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Franco, Antonio; Landfeldt, Björn; Körner, Ulf

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Analysis of Age of Information Threshold Violations

Antonio Franco
antonio.franco@eit.lth.se
Lund University
Department of Electrical and Information Technology
Lund, Sweden

Björn Landfeldt
bjorn.landfeldt@eit.lth.se
Lund University
Department of Electrical and Information Technology
Lund, Sweden

Ulf Körner
ulf.korner@eit.lth.se
Lund University
Department of Electrical and Information Technology
Lund, Sweden

ABSTRACT

We study a scenario where a monitor is interested in the freshest possible update from a remote sensor. The monitor also seeks to minimize the number of updates that exceed a certain freshness threshold, beyond which, the information is deemed to be too old. Previous work has presented results for First Come First Served (FCFS) systems. However, it has been shown that Last Come First Served (LCFS) with preemption is more effective in terms of average Age of Information (AoI); we therefore study an M/G/1 LCFS system with preemption. The generality of the busy time distribution gives the advantage of applicability on any distribution inside the model. For example, one can use a deterministic distribution to study a TDMA system, a gamma distribution to model a routing network, or a more complicated distribution to study a CSMA access scheme. We find a general procedure to derive the exact expression of the outage update probability – i.e. the portion of time updates have information older than a certain threshold. We compare different busy time distributions to the ones already present in literature for equivalent FCFS systems, showing the benefit of using the former discipline. We further study how the variance of the busy time distribution affects the update outage probability. We compare the M/D/1 LCFS with preemption against the M/Γ/1 LCFS with preemption and let the variance of the busy time of the latter vary, while maintaining the same average busy time for both systems. We find that at low thresholds and low loads, higher variance gives an advantage in terms of update outage probability.

CCS CONCEPTS

• **Mathematics of computing → Queueing theory**; • **Computer systems organization** → *Sensor networks*; • **Networks** → Network performance analysis.

KEYWORDS

age of information; analytical; LCFS queues; preemption; status updates

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1 INTRODUCTION

The Internet of Things (IoT), opens up new challenges in many ways and early studies of IoT systems have opened up new ways of thinking about networked systems. A new concept called Age of Information (AoI) [13], was put forward where information is considered as a combination of application specific parameters and network specific parameters. In the AoI view, the 'freshness' of data is considered, which may not follow network delay directly. In a sense, the cross layer nature of AoI, stemming from it being a characteristic of the end-to-end information flow, represents a broader view of information freshness than delay does. To illustrate the difference between network delay and AoI, consider a scenario with a single First Come First Served (FCFS) queueing system. If the data generation rate increases, the queueing delay increases which in turn increases the delay through the system. Decreasing the data generation rate leads to shorter delay through the system, but at the same time, the time between measurements increase, which leads to larger AoI.

In many IoT scenarios such as smart cities [4], a typical application might be sensor nodes continuously measuring and sending data, e.g. using an IEEE 802.11ah Wireless Local Area Network (WLAN) [1]. For example, sensor nodes might be interested in uploading the measured information to a remote unit, for storing or further processing. If the remote server is only interested in the freshest possible piece of the information sent by the sensor node, it is interested in the sensor node trying to minimize the AoI at the receiver.

Since the introduction of the concept of AoI, the performance of this metric has been studied in multiple queueing systems with different queueing disciplines. Last Come First Served (LCFS) has the advantage of not sending stale jobs to the receiver end, thus being preferable when the receiver is interested only in the freshest piece of information. LCFS systems with preemption have been studied for single queueing systems [14, 16, 17], resulting in a significant improvement in terms of average AoI compared to systems with a FIFO or LCFS discipline without preemption.

Some applications may be interested not only to keep the average AoI low, but also to ensure a statistical guarantee that the AoI will not be above a threshold for a certain percentage of the time. We call the percentage of time that the AoI is above a certain threshold update outage probability. In order to find the update outage probability, it is necessary to know the entire distribution of the AoI at the receiver end, being it the survival function of the AoI itself. The first work to address the complete stationary distribution in FCFS queueing systems is in [9], where the authors obtain a general expression for the stationary distribution of the AoI in a G/G/1 FCFS system and several close form expressions for said distribution in derived systems.

Two studies addressed statistical guarantees regarding updates. In [5] the authors proposed an optimization problem for computing an approximation on the bound on the tail of the AoI distribution in D/M/1 FCFS systems. In [7], on the other hand, the authors characterized the delay and peak Age of Information (pAoI) violation probabilities for packets generated according to a Bernoulli process, placed in an FCFS queue, and sent through an Additive White Gaussian Noise (AWGN) channel. This system has also an Automatic repeat request (ARQ) mechanism in place. They also found an optimal block-length that minimizes the aforementioned probabilities.

In this paper we study the update outage probability of an M/G/1 LCFS system with preemption, and compare systems with different busy time distributions to their equivalents using FCFS disciplines. We show that the former substantially outperforms the latter, especially at high loads (i.e. when the source generation rate λ jobs/s approaches the inverse of the average busy time). Also an LCFS system with preemption has the advantage of being stable for loads greater than one, thus making room for further reducing the update outage probability.

Of greater importance, we derive expressions for the deterministic, exponential and gamma distributions. The first models a TDMA system. The gamma distribution, or, more specifically, its special case the Erlang distribution, could be used to model an information stream traveling through multiple hops in a network where each node follows an LCFS discipline with preemption with exponentially distributed busy times. Additionally, we study how the variance of the busy time distribution affects the update outage probability by comparing the M/D/1 LCFS with preemption against the M/Γ/1 LCFS with preemption by varying the variance of the busy time of the latter while maintaining the same average busy time for both systems. We discover instances where having higher variances in the busy time is beneficial for reducing the probability of violating a threshold on the AoI. Finally, with the general service time distribution our work is applicable to general wireless networks where different standards lead to different access delay distributions [10, 18, 20, 22, 23].

The rest of this paper is subdivided as follows. In Section 2 the scenario is described in detail. In Section 3 a method to derive the expression of the update outage probability is derived for an M/G/1 LCFS system with preemption. In Section 4 the previous expressions are first tested against simulations, compared with the results in [9] and then we study the effect of the variance of the busy time distribution on the update outage probability. Finally in Section 5 conclusions are drawn.

2 MODEL DESCRIPTION

Our model consists of an LCFS M/G/1 queueing system with preemption, sending jobs to a sink. The source generates pieces of information according to an exponential inter-arrival distribution with average rate λ job per seconds, i.e.:

$$f_A(t) = \lambda e^{-\lambda t} \mathrm{H}(t),$$

where $\mathrm{H}(t)$ is the Heaviside step function defined as:

$$\mathrm{H}(t) = \begin{cases} 1 & , t \geq 0 \\ 0 & , t < 0 \end{cases}.$$

The source sends updates about a single information stream i.e. there is only one class of jobs. It is also worth mentioning that for the remainder of the paper the Probability Density Function (PDF) of a random variable X will be expressed as $f_X(x)$, its Cumulative Distribution Function (CDF) as $F_X(x) = \Pr\{X \leq x\}$, its Survival Function $G_X(x) = \Pr\{X > x\} = 1 - F_X(x)$ and its Moment Generating Function (MGF) $\Phi_X(s) = \mathrm{E}\left[e^{sX}\right]$. Also, we will indicate an M/G/1 LCFS system with preemption as M/G/1/1* —since, as we will see, there could be only one job in the system –, and an M/G/1 FCFS system without preemption will be shortened just as M/G/1.

Also, we will refer to the time generated when a job arrives to the server without finding any other job in service as **busy time**, while the time from the arrival of a job to the server and its departure from the system will be referred as **service time**. While in systems without preemption they are the same, in preemptive systems they are different, as we will see in the next paragraphs.

We consider a preemption in which each time a fresher piece of information is generated, the new job takes the place in the server of the previous one already in service. The service time experienced by the new job will be the residual service time of the preempted job. Substituting a job being served models, for example, substituting a frame containing staler information that was already in the transmission buffer, waiting to be sent; this could happen because of a duty cycle, or because there is a back-off mechanism in place (e.g. IEEE 802.11 is used), as in [8]. The substituted job is discarded. Since we consider only one information stream, there are no jobs in the queue; they can only be in service.

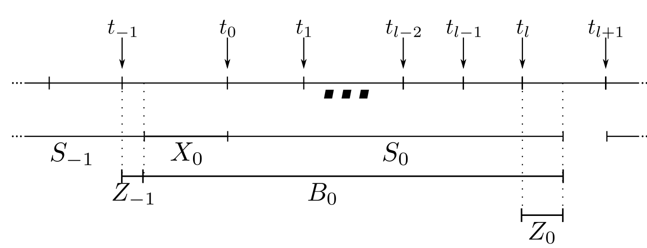

Figure 1: A typical busy/idle period.

In Figure 1 a typical busy/idle period is shown. The busy period S_{-1} ends before the arrival of job 0. Then, when job 0 is generated, a busy period starts again. A number of jobs gets preempted, until job l, because job $l+1$ happens to be generated after the busy period S_0 expires. The effective inter departure time for the two jobs that have survived (i.e. not preempted) is $B_0 = X_0 + S_0$, where X_0 is the idle period. Also, the successful job l will see a service time Z_0.

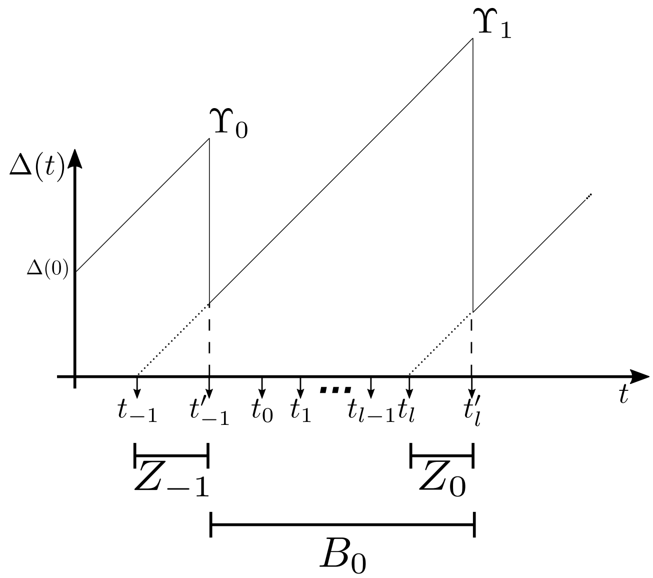

Figure 2: AoI over a typical busy/idle period. t_i is the i-th arrival time, while t'_i is the corresponding departure time.

The above timeline translates into the AoI function $\Delta(t)$ in Figure 2. When job -1 arrives at the sink, after a time Z_{-1}, the AoI will jump to the service time experienced by the latter. Then it will continue to grow with slope 1, until job l arrives, where it again jumps to its service time Z_0. The AoI just before the reception of a meaningful job is called peak AoI (pAoI) – marked as Υ_k, and it is, as seen in the figure, the sum of the effective inter-departure time for job k, B_k and the service time for the previous one, Z_{k-1}. Since the length of the inter-departure time B_k is independent from the the previous service time Z_{k-1}, at steady state:

$$\Phi_\Upsilon(s) = \Phi_B(s)\Phi_Z(s). \tag{1}$$

Also, since our system is ergodic, and we are considering the steady state distributions, we can calculate the CDF of the AoI (represented by the random variable Δ) by using [9, Lemma 1]:

$$F_\Delta(t) = \lambda_e \int_0^t F_Z(y) - F_\Upsilon(y)\mathrm{d}y, \tag{2}$$

where λ_e is the effective departure rate, expressed in jobs per second.

3 DISTRIBUTION OF THE AGE OF INFORMATION

In order to find the complete distribution of the AoI Δ (2), we need the effective departure rate λ_e, the CDF of the service time Z and of the pAoI Υ. In order to express the latter (1), we also need the MGF of the effective inter-departure time B.

We start by finding the distributions of B and Z, given that the busy time is a constant, i.e. t' seconds; our objective is to find an expression conditioned to the busy time being equal to a constant, in order to de-condition it; this in order to find the general form of the distribution of Δ. The busy time distribution will be t' seconds with probability 1, i.e.:

$$f_{S|S=t'}(t) = \delta\left(t - t'\right). \tag{3}$$

where $\delta(t)$ is the Dirac delta function.

We now look at the distribution of the inter-departure time given that the busy time is t' seconds, i.e. $B|S = t'$. Looking at Figure 1 we notice that the inter-departure time B_k is the sum of the preceding idle time X_k and the busy time S_k. The idle time is the residual life of a point between two poissonian arrivals with rate λ jobs per second, so its distribution is also exponential with rate λ jobs per second (See the hippie paradox in [15]). Since there is no queueing delay, the effective departure rate λ_e will be:

$$\lambda_{e|(S=t')} = \frac{1}{\mathrm{E}[B]} = \frac{1}{\mathrm{E}[S] + \mathrm{E}[X]} = \frac{1}{t' + \frac{1}{\lambda}}. \tag{4}$$

Also, since the length of the busy time is independent from the preceding idle time, the steady state distribution of the inter-departure time $B|S = t'$ will be the convolution of (3) with an exponential distribution with rate λ jobs per second, i.e.:

$$f_{B|S=t'}(t) = \lambda \mathrm{e}^{-\lambda(t-t')}\mathrm{H}\left(t - t'\right), \tag{5}$$

with CDF:

$$F_{B|S=t'}(t) = \left(1 - \mathrm{e}^{-\lambda(t-t')}\right)\mathrm{H}\left(t - t'\right), \tag{6}$$

and associated MGF:

$$\Phi_{B|S=t'}(s) = \frac{\lambda}{\lambda - s}\mathrm{e}^{st'}. \tag{7}$$

Looking at the conditioned service time $Z|S = t'$, we notice that, since the maximum length for a service time for a successful job is t' seconds, $Z|S = t'$ is less than t' with probability 1, i.e.:

$$\Pr\left\{Z|S = t' < t|t \geq t'\right\} = F_{Z|S=t'}\left(t : t \geq t'\right) = 1.$$

We also notice that $Z|S = t'$ is the age of a point between two poissonian arrivals with rate λ jobs per second, so its distribution is also exponential with rate λ jobs per second (Again, see the hippie paradox in [15]). By combining those two observations, we can write:

$$F_{Z|S=t'}(t) = \left(1 - \mathrm{e}^{-\lambda t}\right)\mathrm{H}\left(t' - t\right) + \mathrm{H}\left(t - t'\right), \tag{8}$$

with associated PDF:

$$f_{Z|S=t'}(t) = \lambda \mathrm{e}^{-\lambda t}\mathrm{H}\left(t' - t\right) + \mathrm{e}^{-\lambda t'}\delta(t - t').$$

Finally we can calculate the MGF of $Z|S = t'$ as:

$$\begin{aligned}\Phi_{Z|S=t'}(s) =& \lambda \int_0^\infty \mathrm{e}^{-(\lambda-s)t}\mathrm{H}\left(t' - t\right)\mathrm{d}t + \mathrm{e}^{-\lambda t'}\mathrm{e}^{st'} \\ =& \lambda \int_0^{t'} \mathrm{e}^{-(\lambda-s)t}\mathrm{d}t + \mathrm{e}^{-\lambda t'}\mathrm{e}^{st'} \\ =& \frac{\lambda}{\lambda - s} - \frac{\lambda}{\lambda - s}\mathrm{e}^{-t'(\lambda-s)} + \mathrm{e}^{-t'(\lambda-s)}.\end{aligned} \tag{9}$$

At this point one could be tempted to use (7) and (9) in (1) in order to find $F_{\Upsilon|S=t'}(t)$, and then use it, along with (8) and (4) in (2) in order to find $F_{\Delta|S=t'}(t)$; then de-condition it for a particular distribution of t' in order to find the CDF of the AoI. This is not possible in general, since, although B and Z are independent, they are not conditionally independent with respect to S. In order to demonstrate it we will derive the CDF of the pAoI for an M/M/1/1* system and show that is not the same as de-conditioning $F_{\Upsilon|S=t'}(t)$ with:

$$f_S(t) = \mu e^{-\mu t} H(t), \quad (10)$$

where μ^{-1} is the average busy time, expressed in seconds. The proof will be presented in Lemma 3.1. Also notice that both Z and B are directly derived from the busy time S, so is always possible to de-condition them, being all the random variables from which they are derived conditionally independent with respect to S.

3.1 Update outage probability for an M/M/1 LCFS system with preemption

The effective departure rate will be:

$$\lambda_e^M = \frac{1}{E[B]} = \frac{1}{E[S] + E[X]} = \frac{1}{\frac{1}{\mu} + \frac{1}{\lambda}} = \frac{\mu\lambda}{\lambda + \mu}. \quad (11)$$

Also, the distribution of the inter-departure time B will be the sum of two exponential distributions with rates λ and μ respectively. The MGF will then be:

$$\Phi_B^M(s) = \frac{\lambda\mu}{(\lambda - s)(\mu - s)}. \quad (12)$$

The effective service time will be the service time experienced by a successful job. It means is the service time of a job given that the next arrival comes after the remaining busy period, i.e:

$$\begin{aligned} F_Z^M(t) &= \Pr\{S < t | S < A\} = \frac{\Pr\{S < t, S < A\}}{\Pr\{S < A\}} \\ &= \frac{\int_0^\infty \Pr\{S < t, S < a\} f_A(a) da}{\int_0^\infty \Pr\{S < a\} f_A(a) da} \\ &= \frac{\left(\int_0^t F_S^M(a) f_A(a) da + F_S^M(t) \int_t^\infty f_A(a) da\right)}{\int_0^\infty F_S^M(a) f_A(a) da} \\ &= \frac{\lambda + \mu}{\mu}\left(\int_0^t F_S^M(a) f_A(a) da + F_S^M(t)\left[1 - F_A(t)\right]\right) \\ &= 1 - e^{-(\lambda+\mu)t} \end{aligned} \quad (13)$$

and its MGF:

$$\Phi_Z^M(s) = \frac{\lambda + \mu}{\lambda + \mu - s}. \quad (14)$$

We now use (12) and (14) along with (1) in order to find $\Phi_\Upsilon(t)$:

$$\begin{aligned} \Phi_\Upsilon^M(s) &= \Phi_B^M(s)\Phi_Z^M(s) \\ &= \begin{cases} \frac{\lambda+\mu}{\lambda+\mu-s} - \frac{\lambda(\lambda+\mu)}{(\lambda-\mu)(\lambda-s)} + \frac{\mu(\lambda+\mu)}{(\lambda-\mu)(\mu-s)} & , \lambda \neq \mu \\ \frac{2\lambda}{2\lambda-s} + \frac{2\lambda^2}{(\lambda-s)^2} - \frac{2\lambda}{\lambda-s} & , \lambda = \mu \end{cases} \end{aligned}$$

with the associated CDF:

$$F_\Upsilon^M(t) = \begin{cases} 1 - e^{-(\lambda+\mu)t} + \frac{\lambda+\mu}{\lambda-\mu}\left(e^{-\lambda t} - e^{-\mu t}\right) & , \lambda \neq \mu \\ 1 - 2\lambda t e^{-\lambda t} - e^{-2\lambda t} & , \lambda = \mu \end{cases}. \quad (15)$$

Finally, by using (11), (13) and (15) in (2) we find:

$$F_\Delta^M(t) = \begin{cases} 1 - \frac{\lambda}{\lambda-\mu} e^{-\mu t} + \frac{\mu}{\lambda-\mu} e^{-\lambda t} & , \lambda \neq \mu \\ 1 - (\lambda t + 1) e^{-\lambda t} & , \lambda = \mu \end{cases},$$

and its survival function:

$$G_\Delta^M(t) = 1 - F_\Delta^M(t) = \begin{cases} \frac{\lambda}{\lambda-\mu} e^{-\mu t} - \frac{\mu}{\lambda-\mu} e^{-\lambda t} & , \lambda \neq \mu \\ (\lambda t + 1) e^{-\lambda t} & , \lambda = \mu \end{cases}. \quad (16)$$

3.2 A general method to find the update outage probability of an M/G/1 LCFS system with preemption

Lemma 3.1. *The random variable B describing the inter-departure times and the random variable Z describing the service time are not, in general, conditionally independent with respect to the random variable describing the busy time S.*

Proof. If B and Z were conditionally independent with respect to S, then, for every S it must hold:

$$\Upsilon|S = t' = (B + Z)|S = t' = B|S = t' + Z|S = t'.$$

This entails that for every S:

$$\Phi_{\Upsilon|S=t'}(s) = \Phi_{B|S=t'}(s)\Phi_{Z|S=t'}(s),$$

is the MGF of

$$f_{\Upsilon|S=t'}(t) = \mathcal{M}^{-1}\left\{\Phi_{\Upsilon|S=t'}(s)\right\}(t),$$

where $\mathcal{M}^{-1}\{\cdot\}$ is the inverse transform operator. The associated CDF is:

$$F_{\Upsilon|S=t'}(t) = \int_0^t f_{\Upsilon|S=t'}(\hat{t}) d\hat{t}.$$

Then it should be possible to obtain the CDF of Υ for any distribution of S, $f_S(t)$ as:

$$F_\Upsilon(t) = \int_0^\infty F_{\Upsilon|S=t'}(t) f_S(t') dt'. \quad (17)$$

We now choose $f_S(t)$ to be (10). By using Eq (7) and (9) in (1) we can then calculate the MGF of $\Upsilon|S = t'$:

$$\Phi_{\Upsilon|S=t'}(s) = \left(\frac{\lambda}{\lambda - s}\right)^2 e^{st'} - e^{-t'\lambda}\left(\frac{\lambda}{\lambda - s}\right)^2 e^{2st'} + e^{-t'\lambda}\frac{\lambda}{\lambda - s} e^{2st'},$$

associated with the CDF:

$$\begin{aligned} F_{\Upsilon|S=t'}(t) &= \gamma\left(\lambda(t - t'), 2\right) H\left(t - t'\right) \\ &+ e^{-t'\lambda}\left[\left(1 - e^{-\lambda(t-2t')}\right) - \gamma\left(\lambda(t - 2t'), 2\right)\right] H\left(t - 2t'\right) \\ &= \left[e^{-\lambda(t'-t)}\left(\lambda\left(t' - t\right) - 1\right) + 1\right] H\left(t - t'\right) \\ &- \lambda e^{-\lambda(t'-t)}\left(2t' - t\right) H\left(t - 2t'\right), \end{aligned} \quad (18)$$

where $\gamma(x, a)$ is the regularized lower incomplete gamma function defined as:

$$\gamma(x, a) = \frac{1}{\Gamma(a)}\int_0^x t^{a-1} e^{-t} dt.$$

We now de-condition the previous using (10) (supposing $\lambda \neq \mu$) in (17), i.e.:

$$\begin{aligned} F_\Upsilon^M(t) &= \int_0^\infty F_{\Upsilon|S=t'}(t)\mu e^{-\mu t'} dt' \\ &= \frac{e^{-\frac{3t(\lambda+\mu)}{2}}}{(\lambda - \mu)^2}\left(\lambda^2 e^{\frac{3t(\lambda+\mu)}{2}} + \mu^2 e^{\frac{3t(\lambda+\mu)}{2}} - \lambda^2 e^{\frac{t(3\lambda+\mu)}{2}}\right. \\ &\left. - \mu^2 e^{\frac{t(\lambda+3\mu)}{2}} + 2\lambda\mu e^{t(\lambda+\mu)} - 2\lambda\mu e^{\frac{3t(\lambda+\mu)}{2}}\right), \end{aligned}$$

that is different from (15). So we conclude that, in general, B and Z are not conditionally independent with respect to S. □

Also we can write some general expressions for a general PDF of the busy time $f_S(t)$. First if we write the de-conditioning of B using (5):

$$f_B(t) = \int_0^{\infty} \lambda e^{-\lambda(t-t')} H\left(t-t'\right) f_S(t') dt'$$

we notice that this is a convolution between an exponential distribution with rate λ and the busy time distribution, so we can write its MGF as:

$$\Phi_B(t) = \frac{\lambda}{\lambda - s} \Phi_S(s). \tag{19}$$

Now we write the CDF of the de-conditioning of Z using (8):

$$\begin{aligned} F_Z(t) &= \int_t^{\infty} \left(1 - e^{-\lambda t}\right) f_S(t') dt' + \int_0^t f_S(t') dt' \\ &= 1 - e^{-\lambda t} \left(1 - F_S(t)\right). \end{aligned} \tag{20}$$

Similarly, for the PDF:

$$\begin{aligned} f_Z(t) &= \lambda e^{-\lambda t} \int_t^{\infty} f_S(t') dt' + \int_0^{\infty} e^{-\lambda t'} f_S(t') \delta(t-t') dt' \\ &= \lambda e^{-\lambda t} + e^{-\lambda t} f_S(t) - \lambda e^{-\lambda t} \int_0^t f_S(t') dt'. \end{aligned}$$

By using the frequency translation and the time integral properties of the transform, we can finally write that the MGF of Z for a busy time S is:

$$\Phi_Z(s) = \frac{\lambda}{\lambda - s} + \Phi_S(s-\lambda) - \frac{\lambda}{\lambda - s} \Phi_S(s-\lambda). \tag{21}$$

We can now present a general method to find the update outage probability for an M/G/1 LCFS system with preemption. The method is the following:

(1) Given a particular PDF for S, $f_S(t)$, calculate the MGF of the pAoI as the product of (19) and (21)
(2) Anti-transform the previous in order to find its CDF
(3) Use (20), the effective departure rate and the CDF of Υ by substituting them in (2). Find the CDF of the AoI, and, finally, its survival function.

3.3 Update outage probability for some important systems

In this section we will derive the update outage probability for the M/D/1/1* and the M/Γ/1/1* systems. As we will see in the next sections, those two systems are linked by the fact that the gamma distribution approximates the deterministic distribution when letting one of its parameters to infinity.

3.3.1 Update outage probability for an M/D/1 LCFS system with preemption. Since the PDF of the busy time is (3), we can say that the CDF of the pAoI is (18) the CDF of the service time Z is (8) and the effective departure rate is (4). By using the previous in (2) we find:

$$\begin{aligned} F_{\Delta}^{D}(t) = &\frac{1}{t'\lambda + 1} \Bigg[\left(e^{-\lambda t} + \lambda t - 1\right) H\left(t'-t\right) \\ &+ \Bigg(e^{-t'\lambda} + t'\lambda - 1 + e^{-\lambda t} \left(t' e^{t'\lambda} - t e^{t'\lambda}\right) \\ &- \lambda e^{-\lambda t} \left(2 e^{t'\lambda} - 2 e^{\lambda t}\right) \Bigg) H\left(t-t'\right) + e^{-\lambda (t'+t)} \\ &\times \left(e^{2t'\lambda} - e^{\lambda t} - 2t'\lambda e^{2t'\lambda} + \lambda t e^{2t'\lambda}\right) H\left(t - 2t'\right) \Bigg], \end{aligned}$$

and its survival function:

$$\begin{aligned} G_{\Delta}^{D}(t) = &1 - F_{\Delta}^{D}(t) \\ = &1 - \frac{1}{t'\lambda + 1} \Bigg[\left(e^{-\lambda t} + \lambda t - 1\right) H\left(t'-t\right) \\ &+ \Bigg(e^{-t'\lambda} + t'\lambda - 1 + e^{-\lambda t} \left(t' e^{t'\lambda} - t e^{t'\lambda}\right) \\ &- \lambda e^{-\lambda t} \left(2 e^{t'\lambda} - 2 e^{\lambda t}\right) \Bigg) H\left(t-t'\right) + e^{-\lambda (t'+t)} \\ &\times \left(e^{2t'\lambda} - e^{\lambda t} - 2t'\lambda e^{2t'\lambda} + \lambda t e^{2t'\lambda}\right) H\left(t - 2t'\right) \Bigg]. \end{aligned} \tag{22}$$

3.3.2 Update outage probability for an M/Γ/1 LCFS system with preemption. The gamma distribution is an important distribution since it is possible to assign it a mean and let the variance be as little as we want, approximating the deterministic distribution. This property allows to study the effect of the variance to the update outage probability. The busy time distribution is:

$$f_S^{\Gamma}(t) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} t^{\alpha-1} e^{-\beta t} H(t), \tag{23}$$

with mean $E[S] = \frac{\alpha}{\beta}$ s and variance $\mathrm{Var}[S] = \frac{\alpha}{\beta^2}$ s^2. If we assign a mean $E[S] = \frac{1}{\mu}$ s we can write:

$$\frac{\alpha}{\beta} = \frac{1}{\mu} \Rightarrow \mathrm{Var}[S] = \frac{\alpha}{\beta} \cdot \frac{1}{\beta} = \frac{1}{\mu} \cdot \frac{1}{\beta}$$

so when we take the limit for $\beta \to \infty$ and setting $\alpha = \frac{\beta}{\mu}$ in order to maintain the expected value constant:

$$\lim_{\alpha \to \infty} \mathrm{Var}[S] = \frac{1}{\mu} \lim_{\beta \to \infty} \frac{1}{\beta} = 0\,\mathrm{s}^2,$$

while the expected value remains $E[S] = \frac{1}{\mu}$ s.

The effective departure rate is:

$$\lambda_e^{\Gamma} = \frac{1}{\frac{\alpha}{\beta} + \frac{1}{\lambda}} = \frac{\beta\lambda}{\alpha\lambda + \beta}. \tag{24}$$

We use (19) substituting the MGF of (23) in it in order to find the MGF of B:

$$\Phi_B^{\Gamma}(s) = \frac{\lambda}{\lambda - s} \left(\frac{\beta}{\beta - s}\right)^{\alpha}. \tag{25}$$

Similarly, using (21):

$$\Phi^{\Gamma}_{Z}(s) = \frac{\lambda}{\lambda - s} + \left(\frac{\beta}{\beta+\lambda-s}\right)^{\alpha} - \frac{\lambda}{\lambda-s}\left(\frac{\beta}{\beta+\lambda-s}\right)^{\alpha}. \tag{26}$$

We find its CDF using (20) i.e.:

$$F^{\Gamma}_{Z}(t) = 1 - \mathrm{e}^{-\lambda t}\left(1 - \gamma(\beta t, \alpha)\right). \tag{27}$$

Next, the MGF of Υ will be, by multiplying (25) and (26):

$$\begin{aligned}\Phi^{\Gamma}_{\Upsilon}(s) &= \left(\frac{\lambda}{\lambda-s}\right)^{2}\left(\frac{\beta}{\beta-s}\right)^{\alpha} - \left(\frac{\lambda}{\lambda-s}\right)^{2}\left(\frac{\beta}{\beta-s}\right)^{\alpha}\left(\frac{\beta}{\beta+\lambda-s}\right)^{\alpha}\\ &\quad + \frac{\lambda}{\lambda-s}\left(\frac{\beta}{\beta-s}\right)^{\alpha}\left(\frac{\beta}{\beta+\lambda-s}\right)^{\alpha}\\ &= \left(\frac{\lambda}{\lambda-s}\right)^{2}\left(\frac{\beta}{\beta-s}\right)^{\alpha}\\ &\quad - \left(\frac{\beta}{\beta+\lambda}\right)^{\alpha}\left(\frac{\lambda}{\lambda-s}\right)^{2}\left(\frac{\beta}{\beta-s}\right)^{\alpha}\left(\frac{\beta+\lambda}{\beta+\lambda-s}\right)^{\alpha}\\ &\quad + \left(\frac{\beta}{\beta+\lambda}\right)^{\alpha}\frac{\lambda}{\lambda-s}\left(\frac{\beta}{\beta-s}\right)^{\alpha}\left(\frac{\beta+\lambda}{\beta+\lambda-s}\right)^{\alpha}.\end{aligned}$$

The above is the sum of the distributions of three different sums of independent gamma variates (the exponential and Erlang distributions are particular cases of the gamma distribution). If we define Y_l with $l = 1 \dots L$ a gamma variate with parameters (α_l, β_l), the CDF of $Y = \sum_{i=1}^{L} Y_l$ – i.e. the sum of L independent gamma random variables – is [2, Theorem 2]:

$$F_Y(t) = 1 + \left(\prod_{l=1}^{L}\beta_l^{\alpha}\right)\overline{H}^{0,L+1}_{L+1,L+1}\left[\mathrm{e}^{y}\,\middle|\,\begin{matrix}\Xi^{(1)}_{L}, & (1,1,1)\\ \Xi^{(2)}_{L}, & (0,1,1)\end{matrix}\right],$$

where:

$$\begin{aligned}\Xi^{(1)}_{k} &= (1-\beta_1, 1, \alpha_1)(1-\beta_2, 1, \alpha_2)\dots(1-\beta_k, 1, \alpha_k)\\ \Xi^{(2)}_{k} &= (-\beta_1, 1, \alpha_1)(-\beta_2, 1, \alpha_2)\dots(-\beta_k, 1, \alpha_k)\end{aligned}$$

and $\overline{H}$ is the Fox's $\overline{H}$ function [19] defined as:

$$\begin{aligned}&\overline{H}^{m,n}_{p,q}\left[z\,\middle|\,\begin{matrix}(a_j, A_j; \alpha_j)_{1,n} & (a_j, A_j)_{n+1,p}\\ (b_j, B_j)_{1,m} & (b_j, B_j; \beta_j)_{m+1,q}\end{matrix}\right]\\ &= \frac{1}{2\pi i}\int_{\mathcal{V}}\chi(s)z^{-s}\mathrm{d}s,\end{aligned}$$

where:

$$\chi(s) = \frac{(\prod_{j=1}^{m}\Gamma(b_j + B_j s))(\prod_{j=1}^{n}\Gamma(1 - a_j - A_j s)^{\alpha_j})}{(\prod_{j=m+1}^{q}\Gamma(1 - b_j - B_j s)^{\beta_j})(\prod_{j=n+1}^{p}\Gamma(a_j + A_j s))}, \tag{28}$$

and $\mathcal{V}$ is a contour starting at the point $\tau - i\infty$ and terminating at the point $\tau + i\infty$ for some $\tau \in \mathbb{R}$. We notice that the Fox's $\overline{H}$ function could be also viewed as the inverse Mellin transform of (28). The CDF of Υ will then be:

$$\begin{aligned}F^{\Gamma}_{\Upsilon}(t) =& 1 + \lambda^2\beta^{\alpha}\overline{H}^{0,3}_{3,3}\left[\mathrm{e}^{t}\,\middle|\,\begin{matrix}\xi^{(1)}_1\\ \xi^{(2)}_1\end{matrix}\right] - \lambda^2\beta^{2\alpha}\overline{H}^{0,4}_{4,4}\left[\mathrm{e}^{t}\,\middle|\,\begin{matrix}\xi^{(1)}_2\\ \xi^{(2)}_2\end{matrix}\right]\\ &+ \lambda\beta^{2\alpha}\overline{H}^{0,4}_{4,4}\left[\mathrm{e}^{t}\,\middle|\,\begin{matrix}\xi^{(1)}_3\\ \xi^{(2)}_3\end{matrix}\right],\end{aligned} \tag{29}$$

where:

$$\begin{aligned}\xi^{(1)}_1 &= (1-\lambda, 1, 2), (1-\beta, 1, \alpha), (1, 1, 1)\\ \xi^{(2)}_1 &= (-\lambda, 1, 2), (-\beta, 1, \alpha), (0, 1, 1)\\ \xi^{(1)}_2 &= (1-\lambda, 1, 2), (1-\beta, 1, \alpha), (1-\beta-\lambda, 1, \alpha), (1, 1, 1)\\ \xi^{(2)}_2 &= (-\lambda, 1, 2), (-\beta, 1, \alpha), (-\beta-\lambda, 1, \alpha), (0, 1, 1)\\ \xi^{(1)}_3 &= (1-\lambda, 1, 1), (1-\beta, 1, \alpha), (1-\beta-\lambda, 1, \alpha), (1, 1, 1)\\ \xi^{(2)}_3 &= (-\lambda, 1, 1), (-\beta, 1, \alpha), (-\beta-\lambda, 1, \alpha), (0, 1, 1).\end{aligned}$$

In order to find the CDF of Δ, we must combine (24), (27) and (29) in (2):

$$\begin{aligned}F^{\Gamma}_{\Delta}(t) = \frac{\beta\lambda}{\alpha\lambda+\beta}\Bigg(&\frac{\mathrm{e}^{-\lambda t}}{\lambda} - \frac{1}{\lambda} + \int_0^t \mathrm{e}^{-\lambda t'}\gamma(\beta t', \alpha)\mathrm{d}t'\\ &- \lambda^2\beta^{\alpha}\int_0^t \overline{H}^{0,3}_{3,3}\left[\mathrm{e}^{t'}\,\middle|\,\begin{matrix}\xi^{(1)}_1\\ \xi^{(2)}_1\end{matrix}\right]\mathrm{d}t'\\ &+ \lambda^2\beta^{2\alpha}\int_0^t \overline{H}^{0,4}_{4,4}\left[\mathrm{e}^{t'}\,\middle|\,\begin{matrix}\xi^{(1)}_2\\ \xi^{(2)}_2\end{matrix}\right]\mathrm{d}t'\\ &- \lambda\beta^{2\alpha}\int_0^t \overline{H}^{0,4}_{4,4}\left[\mathrm{e}^{t'}\,\middle|\,\begin{matrix}\xi^{(1)}_3\\ \xi^{(2)}_3\end{matrix}\right]\mathrm{d}t'\Bigg)\end{aligned} \tag{30}$$

We solve by parts:

$$\begin{aligned}&\int_0^t \mathrm{e}^{-\lambda t'}\gamma(\beta t', \alpha)\mathrm{d}t'\\ &\quad = \frac{1-\mathrm{e}^{-\lambda t}}{\lambda}\gamma(\beta t, \alpha) - \frac{\beta^{\alpha}}{\Gamma(\alpha)}\int_0^t t'^{\alpha-1}\mathrm{e}^{-\beta t'}\frac{1-\mathrm{e}^{-\lambda t}}{\lambda}\mathrm{d}t'\\ &\quad = \lambda^{-1}\left(\frac{\beta}{(\beta+\lambda)}\right)^{\alpha}\gamma\left((\beta+\lambda)t, \alpha\right) - \lambda^{-1}\mathrm{e}^{-\lambda t}\gamma(\beta t, \alpha).\end{aligned} \tag{31}$$

Then we make use of the integration in time property of the Mellin transform, and that:

$$\frac{1}{s} = \frac{\Gamma(s)}{s\Gamma(s)} = \frac{\Gamma(s)}{\Gamma(1+s)},$$

inserting the latter as an additional term in (28), using a reasoning similar to the proof of [2, Theorem 2], obtaining:

$$\begin{aligned}&\prod_{l=1}^{L}\beta_l^{\alpha}\int_0^t \overline{H}^{0,L+1}_{L+1,L+1}\left[\mathrm{e}^{t'}\,\middle|\,\begin{matrix}\Xi^{(1)}_L, & (1,1,1)\\ \Xi^{(2)}_L, & (0,1,1)\end{matrix}\right]\mathrm{d}t'\\ &\quad = 1 + \prod_{l=1}^{L}\beta_l^{\alpha}\overline{H}^{0,L+2}_{L+2,L+2}\left[\mathrm{e}^{t}\,\middle|\,\begin{matrix}\Xi^{(1)}_L, & (1,1,1), & (1,1,1)\\ \Xi^{(2)}_L, & (0,1,1), & (0,1,1)\end{matrix}\right]\\ &\quad = 1 + \prod_{l=1}^{L}\beta_l^{\alpha}\overline{H}^{0,L+1}_{L+1,L+1}\left[\mathrm{e}^{t}\,\middle|\,\begin{matrix}\Xi^{(1)}_L, & (1,1,2)\\ \Xi^{(2)}_L, & (0,1,2)\end{matrix}\right].\end{aligned}$$

We then use (31) and the above in (30) to finally obtain:

$$F_\Delta^\Gamma(t) = \frac{\beta\lambda}{\alpha\lambda+\beta}\Bigg(\lambda^{-1}\mathrm{e}^{-\lambda t} - \lambda^{-1} + \lambda^{-1}\left(\frac{\beta}{(\beta+\lambda)}\right)^\alpha \gamma\left((\beta+\lambda)t,\alpha\right) - \lambda^{-1}\mathrm{e}^{-\lambda t}\gamma(\beta t,\alpha) - \lambda^2\beta^\alpha \overline{H}_{4,4}^{0,4}\left[\mathrm{e}^t \left| \begin{matrix} \xi_1^{(1)} & ,(1,1,1) \\ \xi_1^{(2)} & ,(0,1,1)\end{matrix}\right.\right] + \lambda^2\beta^{2\alpha}\overline{H}_{5,5}^{0,5}\left[\mathrm{e}^t \left| \begin{matrix} \xi_2^{(1)} & ,(1,1,1) \\ \xi_2^{(2)} & ,(0,1,1)\end{matrix}\right.\right] - \lambda\beta^{2\alpha}\overline{H}_{5,5}^{0,5}\left[\mathrm{e}^t \left| \begin{matrix} \xi_3^{(1)} & ,(1,1,1) \\ \xi_3^{(2)} & ,(0,1,1)\end{matrix}\right.\right]\Bigg),$$

from which we finally obtain the expression for the update outage probability:

$$G_\Delta^\Gamma(t) = 1 - \frac{\beta\lambda}{\alpha\lambda+\beta}\Bigg(\lambda^{-1}\mathrm{e}^{-\lambda t} - \lambda^{-1} + \lambda^{-1}\left(\frac{\beta}{(\beta+\lambda)}\right)^\alpha \gamma\left((\beta+\lambda)t,\alpha\right) - \lambda^{-1}\mathrm{e}^{-\lambda t}\gamma(\beta t,\alpha) - \lambda^2\beta^\alpha \overline{H}_{4,4}^{0,4}\left[\mathrm{e}^t \left| \begin{matrix} \xi_1^{(1)} & ,(1,1,1) \\ \xi_1^{(2)} & ,(0,1,1)\end{matrix}\right.\right] + \lambda^2\beta^{2\alpha}\overline{H}_{5,5}^{0,5}\left[\mathrm{e}^t \left| \begin{matrix} \xi_2^{(1)} & ,(1,1,1) \\ \xi_2^{(2)} & ,(0,1,1)\end{matrix}\right.\right] - \lambda\beta^{2\alpha}\overline{H}_{5,5}^{0,5}\left[\mathrm{e}^t \left| \begin{matrix} \xi_3^{(1)} & ,(1,1,1) \\ \xi_3^{(2)} & ,(0,1,1)\end{matrix}\right.\right]\Bigg). \quad (32)$$

As a side note, by using the results in [2, Corollary 3], if we have an Erlang distribution instead of a gamma distribution (i.e. α is an integer), the update outage probability becomes:

$$G_\Delta^{\mathrm{Er}}(t) = 1 - \frac{\beta\lambda}{\alpha\lambda+\beta}\Bigg(t + \lambda^{-1}\mathrm{e}^{-\lambda t} - \lambda^{-1} + \lambda^{-1}\left(\frac{\beta}{(\beta+\lambda)}\right)^\alpha \gamma\left((\beta+\lambda)t,\alpha\right) - \lambda^{-1}\mathrm{e}^{-\lambda t}\gamma(\beta t,\alpha) - \lambda^2\beta^\alpha \mathcal{G}_{4+\alpha,4+\alpha}^{0,4+\alpha}\left[\mathrm{e}^{-t} \left| \begin{matrix} \psi_1^{(1)} & ,1,1 \\ \psi_1^{(2)} & ,0,0\end{matrix}\right.\right] + \lambda^2\beta^{2\alpha}\mathcal{G}_{4+2\alpha,4+2\alpha}^{0,4+2\alpha}\left[\mathrm{e}^{-t} \left| \begin{matrix} \psi_2^{(1)} & ,1,1 \\ \psi_2^{(2)} & ,0,0\end{matrix}\right.\right] - \lambda\beta^{2\alpha}\mathcal{G}_{3+2\alpha,3+2\alpha}^{0,3+2\alpha}\left[\mathrm{e}^{-t} \left| \begin{matrix} \psi_3^{(1)} & ,1,1 \\ \psi_3^{(2)} & ,0,0\end{matrix}\right.\right]\Bigg),$$

where $\mathcal{G}$ is the Meijer G function (for the definition see [3]), and:

$$\psi_1^{(1)} = (1+\lambda), (1+\lambda), \overbrace{(1+\beta)\dots(1+\beta)}^{\alpha \text{ times}}$$

$$\psi_1^{(2)} = (\lambda), (\lambda), \overbrace{(\beta)\dots(\beta)}^{\alpha \text{ times}}$$

$$\psi_2^{(1)} = (1+\lambda), (1+\lambda), \overbrace{(1+\beta)\dots(1+\beta)}^{\alpha \text{ times}}, \overbrace{(1+\beta+\lambda)\dots(1+\beta+\lambda)}^{\alpha \text{ times}}$$

$$\psi_2^{(2)} = (\lambda), (\lambda), \overbrace{(\beta)\dots(\beta)}^{\alpha \text{ times}}, \overbrace{(\beta+\lambda)\dots(\beta+\lambda)}^{\alpha \text{ times}}$$

$$\psi_3^{(1)} = (1+\lambda), \overbrace{(1+\beta)\dots(1+\beta)}^{\alpha \text{ times}}, \overbrace{(1+\beta+\lambda)\dots(1+\beta+\lambda)}^{\alpha \text{ times}}$$

$$\psi_3^{(2)} = (\lambda), \overbrace{(\beta)\dots(\beta)}^{\alpha \text{ times}}, \overbrace{(\beta+\lambda)\dots(\beta+\lambda)}^{\alpha \text{ times}}.$$

4 NUMERICAL RESULTS

We have conducted simulation studies using OMNeT++ [21]. We fixed the expected value of the busy time $\mathrm{E}[S] = 1$ s and let λ and the threshold vary. The expected value of the busy time is t' s for the M/D/1 systems, μ^{-1} s for the M/M/1 systems and $\frac{\alpha}{\beta}$ s for the M/Γ/1 systems. In the figures we plot against the threshold expressed in seconds, and system load $\rho = \lambda\mathrm{E}[S]$. All the plots involving simulations are presented with 95% confidence intervals, allowing for a sufficient warm-up period before taking measurements. All the plots make use of a black and white printer-friendly and accessible color scheme [6, 12]. Also, for the numerical inversion of the Laplace transforms we used the Python package mpmath [11]. First, we compared (22), (16) and (32) against the simulations (Figure 3, 4 and 5), in order to validate our expressions. The theoretical results are in agreement with the measured update outage probability.

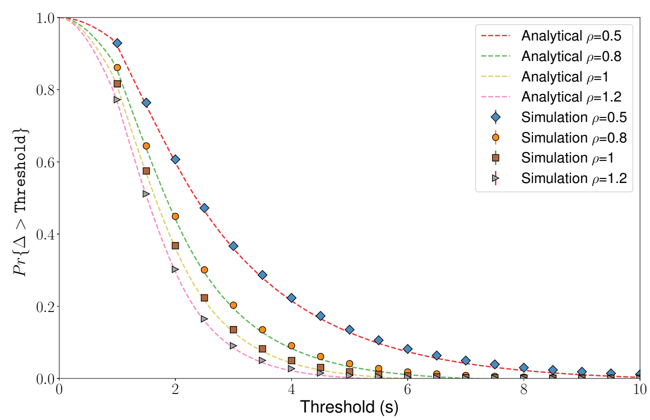

Figure 3: Update outage probability for an M/D/1/1* system (22); Simulation vs analytical.

The comparison of the outage probability for the M/D/1 LCFS with preemption system with the FCFS system is shown in Figure 6. We plot the outage probability when varying ρ between 0.1 and

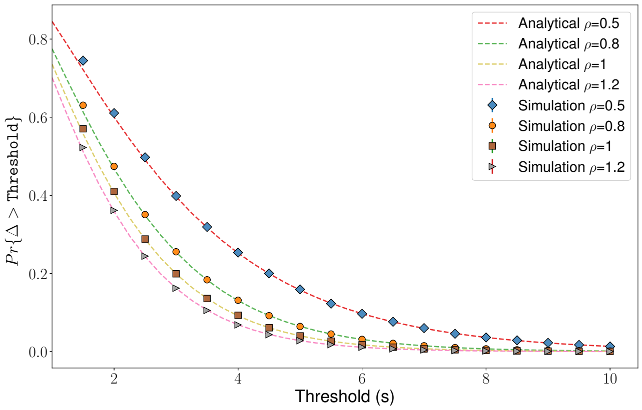

Figure 4: Update outage probability for an M/M/1/1* system (16)**; Simulation vs analytical.**

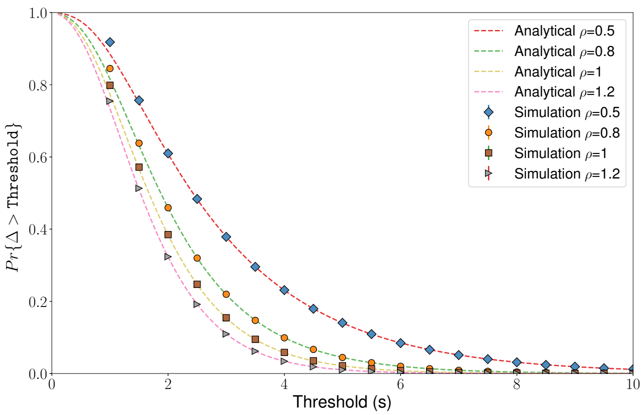

Figure 5: Update outage probability for an M/Γ/1/1* system (32)**; Simulation vs analytical.**

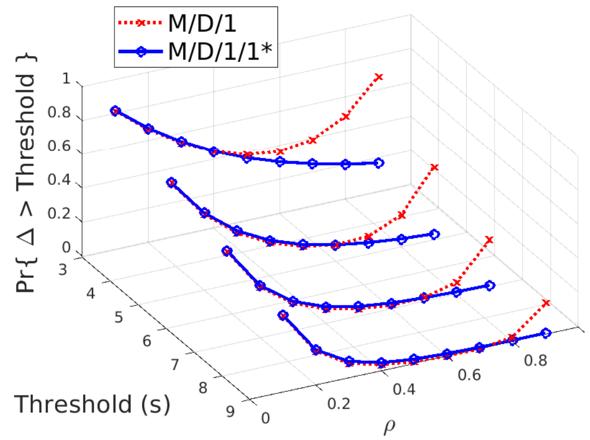

Figure 6: Update outage probability. M/D/1 LCFS with preemption (22) **vs FCFS** (33)**.**

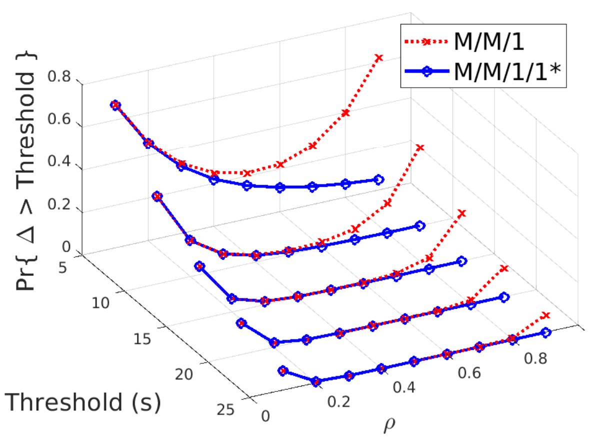

Figure 7: Update outage probability. M/M/1 LCFS with preemption (16) **vs FCFS** (34)**.**

0.9 with different thresholds. We used [9, (20)] with a service time distributed as (3) to find the MGF of the AoI for an FCFS M/D/1 system:

$$\Phi^D_{\Delta_{\text{FCFS}}}(s) = \frac{s\,e^{t'\,s}\,(t'\,\lambda - 1)}{s - \lambda\,e^{-t'\,(\lambda - s)}} - \frac{s\,e^{t'\,s}\,(t'\,\lambda - 1)}{\lambda + s - \lambda\,e^{t'\,s}},$$

then we numerically inverted it in order to find the associated CDF and, finally, its survival function, i.e.:

$$G^D_{\Delta_{FCFS}}(t) = 1 - F^D_{\Delta_{FCFS}}(t) = 1 - \mathcal{L}^{-1}\left\{\frac{1}{s}\Phi^D_{\Delta_{\text{FCFS}}}(-s)\right\}(t), \quad (33)$$

where $\mathcal{L}^{-1}\{\cdot\}$ is the inverse Laplace transform operator.

The comparison of the outage probability for the M/M/1 LCFS with preemption system with the FCFS system [9, Example 11] is shown in Figure 7. We plot the outage probability when varying ρ between 0.1 and 0.9 and the threshold between 5 and 25 seconds. Notice that the formula in the aforementioned paper for the CDF is incorrect. The correct formula is:

$$F^M_{\Delta_{FCFS}}(t) = 1 - e^{-(\lambda-\mu)t} + \lambda t e^{-\mu t} - \frac{\mu}{\lambda - \mu}e^{-\mu t} + \frac{\mu}{\lambda - \mu}e^{-\lambda t}$$

where $\lambda < \mu$. The update outage probability then is:

$$G^M_{\Delta_{FCFS}}(t) = 1 - F^M_{\Delta_{FCFS}}(t) = e^{-(\lambda-\mu)t} - \lambda t e^{-\mu t} + \frac{\mu}{\lambda - \mu}e^{-\mu t} - \frac{\mu}{\lambda - \mu}e^{-\lambda t}. \quad (34)$$

The comparison of the outage probability for the M/Γ/1 LCFS with preemption system with the FCFS system is shown in Figure 8. We plot the outage probability when varying ρ between 0.1 and 0.9 with different thresholds. We used [9, (20)] with a service time distributed as (23) to find the MGF of the AoI for an FCFS M/Γ/1

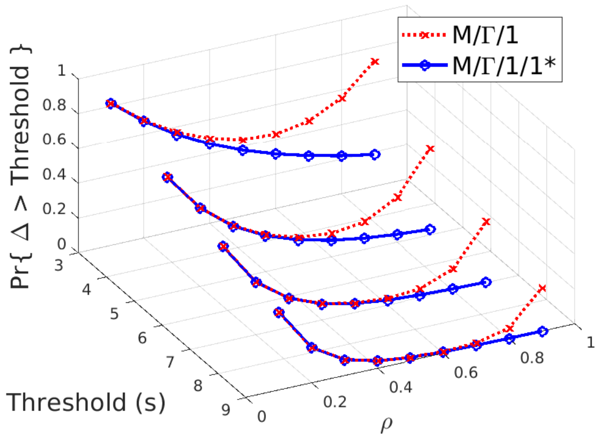

Figure 8: Update outage probability. M/Γ/1 LCFS with preemption (32) **vs FCFS** (35).

system:

$$\Phi^{\Gamma}_{\Delta_{\text{FCFS}}}(s) = \frac{\beta^{\alpha-1}\, s\, (\beta - \alpha\, \lambda)}{(\beta - s)^{\alpha}\, \left(\lambda + s - \frac{\beta^{\alpha}\, \lambda}{(\beta - s)^{\alpha}}\right)} - \frac{\beta^{\alpha-1}\, s\, (\beta - \alpha\, \lambda)\, (\beta + \lambda - s)^{\alpha}}{(\beta - s)^{\alpha}\, \left(s\, (\beta + \lambda - s)^{\alpha} - \beta^{\alpha}\, \lambda\right)},$$

then we numerically inverted it in order to find the associated CDF and, finally, its survival function, i.e.:

$$G^{\Gamma}_{\Delta_{FCFS}}(t) = 1 - F^{\Gamma}_{\Delta_{FCFS}}(t) = 1 - \mathcal{L}^{-1}\left\{\frac{1}{s}\Phi^{\Gamma}_{\Delta_{\text{FCFS}}}(-s)\right\}(t). \quad (35)$$

The results show that the LCFS discipline with preemption significantly outperforms the FCFS discipline in all cases in terms of update outage probability, especially when the server is fully loaded i.e. $\rho \cong 1$, as the queueing delay becomes dominant in the FCFS system.

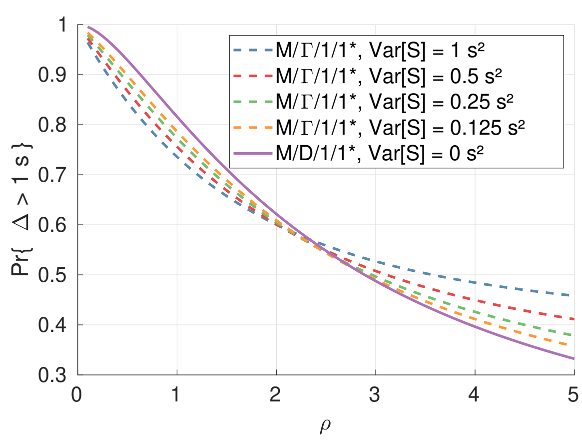

Figure 9: Effect of the variance of S on update outage probability.

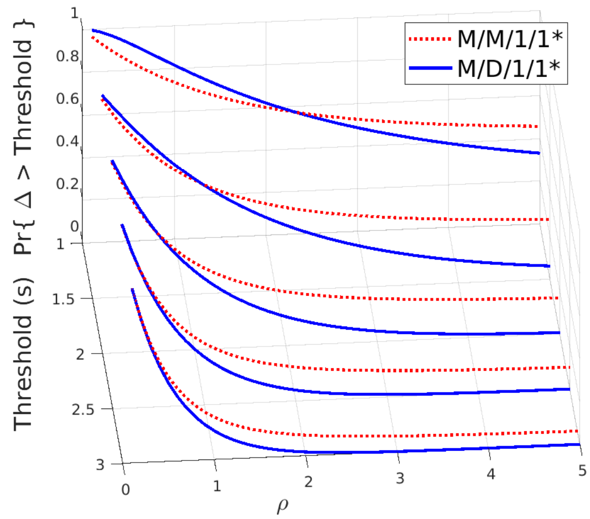

Figure 10: Update outage probability. M/D/1 LCFS with preemption (22) **vs M/M/1 LCFS with preemption** (16).

We now want to investigate the effects of the variance of S on the update outage probability. We already mentioned in Section 3.3.2 that the gamma distribution approximates a deterministic distribution as β approaches infinity, while maintaining the same expected value. As usual we fixed $\mathrm{E}[S] = 1$ s, but we let α and β vary in order to obtain a different variance. In Figure 9 the update outage probability for a threshold of 1 s is presented. The M/D/1/1* system is the solid line, representing the limit when Var[S] is 0 s^2. Various M/Γ/1/1* with decreasing variances are presented as dashed lines. We notice that there is a break-point around a load of $\rho = 2.5$, where the M/D/1/1* system starts to have the lowest outage probability, while for lower loads the more variance seems to give lower outage. This means that, for example, instead of a TDMA system, which is modeled with a deterministic busy time distribution, given a load and a threshold, could be preferable to use a different access system with higher variance, such as a CSMA access system, in order to achieve a lower update outage probability.

In Figure 10, we plotted the outage probability for the M/D/1/1* versus the outage probability for the M/M/1/1* – that is simply the limiting case for a M/Γ/1/1* system with $\alpha = 1$ and $\beta = \mu$ jobs per second, where μ is the one defined in (10) – for different thresholds; we can see that the aforementioned behavior disappears for higher thresholds; this confirms that, when considering the update outage probability for different loads and thresholds, designers should be aware that higher variance does not necessarily mean worse performances.

Finally, as an example, we plotted the outage probability vs both the system load and the the threshold (Figure 11) for the M/D/1/1* system. It is a contour plot, where there are isolevel lines for the updated outage probability every 0.1. For very high loads, the outage update probability starts to fall under 50% only after a couple of inter-generation times. For lower inter-generation rates it becomes slightly worse. Figure 11 is an useful tool to design systems

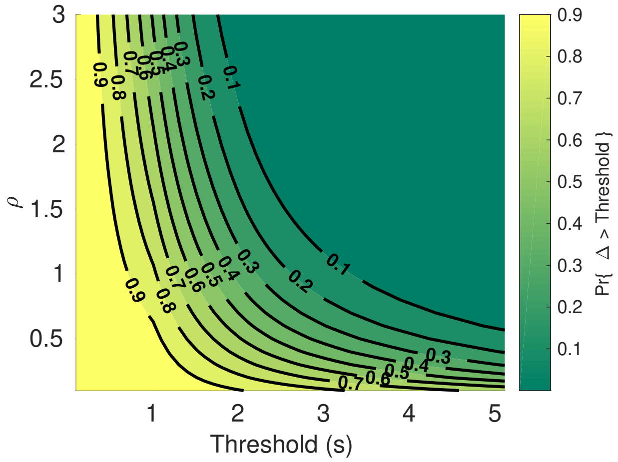

Figure 11: Update outage probability for an M/D/1 LCFS with preemption (22)**, contour plot.**

by varying the system load in order to meet a statistical constraint on the update outage time.

5 CONCLUSIONS

In this paper we studied an M/G/1 LCFS system with preemption. We found a method to derive the closed expressions for the update outage probabilities for any distribution of the busy time and showed they are significantly lower than the ones in equivalent FCFS systems, especially at high loads. We also provided closed form expressions for the complete distribution of the AoI, pAoI and effective inter-generation times at the receiver end for various busy time distributions.

Of greater relevance, we discovered that using a busy time distribution with higher variance, for certain combinations of thresholds and loads, could bring benefits to the overall update outage probability. This means that, for example, for a TDMA system, which is modeled with a deterministic busy time distribution, given a load and a threshold, could, instead, be preferable to use a different access system with higher variance, such as a CSMA access system, in order to achieve a lower update outage probability.

Finally, the system we studied being an M/G/1 system, it is possible to plug in the busy time distribution of an access delay distribution already present in the literature, in order to study the update outage probability in different systems belonging to different wireless communication techniques and standards. This means that any aspect of the access process including channel loss, ARQ and access mechanism can be applied as long as the delay distribution is known.

6 ACKNOWLEDGMENTS

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