|^2_{L^2(\Omega)}.$$

Since $H \geq \kappa/\mu$, we get further

$$\int_\Omega |\nabla f|^2 |\psi|^2\, dx \geq \Big(\frac{d}{2} - \frac{C}{R_0^2}\Big)\mu^{-1}\kappa^2\|f\psi\|^2_{L^2(\Omega)}. \qquad (7.7)$$

175

On the other hand, using (7.5), we estimate the term $\int_\Omega |\nabla f|^2 |\psi|^2 \, dx$ as follows:

$$\int_\Omega |\nabla f|^2 |\psi|^2 \, dx \le 2\beta^2 \kappa H \|f\psi\|_{L^2(\Omega)}^2 + C(R_0)\kappa H \int_{\left\{\sqrt{\kappa H} t(x) < R_0\right\}} |\psi|^2 \, dx,$$

$$(7.8)$$

where $C(R_0)$ is a constant only dependent on R_0. Recall that we are working under the assumption in (7.4). Hence, we combine (7.7) and (7.8), and we divide by κ^2 to get

$$\left(\frac{\mu^{-1}d}{2} - \frac{C\mu^{-1}}{R_0^2} - 2C_1\beta^2\right)\|f\psi\|_{L^2(\Omega)}^2 \le \tilde{C}(R_0) \int_{\left\{\sqrt{\kappa H} t(x) < R_0\right\}} |\psi|^2 \, dx,$$

where C_1 is the value in (7.4). We choose β small so that $\mu^{-1}d - 4C_1\beta^2 > 0$ (that is $\beta < 1/2\sqrt{\mu^{-1}d/C_1}$). Consequently, for R_0 sufficiently large, we get the existence of $\hat{C} = C(R_0, \beta) > 0$ such that

$$\|f\psi\|_{L^2(\Omega)}^2 \le \hat{C} \int_{\left\{\sqrt{\kappa H} t(x) < R_0\right\}} |\psi|^2 \, dx. \tag{7.9}$$

Plug (7.8) and (7.9) in (7.6) to complete the proof. $\qquad\square$

8 Equality of global and local fields

We consider the global and local critical fields $\overline{H}_{C_3}(\kappa)$, $\underline{H}_{C_3}(\kappa)$, $\overline{H}_{C_3}^{\text{loc}}(\kappa)$ and $\underline{H}_{C_3}^{\text{loc}}(\kappa)$ defined in (1.5), (1.6), (1.7), and (1.8) respectively.

Theorem 8.1. *Let $\kappa > 0$. Under Assumption 1.1, the following relations hold:*

$$\overline{H}_{C_3}(\kappa) \ge \overline{H}_{C_3}^{\text{loc}}(\kappa), \qquad \underline{H}_{C_3}(\kappa) \ge \underline{H}_{C_3}^{\text{loc}}(\kappa). \tag{8.1}$$

Proof. First, we prove the left inequality in (8.1). Let $H < \overline{H}_{C_3}^{\text{loc}}(\kappa)$, hence there exists $H_0 > H$ such that

$$\lambda(\kappa H_0) - \kappa^2 < 0, \tag{8.2}$$

where $\lambda(\kappa H_0)$ is the value in (4.4). It suffices to prove that $H < \overline{H}_{C_3}(\kappa)$. Let ψ_0 be a normalized ground-state of $\mathcal{P}_{\kappa H_0, \mathbf{F}}$ in (4.1). Let $t > 0$, we have

$$\mathcal{E}_{\kappa, H_0}(t\psi_0, \mathbf{F}) = t^2(\lambda(\kappa H_0) - \kappa^2) + \frac{\kappa^2}{2} t^4 \|\psi_0\|_{L^4(\Omega)}^4.$$

Choose t such that $t^2 < 2(\kappa^2 - \lambda(\kappa H_0))/\kappa^2\|\psi_0\|^4_{L^4(\Omega)}$, and use (8.2) to get

$$\mathscr{E}_{\kappa,H_0}(t\psi_0, \mathbf{F}) < 0.$$

This reveals the existence of a non-trivial minimizer of $\mathscr{E}_{\kappa,H_0}$. Recalling the definition of $\overline{H}_{C_3}(\kappa)$, we get that $H < \overline{H}_{C_3}(\kappa)$ which yields the claim.

Secondly, to derive the right inequality in (8.1), we proceed as in the argument above to get that $\mathscr{E}_{\kappa,H}$ has a non-trivial minimizer, for all $H < \underline{H}^{loc}_{C_3}(\kappa)$. Consequently, assuming that $\underline{H}^{loc}_{C_3}(\kappa) > \underline{H}_{C_3}(\kappa)$ contradicts the definition of $\underline{H}_{C_3}(\kappa)$. □

With Theorem 8.1 and the equality of the local critical fields in hand (see Remark 6.6), it remains to prove the equality of the local and global upper fields in order to establish the equality of the global and local fields. This together with Proposition 6.5 and Proposition 6.7 will complete the proof of Theorem 1.5. To this end, we follow similar steps as in [BNF07, Theorem 1.7] and use the following additional result:

Theorem 8.2. *Given $a \in [-1, 1)\backslash\{0\}$, there exist positive constants κ_0, C and δ such that if $\kappa \geq \kappa_0$ and $(\psi, \mathbf{A})$ is a solution of* (1.4) *for $H > 1/|a|\kappa$ then*

$$\int_\Omega \left(|\psi|^2 + \frac{1}{\kappa H}|(\nabla - i\kappa H\mathbf{A})\psi|^2\right) \exp\left(2\delta\sqrt{\kappa H}\,\text{dist}(x, \partial\Omega \cup \Gamma)\right) dx$$

$$\leq C\int_{\Omega\cap\{\text{dist}(x,\partial\Omega\cup\Gamma)<\frac{1}{\sqrt{\kappa H}}\}} |\psi|^2\,dx.$$

Theorem 8.2 displays certain Agmon-type estimates established in [AK16, Theorems 1.5 & 7.3]. These estimates reveal the exponential decay of the order parameter and the GL energy in the bulk of Ω_1 and Ω_2, in a certain regime of the intensity of the applied magnetic field.

Proof of Theorem 1.5. Let $\kappa > 0$. $\overline{H}_{C_3}(\kappa) \geq \overline{H}^{loc}_{C_3}(\kappa)$ was proved in Theorem 8.1. Next, we prove that $\overline{H}_{C_3}(\kappa) \leq \overline{H}^{loc}_{C_3}(\kappa)$. Assume that $\overline{H}_{C_3}(\kappa) > \overline{H}^{loc}_{C_3}(\kappa)$, then the definitions of $\overline{H}_{C_3}(\kappa)$ and $\overline{H}^{loc}_{C_3}(\kappa)$ ensure the existence of $H > 0$ satisfying:

1. $\overline{H}^{loc}_{C_3}(\kappa) < H \leq \overline{H}_{C_3}(\kappa)$.

2. $\lambda(\kappa H) \geq \kappa^2$.

3. The GL functional $\mathscr{E}_{\kappa,H}$ in (1.1) admits a non-trivial minimizer $(\psi, \mathbf{A})$.

In particular, $(\psi, \mathbf{A})$ satisfies

$$\kappa^2 \|\psi\|^2_{L^2(\Omega)} > Q_{\kappa H, \mathbf{A}}(\psi),$$

where $Q_{\kappa H, \mathbf{A}}$ is the quadratic form in (4.2). We define $\Delta = \kappa^2 \|\psi\|^2_{L^2(\Omega)} - Q_{\kappa H, \mathbf{A}}(\psi)$. An integration in the first GL equation of (1.4) gives

$$\|\psi\|^4_{L^4(\Omega)} = \frac{\Delta}{\kappa^2}. \tag{8.3}$$

Furthermore, the assumption that $H > \overline{H}^{loc}_{C_3}(\kappa)$ and the asymptotics of $\overline{H}^{loc}_{C_3}(\kappa)$ in Proposition 6.7 assert that we are working under the conditions of Theorem 8.2 and allow us to write

$$\|\psi\|^2_{L^2(\Omega)} \le C \int_{\Omega \cap \{\mathrm{dist}(x, \partial\Omega \cup \Gamma) < \frac{1}{\sqrt{\kappa H}}\}} |\psi|^2 \, dx$$

$$\le C \|\psi\|^2_{L^4(\Omega)} \Big(\int_{\Omega \cap \{\mathrm{dist}(x, \partial\Omega \cup \Gamma) < \frac{1}{\sqrt{\kappa H}}\}} dx \Big)^{\frac{1}{2}} \le C \kappa^{-\frac{3}{2}} \Delta^{\frac{1}{2}}. \tag{8.4}$$

The last inequality follows from (8.3). Since $\psi \ne 0$ then, using the min-max principle and Cauchy-Schwarz inequality, we can estimate

$$0 < \Delta \le \big(\kappa^2 - (1-\delta)\lambda(\kappa H) \big) \|\psi\|^2_{L^2(\Omega)} + C \delta^{-1}(\kappa H)^2 \|\mathbf{A} - \mathbf{F}\|^2_{L^4(\Omega)} \|\psi\|^2_{L^4(\Omega)}, \tag{8.5}$$

for any $\delta \in (0, 1)$. By the Sobolev estimates in $\mathbb{R}^2$ and the curl-div estimates (see [FH10, Proposition D.2.1]), we have

$$\|\mathbf{A} - \mathbf{F}\|_{L^4(\Omega)} \le C \|\mathbf{A} - \mathbf{F}\|_{H^1(\Omega)} \le C \| \mathrm{curl}(\mathbf{A} - \mathbf{F}) \|_{L^2(\Omega)}.$$

Consequently, since $\mathscr{E}_{\kappa,H}(\psi, \mathbf{A}) \le 0$ we conclude that

$$(\kappa H)^2 \|\mathbf{A} - \mathbf{F}\|^2_{L^4(\Omega)} \le C(\kappa H)^2 \| \mathrm{curl}(\mathbf{A} - \mathbf{F}) \|^2_{L^2(\Omega)} \le C\Delta. \tag{8.6}$$

Choose $\delta = \Delta^{1/2} \kappa^{-3/4}$. The hypothesis on H and the definition of $\overline{H}_{C_3}(\kappa)$ together with Theorem 5.3 ensure that $H \le C_1 \kappa$, where C_1 is the constant in the aforementioned theorem. We use this upper bound of H and Proposition 4.7, and we insert (8.3), (8.4), and (8.6) in (8.5) to get

$$0 < \Delta \le \big(\kappa^2 - \lambda(\kappa H) \big) \|\psi\|^2_{L^2(\Omega)} + C\Delta \kappa^{-\frac{1}{4}}.$$

When κ is big, $1 - C\kappa^{-1/4} > 0$. Therefore, since $\lambda(\kappa H) \geq \kappa^2$ we get

$$0 < (1 - C\kappa^{-\frac{1}{4}})\Delta \leq (\kappa^2 - \lambda(\kappa H))\|\psi\|^2_{L^2(\Omega)} \leq 0,$$

which is absurd. This means that $\overline{H}_{C_3}(\kappa) \leq \overline{H}^{\text{loc}}_{C_3}(\kappa)$. $\qquad\square$

Acknowledgements

A   Some spectral properties of the model operator $\mathscr{H}_{\alpha,a}$

Let $\alpha \in (0, \pi)$ and $a \in [-1, 1)\backslash\{0\}$. Recall the operator $\mathscr{H}_{\alpha,a}$ defined on $\mathbb{R}^2_+$ in Section 3. This appendix is devoted to the establishment some spectral properties of this operator, presented in the aforementioned section. In particular, we prove the claim in Theorem 3.1 that the bottom of the essential spectrum of $\mathscr{H}_{\alpha,a}$ is equal to $|a|\Theta_0$.

Recall the set $\mathscr{M}_r$ defined in (3.11). A central step in proving Theorem 3.1 is to establish Theorem A.1 below.

Theorem A.1. *The essential spectrum of the Neumann realization of the operator $\mathscr{H}_{\alpha,a}$ defined in* (3.2) *satisfies*

$$\inf \text{sp}_{\text{ess}} \mathscr{H}_{\alpha,a} = \Sigma\mathscr{H}_{\alpha,a},$$

where

$$\Sigma\mathscr{H}_{\alpha,a} = \lim_{r\to+\infty} \Sigma(\mathscr{H}_{\alpha,a}, r)$$

and

$$\Sigma(\mathscr{H}_{\alpha,a}, r) = \inf_{\substack{u \in \mathscr{M}_r \\ u \neq 0}} \frac{\|(\nabla - i\mathbf{A}_{\alpha,a})u\|^2_{L^2(\mathbb{R}^2_+)}}{\|u\|^2_{L^2(\mathbb{R}^2_+)}}.$$

Remark A.2. The function $r \mapsto \Sigma(\mathscr{H}_{\alpha,a}, r)$ is increasing on $\mathbb{R}_+$. Indeed, if a function $u \in \mathscr{M}_r$ then $u \in \mathscr{M}_\rho$ for $\rho < r$. Consequently, the limit $\Sigma\mathscr{H}_{\alpha,a}$ exists and belongs to $(0, +\infty]$, having $\Sigma(\mathscr{H}_{\alpha,a}, r)$ positive.

The following lemma is needed in the proof of Theorem A.1.

Lemma A.3. *Let (u_n) be a Weyl sequence of the operator $\mathscr{H}_{\alpha,a}$. For all $r > 0$, (u_n) converges to zero in $L^2(B_r^+)$.*

Proof. A Weyl sequence (u_n) is included in Dom $\mathscr{H}_{\alpha,a}$ and satisfies:

$$\|u_n\|_{L^2(\mathbb{R}_+^2)} = 1, \quad u_n \rightharpoonup 0 \quad \text{and} \quad \|\mathscr{H}_{\alpha,a}u_n - \lambda u_n\|_{L^2(\mathbb{R}_+^2)} \to 0, \quad \text{(A.1)}$$

where λ is the scalar associated to (u_n). First, we prove the boundedness of (u_n) in $H^1(B_r^+)$. Using Cauchy-Schwarz inequality, we have

$$\langle \mathscr{H}_{\alpha,a}u_n, u_n \rangle - \lambda = \langle \mathscr{H}_{\alpha,a}u_n - \lambda u_n, u_n \rangle \leq \|\mathscr{H}_{\alpha,a}u_n - \lambda u_n\|_{L^2(\mathbb{R}_+^2)}. \quad \text{(A.2)}$$

The third property satisfied by (u_n) in (A.1) assures the existence of $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\|\mathscr{H}_{\alpha,a}u_n - \lambda u_n\|_{L^2(\mathbb{R}_+^2)} \leq 1$. Implementing this inequality in (A.2), we get for $n \geq n_0$

$$\langle \mathscr{H}_{\alpha,a}u_n, u_n \rangle - \lambda \leq 1. \quad \text{(A.3)}$$

Having $u_n \in$ Dom $\mathscr{H}_{\alpha,a}$, we integrate by parts in (A.3) to get

$$\|(\nabla - i\mathbf{A}_{\alpha,a})u_n\|_{L^2(\mathbb{R}_+^2)}^2 + \|u_n\|_{L^2(\mathbb{R}_+^2)}^2 \leq \lambda + 2.$$

Particularly,

$$\|(\nabla - i\mathbf{A}_{\alpha,a})u_n\|_{L^2(B_r^+)}^2 + \|u_n\|_{L^2(B_r^+)}^2 \leq \lambda + 2. \quad \text{(A.4)}$$

Thus, there exists $C > 0$ dependent on r such that

$$\|\nabla u_n\|_{L^2(B_r^+)}^2 + \|u_n\|_{L^2(B_r^+)}^2 \leq \lambda + C,$$

having $\mathbf{A}_{\alpha,a}$ bounded in B_r^+. Hence (u_n) is bounded in $H^1(B_r^+)$.

Next, we prove that the sequence (u_n) converges to zero in $L^2(B_r^+)$. Suppose not, then there exist $\varepsilon > 0$ and a subsequence (u_{n_j}) of (u_n) such that

$$\|u_{n_j}\|_{L^2(B_r^+)} > \varepsilon. \quad \text{(A.5)}$$

The boundedness of (u_n) in $H^1(B_r^+)$ and the compact injection of $H^1(B_r^+)$ into $L^2(B_r^+)$ imply that (u_{n_j}) is convergent in $L^2(B_r^+)$, along a subsequence. The second property in (A.1) assures that the limit of this subsequence is zero, which contradicts (A.5). □

Proof of Theorem A.1. First we prove

$$\Sigma \mathscr{H}_{\alpha,a} \le \inf \text{sp}_{\text{ess}}(\mathscr{H}_{\alpha,a}). \tag{A.6}$$

Let $\lambda \in \text{sp}_{\text{ess}}(\mathscr{H}_{\alpha,a})$. Recalling the definition of $\Sigma \mathscr{H}_{\alpha,a}$, it suffices to prove that $\Sigma(\mathscr{H}_{\alpha,a}, r) \le \lambda$ for all $r > 0$. We consider the Weyl sequence (u_n) associated to λ, and localize this sequence outside B_r^+ by using a truncation function $\chi \in \mathscr{C}^\infty(\mathbb{R}_+^2, [0,1])$ satisfying for $\rho > r$

$$\chi(x) = 1 \text{ in } B_\rho^{\complement} \cap \mathbb{R}_+^2, \qquad \text{and } \chi(x) = 0 \text{ in } B_r^+. \tag{A.7}$$

Note that $\chi u_n \in \mathscr{M}_r$. The triangle inequality gives

$$\|(\nabla - i\mathbf{A}_{\alpha,a})\chi u_n\|_{L^2(\mathbb{R}_+^2)} \le \|\chi(\nabla - i\mathbf{A}_{\alpha,a})u_n\|_{L^2(\mathbb{R}_+^2)} + \|u_n|\nabla\chi|\|_{L^2(\mathbb{R}_+^2)}. \tag{A.8}$$

Using the properties of χ in (A.7), we have

$$\|u_n|\nabla\chi|\|_{L^2(\mathbb{R}_+^2)}^2 = \int_{B_\rho \cap \mathbb{R}_+^2} |u_n|^2 |\nabla\chi|^2 \, dx \le C^2 \int_{B_\rho \cap \mathbb{R}_+^2} |u_n|^2 \, dx.$$

But (u_n) converges to zero in $L^2(B_\rho \cap \mathbb{R}_+^2)$ by Lemma A.3, then for any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for $n \ge n_0$,

$$\int_{B_\rho \cap \mathbb{R}_+^2} |u_n|^2 \, dx \le \frac{\varepsilon^2}{C^2}.$$

Hence,

$$\|u_n|\nabla\chi|\|_{L^2(\mathbb{R}_+^2)} \le \varepsilon. \tag{A.9}$$

On the other hand, the properties of (u_n) and χ in (A.1) and (A.7) respectively, together with an integration by parts, ensure the existence of $n_1 \ge n_0$ such that for all $n \ge n_1$,

$$\|\chi(\nabla - i\mathbf{A}_{\alpha,a})u_n\|_{L^2(\mathbb{R}_+^2)}^2 \le \|(\nabla - i\mathbf{A}_{\alpha,a})u_n\|_{L^2(\mathbb{R}_+^2)}^2 \le \lambda + \varepsilon.$$

Put the above inequality together with (A.9) into (A.8) to get

$$\|(\nabla - i\mathbf{A}_{\alpha,a})\chi u_n\|_{L^2(\mathbb{R}_+^2)}^2 \le \lambda + C\varepsilon. \tag{A.10}$$

Next, we prove that for n sufficiently large

$$\frac{1}{\|\chi u_n\|_{L^2(\mathbb{R}_+^2)}^2} \le 1 + \varepsilon. \tag{A.11}$$

We have

$$1 = \|u_n\|^2_{L^2(\mathbb{R}^2_+)} \geq \|\chi u_n\|^2_{L^2(\mathbb{R}^2_+)} = \int_{B_\rho \cap \mathbb{R}^2_+} |\chi u_n|^2 \, dx + \int_{B_\rho^C \cap \mathbb{R}^2_+} |u_n|^2 \, dx$$

$$= \int_{\mathbb{R}^2_+} |u_n|^2 \, dx + \int_{B_\rho \cap \mathbb{R}^2_+} (\chi^2 - 1)|u_n|^2 \, dx$$

$$\geq 1 - \int_{B_\rho \cap \mathbb{R}^2_+} |u_n|^2 \, dx. \qquad \text{(A.12)}$$

In light of Lemma A.3, we introduce $\lim_{n \to +\infty}$ on (A.12) to get the convergence of $\|\chi u_n\|^2_{L^2(\mathbb{R}^2_+)}$ to 1 as n tends to $+\infty$, which proves (A.11). The inequalities in (A.10) and (A.11) imply the existence of $n_2 \in \mathbb{N}$ and a positive constant C, independent of ε, such that for all $n \geq n_2$,

$$\frac{\|(\nabla - i\mathbf{A}_{\alpha,a})\chi u_n\|^2_{L^2(\mathbb{R}^2_+)}}{\|\chi u_n\|^2_{L^2(\mathbb{R}^2_+)}} \leq \lambda + C\varepsilon.$$

Then by the definition of $\Sigma(\mathscr{H}_{\alpha,a}, r)$, we get for any $\lambda \in \mathrm{sp}_{\mathrm{ess}}(\mathscr{H}_{\alpha,a})$

$$\Sigma(\mathscr{H}_{\alpha,a}, r) \leq \lambda + C\varepsilon.$$

Taking ε to zero establishes (A.6).

Now we prove that

$$\Sigma\mathscr{H}_{\alpha,a} \geq \inf \mathrm{sp}_{\mathrm{ess}}(\mathscr{H}_{\alpha,a}). \qquad \text{(A.13)}$$

Let $\mu < \inf \mathrm{sp}_{\mathrm{ess}}(\mathscr{H}_{\alpha,a})$ and $\varepsilon > 0$. By Remark A.2, it is sufficient to establish the existence of $r_\varepsilon > 0$ such that

$$\Sigma(\mathscr{H}_{\alpha,a}, r_\varepsilon) \geq \mu - \mathcal{O}(\varepsilon). \qquad \text{(A.14)}$$

By the min-max principle, the previous inequality trivially holds if $\mu < \inf \mathrm{sp}(\mathscr{H}_{\alpha,a})$. Assume now that $\inf \mathrm{sp}(\mathscr{H}_{\alpha,a}) \leq \mu < \inf \mathrm{sp}_{\mathrm{ess}}(\mathscr{H}_{\alpha,a})$. Let $q_{\alpha,a}$ be the quadratic form associated to $\mathscr{H}_{\alpha,a}$, and $\mathbb{1}_{(-\infty,\mu]}(\mathscr{H}_{\alpha,a})$ be the spectral projection operator corresponding to this operator, that has finite rank (since we are below the essential spectrum). There exists a finite orthonormal system of normalized eigenfunctions $(v_i) \in L^2(\mathbb{R}^2_+)$ such that

$$\mathbb{1}_{(-\infty,\mu]}(\mathscr{H}_{\alpha,a}) = \sum_i \langle ., v_i \rangle v_i.$$

For all $x \in \mathbb{R}_+^2$ and $\varphi \in L^2(\mathbb{R}_+^2)$, we have

$$| \mathbb{1}_{(-\infty,\mu]}(\mathscr{H}_{\alpha,a})\varphi|^2(x) = \sum_i |\langle \varphi, v_i \rangle|^2 |v_i(x)|^2 \leq \|\varphi\|^2_{L^2(\mathbb{R}_+^2)} \sum_i |v_i(x)|^2.$$

Since the sum is over a finite set and (v_i) are in $L^2(\mathbb{R}_+^2)$, then the dominated convergence theorem asserts that, for all $\varepsilon > 0$, there exists r_ε such that

$$\text{for all } \varphi \in L^2(\mathbb{R}_+^2), \int_{|x| \geq r_\varepsilon} | \mathbb{1}_{(-\infty,\mu]}(\mathscr{H}_{\alpha,a})\varphi|^2(x) \, dx \leq \varepsilon \|\varphi\|^2_{L^2(\mathbb{R}_+^2)}.$$

Hence for all $\varphi \in \mathscr{M}_{r_\varepsilon}$, it holds

$$\| \mathbb{1}_{(-\infty,\mu]}(\mathscr{H}_{\alpha,a})\varphi\|^2_{L^2(\mathbb{R}_+^2)} \leq \varepsilon \|\varphi\|^2_{L^2(\mathbb{R}_+^2)}. \tag{A.15}$$

Using the properties of the spectral projections, we have for all $\varphi \in \mathscr{M}_{r_\varepsilon}$,

$$q_{\alpha,a}(\varphi) = q_{\alpha,a}\big(\mathbb{1}_{(-\infty,\mu]}(\mathscr{H}_{\alpha,a})\varphi\big) + q_{\alpha,a}\big((I - \mathbb{1}_{(-\infty,\mu]}(\mathscr{H}_{\alpha,a}))\varphi\big).$$

The min-max principle and the definition of $\mathbb{1}_{(-\infty,\mu]}(\mathscr{H}_{\alpha,a})$ ensure the boundedness of $q_{\alpha,a}\big((I - \mathbb{1}_{(-\infty,\mu]}(\mathscr{H}_{\alpha,a}))\varphi\big)$ from below by $\mu\|(I - \mathbb{1}_{(-\infty,\mu]}(\mathscr{H}_{\alpha,a}))\varphi\|^2$ (for $\varphi \neq 0$). In addition, $q_{\alpha,a}\big(\mathbb{1}_{(-\infty,\mu]}(\mathscr{H}_{\alpha,a})\varphi\big)$ is non negative, then for all $\varphi \in \mathscr{M}_{r_\varepsilon}$,

$$q_{\alpha,a}(\varphi) \geq \mu\|(I - \mathbb{1}_{(-\infty,\mu]}(\mathscr{H}_{\alpha,a}))\varphi\|^2_{L^2(\mathbb{R}_+^2)}. \tag{A.16}$$

On the other hand, we have

$$\|(I - \mathbb{1}_{(-\infty,\mu]}(\mathscr{H}_{\alpha,a}))\varphi\|^2_{L^2(\mathbb{R}_+^2)} = \|\varphi\|^2_{L^2(\mathbb{R}_+^2)} - \| \mathbb{1}_{(-\infty,\mu]}(\mathscr{H}_{\alpha,a})\varphi\|^2_{L^2(\mathbb{R}_+^2)}.$$

Hence, by (A.15) we get

$$\|(I - \mathbb{1}_{(-\infty,\mu]}(\mathscr{H}_{\alpha,a}))\varphi\|^2_{L^2(\mathbb{R}_+^2)} \geq (1 - \varepsilon)\|\varphi\|^2_{L^2(\mathbb{R}_+^2)}.$$

We use the above inequality together with (A.16) to obtain

$$\frac{q_{\alpha,a}(\varphi)}{\|\varphi\|^2_{L^2(\mathbb{R}_+^2)}} \geq \mu(1 - \varepsilon).$$

Since $\varphi \in \mathscr{M}_{r_\varepsilon}$ is arbitrary, then

$$\Sigma(\mathscr{H}_{\alpha,a}, r_\varepsilon) \geq \mu(1 - \varepsilon).$$

This establishes (A.14) and consequently (A.13). □

Next, we give the proof of Lemma 3.7 which will also be used in the proof of Theorem 3.1 below.

Proof of Lemma 3.7. The main tool is a partition of unity that divides $\mathbb{R}_+^2$ into three sectors, which allows us to use spectral properties of some explored operators in Sections 2.1 and 2.2. One can find a partition of unity $(\hat{\chi}_j)$ for the interval $[0, \pi]$ satisfying

$$\operatorname{supp} \hat{\chi}_1 \subset \left[0, \frac{2}{3}\alpha\right], \quad \operatorname{supp} \hat{\chi}_2 \subset \left[\frac{1}{3}\alpha, \frac{1}{2}\alpha + \frac{\pi}{2}\right], \quad \operatorname{supp} \hat{\chi}_3 \subset \left[\frac{3}{4}\alpha + \frac{\pi}{4}, \pi\right],$$

$$\sum_{j=1}^{3} \hat{\chi}_j^2(\theta) = 1, \quad \sum_{j=1}^{3} |\hat{\chi}_j'^2(\theta)| \le C, \quad \forall \theta \in [0, \pi],$$

where C is a constant dependent on α, but independent of a. Let $r > 0$. We define the truncation functions in polar coordinates

$$\forall (\rho, \theta) \in \mathbb{R}_+ \times (0, \pi), \quad \chi_j^{r,\text{pol}}(\rho, \theta) = \hat{\chi}_j(\theta),$$

for $j \in \{1, 2, 3\}$. The associated functions in the Cartesian coordinates are defined by:

$$\chi_j^r(x_1, x_2) = \chi_j^{r,\text{pol}}(\rho, \theta), \quad (x_1, x_2) \in \mathbb{R}_+^2.$$

Consider a non-zero function $\varphi \in \mathcal{M}_r$. The IMS localization formula ensures that

$$\|(\nabla - i\mathbf{A}_{\alpha,a})\varphi\|_{L^2(\mathbb{R}_+^2)}^2 = \sum_{j=1}^{3} \|(\nabla - i\mathbf{A}_{\alpha,a})(\chi_j^r\varphi)\|_{L^2(\mathbb{R}_+^2)}^2 - \sum_{j=1}^{3} \|\varphi|\nabla\chi_j^r|\|_{L^2(\mathbb{R}_+^2)}^2. \quad \text{(A.17)}$$

We first evaluate the term $\sum_{j=1}^{3} \|\varphi|\nabla\chi_j^r|\|_{L^2(\mathbb{R}_+^2)}^2$. For $(x_1, x_2) \in \mathbb{R}_+^2$, we have

$$|\nabla\chi_j^r(x_1, x_2)|^2 = \left|\partial_\rho \chi_j^{r,\text{pol}}(\rho, \theta)\right|^2 + \frac{1}{\rho^2}\left|\partial_\theta \chi_j^{r,\text{pol}}(\rho, \theta)\right|^2 = \frac{1}{\rho^2}\left|\partial_\theta \chi_j^{r,\text{pol}}(\rho, \theta)\right|^2.$$

By the construction of χ_j^r and due to the support of φ, we get

$$\sum_{j=1}^{3} \|\varphi|\nabla\chi_j^r|\|_{L^2(\mathbb{R}_+^2)}^2 \le \frac{C}{r^2}\|\varphi\|_{L^2(\mathbb{R}_+^2)}^2, \quad \text{(A.18)}$$

for some $C = C(\alpha)$. Next, we bound $\sum_{j=1}^{3} \|(\nabla - i\mathbf{A}_{\alpha,a})(\chi_j^r\varphi)\|_{L^2(\mathbb{R}_+^2)}^2$. The idea is to extend the functions $\chi_j^r\varphi$ by zero, to refer to the operators introduced in the

sections 2.1 and 2.2. Notice that $\operatorname{curl} \mathbf{A}_{\alpha,a} = \operatorname{curl} \mathbf{A}_0 = 1$ in the support of $\chi_1^r \varphi$, where $\mathbf{A}_0$ is the vector potential defined in (2.1). Hence, extending $\chi_1^r \varphi$ by zero in the half-plane $\mathbb{R}_+^2$, and performing a suitable change of gauge, we get by the min-max principle

$$\frac{\|(\nabla - i\mathbf{A}_{\alpha,a})(\chi_1^r \varphi)\|_{L^2(\mathbb{R}_+^2)}^2}{\|\chi_1^r \varphi\|_{L^2(\mathbb{R}_+^2)}^2} \geq \inf_{\substack{u \in \operatorname{Dom} Q_{b=1,\mathbb{R}_+^2} \\ u \neq 0}} \frac{\|(\nabla - i\mathbf{A}_0)u\|_{L^2(\mathbb{R}_+^2)}^2}{\|u\|_{L^2(\mathbb{R}_+^2)}^2} = \Theta_0. \quad \text{(A.19)}$$

(see Section 2.1). Proceeding similarly and using a simple scaling, we get

$$\frac{\|(\nabla - i\mathbf{A}_{\alpha,a})(\chi_3^r \varphi)\|_{L^2(\mathbb{R}_+^2)}^2}{\|\chi_3^r \varphi\|_{L^2(\mathbb{R}_+^2)}^2} \geq |a|\Theta_0. \quad \text{(A.20)}$$

Finally, we extend $\chi_2^r \varphi$ by zero in $\mathbb{R}^2$, and we perform a rotation of domain (by angle $\pi/2 - \alpha$) and a suitable change of gauge to get

$$\frac{\|(\nabla - i\mathbf{A}_{\alpha,a})(\chi_2^r \varphi)\|_{L^2(\mathbb{R}_+^2)}^2}{\|\chi_2^r \varphi\|_{L^2(\mathbb{R}_+^2)}^2} \geq \beta_a, \quad \text{(A.21)}$$

where β_a is the ground-state energy of the operator $\mathscr{L}_a$ in (2.10). Gathering results in (A.19), (A.20) and (A.21) yields

$$\sum_{j=1}^{3} \|(\nabla - i\mathbf{A}_{\alpha,a})(\chi_j^r \varphi)\|_{L^2(\mathbb{R}_+^2)}^2 \geq |a|\Theta_0 \|\varphi\|_{L^2(\mathbb{R}_+^2)}^2. \quad \text{(A.22)}$$

The last inequality follows from the fact that $a \in [-1,1) \setminus \{0\}$, $\beta_a \geq |a|\Theta_0$ (see Section 2.2) and $\sum_{j=1}^{3} |\chi_j^r|^2 = 1$ in $\mathbb{R}_+^2$. Implementing (A.18) and (A.22) in (A.17) completes the proof. □

Proof of Theorem 3.1. We can equivalently prove that $\Sigma \mathcal{H}_{\alpha,a} = |a|\Theta_0$, now that we have Theorem A.1 in hand. This is done in two steps:

Step 1. We prove $\Sigma \mathcal{H}_{\alpha,a} \geq |a|\Theta_0$. Let $r > 0$, recall the definition of $\Sigma(\mathcal{H}_{\alpha,a}, r)$. In light of Lemma 3.7, we get the following lower bound:

$$\Sigma(\mathcal{H}_{\alpha,a}, r) \geq |a|\Theta_0 - \frac{C}{r^2}.$$

Taking $r \to +\infty$ in the inequality above establishes Step 1.

Step 2. We prove $\Sigma \mathscr{H}_{\alpha,a} \leq |a| \Theta_0$. Let $\varepsilon > 0$ and $r > 0$. The Neumann realization of the operator $-(\nabla - i a \mathbf{A}_0)^2$ in the half-plane $\mathbb{R}_+^2$ admits $|a|\Theta_0$ as a ground-state energy. Hence, the min-max principle together with a standard limiting argument ensure the existence of a constant $r > 0$ and a function f, belonging to the form domain of $(\nabla - i a \mathbf{A}_0)^2$ and vanishing outside $B(0, r)$, such that

$$|a|\Theta_0 \leq \frac{\|(\nabla - i a \mathbf{A}_0)f\|_{L^2(\mathbb{R}_+^2)}^2}{\|f\|_{L^2(\mathbb{R}_+^2)}^2} \leq |a|\Theta_0 + \varepsilon.$$

Notice that $\operatorname{curl} \mathbf{A}_{\alpha,a} = \operatorname{curl} a \mathbf{A}_0 = a$ in the set D_α^2 defined in (3.1). Hence, one may perform a translation and a change of gauge to obtain from f a function v, supported in $B_r^{\complement} \cap D_\alpha^2$ and satisfying

$$\frac{\|(\nabla - i a \mathbf{A}_0)f\|_{L^2(\mathbb{R}_+^2)}^2}{\|f\|_{L^2(\mathbb{R}_+^2)}^2} = \frac{\|(\nabla - i \mathbf{A}_{\alpha,a})v\|_{L^2(\mathbb{R}_+^2)}^2}{\|v\|_{L^2(\mathbb{R}_+^2)}^2}.$$

Consequently,

$$\Sigma(\mathscr{H}_{\alpha,a}, r) \leq \frac{\|(\nabla - i \mathbf{A}_{\alpha,a})v\|_{L^2(\mathbb{R}_+^2)}^2}{\|v\|_{L^2(\mathbb{R}_+^2)}^2} \leq |a|\Theta_0 + \varepsilon.$$

Take successively ε to zero and r to $+\infty$ to complete the proof of Step 2. □

Proof of Lemma 3.4. Let $h \in \mathbb{R}$ such that $a + h \in [-1, 1) \backslash \{0\}$. We prove that $\lim_{h \to 0} \mu(\alpha, a + h, r) = \mu(\alpha, a, r)$. Let $u \in \mathscr{D}_r$ such that $\|u\|_{L^2(\mathbb{R}_+^2)} = 1$. We extend u by zero outside the ball B_r and we use the min-max principle together with Cauchy's inequality to write,

$$\mu(\alpha, a, r) \leq q_{\alpha,a}(u) \leq (1 + |h|)q_{\alpha,a+h}(u) + C|h|^{-1} \int_{B_r} h^2(x_1^2 + x_2^2)|u|^2 \, dx$$

$$\leq (1 + |h|)q_{\alpha,a+h}(u) + C(r)|h|,$$

where $C(r)$ is a constant solely dependent on r. Again the min-max principle gives

$$\mu(\alpha, a, r) \leq (1 + |h|)\mu(\alpha, a + h, r) + C(r)|h|.$$

Taking h to zero, we get $\mu(\alpha, a, r) \leq \liminf_{h \to 0} \mu(\alpha, a + h, r)$.

In a similar fashion, we establish that $\mu(\alpha, a, r) \geq \limsup_{h \to 0} \mu(\alpha, a + h, r)$. □

B   Change of variables

B.1   Frenet coordinates

In this section we assume that the set Γ consists of a simple smooth curve that transversely intersects the boundary of Ω in two points. In the general case, Γ consists of a finite number of (disjoint) such curves. We may reduce to the simple case above by working on each component separately. We introduce some *Frenet coordinates* which are valid in a tubular neighbourhood of Γ. These coordinates are known in the literature. We list below some of their basic properties. For more details, see [FH10, Appendix F] and [AKPS19].

Let $[-|\Gamma|/2, |\Gamma|/2] \ni s \longmapsto M(s) \in \Gamma$ be the arc length parametrization of Γ. Let $T(s)$ be a unit tangent vector to Γ at the point $M(s)$, and $\nu(s)$ be the unit normal of Γ at the point $M(s)$, pointed toward Ω_1. The orientation of the parametrization M is fixed as follows:

$$\det\big(T(s), \nu(s)\big) = 1.$$

The curvature k_r of Γ is defined by $T'(s) = k_r(s)\nu(s)$. For $t_0 > 0$, we define the transformation

$$\Phi : \left(-\frac{|\Gamma|}{2}, \frac{|\Gamma|}{2}\right) \times (-t_0, t_0) \ni (s, t) \longmapsto M(s) + t\nu(s) \in \mathbb{R}^2.$$

For a sufficiently small t_0, Φ is a diffeomorphism from $\big(-|\Gamma|/2, |\Gamma|/2\big) \times (-t_0, t_0)$ to $\Gamma(t_0)$, where $\Gamma(t_0) := \mathrm{Im}$. The Jacobian of Φ is

$$a(s, t) = J_\Phi(s, t) = 1 - t\,k_r(s). \tag{B.1}$$

The inverse, Φ^{-1}, of Φ defines a system of coordinates for the tubular neighbourhood $\Gamma(t_0)$ of Γ,

$$\Phi^{-1}(x) = \big(s(x), t(x)\big).$$

Note that since the curvature is bounded, then (B.1) implies the existence of $C > 0$ such that

$$\big|J_{\Phi^{-1}}(x) - 1\big| \leq C\ell \qquad \text{and} \qquad \big|J_\Phi(s, t) - 1\big| \leq C\ell, \tag{B.2}$$

where $x \in B(\ell) \subset \Gamma(t_0)$, $B(\ell)$ is a ball of radius ℓ, and $(s, t) = \big(s(x), t(x)\big)$.

To each function $u \in H_0^1\big(\Gamma(t_0)\big)$, we associate the function $\tilde{u} \in H^1\big(\big(- |\Gamma|/2, |\Gamma|/2\big) \times (-t_0, t_0)\big)$ as follows:

$$\tilde{u}(s, t) = u\big(\Phi(s, t)\big).$$

We also associate to any vector potential $\mathbf{E} = (E_1, E_2) \in H_{\mathrm{loc}}^1(\mathbb{R}^2, \mathbb{R}^2)$, the vector field $\tilde{\mathbf{E}} = (\tilde{E}_1, \tilde{E}_2) \in H^1\big(\big(- |\Gamma|/2, |\Gamma|/2\big) \times (-t_0, t_0), \mathbb{R}^2\big)$, where

$$\tilde{E}_1(s, t) = a(s, t) E\big(\Phi(s, t)\big) \cdot T(s) \quad \text{and} \quad \tilde{E}_2(s, t) = E\big(\Phi(s, t)\big) \cdot \nu(s). \quad \text{(B.3)}$$

We have the following change of variable formulae:

$$\int_{\Gamma(t_0)} \big|(\nabla - i\mathbf{E})u\big|^2 \, dx = \int_{-\frac{|\Gamma|}{2}}^{\frac{|\Gamma|}{2}} \int_{-t_0}^{t_0} \Big(a^{-2}\big|(\partial_s - i\tilde{E}_1)\tilde{u}\big|^2 + \big|(\partial_t - i\tilde{E}_2)\tilde{u}\big|^2\Big) a \, ds \, dt.$$
$$\text{(B.4)}$$

Finally, we present the following gauge transformation lemma:

Lemma B.1. *Let $a \in [-1, 1)\backslash\{0\}$ and $\overline{B_\ell} \subset (-|\Gamma|/2, |\Gamma|/2) \times (-t_0, t_0)$ be a ball of radius ℓ such that $\Phi(B_\ell) \subset \Omega$. If $\mathbf{E}$ is a vector potential in $H^1(\Omega, \mathbb{R}^2)$ with $\mathrm{curl}\, \mathbf{E} = \mathbb{1}_{\Omega_1} + a\mathbb{1}_{\Omega_2}$, then there exists a function $\omega_\ell \in H^2(B_\ell)$ such that the vector potential $\tilde{\mathbf{E}}_{\mathrm{g}} := \tilde{\mathbf{E}} - \nabla_{s,t}\omega_\ell$, defined in B_ℓ, satisfies*

$$\big(\tilde{E}_{\mathrm{g}}\big)_1(s, t) = \begin{cases} -\big(t - \frac{t^2}{2} k_r(s)\big), & \text{if } t > 0 \\ -a\big(t - \frac{t^2}{2} k_r(s)\big), & \text{if } t < 0 \end{cases}; \qquad \big(\tilde{E}_{\mathrm{g}}\big)_2(s, t) = 0.$$

B.2 Coordinates near $\Gamma \cap \partial\Omega$

In this section we will explicitly define the diffeomorphism Ψ introduced in Section 4.2. The construction of Ψ below is inspired by [Bon03, Lemma 14.3].

For $j \in \{1, ..., n\}$, consider $\mathsf{p}_j \in \Gamma \cap \partial\Omega$ and α_j the corresponding angle introduced in Notation 1.2. We choose a system of coordinates such that p_j is the origin, there exists a neighbourhood of p_j where $\partial\Omega$ and Γ coincide respectively with the representative curves of two smooth monotonous functions f_1 and f_2, defined in an interval $(-r_j, r_j)$ for a small $r_j > 0$ and the following is satisfied:

$$f_1(0) = 0, \ f_2(0) = 0, \ f_1'(0) = -\tan\frac{\alpha_j}{2}, \ f_2'(0) = \tan\frac{\alpha_j}{2},$$

$$\Omega \cap B(0, r_j) = E \cap B(0, r_j),$$
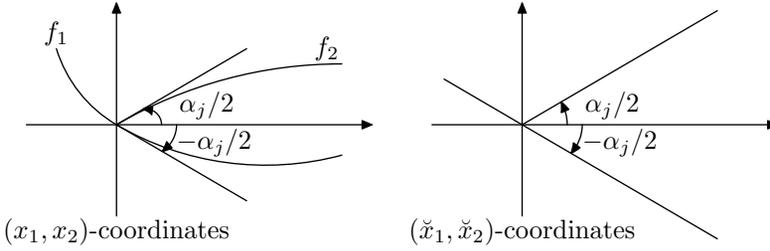
Figure 4: Change of coordinates from (x_1, x_2) to $(\check{x}_1, \check{x}_2)$.

$$E := \{(x_1, x_2) \; : \; x_1 \geq 0 \text{ and } f_1(x_1) < x_2 \leq f_2(x_1)\}$$
$$\cup \{(x_1, x_2) \; : \; x_2 \geq 0 \text{ and } f_1^{-1}(x_2) < x_1 \leq f_2^{-1}(x_2)\}.$$

We define the diffeomorphism $\check{\Psi}$ in $B(0, r_j)$ by (see Figure 4)

$$\check{\Psi}(x_1, x_2) = \left(\frac{f_2(x_1) - f_1(x_1)}{2 \tan \frac{\alpha_j}{2}}, x_2 - \frac{f_2(x_1) + f_1(x_1)}{2} \right) := (\check{x}_1, \check{x}_2).$$

By performing a rotation of axes of an angle $-\alpha_j/2$, we can define out of $\check{\Psi}$ a diffeomorphism Ψ satisfying the desired conditions in Section 4.2.

C Regularity properties

Let $b > 0$. Recall the operator $\mathcal{P}_{b,\mathbf{F}}$ and the associated quadratic form $Q_{b,\mathbf{F}}$, introduced in (4.1) and (4.2) respectively, where $\mathbf{F}$ is the vector potential in $H^1_{\text{div}}(\Omega)$ satisfying curl $\mathbf{F} = B_0 = \mathbb{1}_{\Omega_1} + a\mathbb{1}_{\Omega_2}$, $a \in [-1, 1)\backslash\{0\}$. In this section we prove the claim in (4.3) that the corresponding domains of $\mathcal{P}_{b,\mathbf{F}}$ and $Q_{b,\mathbf{F}}$ are independent of the parameter b.

A key-ingredient of the argument is the boundedness of the field $\mathbf{F}$. This boundedness is known for smooth fields, but it should be ensured for our potential with the piecewise-constant field B_0. As will be seen below, the fact that $\mathbf{F} \in H^1_{\text{div}}(\Omega)$ and $B_0 \in L^p(\Omega)$, for $p \in [1, \infty]$, is sufficient for our needs.

Theorem C.1. *Let* $a \in [-1, 1)\backslash\{0\}$ *and* $\mathbf{F} \in H^1_{\text{div}}(\Omega)$ *be such that* curl $\mathbf{F} = \mathbb{1}_{\Omega_1} + a\mathbb{1}_{\Omega_2}$, *then* $\mathbf{F} \in L^\infty(\Omega)$.

Proof. Since $\mathbf{F} \in H^1_{\text{div}}(\Omega)$ and curl $\mathbf{F} = B_0 \in L^2(\Omega)$ then $\mathbf{F} = (-\partial_{x_2} u, \partial_{x_1} u)$, where u is the unique solution in $H^1_0(\Omega) \cap H^2(\Omega)$ of the Dirichlet problem for

189

the Laplacian $-\Delta u = B_0$ (see [FH10, Propositions D.2.1 & D.2.5] and [GT00, Theorem 9.15]).

Now, notice that $B_0 \in L^p(\Omega)$, for all $p \in [1, +\infty]$. Consequently, for a fixed $p \in [2, +\infty)$ there exists a unique $v \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ satisfying $-\Delta v = B_0$ ([GT00, Theorem 9.15]). But $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega) \subset H_0^1(\Omega) \cap H^2(\Omega)$, thus $v = u$ and $\mathbf{F} = (-\partial_{x_2} v, \partial_{x_1} v)$. Pick $p = 4$, [GT00, (7.30)] asserts that $v \in \mathscr{C}^1(\Omega)$ and $\partial_{x_1} v, \partial_{x_2} v \in L^\infty(\Omega)$. This completes the proof. $\qquad\square$

Proof of (4.3). With $\mathbf{F} \in L^\infty(\Omega)$ in hand, the proof is easy to establish. We will only derive the operator domain result in (4.3). Let $u \in \mathrm{Dom}\,\mathscr{P}_{b,\mathbf{F}}$. We have

$$\Delta u = (\nabla - i\,b\mathbf{F})^2 u + 2i\,b\mathbf{F} \cdot \nabla u + |b|^2 |\mathbf{F}|^2 u.$$

Since $\mathbf{F} \in H_{\mathrm{div}}^1(\Omega) \cap L^\infty(\Omega)$, we get that $\Delta u \in L^2(\Omega)$ and $\nabla u \cdot \nu_{|\partial\Omega} = 0$. This ensures that $u \in H^2(\Omega)$ (see [FH10, Theorem E.4.7]). One can similarly establish the opposite inclusion; $\{u \in H^2(\Omega) \ : \ \nabla u \cdot \nu|_{\partial\Omega} = 0\} \subset \mathrm{Dom}\,\mathscr{P}_{b,\mathbf{F}}$. $\qquad\square$

Bibliography

[Agm14]   S. Agmon. *Lectures on Exponential Decay of Solutions of Second-Order Elliptic Equations: Bounds on Eigenfunctions of N-Body Schrodinger Operations.(MN-29).* Princeton University Press, 2014.

[AH07]   Y. Almog and B. Helffer. The distribution of surface superconductivity along the boundary: On a conjecture of X.B. Pan. *SIAM J. Math. Anal.*, 38(6):1715–1732, 2007.

[AK16]   W. Assaad and A. Kachmar. The influence of magnetic steps on bulk superconductivity. *Discrete Contin. Dyn. Syst. Ser. A*, 36:6623–6643, 2016.

[AKPS19]   W. Assaad, A. Kachmar, and M. Persson-Sundqvist. The distribution of superconductivity near a magnetic barrier. *Comm. Math. Phys.*, 366(1):269–332, 2019.

[BN12]   V. Bonnaillie-Noël. Harmonic oscillators with Neumann condition on the half-line. *Commun. Pure Appl. Math*, 11(6):2221–2237, 2012.

[BND06]   V. Bonnaillie-Noël and M. Dauge. Asymptotics for the low-lying eigenstates of the Schrödinger operator with magnetic field near corners. 7(5):899–931, 2006.

[BNDMV07] V. Bonnaillie-Noël, M. Dauge, D. Martin, and G. Vial. Computations of the first eigenpairs for the Schrödinger operator with magnetic field. *Comput. Methods Appl. Mech. Engrg.*, 196(37):3841–3858, 2007.

[BNF07]   V. Bonnaillie-Noël and S. Fournais. Superconductivity in domains with corners. *Rev. Math. Phys.*, 19(06):607–637, 2007.

[Bon03]   V. Bonnaillie. *Analyse mathématique de la supraconductivité dans un domaine à coins: méthodes semi-classiques et numériques.* PhD thesis, Université Paris Sud-Paris XI, 2003.

[Bon05]   V. Bonnaillie. On the fundamental state energy for a Schrödinger operator with magnetic field in domains with corners. *Asymptot. Anal.*, 41(3-4):215–258, 2005.

[BS98]   A. Bernoff and P. Sternberg. Onset of superconductivity in decreasing fields for general domains. *J. Math. Phys.*, 39(3):1272–1284, 1998.

[CFKS09]   H.L. Cycon, R.G. Froese, W. Kirsch, and B. Simon. *Schrödinger Operators: With Application to Quantum Mechanics and Global Geometry.* Springer, 2009.

[CG17]   M. Correggi and E. L. Giacomelli. Surface superconductivity in presence of corners. *Rev. Math. Phys.*, 29(2):1750005, 2017.

[CR14]   M. Correggi and N. Rougerie. On the Ginzburg–Landau functional in the surface superconductivity regime. *Commun. Math. Phys.*, 332(3):1297–1343, 2014.

[DR13]   N. Dombrowski and N. Raymond. Semiclassical analysis with vanishing magnetic fields. *J. Spectr. Theory*, 3(3):423–464, 2013.

[ELPO18]   P. Exner, V. Lotoreichik, and A. Pérez-Obiol. On the bound states of magnetic Laplacians on wedges. *Rep. Math. Phys.*, 82(2):161–185, 2018.

[Erd97]   László Erdős. Dia-and paramagnetism for nonhomogeneous magnetic fields. *J. Math. Phys.*, 38(3):1289–1317, 1997.

[FH06]   S. Fournais and B. Helffer. On the third critical field in Ginzburg–Landau theory. *Commun. Math. Phys.*, 266(1):153–196, 2006.

[FH07]   S. Fournais and B. Helffer. Strong diamagnetism for general domains and application. *Ann. Inst. Fourier*, 57(7):2389–2400, 2007.

[FH09]   S. Fournais and B. Helffer. On the Ginzburg–Landau critical field in three dimensions. *Comm. Pure Appl. Math.*, 62(2):215–241, 2009.

[FH10]   S. Fournais and B. Helffer. *Spectral Methods in Surface Superconductivity*, volume 77. Springer Science & Business Media, 2010.

[FP11]   S. Fournais and M. Persson. Strong diamagnetism for the ball in three dimensions. *Asymptot. Anal.*, 72(1-2):77–123, 2011.

[FPS15]   S. Fournais and M. Persson-Sundqvist. Lack of diamagnetism and the Little–Parks effect. *Commun. Math. Phys.*, 337(1):191–224, 2015.

[GP99]   T. Giorgi and D. Phillips. The breakdown of superconductivity due to strong fields for the Ginzburg–Landau model. *SIAM J. Math. Anal.*, 30:341–359, 1999.

[GR15]   I.S. Gradshteyn and I.M. Ryzhik. *Table of Integrals, Series, and Products*. Elsevier, Academic Press, Amsterdam, 2015.

[GT00]   D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. springer, 2000.

[Hel99]   B. Helffer. *Spectral Theory and Applications*. Cours de D.E.A, Université Paris Sud, Orsay, 1999.

[HFPS11]   B. Helffer, S. Fournais, and M. Persson-Sundqvist. Superconductivity between H_{C_2} and H_{C_3}. *J. Spectr. Theory*, 1(3):273–298, 2011.

[HK18]   B. Helffer and A. Kachmar. The density of superconductivity in domains with corners. *Lett. Math. Phys.*, 108(9):2169–2187, 2018.

[HM96]   B. Helffer and A. Mohamed. Semiclassical analysis for the ground state energy of a Schrödinger operator with magnetic wells. *J. Funct. Anal.*, 138(1):40–81, 1996.

[HM01]   B. Helffer and A. Morame. Magnetic bottles in connection with superconductivity. *J. Funct. Anal.*, 185(2):604–680, 2001.

[HP03]   B. Helffer and X.B. Pan. Upper critical field and location of surface nucleation of superconductivity. *Ann. Inst. H.Poincaré Anal. Non Linéaire*, 20(1):145–181, 2003.

[HPRS16]   P.D. Hislop, N. Popoff, N. Raymond, and M. Sundqvist. Band functions in the presence of magnetic steps. *Math. Models Methods Appl. Sci.*, 26(1):161–184, 2016.

[HS15]   P.D. Hislop and E. Soccorsi. Edge states induced by Iwatsuka Hamiltonians with positive magnetic fields. *J. Math. Anal. Appl.*, 422(1):594–624, 2015.

[Iwa85]   A. Iwatsuka. Examples of absolutely continuous Schrödinger operators in magnetic fields. *Publ. Res. I. Math. Sci.*, 21(2):385–401, 1985.

[Jad01]   H.T. Jadallah. The onset of superconductivity in a domain with a corner. *J. Math. Phys.*, 42(9):4101–4121, 2001.

[Kat66]   T. Kato. *Perturbation Theory for Linear Operators*. Springer-Verlag, New York, 1966.

[Lei83]   H. Leinfelder. Gauge invariance of schrödinger operators and related spectral properties. *J. Operator Theory*, 9(1):163–179, 1983.

[LP62]   W.A. Little and R.D. Parks. Observation of quantum periodicity in the transition temperature of a superconducting cylinder. *Phys. Rev. Lett.*, 9(1):9–12, 1962.

[LP99]   K. Lu and X.B. Pan. Estimates of the upper critical field for the Ginzburg–Landau equations of superconductivity. *Physica D*, 127(1):73–104, 1999.

[LP00]   K. Lu and X.B. Pan. Gauge invariant eigenvalue problems in $\mathbb{R}^2$ and $\mathbb{R}^2_+$. *Trans. Amer. Math. Soc.*, 352(3):1247–1276, 2000.

[Pan02]   X.B. Pan. Surface superconductivity in applied magnetic fields above H_{C_2}. *Comm. Math. Phys.*, 228(2):327–370, 2002.

[Per60]   A. Persson. Bounds for the discrete part of the spectrum of a semi-bounded Schrödinger operator. *Math. Scand.*, 8(1):143–153, 1960.

[Ray09]   N. Raymond. Sharp asymptotics for the Neumann Laplacian with variable magnetic field: case of dimension 2. *Ann. Henri Poincaré*, 10(1):95–122, 2009.

[SJG63]   D. Saint James and P.G. de Gennes. Onset of superconductivity in decreasing fields. *Phys. Lett.*, 7(5):306–308, 1963.

[SS03]   E. Sandier and S. Serfaty. The decrease of bulk superconductivity close to the second critical field in the Ginzburg–Landau model. *SIAM J. Math. Anal.*, 34(4):939–956, 2003.