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Bias-corrected Common Correlated Effects Pooled estimation in dynamic panels

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Abstract

This paper extends the Common Correlated Effects Pooled (CCEP) estimator to homogeneous dynamic panels. In this setting CCEP suffers from a large bias when the time span (T) of the dataset is fixed. We develop a bias-corrected CCEP estimator that is consistent as the number of cross-sectional units (N) tends to infinity, for T fixed or growing large, provided that the specification is augmented with a sufficient number of cross-sectional averages, and lags thereof. Monte Carlo experiments show that the correction offers strong improvements in terms of bias and variance. We apply our approach to estimate the dynamic impact of temperature shocks on aggregate output growth.

Keywords: Factor augmented regression, multi-factor error structure, common correlated effects, dynamic panel bias

1 Introduction

Error cross-sectional dependence is one of the major themes in recent panel data econometrics. It is well documented that neglecting such dependencies can distort inference or even lead to inconsistent estimates (see Andrews, 2005; Sarafidis and Robertson, 2009; Sarafidis and Wansbeek, 2012, for details). One of the leading approaches to model cross-sectional dependence is by assuming a multi-factor error structure, in which cross-section units are simultaneously influenced by a limited number of unobserved common factors, to which they can respond with different intensities. The common factors may reflect business cycle fluctuations, technological progress, risk and liquidity premia or other global trends and shocks that affect all cross-sectional units in the panel with a potentially differential impact across units arising from differences in institutions, absorptive capacity, technological rigidities, innate ability, preferences, risk aversion, social background, etc. (see for instance Ahn et al., 2001; Moon and Perron, 2007; Eberhardt and Teal, 2011; Sarafidis and Wansbeek, 2012, for examples). Not accounting for unobserved global variables or shocks results in inconsistent estimates when the omitted factors are correlated with the included regressors.

A popular estimation technique for panel data models with a multi-factor error structure is the Common Correlated Effects (CCE) estimator introduced by Pesaran (2006). This consists of augmenting the model with the cross-sectional averages (CSA) of the observed variables such that asymptotically - as the cross-sectional dimension $N \to \infty$ - the effect of the common factors is eliminated. Both a mean group and a pooled version are suggested, depending on whether the slope coefficients are assumed to be heterogeneous (variable) or homogeneous (constant) over cross-sectional units. Under the more general assumption of slope heterogeneity, the mean group (CCEMG) estimator is calculated as the average of the individual CCE slope coefficient estimates. The pooled (CCEP) estimator yields efficiency gains when the slope coefficients are homogeneous over cross-sectional units. Under the appropriate set of assumptions, the CCEMG and the CCEP estimators are consistent as $N \to \infty$ for either the time series dimension T fixed or $T \to \infty$. Building on the results in Pesaran (2006), the CCE approach is shown to be robust to various generalizations (see e.g. Kapetanios et al., 2011; Pesaran and Tosetti, 2011; Chudik et al., 2011; Harding and Lamarche, 2011). The computational straightforwardness of the CCE approach in combination with its robustness has led to numerous applications in many areas of economics and beyond.

The CCE methodology is well developed in the static model but was originally not intended for use in dynamic settings. Dynamic models are, however, common in practice since many (economic) variables tend to react slowly to changes in their determinants and hence display considerable persistence over time. Typically a lagged dependent variable is added to the empirical specification to account for these dynamics. However, this has important consequences for the properties of the CCE estimators. A first complication arises in the approximation of the common factors. Chudik and Pesaran (2015) show that the combination of dynamics and coefficient heterogeneity requires that an infinite number of lagged CSA should be added to the model to eliminate the factors. As this is not feasible in finite samples, they suggest to let the number of CSA grow with T. The implications of dynamics for approximating the common factors in models with homogeneous slope coefficients have not yet been studied. Secondly, Everaert and De Groote (2016) show that in a dynamic setting the CCEP estimator is inconsistent as $N \to \infty$ with T fixed, and that its asymptotic bias tends to be much larger than the standard dynamic panel data bias (Nickell, 1981) of the FE estimator in the absence of common factors. Especially when persistence is high, the CCEP estimator remains notably biased even for a moderately long time dimension T up to 50 periods. Monte Carlo simulations further show that the small sample properties of the CCEP estimator are not very sensitive to the size of N. Similar results are obtained by Chudik and Pesaran (2015) for the CCEMG estimator. Hence, in dynamic panels it is mainly the time series dimension that should be sufficiently large to allow for reliable CCE estimation and inference. This is problematic especially for estimating micro-level dynamic models where N tends to be large and T is typically (very) small, for instance when estimating dynamic employment equations with firm-level data (see e.g. Carlsson et al., 2013; Eriksson and Stadin, 2017), but also in macro-level panels, where although T tends to be larger than or similar to N the available time span is in many cases smaller than what is needed to make the bias negligibly small. In an attempt to reduce the small T bias, Chudik and Pesaran (2015) suggest the recursive mean adjustment of So and Shin (1999) or the split-panel jackknife of Dhaene and Jochmans (2015). Although these approaches succeed in mitigating the bias, they are unable to resolve the issue for short-T panels. Despite these two important complications, the CCE approach is increasingly used to estimate dynamic panel data models with common factors in a variety of empirical settings, including - among many others - development economics (Temple and Van de Sijpe, 2017); economic growth (Minniti and Venturini, 2017); international economics (Wu and Wu, 2018); the economics of inequality (Madsena et al., 2018); environmental economics (Tao, 2018).

This paper considers the CCEP approach to estimate a homogeneous dynamic panel data model. We first show that, in contrast to the heterogeneous slope model, only a finite number of lagged CSA are required to eliminate the factors from the error terms. We next remove the finite T bias of the CCEP estimator by deriving a bias-corrected alternative (referred to as CCEPbc) based on large N analytical bias expressions allowing for multiple common factors and exogenous variables. We show that, when correctly specified, the resulting estimator is consistent as $N \to \infty$ with T fixed or $T \to \infty$. Monte Carlo simulations show that CCEPbc provides considerable improvements (both in terms of bias and variance) over the original CCEP estimator and is practically unbiased in all of the considered settings. Moreover, CCEPbc is found to outperform both (i) alternative bias-adjusted CCEP estimators and (ii) the bias-corrected least squares with interactive fixed effects estimator of Moon and Weidner (2017), which is the main alternative to the CCEP methodology in dynamic panels. We further find that a (bootstrap) hypothesis test based on the CCEPbc estimator has an actual size close to the desired nominal level, even when T is small.

The remainder of this paper is structured as follows. Section 2 outlines the model and assumptions. In Section 3 we extend the CCEP estimator to homogeneous dynamic panel data models and derive an expression for its finite T inconsistency that will be used in Section 4 to construct a bias-corrected CCEP estimator. Monte Carlo simulation results are presented in Section 5. In Section 6 we use our CCEPbc approach to estimate the dynamic impact of temperature shocks on aggregate output growth in a panel of 125 countries. Section 7 concludes. Proofs and additional results are collected in an Online Supplement.

Before proceeding we introduce some notation that will be used throughout the paper. For a $T \times c$ matrix $\mathbf{A}$, $\|\mathbf{A}\| = [tr(\mathbf{A}\mathbf{A}')]^{1/2}$ denotes the Euclidian (Frobenius) matrix norm, $tr(\cdot)$ the trace, $rk(\cdot)$ the rank, $vec(\cdot)$ is the vectorization operator and $(\mathbf{A}'\mathbf{A})^{\dagger}$ is the Moore-Penrose pseudo-inverse of $\mathbf{A}'\mathbf{A}$. A $-p$ subscript corresponds to the p-period lag of the respective variable or matrix so that $\mathbf{A}_{-p} = L^p\mathbf{A}$, where L is the lag operator.

2 Model and assumptions

Consider the following first-order dynamic panel data model

$$\mathrm{y}_{it} = \alpha_i + \rho \mathrm{y}_{i,t-1} + \mathbf{x}_{it}'\boldsymbol{\beta} + e_{it}, \tag{1}$$

$$e_{it} = \boldsymbol{\gamma}_i'\mathbf{f}_t + \varepsilon_{it}, \tag{2}$$

for $i = 1, ..., N$ and $t = 1, ..., T$ and where y_{it} is the observation on the dependent variable for unit i at time t, α_i is an unobserved individual effect, $\mathbf{x}_{it}$ an individual-specific $k_x \times 1$ column vector of strictly exogenous regressors and e_{it} a multi-factor error term that is composed of an $m \times 1$ vector of unobserved common factors $\mathbf{f}_t$ with heterogeneous factor loadings $\boldsymbol{\gamma}_i$ and an idiosyncratic error term ε_{it}. The unknown parameters ρ and $\boldsymbol{\beta}$ are assumed to be homogeneous over cross-sectional units i and bounded by a finite constant. For notational convenience we assume y_{i0} known.

Following Pesaran et al. (2013) we also exploit information regarding the unobserved common factors that is shared by variables other than y_{it} and $\mathbf{x}_{it}$. To this end consider a $k_g \times 1$ vector of individual-specific strictly exogenous covariates $\mathbf{g}_{it}$ that have no effect on the dependent variable y_{it} but that are driven by the same factors $\mathbf{f}_t$ that affect y_{it}. The individual-specific covariates and other variables are collected in the $k \times 1$ column vector $\mathbf{z}_{it} = [\mathbf{x}_{it}', \mathbf{g}_{it}']'$, with $k = k_x + k_g$, and are assumed to be generated as

$$\mathbf{z}_{it} = \begin{bmatrix} \mathbf{x}_{it} \\ \mathbf{g}_{it} \end{bmatrix} = \mathbf{c}_{\mathrm{z},i} + \sum_{l=1}^{p} \boldsymbol{\lambda}_l \mathbf{z}_{i,t-l} + \boldsymbol{\Gamma}_i'\mathbf{f}_t + \mathbf{v}_{it}, \tag{3}$$

where $\mathbf{c}_{\mathrm{z},i}$ is a $k \times 1$ column vector of unobserved individual effects, p denotes the autoregressive order of $\mathbf{z}_{it}$, $\boldsymbol{\lambda}_l$ is a $k \times k$ matrix of coefficients corresponding to lags $l = 1, \ldots, p$ of $\mathbf{z}_{it}$, $\boldsymbol{\Gamma}_i$ is a $m \times k$ matrix of factor loadings and $\mathbf{v}_{it}$ a $k \times 1$ vector of idiosyncratic errors. The assumption that p is equal for all variables in $\mathbf{z}_{it}$ is for notational convenience only and can easily be relaxed within the current notation by interpreting p as the maximum lag length and setting some of the parameters in $\boldsymbol{\lambda}_l$ equal to zero.

We make the following assumptions:

Assumption 1. *(Idiosyncratic errors) The ε_{it} and $\mathbf{v}_{it}$ are i.i.d. across i and t with $E(\varepsilon_{it}\mathbf{v}_{js}) = \mathbf{0}_{k\times 1}$, $E(\varepsilon_{it}^4) < \infty$ and $E(\|\mathbf{v}_{it}\|^4) < \infty$ for all i, j, t and s. In particular,*

$$\varepsilon_{it} \sim \mathrm{IID}(0, \sigma_\varepsilon^2), \qquad \mathbf{v}_{it} \sim \mathrm{IID}(\mathbf{0}_{k\times 1}, \boldsymbol{\Omega}_{\mathrm{v}}),$$

with $\sigma_\varepsilon^2 > 0$ and $\boldsymbol{\Omega}_{\mathrm{v}}$ a positive definite $k \times k$ matrix.

Assumption 2. *(Common factors) The* $\mathbf{f}_t$ *are covariance stationary with absolute summable autocovariances,* $E(\|\mathbf{f}_t\|^4) < \infty$ *and they are distributed independently of* ε_{is}, $\mathbf{v}_{is}$, $\boldsymbol{\gamma}_i$ *and* $\boldsymbol{\Gamma}_i$ *for all* i, t *and* s.

Assumption 3. *(Factor loadings) The* $\boldsymbol{\gamma}_i$ *and* $\boldsymbol{\Gamma}_i$ *are i.i.d. across* i*, independent of* ε_{jt}, $\mathbf{v}_{jt}$ *and* $\mathbf{f}_t$ *for all* i, j *and* t*, with* $E(\|\boldsymbol{\gamma}_i\|^4) < \infty$ *and* $E(\|\boldsymbol{\Gamma}_i\|^4) < \infty$*. In particular,*

$$\begin{aligned}\boldsymbol{\gamma}_i &= \boldsymbol{\gamma} + \boldsymbol{\eta}_i, \qquad \boldsymbol{\eta}_i \sim IID(\mathbf{0}_{m\times 1}, \boldsymbol{\Omega}_\eta), \qquad (4)\\ \boldsymbol{\Gamma}_i &= \boldsymbol{\Gamma} + \boldsymbol{\nu}_i, \qquad vec(\boldsymbol{\nu}_i) \sim IID(\mathbf{0}_{mk\times 1}, \boldsymbol{\Omega}_\nu), \qquad (5)\end{aligned}$$

where $E(\|\boldsymbol{\eta}_i' \otimes \boldsymbol{\nu}_i'\|) \geq 0$ *and* $\boldsymbol{\Omega}_\eta$, $\boldsymbol{\Omega}_\nu$ *are bounded* $m\times m$ *and* $km\times km$ *matrices respectively.*

Assumption 4. *(Rank condition) The* $(1+k)\times m$ *matrix* $\mathbf{C} = [\boldsymbol{\gamma}, \boldsymbol{\Gamma}]'$ *has* $rk(\mathbf{C}) = m$.

Assumption 5. *(Stationarity)* $|\rho| < 1$ *and the the elements in* $\boldsymbol{\lambda}_l$ *are such that* $\boldsymbol{\lambda}(L) = \mathbf{I}_k - \sum_{l=1}^{p} \boldsymbol{\lambda}_l L^l$ *is invertible. The process of* y_{it} *was initiated in the infinite past.*

For future reference, we let $k_w = 1 + k_x$ and stack the model in eqs.(1)-(2) over time as

$$\mathbf{y}_i = \alpha_i \boldsymbol{\iota}_T + \mathbf{w}_i \boldsymbol{\delta} + \mathbf{F}\boldsymbol{\gamma}_i + \boldsymbol{\varepsilon}_i, \qquad (6)$$

where $\boldsymbol{\delta} = [\rho, \boldsymbol{\beta}']'$ and $\mathbf{w}_i = [\mathbf{y}_{i,-1}, \mathbf{X}_i]$ are $k_w \times 1$ and $T \times k_w$, and $\mathbf{X}_i = [\mathbf{x}_{i1}, \ldots, \mathbf{x}_{iT}]'$, $\mathbf{y}_i = [y_{i1}, \ldots, y_{iT}]'$, $\mathbf{y}_{i,-1} = [y_{i0}, \ldots, y_{i,T-1}]'$, $\mathbf{F} = [\mathbf{f}_1, \ldots, \mathbf{f}_T]'$, $\boldsymbol{\varepsilon}_i = [\varepsilon_{i1}, \ldots, \varepsilon_{iT}]'$ and $\boldsymbol{\iota}_T$ is a $T \times 1$ column vector of ones. Similarly specify $\mathbf{G}_i = [\mathbf{g}_{i1}, \ldots, \mathbf{g}_{iT}]'$ and $\mathbf{Z}_i = [\mathbf{X}_i, \mathbf{G}_i]$.

3 CCEP estimation in dynamic panels

Pesaran (2006) developed the CCE approach in a static model with strictly exogenous regressors and showed that under Ass.4 the differential effects of the unobserved factors can be eliminated as $N \to \infty$ by augmenting the model with the CSA of the observables. In this section, we first review whether the CSA still serve as suitable proxies for the factors in homogeneous dynamic panels. We next show that, in contrast to the static case, the CCEP estimator is inconsistent when $N \to \infty$ and T fixed by deriving its bias expression that will be used in Section 4 to construct a bias-corrected CCEP estimator.

3.1 Cross-sectional averages as proxies for the common factors

Rewriting eqs.(1)-(3) as

$$\begin{aligned}\rho(L)\, y_{it} &= \alpha_i + \mathbf{x}_{it}'\boldsymbol{\beta} + \boldsymbol{\gamma}_i'\mathbf{f}_t + \varepsilon_{it},\\ \boldsymbol{\lambda}(L)\, \mathbf{z}_{it} &= \mathbf{c}_{z,i} + \boldsymbol{\Gamma}_i'\mathbf{f}_t + \mathbf{v}_{it},\end{aligned}$$

where $\rho(L) = 1 - \rho L$ and $\boldsymbol{\lambda}(L) = \mathbf{I}_k - \sum_{l=1}^{p} \boldsymbol{\lambda}_l L^l$, and taking CSA yields

$$\begin{aligned}\rho(L)\, \bar{y}_t &= \bar{\alpha} + \bar{\mathbf{x}}_t'\boldsymbol{\beta} + \boldsymbol{\gamma}'\mathbf{f}_t + O_p(N^{-1/2}), \qquad (7)\\ \boldsymbol{\lambda}(L)\, \bar{\mathbf{z}}_t &= \bar{\mathbf{c}}_z + \boldsymbol{\Gamma}'\mathbf{f}_t + O_p(N^{-1/2}), \qquad (8)\end{aligned}$$

with the affix notation on $\bar{y}_t$ used to denote the CSA $\bar{y}_t = \frac{1}{N}\sum_{i=1}^{N} y_{it}$ and similarly for all other series. Under Ass.4 that $\mathbf{C}$ has full column rank, we can solve for $\mathbf{f}_t$ to obtain

$$\mathbf{f}_t = \left(\mathbf{C}'\mathbf{C}\right)^{-1}\mathbf{C}'\left(\begin{bmatrix}\rho(L) & -(\boldsymbol{\beta}^*)' \\ 0 & \boldsymbol{\lambda}(L)\end{bmatrix}\begin{bmatrix}\bar{y}_t \\ \bar{\mathbf{z}}_t\end{bmatrix} - \begin{bmatrix}\bar{\alpha} \\ \bar{\mathbf{c}}_z\end{bmatrix}\right) + O_p\left(\frac{1}{\sqrt{N}}\right), \tag{9}$$

with $\boldsymbol{\beta}^* = [\boldsymbol{\beta}', \mathbf{0}'_{k_g\times 1}]'$. Eq.(9) shows that as $N \to \infty$ the factors can be approximated by the CSA of y_{it} and $\mathbf{z}_{it}$ as well as a finite number of their lags determined by the orders of the polynomials $\rho(L)$ and $\boldsymbol{\lambda}(L)$. This result differs from the heterogeneous dynamic model considered by Chudik and Pesaran (2015) who find that an infinite number of lags is required in this case.

The intuition behind the above result is that in the presence of dynamics the lags are needed to separate the contemporaneous factor from its past realizations within the CSA. This is necessary to approximate $\mathbf{f}_t$ in function of observables as $N \to \infty$. To see this, consider the simple case of model (1)-(2) with one factor and $\boldsymbol{\beta} = \mathbf{0}$. The CSA of y_{it} can then be written as

$$\bar{y}_t = \bar{\alpha} + \bar{\gamma} f_t + \bar{\varepsilon}_t + \rho\left(\frac{\bar{\alpha}}{1-\rho} + \bar{\gamma} f^{+}_{t-1} + \bar{\varepsilon}^{+}_{t-1}\right), \tag{10}$$

$$= \frac{\bar{\alpha}}{1-\rho} + \gamma\left[f_t + \rho f^{+}_{t-1}\right] + O_p(N^{-1/2}), \tag{11}$$

so that it is not only a function of the factors at time t, a constant and an $O_p(N^{-1/2})$ term, but also of the past realizations of the factors through $f^{+}_{t-1} = \sum_{l=0}^{\infty}\rho^l f_{t-l-1}$. Solving the contemporaneous factor f_t from (11) would therefore still depend on the unobservable f^{+}_{t-1} so a proxy cannot be constructed from it. However, noting that the term inside the brackets of (10) equals $\bar{y}_{t-1}$, subtracting $\rho\bar{y}_{t-1}$ from (10) yields

$$\bar{y}_t - \rho\bar{y}_{t-1} = \bar{\alpha} + \gamma f_t + O_p(N^{-1/2}), \tag{12}$$

so that the past factor realizations are cut out and this equation can be solved for f_t as a function of observables, estimable parameters and an $O_p(N^{-1/2})$ term. The combination of observables can then be used to project out the factors at time t as $N \to \infty$. A similar reasoning holds for $\mathbf{z}_{it}$ as well. This clearly illustrates the difference with the static case in Pesaran (2006), where the absence of dynamics implies that $\rho = 0$ so that the CSA do not contain the past factors and, hence, lags are not required to separate them from f_t.

REMARK 1. The requirement that we have to know the order of $\boldsymbol{\lambda}(L)$ may be unfortunate in practice as p is typically unknown (and may also differ over variables included in $\mathbf{z}_{it}$). Decisions on p imply assumptions about the autoregressive order of $\mathbf{z}_{it}$ that may be hard to verify since the observed persistence in $\mathbf{z}_{it}$ may stem from serially correlated factors $\mathbf{f}_t$ or from $\boldsymbol{\lambda}(L) \neq \mathbf{I}_k$. However, as more time series observations become available the factor approximation should not suffer from including too many lags $p^* > p$ of $\bar{\mathbf{z}}_t$. Hence, in practice it may be convenient to choose $p^* = \lfloor T^{1/3} \rceil$ as in Chudik and Pesaran (2015), with $\lfloor x \rceil$ denoting the integer part of x, to make the CCEP estimator robust to misspecification of p while ensuring that the number of lags does not increase too fast in T and sufficient degrees of freedom are available.

3.2 Dynamic CCEP estimator

In light of the discussion in the previous section, construct the following $T \times c$ matrix $\mathbf{Q} = [\boldsymbol{\iota}_T, \bar{\mathbf{y}}, \bar{\mathbf{y}}_{-1}, \bar{\mathbf{Z}}, \ldots, \bar{\mathbf{Z}}_{-p^*}]$ and augment the model in eq.(6) as

$$\mathbf{y}_i = \mathbf{w}_i\boldsymbol{\delta} + \mathbf{Q}\boldsymbol{\kappa}_i + \mathbf{e}_i, \tag{13}$$

where the CSA in $\mathbf{Q}$ serve to control for the common factors absorbed in the error terms $\mathbf{e}_i$, and $\boldsymbol{\kappa}_i$ are parameters to be estimated along with the slope coefficients of interest $\boldsymbol{\delta} = [\rho, \boldsymbol{\beta}']'$. Assuming that $T \geq k_w + c$ (estimability) and setting pooling weights to N^{-1}, the dynamic CCEP estimator for $\boldsymbol{\delta}$ in eq.(13) is

$$\widehat{\boldsymbol{\delta}} = \left(\sum_{i=1}^{N} \mathbf{w}_i'\mathbf{M}\mathbf{w}_i\right)^{-1} \sum_{i=1}^{N} \mathbf{w}_i'\mathbf{M}\mathbf{y}_i, \tag{14}$$

where $\mathbf{M} = \mathbf{I}_T - \mathbf{H}$ and $\mathbf{H} = \mathbf{Q}(\mathbf{Q}'\mathbf{Q})^{\dagger}\mathbf{Q}'$ is the projection on $\mathbf{Q}$.

The dynamic CCEP estimator in eq.(14) controls, as $N \to \infty$, for the unobserved factors provided that the rank condition (Ass.4) holds and the model is augmented with a sufficient number ($p^* \geq p$) of lagged CSA. However, despite controlling for the factors, the inclusion of the CSA induces a new finite T bias term. The following theorem provides an analytical expression of the asymptotic bias of the dynamic CCEP estimator for $N \to \infty$ and T fixed conditional on the factors and CSA.

Theorem 1. *Suppose that $p^* \geq p$ and Ass.1-5 hold, and let $\mathcal{C}$ be the σ-algebra generated by the common factors and $[\bar{\mathbf{y}}, \bar{\mathbf{Z}}, \ldots, \bar{\mathbf{y}}_{-p^*}, \bar{\mathbf{Z}}_{-p^*}]$. The CCEP estimator in eq.(14) is inconsistent as $N \to \infty$ and T fixed with its asymptotic bias conditional on $\mathcal{C}$ given by*

$$\operatorname*{plim}_{N\to\infty} \widehat{\boldsymbol{\delta}} = \mathbf{m}(\boldsymbol{\delta}) = \boldsymbol{\delta} - \frac{\sigma_\varepsilon^2}{T}\boldsymbol{\Sigma}^{-1}\boldsymbol{\upsilon}(\rho, \mathbf{H}), \tag{15}$$

with $\boldsymbol{\upsilon}(\rho, \mathbf{H}) = \upsilon(\rho, \mathbf{H})\mathbf{q}_1$, $\mathbf{q}_1 = [1, \mathbf{0}_{1\times k_x}]'$, $\upsilon(\rho, \mathbf{H}) = \sum_{t=1}^{T-1} \rho^{t-1} \sum_{s=t+1}^{T} h_{s,s-t}$ and $h_{s,s-t}$ is the element on row s and column $s-t$ of $\mathbf{H}$. $\boldsymbol{\Sigma} = \lim_{N\to\infty} \frac{1}{NT} \sum_{i=1}^{N} \mathbf{w}_i'\mathbf{M}\mathbf{w}_i$ is given in (C-7) of the Online Supplement. Letting $\mathbf{S}_{\mathrm{x}} = [\mathbf{0}_{k_x\times 1}, \mathbf{I}_{k_x}]'$, eq.(15) can be decomposed as

$$\operatorname*{plim}_{N\to\infty} (\widehat{\rho} - \rho) = -\frac{1}{T}\frac{\sigma_\varepsilon^2}{\sigma_{\breve{\mathrm{y}}_{-1}}^2}\upsilon(\rho, \mathbf{H}), \tag{16}$$

$$\operatorname*{plim}_{N\to\infty} \left(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) = -\boldsymbol{\zeta} \operatorname*{plim}_{N\to\infty}(\widehat{\rho} - \rho), \tag{17}$$

with $\sigma_{\breve{\mathrm{y}}_{-1}}^2 = \operatorname{plim}_{N\to\infty} \frac{\breve{\mathbf{y}}_{-1}'\breve{\mathbf{y}}_{-1}}{NT}$, $\boldsymbol{\zeta} = (\mathbf{S}_{\mathrm{x}}'\boldsymbol{\Sigma}\mathbf{S}_{\mathrm{x}})^{-1}\mathbf{S}_{\mathrm{x}}'\boldsymbol{\Sigma}\mathbf{q}_1$ and where $\breve{\mathbf{y}}_{-1} = \mathbb{M}_x[\mathbf{y}_{1,-1}', \ldots, \mathbf{y}_{N,-1}']'$, $\mathbb{M}_x = \mathbb{M} - \breve{\mathbf{X}}(\breve{\mathbf{X}}'\breve{\mathbf{X}})^{-1}\breve{\mathbf{X}}'$, $\breve{\mathbf{X}} = \mathbb{M}[\mathbf{X}_1', \ldots, \mathbf{X}_N']'$ and $\mathbb{M} = \mathbf{I}_N \otimes \mathbf{M}$.

Theorem 1 extends the results in Everaert and De Groote (2016), who consider a model with one factor and no additional covariates, to a model with multiple factors and exogenous regressors. Also in this more general setting, the asymptotic bias of the CCEP estimator is not caused by the factor structure but it is induced by projecting the data on $\mathbf{Q}$ as this induces weak endogeneity in the transformed lagged dependent variable and hence inconsistency of the autoregressive parameter $\widehat{\rho}$ for finite T, as shown in eq.(16). Concerning the coefficients of the exogenous regressors, eq.(17) reveals that the bias of $\widehat{\boldsymbol{\beta}}$ is a fraction $-\boldsymbol{\zeta}$ of the bias of $\widehat{\rho}$, with $\boldsymbol{\zeta}$ being the CCEP estimates (as $N \to \infty$) when regressing $\mathbf{y}_{i,-1}$ on $\mathbf{X}_i$. As such, the direction of the distortion in $\widehat{\boldsymbol{\beta}}$ is determined by the correlation between $\mathbf{y}_{i,-1}$ and $\mathbf{X}_i$ given by $\boldsymbol{\zeta}$, but it is the inconsistency of $\widehat{\rho}$ that creates bias for the entire coefficient vector. The inconsistency in $\widehat{\rho}$ is therefore the principal driver of the overall bias, and we study it in more detail in Section A.2 of the Online Supplement. The most important conclusions of that analysis are:

- The asymptotic bias is expected to be negative for $\rho > 0$.
- The asymptotic bias is a stochastic variable because it depends on the projection matrix $\mathbf{H}$, which is a random matrix even as $N \to \infty$.
- The absolute value of the asymptotic bias is, ceteris paribus, increasing in the persistence ρ and the number of CSA (columns of $\mathbf{Q}$), and decreasing in T and in the importance of the factors when there is more than one factor.

The key practical implication of Theorem 1 is that there is a trade-off associated with augmenting the model with the CSA. On the one hand, to control for the unobserved common factors, a sufficient number of CSA should be included such that the rank condition is satisfied and $p^* > p$. Simulation evidence further suggests that even when the rank condition is satisfied, using additional CSA improves factor approximation in finite N samples (see Section 5). On the other hand, in finite T settings, the augmentation generates a bias term that increases in magnitude with the number of CSA. As such, whereas adding CSA is beneficial to treat the common factor problem, it can simultaneously be detrimental for the finite T properties of the CCEP estimator. Our objective in the next section is to resolve this trade-off by removing the bias induced by projecting out the CSA.

4 Bias-corrected dynamic CCEP

In what follows we develop a bias-corrected CCEP estimator based on the analytical bias expression for $N \to \infty$ and T fixed presented in eq.(15) of Theorem 1, and derive its asymptotic distribution.

4.1 Bias correction procedure

The CCEPbc estimator $\widehat{\boldsymbol{\delta}}_{bc}$ can be obtained as the vector $\boldsymbol{\delta}_0$ that satisfies

$$\widehat{\boldsymbol{\delta}} - \widehat{\mathbf{m}}(\boldsymbol{\delta}_0) = \mathbf{0}_{k_w \times 1}, \tag{18}$$

with $\widehat{\mathbf{m}}(\cdot)$ the feasible version of the asymptotic bias expression in eq.(15),

$$\widehat{\mathbf{m}}(\boldsymbol{\delta}_0) = \boldsymbol{\delta}_0 - T^{-1}\widehat{\sigma}^2_{\varepsilon}(\boldsymbol{\delta}_0)\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\upsilon}(\rho_0, \mathbf{H}), \tag{19}$$

where $\mathbf{\Sigma}$ is replaced by its sample analog $\widehat{\mathbf{\Sigma}} = \frac{1}{NT}\sum_{i=1}^{N} \mathbf{w}_i'\mathbf{M}\mathbf{w}_i$ and the unknown variance σ_ε^2 is substituted by the function

$$\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta}_0) = \frac{1}{N(T-c)} \sum_{i=1}^{N} \left\| \mathbf{M}\left(\mathbf{y}_i - \mathbf{w}_i\boldsymbol{\delta}_0\right)\right\|^2. \tag{20}$$

The traditional estimator for σ_ε^2 based on the uncorrected CCEP error terms $\widehat{\mathbf{e}}_i = \mathbf{y}_i - \mathbf{w}_i\widehat{\boldsymbol{\delta}}$ is inconsistent for finite T due to the inconsistency of $\widehat{\boldsymbol{\delta}}$, but by constructing $\widehat{\sigma}_\varepsilon^2(\cdot)$ as a function of the parameters of interest, solving (18) implies that we use a bias-adjusted estimator for σ_ε^2 as well. In summary, the CCEPbc estimator is

$$\widehat{\boldsymbol{\delta}}_{bc} = \arg\min_{\boldsymbol{\delta}_0 \in \chi} \frac{1}{2}\left\| \widehat{\boldsymbol{\delta}} - \widehat{\mathbf{m}}(\boldsymbol{\delta}_0)\right\|^2, \tag{21}$$

with $\chi \subseteq \mathbb{R}^{k_w}$. This optimization problem is easily managed by standard numerical solvers and requires very little additional programming besides computing the CCEP estimates $\widehat{\boldsymbol{\delta}}$. The solution $\widehat{\boldsymbol{\delta}}_{bc}$ is equivalent to the vector of parameters that follows from inverting $\widehat{\boldsymbol{\delta}} = \widehat{\mathbf{m}}(\boldsymbol{\delta})$ so that we can alternatively write the CCEPbc estimator as $\widehat{\boldsymbol{\delta}}_{bc} = \widehat{\mathbf{m}}^{-1}(\widehat{\boldsymbol{\delta}})$. Notice how eq.(21) implies that the bias adjustment can be seen as a minimum distance estimator, or a GMM approach that employs the bias-corrected orthogonality conditions in (18) to estimate the population parameters. To make this point explicit, straightforward manipulations in (18) give

$$\widehat{\boldsymbol{\delta}} - \widehat{\mathbf{m}}(\boldsymbol{\delta}_0) = (\boldsymbol{\delta} - \boldsymbol{\delta}_0) + \widehat{\mathbf{\Sigma}}^{-1}\left[\frac{1}{NT}\sum_{i=1}^{N} \mathbf{w}_i'\mathbf{M}(\mathbf{F}\boldsymbol{\gamma}_i + \boldsymbol{\varepsilon}_i) + \mathbf{b}_\varepsilon(\boldsymbol{\delta}_0)\right] = \mathbf{0}_{k_w \times 1}, \tag{22}$$

with $\mathbf{b}_\varepsilon(\boldsymbol{\delta}_0) = T^{-1}\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta}_0)\boldsymbol{\upsilon}(\rho_0, \mathbf{H})$. This shows that the moment conditions underlying CCEPbc in eq.(21) are identical to those of the CCEP estimator, except for the $\mathbf{b}_\varepsilon(\boldsymbol{\delta}_0)$ term which corrects for the finite T bias. In this sense our approach is similar in spirit to ideas presented in Chudik and Pesaran (2017). Also note that Bun and Carree (2005) use a similar approach to obtain a bias-corrected FE estimator in dynamic panel data models without common factors.

REMARK 2. The CCEPbc estimator outlined above is a generally applicable method in the sense that it does not require the number of factors to be known. In the single factor setting, eqs.(16)-(17) can be simplified to obtain more efficient restricted bias corrections. We present two alternative restricted CCEPbc estimators in Section A.3 of the Online Supplement.

4.2 Asymptotic properties and inference

The CCEPbc estimator presented in eq.(21) builds on the orthogonality conditions in eq.(22) to estimate the population parameters of interest. We show in theorem 3 of the Online Supplement that these moment conditions are satisfied as $N \to \infty$ at $\boldsymbol{\delta}_0 = \boldsymbol{\delta}$ and that the CCEPbc estimator is thus consistent as $N \to \infty$ and T fixed

$$\widehat{\boldsymbol{\delta}}_{bc} - \boldsymbol{\delta} \longrightarrow^p \mathbf{0}_{k_w \times 1}. \tag{23}$$

As such, in dynamic models the proposed correction restores the large N finite T consistency of the CCEP estimator established by Pesaran (2006) in a static setting.

The finite T distribution of the CCEPbc estimator is generally intractable due to the presence of nuisance parameters, unless one makes the very stringent assumption that $m = 1 + k$ (see e.g. Karabiyik et al., 2017, for more details). In the next theorem, we establish asymptotic normality for the CCEPbc estimator in the general $m \leq 1 + k$ case letting $(N, T) \to \infty$.

Theorem 2. *Let Ass.1-5 hold and suppose that $p^* \geq p$ and $\chi \subseteq \mathbb{R}^{k_w}$ is compact such that $|\rho_0| < 1$ with $\boldsymbol{\delta} \in \chi$. Then, as $(N, T) \to \infty$ it holds that $\widehat{\boldsymbol{\delta}}_{bc} \longrightarrow^p \boldsymbol{\delta}$, and provided $T/N \to 0$*

$$\sqrt{NT}(\widehat{\boldsymbol{\delta}}_{bc} - \boldsymbol{\delta}) \xrightarrow{d} \mathcal{N}\left(\mathbf{0}_{k_w \times 1}, \dot{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\Phi}\dot{\boldsymbol{\Sigma}}^{-1}\right), \tag{24}$$

with $\dot{\boldsymbol{\Sigma}}$ and $\boldsymbol{\Phi}$ defined in eqs.(D-30) and (D-53) of the Online Supplement respectively.

Theorem 2 establishes that the CCEPbc estimator is asymptotically normally distributed as $(N, T) \to \infty$ and that it enables unbiased inference provided $T/N \to 0$. This requirement on the relative growth rate of N and T stems from estimating the factors with the CSA and it is identical to what Pesaran (2006) and Karabiyik et al. (2017) require for unbiased inference with CCEP in the static model.

The asymptotic variance in eq.(24) can consistently, as $(N, T) \to \infty$, be estimated by

$$\widehat{\boldsymbol{\Omega}} = (\widehat{\boldsymbol{\Delta}}'\widehat{\boldsymbol{\Delta}})^{-1}\widehat{\boldsymbol{\Delta}}'\widehat{\boldsymbol{\Phi}}\widehat{\boldsymbol{\Delta}}(\widehat{\boldsymbol{\Delta}}'\widehat{\boldsymbol{\Delta}})^{-1}, \tag{25}$$

where $\widehat{\boldsymbol{\Delta}} = \mathbf{J}_{\mathrm{a}}(\widehat{\boldsymbol{\delta}}_{bc})$ and $\mathbf{J}_{\mathrm{a}}(\cdot)$ is the Jacobian presented in eq.(A-2), and with $\widehat{\boldsymbol{\Phi}} = \frac{1}{NT}\sum_{i=1}^{N}\widehat{\mathbf{q}}_i\widehat{\mathbf{q}}_i'$, $\widehat{\mathbf{q}}_i = \mathbf{w}_i'\mathbf{M}\widehat{\mathbf{e}}_i + \widehat{\sigma}_\varepsilon^2(\widehat{\boldsymbol{\delta}}_{bc})\boldsymbol{\upsilon}(\widehat{\rho}_{bc}, \mathbf{H})$ and $\widehat{\mathbf{e}}_i = \mathbf{y}_i - \mathbf{w}_i\widehat{\boldsymbol{\delta}}_{bc}$.

In practice, in particular in settings when T is not large, the bootstrap provides a convenient alternative to (25) for estimating the finite sample variance of $\widehat{\boldsymbol{\delta}}_{bc}$. To that end, we follow Kapetanios (2008) and obtain bootstrap samples by resampling whole cross-sectional units with replacement from the original dataset. In particular, let $\mathcal{B}^0 = [\mathbf{a}_1, \ldots, \mathbf{a}_N]$ be the original dataset, with $\mathbf{a}_i = [\mathbf{d}_{i,-p^*}, \ldots, \mathbf{d}_{iT}]'$ and $\mathbf{d}_{it} = [\mathrm{y}_{it}, \mathbf{z}_{it}']'$. Bootstrap sample $j = 1, \ldots, J$ is generated by drawing N indices with replacement from $(1, \ldots, N)$, and collecting the $\mathbf{a}_i$ corresponding to these indices in $\mathcal{B}^j$. This resampling scheme is valid as $N \to \infty$ and preserves both the dynamics and the assumed factor structure in the data. The distribution of $\widehat{\boldsymbol{\delta}}_{bc}$ is then simulated by applying CCEPbc to each of the J bootstrap datasets $[\mathcal{B}^1, \ldots, \mathcal{B}^J]$ to obtain the corresponding coefficient vectors $[\widehat{\boldsymbol{\delta}}^b_{bc,1}, \ldots, \widehat{\boldsymbol{\delta}}^b_{bc,J}]$. Inference can then be made using the bootstrapped variance-covariance matrix

$$\widehat{\boldsymbol{\Omega}}_b = \lim_{J\to\infty}\frac{1}{J-1}\sum_{j=1}^{J}\left(\widehat{\boldsymbol{\delta}}^b_{bc,j} - \bar{\boldsymbol{\delta}}^b_{bc}\right)\left(\widehat{\boldsymbol{\delta}}^b_{bc,j} - \bar{\boldsymbol{\delta}}^b_{bc}\right)', \tag{26}$$

with $\bar{\boldsymbol{\delta}}^b_{bc} = \frac{1}{J}\sum_{j=1}^{J}\widehat{\boldsymbol{\delta}}^b_{bc,j}$ the average of the estimates over the J samples.

REMARK 3. Lemmas 14-15 in the Online Supplement show that, in contrast to the CCEPbc estimator, the asymptotic distribution of the uncorrected CCEP estimator in eq.(14) features bias terms unless both $N/T \to 0$ (due to the finite T bias in theorem 1) and $T/N \to 0$ (due to estimation of the factors). As this is clearly a contradiction, bias correction is crucial for reliable inference with the CCEP approach despite that the estimator is consistent as $(N, T) \to \infty$ in the dynamic model.

5 Monte Carlo Simulation

In this section we use Monte Carlo simulations to investigate the small sample properties of our bias-corrected CCEP estimator and compare its performance to the original CCEP estimator and a number of alternative methods proposed in the literature.

5.1 Design

We generate data for y_{it} and $\mathbf{z}_{it}$ according to the model in eqs.(1)-(3) assuming a single explanatory variable x_{it} ($k_x = 1$) and one additional variable g_{it} ($k_g = 1$) that has no impact on y_{it} but provides additional information about the common factors. We set $\beta = 1 - \rho$ to normalize the long-run impact of x_{it} to one and assume $\boldsymbol{\lambda}(L) = (1 - \lambda L)\mathbf{I}_2$ which restricts the autoregressive order of x_{it} and g_{it} to be at most one ($p = 1$). This implies that the one period lagged CSA $\bar{\mathrm{x}}_{t-1}$ should be added to the CCE orthogonalization matrix in settings where $\lambda \neq 0$ (and preferably also $\bar{\mathrm{g}}_{t-1}$ when g_{it} is used as an additional variable).

The m common factors are generated as

$$\mathrm{f}_{jt} \quad = \quad \theta \mathrm{f}_{j,t-1} + \mu_{jt},$$

with $\mu_{jt} \sim N\left(0, (1 - \theta^2)/m\right)$ for every $j = 1, \ldots, m$. The reason for dividing the variance by m is to prevent the factors from dominating the model as their number m rises. We will conduct experiments with $m = 1$ and $m = 2$.

The fixed effects are generated as $\alpha_i \sim N\left(0, \sigma_\alpha^2\right)$ and $\mathbf{c}_{\mathrm{z},i} \sim N\left(\mathbf{0}, \sigma_{\mathrm{c}}^2 \mathbf{I}_2\right)$ and the idiosyncratic errors as $\varepsilon_{it} \sim N\left(0, 1 - \rho^2\right)$ and $\mathbf{v}_{it} \sim N\left(\mathbf{0}, (1 - \lambda^2)\mathbf{I}_2\right)$. The variance parameters σ_α^2 and σ_{c}^2 are set such that the contributions of the fixed effects to the variance of y_{it} and $\mathbf{z}_{it}$ equal that of their respective idiosyncratic innovations (ε_{it} and $\mathbf{v}_{it}$). The factor loadings in the DGPs of y_{it}, x_{it} and g_{it} are generated as

$$\mathbf{C}_i = \begin{bmatrix} \boldsymbol{\gamma}_i' \\ \boldsymbol{\Gamma}_i^{\mathrm{x}\prime} \\ \boldsymbol{\Gamma}_i^{\mathrm{g}\prime} \end{bmatrix} = \begin{bmatrix} \gamma_{1,i} & \gamma_{2,i} \\ \Gamma_{1,i}^{\mathrm{x}} & \Gamma_{2,i}^{\mathrm{x}} \\ \Gamma_{1,i}^{\mathrm{g}} & \Gamma_{2,i}^{\mathrm{g}} \end{bmatrix} \sim \mathrm{IIDU} \begin{bmatrix} [0, \gamma_u] & [0, \gamma_u - 3/5] \\ [0, 1] & [0, 0.2] \\ [-0.6, 0] & [-1.4, 0] \end{bmatrix},$$

when $m = 2$ or with the second column set to zero in case $m = 1$. The upper bound γ_u is calibrated such that the relative importance of the factors and the idiosyncratic errors in the total variance of y_{it}, denoted RI, is either 1 or 3. $RI = 1$ corresponds to cases where the factors have a normal influence on y_{it} whereas $RI = 3$ is a scenario where the factors are very influential. The specific values for the upper and lower bounds of the uniform distributions for the loadings in $\mathbf{C}_i$ are sufficiently different to ensure that the rank condition is satisfied and that the full set of CSA contains enough independent information about the common factors.

Experiments are conducted for combinations of the following parameter values: $\rho \in \{0.4; 0.8\}$, $RI \in \{1; 3\}$ and $\lambda \in \{0; 0.6\}$. The autoregressive parameter θ in the DGP of the factors is set to 0.6 in all experiments to account for the fact that factors are often persistent in practice. We consider $\rho = 0.8$, $\lambda = 0$, $m = 1$ and $RI = 1$ our baseline scenario. This is a challenging setting for our bias-correction procedure as the large autoregressive parameter ρ will result in a considerable bias for the CCEP estimator. We generate datasets with $N = (25, 50, 100, 500, 1000, 5000)$ and $T = (10, 15, 20, 30, 50, 100)$. As such, next to a typical macro panel dimension (N small and T small to moderate) we also consider a more micro panel perspective (N large and T small). In order to conserve space we will report only a few relevant combinations of N and T in each table.

We initialize $\mathrm{y}_{i,-50}$, $\mathbf{z}_{i,-50}$ and $\mathrm{f}_{j,-50}$ at zero and discard the first 50 observations to neutralize initial conditions. We generate 2000 datasets for each combination of N and T and calculate performance measures including median bias, root mean squared error (rmse) and actual size. Although analytical variance expressions are available for some estimators, to make fair comparisons we obtain standard errors using a bootstrap approach for each of the considered estimators. Following Kapetanios (2008) we resample cross-sectional units as a whole as described in Subsection 4.2. The advantage of this scheme is that it preserves both the persistence and the cross-sectional dependence in the data and is valid even when T is small. We calculate actual test size using bootstrap standard errors based on 150 bootstrap samples. The reported actual size is the false rejection probability of a t-test at the 5% nominal significance level. Results for the CCEPbc estimator with standard errors estimated using (25) are available upon request.

We summarize and discuss our main findings below. We start with some baseline results for estimating ρ and β using various estimators and sample sizes. Next, we focus on a number of interesting aspects with respect to estimating ρ by considering changes to the baseline design and alternative setups for the bias corrections. Since differences between estimators are more pronounced for large N we mostly report tables for $N = 500$ in the main text. Small N versions ($N = 25$) are provided in Section E of the Online Supplement, while in Section F we fix $T = 10$ and plot the behavior of CCEPbc as N grows very large to assess its behavior as $N \to \infty$.

5.2 Baseline results

We start our discussion with a comparison of the performance of our CCEPbc estimator to various alternative estimators in the baseline scenario where $\rho = 0.8$, $\lambda = 0$, $m = 1$ and $RI = 1$. The CCEP estimator is included as the benchmark estimator. Inspired by Chudik and Pesaran (2015), we also consider two alternative bias-corrected CCEP estimators as direct comparisons to our approach, i.e., the recursive mean adjustment (denoted CCEPrm) proposed by So and Shin (1999) and the split-panel jackknife correction (denoted CCEPjk) of Dhaene and Jochmans (2015). We find that CCEPrm provides no improvement over CCEP in any scenario so we exclude it from the tables. In our baseline scenario, the CCEP estimator and the various bias corrections thereof make no use of the additional g_{it} variable or lags of the exogenous variables (which is in line with $\lambda = 0$) in the orthogonalization matrix. Finally, we consider Moon and Weidner's (2017) bias-corrected version of the least squares with interactive effects estimator of Bai (2009). This estimator (denoted FLSbc) is implemented selecting the correct number of factors (2 in our baseline scenario due to the presence of fixed effects) and a bandwidth for the bias correction equal to 4 (which is the optimal choice based on the simulation results of Moon and Weidner for high persistence settings).

The results in Table 1 show that the original CCEP estimator has a severe negative

Table 1: Monte Carlo results for ρ and β : baseline design

Results for $\widehat{\rho}$													
		bias				*rmse*				*size*			
Estimator	(N,T)	10	20	30	50	10	20	30	50	10	20	30	50
CCEP	25	-0.385	-0.176	-0.109	-0.061	0.417	0.188	0.118	0.067	0.90	0.92	0.90	0.81
	100	-0.391	-0.176	-0.112	-0.062	0.417	0.185	0.115	0.064	1.00	1.00	1.00	1.00
	500	-0.397	-0.183	-0.113	-0.062	0.421	0.189	0.116	0.063	1.00	1.00	1.00	1.00
	5000	-0.396	-0.179	-0.111	-0.062	0.417	0.186	0.114	0.063	1.00	1.00	1.00	1.00
CCEPbc	25	-0.004	0.000	0.000	0.000	0.151	0.064	0.038	0.022	0.06	0.08	0.06	0.06
	100	-0.003	0.001	0.000	-0.001	0.100	0.031	0.017	0.011	0.08	0.04	0.04	0.06
	500	0.000	0.001	0.000	0.000	0.057	0.014	0.008	0.005	0.06	0.04	0.05	0.04
	5000	0.000	0.000	0.000	0.000	0.015	0.004	0.002	0.002	0.02	0.05	0.05	0.05
CCEPjk	25	0.027	0.037	0.028	0.014	0.358	0.124	0.074	0.037	0.40	0.31	0.30	0.25
	100	0.044	0.045	0.032	0.015	0.325	0.110	0.063	0.028	0.51	0.53	0.55	0.46
	500	0.031	0.036	0.031	0.016	0.315	0.108	0.058	0.025	0.63	0.72	0.78	0.73
	5000	0.045	0.040	0.035	0.016	0.312	0.105	0.059	0.024	0.66	0.84	0.92	0.90
FLSbc	25	-0.261	-0.067	-0.029	-0.012	0.276	0.089	0.054	0.033	0.37	0.04	0.04	0.04
	100	-0.271	-0.076	-0.038	-0.019	0.271	0.084	0.043	0.022	0.97	0.72	0.49	0.27
	500	-0.280	-0.079	-0.038	-0.018	0.270	0.083	0.041	0.020	0.99	0.99	1.00	0.98
	5000	-0.283	-0.077	-0.037	-0.018	0.270	0.081	0.040	0.019	1.00	1.00	1.00	1.00
Results for $\widehat{\beta}$													
CCEP	25	-0.033	-0.011	-0.006	-0.002	0.058	0.033	0.024	0.018	0.09	0.07	0.06	0.06
	100	-0.033	-0.010	-0.005	-0.002	0.042	0.018	0.013	0.009	0.29	0.11	0.07	0.06
	500	-0.033	-0.011	-0.005	-0.002	0.038	0.014	0.008	0.004	0.73	0.40	0.18	0.09
	5000	-0.033	-0.010	-0.005	-0.002	0.036	0.012	0.005	0.002	0.97	0.94	0.76	0.33
CCEPbc	25	-0.001	0.000	-0.001	0.000	0.052	0.031	0.024	0.018	0.04	0.05	0.07	0.06
	100	0.001	0.000	0.000	0.000	0.026	0.015	0.012	0.009	0.05	0.05	0.05	0.06
	500	0.000	0.000	0.000	0.000	0.012	0.007	0.005	0.004	0.04	0.05	0.05	0.05
	5000	0.000	0.000	0.000	0.000	0.004	0.002	0.002	0.001	0.04	0.06	0.04	0.06
CCEPjk	25	0.014	0.012	0.006	0.003	0.097	0.041	0.029	0.020	0.25	0.13	0.11	0.08
	100	0.018	0.013	0.008	0.003	0.056	0.025	0.016	0.011	0.33	0.22	0.14	0.10
	500	0.019	0.011	0.008	0.003	0.044	0.017	0.011	0.006	0.52	0.43	0.35	0.17
	5000	0.019	0.012	0.008	0.004	0.041	0.016	0.009	0.004	0.69	0.81	0.84	0.70
FLSbc	25	-0.016	0.001	0.000	0.001	0.051	0.036	0.029	0.022	0.04	0.03	0.03	0.02
	100	-0.021	-0.003	-0.002	0.000	0.032	0.016	0.012	0.009	0.18	0.05	0.03	0.03
	500	-0.023	-0.004	-0.002	-0.001	0.027	0.009	0.006	0.004	0.62	0.15	0.08	0.06
	5000	-0.022	-0.004	-0.002	-0.001	0.025	0.006	0.003	0.001	0.89	0.54	0.25	0.09

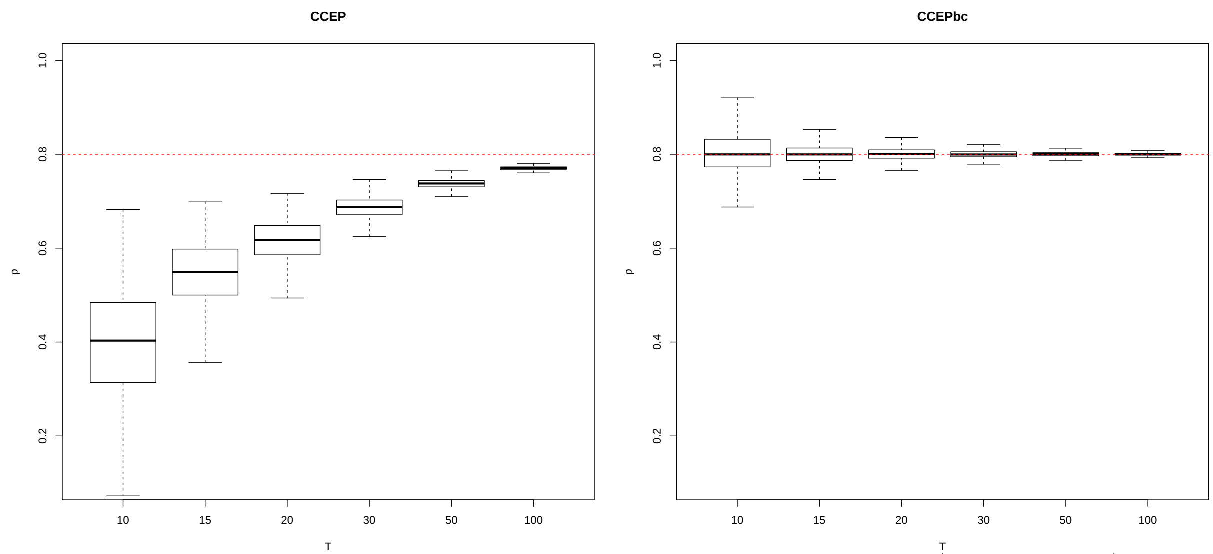
Note: (*i*) Reported are simulation results for estimating ρ and β in the baseline case ($\rho = 0.8$, $\beta = 0.2$, $\lambda = 0$, $m = 1$). The factor has a contribution to the variance of the dependent variable that is equal to that of the idiosyncratic errors ($RI = 1$). (*ii*) CCEPbc is the bias-corrected CCEP estimator. CCEPjk is the jacknife CCEP correction and FLSbc is the bias-adjusted least squares with interactive effects estimator supplied with the correct number of factors ($m + 1$). CCEP estimators do not use $\bar{g}_t$ and include no lags of $\bar{x}_t$. (*iii*) The *size* column reports actual test size for t-tests based on bootstrap standard errors estimated with 150 bootstrap samples.

small T bias for ρ of which a fraction is carried over to the estimates for β. When $T = 10$, the bias for $\widehat{\rho}$ amounts to -0.4, while the more moderate time series dimensions of $T = 20$ and $T = 30$ still result in biases of -0.18 and -0.11, respectively. Even for $T = 50$, the bias of -0.06 should not be neglected as this implies seriously distorted inference. Figure 1 further visualizes this in a setting with $N = 500$ and shows that even for $T = 100$ the CCEP estimator will suffer from some bias and hence unreliable inference. Although the CCE approach relies on $N \to \infty$, the results show that biases are more or less stable over alternative values of N. Experiments for $\rho = 0.4$ (see Table E-1 in the Online Supplement) confirm that the absolute value of the bias of the CCEP estimator is increasing in ρ.

The main takeaway from Table 1 is that our bias-corrected CCEP estimator is (nearly) unbiased in all of the considered sample sizes and hence offers a strong improvement over the original CCEP estimator. Interestingly, CCEPbc also provides a considerable variance reduction whenever $N > 25$. This is due to the fact that the bias of the CCEP estimator is stochastic, as discussed below Theorem 1, which contributes to its variance. The combination of bias removal and variance reduction implies that the rmse of the CCEPbc estimator is always much lower than that of the CCEP estimator, even for moderately large T. The behavior of CCEPbc for $N = 500$ and varying T is also visualized in Figure 1, showing that in contrast to the CCEP estimator our corrected version is correctly centered. In Figure F-1 of the Online Supplement we set $T = 10$ and let the cross-section size N grow large to illustrate the behavior of CCEPbc as $N \to \infty$ and T fixed. The plot reveals that the corrected estimator is indeed consistent as $N \to \infty$, which is clearly not the case for the uncorrected estimator. CCEPbc also offers substantial improvements regarding inference. In contrast to the CCEP, its actual size is always close to the nominal 5% level. As all of these findings hold for each of the considered sample sizes, the CCEPbc is not only an appropriate small T estimator but should also be preferred over CCEP for larger values of T. Moreover, Table 1 shows that the performance of the CCEPbc estimator is not too sensitive to the size of N. As such, it is even applicable in a sample as small as $N = 25$ and $T = 10$.

The alternative bias-adjusted estimators offer some alleviation of the bias but appear less effective compared to CCEPbc. The FLSbc still has a considerable negative small T bias for ρ, while the CCEPjk is able to remove a lot of bias but at the cost of a much larger variance. Accordingly, these alternatives have a much larger rmse compared to CCEPbc, which should be preferred even for larger T due the more effective correction. Since the bias for $\widehat{\beta}$ is a fraction of that for $\widehat{\rho}$, also $\widehat{\beta}$ is not correctly centered for the alternative estimators and the test size for this coefficient is generally distorted, whereas in the case of CCEPbc it is at the desired 5% level. Similar results are obtained in the low persistence scenario (see Table E-1 in the Online Supplement), but differences between estimators are smaller since there is less bias to correct for.

Figure 1: Monte Carlo results for ρ: comparison of CCEP and CCEPbc over T for $N = 500$



Notes: Reported are simulation results for estimating ρ in the baseline case when $N = 500$ (see notes Table 1). Dotted red lines indicate the population parameter value ($\rho = 0.8$). The boxplot 'whiskers' extend to the most extreme data point which is no more than 1.5 times the interquartile range from the box.

Table 2: Monte Carlo results for ρ : number and strength of factors ($N = 500$)

	bias	*rmse*	*size*	*bias*	*rmse*	*size*	*bias*	*rmse*	*size*	*bias*	*rmse*	*size*	*bias*	*rmse*	*size*	*bias*	*rmse*	*size*
	one factor																	
	$RI = 1$									$RI = 3$								
	$T = 10$			$T = 20$			$T = 30$			$T = 10$			$T = 20$			$T = 30$		
CCEP	−0.397	0.421	1.00	−0.183	0.189	1.00	−0.113	0.116	1.00	−0.399	0.424	1.00	−0.182	0.189	1.00	−0.112	0.116	1.00
CCEPbc	0.000	0.057	0.06	0.001	0.014	0.04	0.000	0.008	0.05	0.001	0.055	0.06	0.000	0.014	0.04	0.000	0.008	0.04
CCEPjk	0.031	0.315	0.63	0.036	0.108	0.72	0.031	0.058	0.78	0.037	0.312	0.63	0.040	0.108	0.73	0.031	0.059	0.79
CCEP(+g)	−0.406	0.431	1.00	−0.183	0.191	1.00	−0.113	0.116	1.00	−0.408	0.433	1.00	−0.183	0.190	1.00	−0.112	0.116	1.00
CCEPbc(+g)	0.000	0.066	0.06	0.000	0.014	0.02	0.000	0.008	0.04	0.000	0.064	0.06	0.000	0.014	0.03	0.000	0.008	0.04
CCEPjk(+g)	-	-	-	0.041	0.116	0.41	0.033	0.060	0.53	-	-	-	0.044	0.116	0.42	0.033	0.061	0.56
FLSbc	−0.280	0.270	0.99	−0.079	0.083	0.99	−0.038	0.041	1.00	−0.259	0.248	0.99	−0.064	0.072	1.00	−0.035	0.038	1.00
	two factors																	
	$RI = 1$									$RI = 3$								
	$T = 10$			$T = 20$			$T = 30$			$T = 10$			$T = 20$			$T = 30$		
CCEP	−0.408	0.431	1.00	−0.190	0.197	1.00	−0.117	0.120	1.00	−0.402	0.425	1.00	−0.181	0.188	1.00	−0.109	0.112	1.00
CCEPbc	0.000	0.058	0.06	0.000	0.014	0.04	0.000	0.008	0.04	0.006	0.070	0.10	0.006	0.020	0.08	0.005	0.012	0.12
CCEPjk	0.035	0.311	0.62	0.034	0.107	0.72	0.031	0.059	0.79	0.034	0.316	0.64	0.045	0.116	0.73	0.037	0.063	0.76
CCEP(+g)	−0.447	0.471	1.00	−0.209	0.216	1.00	−0.128	0.131	1.00	−0.416	0.443	1.00	−0.190	0.196	1.00	−0.115	0.118	1.00
CCEPbc(+g)	0.000	0.070	0.08	0.000	0.016	0.03	0.000	0.008	0.05	0.000	0.067	0.06	0.001	0.014	0.04	0.000	0.008	0.05
CCEPjk(+g)	-	-	-	0.036	0.122	0.50	0.034	0.067	0.60	-	-	-	0.049	0.124	0.52	0.035	0.064	0.60
FLSbc	−0.532	0.525	1.00	−0.204	0.199	1.00	−0.097	0.098	1.00	−0.518	0.504	1.00	−0.170	0.170	1.00	−0.066	0.071	0.98

Notes: (*i*) Data for this experiment are generated with $\rho = 0.8$, $\beta = 0.2$ and $\lambda = 0$. $RI = (1, 3)$ represents factors that have a contribution to the total variance of the dependent variable that is equal to, or respectively 3 times that of the idiosyncratic errors. We display results for estimating ρ with $N = 500$. (*ii*) CCEP is the Pooled CCE estimator and CCEPbc its bias-corrected version. CCEPjk represents the jackknife corrected CCEP and FLSbc is the bias-adjusted least squares with interactive effects estimator supplied with the correct number of factors $(m + 1)$. CCEP-type estimators with suffix '(+g)' indicate that $\bar{g}_t$ was included in the orthogonalization matrix. No lags of $\bar{x}_t$ and $\bar{g}_t$ are employed. (*iii*) The reported actual test size (*size*) is for a t-test using bootstrap standard errors based on 150 samples.

5.3 Number of factors and their strength

In this section we analyze the performance of CCEPbc when varying the number of factors (m is 1 and 2) and their strength (RI is 1 and 3). Table 2 reports simulation results for $N = 500$. Small N results are provided in Table E-2 of the Online Supplement. Next to the CCEP estimator and its bias corrections that do not use the CSA of g_{it} when approximating the factors, we now also include CCEP variants that do use g_{it} and denote them by adding the (+g) suffix.

The results in Table 2 show that the performance of CCEP and of its bias corrections is not very sensitive to the number of factors or their strength. Only when we drive up the factor strength in the presence of two factors (see the lower right panel of Table 2), we note a slight increase in the bias of our CCEPbc approach. Table 3 further summarizes the behavior of CCEPbc for various sizes of N and T with two strong factors. The top panel reveals that even though the small T bias clearly decreases as N grows, it results in distorted inference unless N is much larger than T. The explanation for this finding is that even though the rank condition is exactly satisfied (2 observables for 2 factors) the information in $\bar{\mathrm{y}}_t$ and $\bar{\mathrm{x}}_t$ may not be sufficiently distinct to effectively remove two strong factors in finite N settings. In this case CCEP will have an additional finite N bias term which is not taken into account by our CCEPbc estimator.

Although the remaining bias in the presence of two strong factors disappears as N increases further (see Figure F-4 in the Online Supplement), we find that the inclusion of $\bar{\mathrm{g}}_t$ is a highly effective solution in finite samples. The additional information on the factors that is added through including $\bar{\mathrm{g}}_t$ yields a notable improvement in the finite N performance of the CCEPbc approach in the lower right panel of Table 2. This is further demonstrated in the lower panel of Table 3 which shows that the CCEPbc(+g) estimator suffers less bias compared to CCEPbc and has an adequate actual size for all combinations of N and T.

The above discussion shows that additional covariates can have a beneficial effect on CCE-type estimators when factors are very influential in the model, even in cases where the rank condition is already satisfied. However, comparing the bias of the CCEP estimator to that of CCEP(+g) in Table 2 also confirms our theoretical finding that adding more CSA to the orthogonalization matrix increases the bias of the uncorrected CCEP estimator. Fortunately, the CCEPbc adjustment is effective in removing this bias. For less influential factors ($RI = 1$) the only downside is a relative loss in efficiency compared to not using g_{it}. Finally, comparing CCEP(+g) over different factor strengths confirms our claim (see discussion below Theorem 1) that more influential factors (i.e., increasing RI from 1 to 3) do not change the bias in the one factor case (upper panel) but it will reduce the bias when more than one factor is present (lower panel).

5.4 Dynamics in $\mathbf{z}_{it}$

In this section we allow for dynamics in $\mathbf{z}_{it}$ (setting $\lambda = 0.6$) to analyze the importance of including lagged CSA to adequately capture the common factors. Table 4 reports the main results in a setting where factors are strong ($RI = 3$) and $N = 500$. Results for $N = 25$ are reported in Table E-3 of the Online Supplement. We let CCEP and CCEPbc

Table 3: Monte Carlo results for ρ : CCEPbc estimators with two highly influential factors

	bias					*size*				
(N,T)	10	20	30	50	100	10	20	30	50	100
					CCEPbc					
25	0.014	0.019	0.017	0.017	0.016	0.08	0.12	0.09	0.16	0.27
100	0.012	0.013	0.012	0.012	0.011	0.13	0.10	0.14	0.24	0.42
500	0.006	0.006	0.005	0.005	0.004	0.10	0.08	0.12	0.17	0.28
5000	0.001	0.001	0.001	0.001	0.001	0.04	0.06	0.06	0.06	0.08
					CCEPbc(+g)					
25	0.006	0.008	0.006	0.006	0.006	0.05	0.08	0.05	0.06	0.09
100	0.004	0.003	0.002	0.002	0.002	0.07	0.04	0.05	0.06	0.06
500	0.000	0.001	0.000	0.001	0.000	0.06	0.04	0.05	0.05	0.05
5000	0.000	0.000	0.000	0.000	0.000	0.02	0.05	0.05	0.05	0.05

Notes: (*i*) Reported are simulation results for estimation and inference on the ρ coefficient. Data for this experiment are generated with $\rho = 0.8$, $\beta = 0.2$, $m = 2$ and $\lambda = 0$. Factors have a contribution to the total variance of the dependent variable that is 3 times that of the idiosyncratic errors ($RI = 3$). (*ii*) CCEPbc is the unrestricted corrected CCEP estimator. The '(+g)' indicates that $\bar{\mathrm{g}}_t$ was included in the orthogonalization matrix. No lags of $\bar{\mathrm{x}}_t$ and $\bar{\mathrm{g}}_t$ are used. (*iii*) The test size (*size*) is for a t-test using bootstrap standard errors based on 150 samples.

with suffix notation $\_p_1$ denote the estimators that are correctly specified with one lag of $\bar{\mathbf{Z}} = [\bar{\mathbf{X}}, \bar{\mathbf{G}}]$ added to the orthogonal projection matrix $\mathbf{M}$. The suffix notation $\_p_T$ is used to indicate the inclusion of $p_T = \lfloor T^{1/3} \rfloor$ lags while $\_p_0$ denotes the misspecified variant without lags of $\bar{\mathbf{Z}}$. We report results for CCEP-type estimators that add the CSA of g_{it} to avoid that the results are driven by using an insufficient number of covariates to proxy for the common factors. The correctly specified FLSbc and jackknife correction are included as alternative estimators. Note that some estimators cannot be implemented when $T = 10$ due to insufficient degrees of freedom (because of the larger number of CSA used for orthogonalization).

The simulation results for the misspecified CCEPbc$_p_0$ estimator reveal that it performs well when $m = 1$ but that it is not correctly centered when $m = 2$, despite the use of $\bar{\mathrm{g}}_t$. Especially when T is large, the bias that remains in the latter case results in large size distortions. This suggests that the lag of $\bar{\mathrm{y}}_t$ holds enough information to deal with the unobserved components in the single factor case but that it is not sufficient to control for multiple strong factors without lags of $\bar{\mathrm{x}}_t$ (and $\bar{\mathrm{g}}_t$). The correctly specified CCEPbc$_p_1$ estimator instead performs much better, with an adequate size for all values of T. This confirms that the approximation of the factors requires the number of lagged CSA to be equal to the AR lag order (p) of the exogenous variables. When p is unknown, we have suggested to follow the approach of Chudik and Pesaran (2015) and specify the number of lags as $p^* = \lfloor T^{1/3} \rceil$ to let them grow with T as a precaution against misspecification. As this implies orthogonalization on a large number of CSA, the resulting bias of the uncorrected CCEP$_p_T$ estimator is very large. CCEPbc$_p_T$ is however highly effective in removing the distortions and has an adequate size. The price paid for this robustness is that the larger number of CSA translates in a substantially higher variance compared to the cor-

Table 4: Monte Carlo results for ρ : dynamics in $\mathbf{z}_{it}$ with strong factors ($N = 500$)

	bias	*rmse*	*size*	*bias*	*rmse*	*size*	*bias*	*rmse*	*size*	*bias*	*rmse*	*size*
	one factor											
	$T = 10$			$T = 20$			$T = 30$			$T = 50$		
CCEP_p_0(+g)	−0.600	0.610	0.99	−0.253	0.261	1.00	−0.146	0.150	1.00	−0.076	0.078	1.00
CCEP_p_1(+g)	−0.685	0.713	0.95	−0.271	0.280	1.00	−0.152	0.157	1.00	−0.078	0.079	1.00
CCEP_p_T(+g)	-	-	-	−0.336	0.349	0.99	−0.203	0.210	1.00	−0.091	0.093	1.00
CCEPbc_p_0(+g)	0.000	0.090	0.05	0.000	0.017	0.03	−0.001	0.009	0.04	0.000	0.005	0.04
CCEPbc_p_1(+g)	−0.001	0.140	0.03	0.001	0.020	0.02	0.000	0.009	0.03	0.000	0.005	0.04
CCEPbc_p_T(+g)	-	-	-	0.001	0.029	0.02	−0.001	0.013	0.03	0.000	0.006	0.03
CCEPjk_p_1(+g)	-	-	-	0.140	0.227	0.27	0.088	0.124	0.43	0.039	0.050	0.58
FLSbc	−0.269	0.257	0.98	−0.058	0.065	0.99	−0.029	0.032	0.99	−0.014	0.015	0.95
	two factors											
	$T = 10$			$T = 20$			$T = 30$			$T = 50$		
CCEP_p_0(+g)	−0.655	0.661	1.00	−0.290	0.296	1.00	−0.170	0.174	1.00	−0.091	0.092	1.00
CCEP_p_1(+g)	−0.779	0.794	0.99	−0.320	0.328	1.00	−0.179	0.183	1.00	−0.089	0.090	1.00
CCEP_p_T(+g)	-	-	-	−0.400	0.409	1.00	−0.246	0.251	1.00	−0.105	0.107	1.00
CCEPbc_p_0(+g)	−0.034	0.096	0.13	−0.018	0.028	0.24	−0.013	0.017	0.35	−0.009	0.011	0.45
CCEPbc_p_1(+g)	0.007	0.155	0.06	0.000	0.021	0.03	−0.001	0.010	0.04	0.000	0.005	0.05
CCEPbc_p_T(+g)	-	-	-	0.001	0.033	0.03	−0.001	0.014	0.04	0.000	0.006	0.05
CCEPjk_p_1(+g)	-	-	-	0.134	0.233	0.48	0.106	0.146	0.71	0.050	0.062	0.81
FLSbc	−0.528	0.519	1.00	−0.174	0.172	1.00	−0.069	0.073	0.99	−0.021	0.023	0.97

Notes: (*i*) Reported are simulation results for estimating the ρ coefficient. Data for this experiment are generated with $\rho = 0.8$, $\beta = 0.2$ and $\lambda = 0.6$. The contribution of the factors to the total variance of the dependent variable is 3 times that of the idiosyncratic errors ($RI = 3$). We display results for estimating ρ with $N = 500$. (*ii*) CCEP is the Pooled CCE estimator and CCEPbc its unrestricted bias-correction. CCEPjk represents the jacknife corrected CCEP and FLSbc is the bias-corrected least squares with interactive effects estimator supplied with the correct number of factors ($m + 1$). All CCEP estimators additionally include $\bar{\mathrm{g}}_t$ to project out the factors. CCEP estimators with a p_0, p_1 or p_T suffix respectively include no, one or $\lfloor T^{1/3} \rfloor$ lags of $\bar{\mathrm{x}}_t$ and $\bar{\mathrm{g}}_t$ in the orthogonalization matrix. (*iii*) The reported test size (*size*) is for a t-test using bootstrap standard errors based on 150 samples.

rectly specified CCEPbc_p_1. As expected, this difference disappears as T grows. Results for small N (see Table E-3 in the Online Supplement) are highly similar (with marginally larger biases) but whenever bias remains it has a much smaller impact on inference.

6 Temperature shocks and economic growth

In this section we apply our bias-corrected CCEP estimator to identify the dynamic effects of temperature shocks on aggregate output growth. In line with the recent literature (see e.g., Dell et al., 2012; Colacito et al., 2018) we consider the benchmark dynamic model

$$g_{it} = \alpha_i + \rho g_{i,t-1} + \beta_1 T_{it} + \beta_2 T_{i,t-1} + u_{it}, \tag{27}$$

where g_{it} is per-capita real output growth and T_{it} is temperature. The lagged dependent variable $g_{i,t-1}$ is included to capture output growth persistence, while lagged temperature $T_{i,t-1}$ is added to discriminate between permanent and transitory output effects. The contemporaneous impact of a transitory 1°C rise in temperature on output growth is measured by β_1. If $\beta_2 = -\beta_1$, the impact on output growth is reversed in the next period (or periods if $\rho > 0$) such that the level of output (eventually) bounces back, i.e., the cumulative growth effect $(\beta_1 + \beta_2)/(1 - \rho) = 0$. There is no (complete) reversal if $\beta_2 \neq -\beta_1$, which implies that the level of output is permanently affected by a transitory temperature shock. Typically no additional variables are included because most economic and political variables are potentially affected by weather variables, such that including them as controls implies that the estimates do not capture all relevant channels through which weather affects the economy.

The strategy in the recent climate-economy literature (see Dell et al., 2014, for an overview) is to exploit random variation in weather events over time within countries to identify its causal effects. Country fixed effects α_i are included to isolate weather effects from time-invariant characteristics, while time fixed effects (possibly interacted with region dummies) are added to neutralize common shocks. In panels with a relatively short time span, the latter avoids that the estimates pick up spurious correlation between global trends in weather and growth. However, time fixed effects impose a homogeneous reaction (within regions) to common shocks. We allow for a more general heterogeneous response by letting u_{it} take the multi-factor structure specified in eq.(2).

Data are taken from Dell et al. (2012), who have collected yearly output growth and annual average temperatures for an unbalanced panel of 125 countries over the period 1961-2003. We follow their approach of allowing the temperature effects to be different for 'rich' and 'poor' countries (defined as having above respectively below-median PPP-adjusted per capita GDP). We further split the time dimension into two subperiods since weather effects may have become either larger (due to intensification) or smaller (due to adaptation) in recent years (Dell et al., 2014). This results in a balanced sample of 93 countries over the period 1962-1982 and 118 countries over the period 1983-2003. As this makes the time series dimension relatively short ($T = 21$), at least much smaller than N, this is the ideal setting to illustrate our CCEPbc estimator.

Estimation results are presented in Table 5. Beginning with the left panel for the first part of the sample, the FE estimates in column (1) confirm the finding of Dell et al. (2012)

that temperature shocks have a significantly negative effect on output growth only in poor countries, where a transitory 1°C rise in temperature reduces output growth in the same year by about 2 percentage points. Moreover, output does not significantly bounce back in the year after the shock, resulting in a 1.67% permanent decrease in output. The CCEP estimates in column (2) show a highly similar contemporaneous impact, but the coefficient on $T_{i,t-1}$ increases substantially, even to the extent that temperature shocks only have a temporary impact on output. The bounce-back effect is, however, only significant at the 10% level of significance. Theorem 1 implies that the CCEP estimates of ρ and β_2 are expected to be downward biased, while β_1 should be unbiased. This is because $g_{i,t-1}$ is not correlated with future temperature shocks (which show no significant persistence) and negatively correlated with current shocks, such that the CCEP estimates $\boldsymbol{\zeta}$ of $g_{i,t-1}$ on T_{it} and $T_{i,t-1}$ in eq.(17) are expected to show a zero value for T_{it} and a negative value for $T_{i,t-1}$. The CCEPbc estimation results reported in column (3) indeed show an upward adjustment of the coefficients on $g_{i,t-1}$ and $T_{i,t-1}$. In particular, the coefficient on $T_{i,t-1}$ turns significant at the 5% level of significance, reinforcing the finding that temperature shocks only have a transitory impact on output.

Table 5: Temperature shocks and economic growth

	N=93, 1962-1982			N=118, 1983-2003		
	FE (1)	CCEP (2)	CCEPbc (3)	FE (4)	CCEP (5)	CCEPbc (6)
$g_{i,t-1}$	0.17	0.15	0.24	0.18	0.07	0.22
	(0.07)**	(0.08)*	(0.08)***	(0.08)**	(0.06)	(0.07)***
Rich countries						
T_{it}	0.17	0.47	0.48	0.13	0.47	0.44
	(0.45)	(0.53)	(0.52)	(0.20)	(0.39)	(0.37)
$T_{i,t-1}$	−0.30	−0.35	−0.39	0.44	0.09	0.08
	(0.33)	(0.55)	(0.54)	(0.22)**	(0.34)	(0.32)
Poor countries						
T_{it}	−2.08	−1.94	−1.93	−1.26	−1.11	−1.24
	(0.57)***	(0.79)**	(0.80)**	(0.45)***	(0.66)*	(0.68)*
$T_{i,t-1}$	0.69	1.76	1.84	0.83	0.30	0.57
	(0.69)	(0.91)*	(0.92)**	(0.33)**	(0.66)	(0.68)
Implied cumulative growth effects	$(\beta_1+\beta_2)/(1-\rho)$					
Rich countries	−0.16	0.14	0.12	0.70	0.60	0.66
	(0.83)	(0.98)	(1.05)	(0.37)*	(0.64)	(0.68)
Poor countries	−1.67	−0.21	−0.12	−0.53	−0.87	−0.87
	(0.75)**	(1.17)	(1.27)	(0.44)	(0.85)	(0.94)

Notes: The dependent variable g_{it} is the growth rate of per-capita real GDP, T_{it} is the average annual temperature. Both subsamples include a balanced sample of countries. The FE specifications include country, region×year and poor×year fixed effects (see Dell et al., 2012, for region composisitions). Rich and Poor are defined as countries having above respectively below-median PPP-adjusted per capita GDP in the first year of the sample. The CCEP estimators use the contemporaneous and one-year lagged CSA of g_{it}, Rich×T_{it} and Poor×T_{it}. Bootstrapped standard deviations are reported in brackets. ***/**/* denote significance at the 1/5/10 percent level respectively.

Turning to the results for the second part of the sample reported in columns (4)-(6), temperature shocks again only have a negative impact in poor countries, but this now turns out to be more moderate. The contemporaneous impact decreases from roughly -2 to around -1.2, suggesting that there may be some adaptation in more recent years. Note that the FE estimates now reveal a significant bounce-back effect while this is not the case for the CCEP and CCEPbc estimators. However, the three estimators agree that there is no significant permanent impact of temperature shocks on output. Concerning our bias-correction method, it is interesting to note that the CCEP estimate for ρ reported in column (5) is only 0.07 and not significant, while its bias-corrected estimate in column (6) is 0.22 and highly significant. Moreover, the coefficient on $T_{i,t-1}$ roughly doubles when bias-correcting the CCEP estimator, but given the relatively large standard error it is not significant.

7 Conclusion

In this article we extend the CCEP estimator designed by Pesaran (2006) to dynamic homogeneous panel data models and develop a bias-corrected version that eliminates its finite T bias. We first show that in homogeneous dynamic panels, the unobserved common factors can be effectively approximated by CSA of the observed data provided that a sufficient number of observables is available (rank condition) and an appropriate number of lagged CSA is added to the model. This number of lags should coincide with the autoregressive order of the observed data. We next derived the asymptotic bias expression for $N \to \infty$ of the CCEP estimator and used this to devise a bias-corrected estimator. We show that the resulting CCEPbc estimator is consistent as $N \to \infty$, both for T fixed or $T \to \infty$.

Extensive Monte Carlo experiments show that, when appropriately specified, CCEPbc performs very well and is superior to the original CCEP estimator and to alternative corrections available in the literature. More specifically, CCEPbc is found to be nearly unbiased across all of the sample sizes and designs we considered. Hence, it offers a strong improvement over the severely biased CCEP estimator. This is especially the case when T is small but even holds true for large T. Interestingly, CCEPbc also provides a notable variance reduction compared to the original CCEP estimator. This is due to the fact that the stochastic bias of the latter also drives up its variance. Moreover, using bootstrapped standard errors, the actual size of CCEPbc was found to be close to the 5% nominal level. The Monte Carlo simulations further show that it is important to include a sufficient number of CSA of observables in the model. First, the number of observables is important to satisfy the rank condition, but even when this already holds it is beneficial in terms of bias correction and inference to add CSA of additional observables when these hold information about highly influential common factors. Second, the simulation results confirm our theoretical finding that lagged CSA should be added to the model in line with the autoregressive order of the observables. In case the autoregressive order is unknown, letting the number of lags grow with T was found to be a robust approach.

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References

Ahn, S. C., Hoon Lee, Y., and Schmidt, P. (2001). GMM estimation of linear panel data models with time-varying individual effects. Journal of Econometrics, 101(2):219–255.

Andrews, D. W. K. (2005). Cross-section regression with common shocks. Econometrica, 73(5):1551–1585.

Bai, J. (2009). Panel Data Models with Interactive Fixed Effects. Econometrica, 77(4):1229–1279.

Bun, M. and Carree, M. (2005). Bias-corrected estimation in dynamic panel data models. Journal of Business and Economic Statistics, 23(2):200–210.

Carlsson, M., Eriksson, S., and Gottfries, N. (2013). Product market imperfections and employment dynamics. Oxford Economic Papers, 65(2):447–470.

Chudik, A., Pesaran, M., and Tosetti, E. (2011). Weak and strong cross-section dependence and estimation of large panels. The Econometrics Journal, 14(1):C45–C90.

Chudik, A. and Pesaran, M. H. (2015). Common correlated effects estimation of heterogeneous dynamic panel data models with weakly exogenous regressors. Journal of Econometrics, 188(2):393 – 420. Heterogeneity in Panel Data and in Nonparametric Analysis in honor of Professor Cheng Hsiao.

Chudik, A. and Pesaran, M. H. (2017). A Bias-Corrected Method of Moments Approach to Estimation of Dynamic Short-T Panels. USC-INET Research Paper 17-26, USC Dornsife Institute for New Economic Thinking.

Colacito, R., Hoffmann, B., and Phan, T. (2018). Temperatures and growth: A panel analysis of the united states. Working Paper 18-09, Federal Reserve Bank Of Richmond.

Dell, M., Jones, B., and Olken, B. (2012). Temperature shocks and economic growth: Evidence from the last half century. American Economic Journal: Macroeconomics, 4(3):66–95.

Dell, M., Jones, B., and Olken, B. (2014). What do we learn from the weather? the new climate-economy literature. Journal of Economic Literature, 52(3):740–798.

Dhaene, G. and Jochmans, K. (2015). Split-panel Jackknife Estimation of Fixed-effect Models. Review of Economic Studies, 82(3):991–1030.

Eberhardt, M. and Teal, F. (2011). Econometrics for Grumblers: A New Look at the Literature on Cross-Country Growth Empirics. Journal of Economic Surveys, 25(1):109–155.

Eriksson, S. and Stadin, K. (2017). What are the determinants of hiring? the importance of product market demand and search frictions. Applied Economics, 49(50):5144–5165.

Everaert, G. and De Groote, T. (2016). Common correlated effects estimation of dynamic panels with cross-sectional dependence. Econometric Reviews, 35(3):428–463.

Feng, X. and Zhang, Z. (2007). The rank of a random matrix. Applied Mathematics and Computation, 185(1):689 – 694.

Harding, M. and Lamarche, C. (2011). Least squares estimation of a panel data model with multifactor error structure and endogenous covariates. Economics Letters, 111(3):197–199.

Kapetanios, G. (2008). A bootstrap procedure for panel data sets with many cross-sectional units. Econometrics Journal, 11(2):377–395.

Kapetanios, G., Pesaran, M., and Yamagata, T. (2011). Panels with non-stationary multifactor error structures. Journal of Econometrics, 160(2):326–348.

Karabiyik, H., Reese, S., and Westerlund, J. (2017). On the role of the rank condition in cce estimation of factor-augmented panel regressions. Journal of Econometrics, 197(1):60 – 64.

Madsena, J., Minnitib, A., and Venturini, F. (2018). Assessing pikettys second law of capitalism. Oxford Economic Papers, 70(1):1–21.

Minniti, A. and Venturini, F. (2017). The long-run growth effects of R&D policy. Research Policy, 46:316–326.

Moon, H. R. and Weidner, M. (2017). Dynamic Linear Panel Regression Models With Interactive Fixed Effects. Econometric Theory, 33(01):158–195.

Moon, H. R. M. and Perron, B. (2007). An Empirical Analysis of Nonstationarity in a Panel of Interest Rates with Factors. Journal, 22:383–400.

Newey, W. K. and McFadden, D. (1994). Chapter 36 large sample estimation and hypothesis testing. volume 4 of Handbook of Econometrics, pages 2111 – 2245. Elsevier.

Nickell, S. (1981). Biases in Dynamic Models with Fixed Effects. Econometrica, 49(6):1417–1426.

Pesaran, M. (2006). Estimation and Inference in Large Heterogeneous Panels with a Multifactor Error Structure. Econometrica, 74(4):967–1012.

Pesaran, M., Smith, L., and Yamagata, T. (2013). Panel Unit Root Tests in the Presence of a Multifactor Error Structure. Journal of Econometrics, 175:94–115.

Pesaran, M. and Tosetti, E. (2011). Large panels with common factors and spatial correlation. Journal of Econometrics, 161(2):182–202.

Sarafidis, V. and Robertson, D. (2009). On the Impact of Error Cross-Sectional Dependence in Short Dynamic Panel Estimation. Econometrics Journal, 12(1):62–81.

Sarafidis, V. and Wansbeek, T. (2012). Cross-Sectional Dependence in Panel Data Analysis. Econometric Reviews, 31(5):483–531.

So, B. and Shin, D. (1999). Recursive mean adjustment in time-series inferences. Statistics & Probability Letters, 43(1):65 – 73.

Tao, X. (2018). Investigating Environmental Kuznets Curve in China-Aggregation bias and policy implications. Energy policy, 114(315-322).

Temple, J. and Van de Sijpe, N. (2017). Foreign aid and domestic absorption. Journal of International Economics, 108(431-443).

Wu, J.-W. and Wu, J.-L. (2018). Does the launch of the euro hinder the current account adjustment of the eurozone? Economic Inquiry, 56(2):1116–1135.

Supplement to "Bias-corrected Common Correlated Effects Pooled estimation in dynamic panels"

by Ignace De Vos and Gerdie Everaert

Section A of this supplement (i) provides the Jacobian matrix for the CCEPbc estimator, (ii) provides an additional discussion on the asymptotic bias of the CCEP estimator for the autoregressive parameter ρ, (iii) develops two restricted CCEPbc estimators for the single factor setting and (vi) reports Monte Carlo evidence comparing the performance of the restricted and unrestricted CCEPbc estimators. Section B introduces important notation and preliminary results for the proofs presented in Sections C and D. Section C presents proofs for $N \to \infty$ and fixed T, and Section D presents proofs for $(N,T) \to \infty$. Section E contains additional Monte Carlo simulation results for the unrestricted CCEPbc estimator.

A Additional results and discussions

A.1 Jacobian

Consider that the CCEPbc estimator in eq.(21) is equivalent to

$$\widehat{\boldsymbol{\delta}}_{bc} = \underset{\boldsymbol{\delta}_0 \in \chi}{\arg\min} \frac{1}{2} \left\| \boldsymbol{\varphi}(\boldsymbol{\delta}_0) \right\|^2, \tag{A-1}$$

with $\boldsymbol{\varphi}(\boldsymbol{\delta}_0)$ given by

$$\boldsymbol{\varphi}(\boldsymbol{\delta}_0) = \frac{1}{NT}\sum_{i=1}^{N} \mathbf{w}_i' \mathbf{M} \mathbf{y}_i - \widehat{\boldsymbol{\Sigma}}\boldsymbol{\delta}_0 + \frac{1}{T}\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta}_0)\boldsymbol{\upsilon}(\rho_0),$$

and $\boldsymbol{\upsilon}(\rho_0) = \upsilon(\rho_0, \mathbf{H})\mathbf{q}_1$. As such, the CCEPbc estimator employs the orthogonality condition $\boldsymbol{\nabla}(\boldsymbol{\delta}_0) = \mathbf{0}$, with $\boldsymbol{\nabla}(\boldsymbol{\delta}_0)$ the gradient evaluated at $\boldsymbol{\delta}_0$,

$$\boldsymbol{\nabla}(\boldsymbol{\delta}_0) = \mathbf{J}_{\mathrm{a}}(\boldsymbol{\delta}_0)'\boldsymbol{\varphi}(\boldsymbol{\delta}_0),$$

and $\mathbf{J}_{\mathrm{a}}(\boldsymbol{\delta}_0)$ is the $k_w \times k_w$ Jacobian matrix in the sample evaluated at $\boldsymbol{\delta}_0$,

$$\mathbf{J}_{\mathrm{a}}(\boldsymbol{\delta}_0) = \frac{1}{T}\left[\left(\boldsymbol{\upsilon}(\rho_0) \otimes \dot{\boldsymbol{\sigma}}'\right) + \left(\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta}_0)\mathbf{q}_1 \otimes \dot{\boldsymbol{\upsilon}}'\right) \right] - \widehat{\boldsymbol{\Sigma}}, \tag{A-2}$$

with

$$\dot{\boldsymbol{\sigma}} = \frac{\partial \widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta}_0)}{\partial \boldsymbol{\delta}_0} = 2\frac{T}{T-c}\widehat{\boldsymbol{\Sigma}}\left(\boldsymbol{\delta}_0 - \widehat{\boldsymbol{\delta}}\right), \tag{A-3}$$

$$\dot{\boldsymbol{\upsilon}} = \frac{\partial \boldsymbol{\upsilon}(\rho_0)}{\partial \boldsymbol{\delta}_0} = \left(\sum_{t=1}^{T-1} (t-1)\rho_0^{t-2} \sum_{s=t+1}^{T} h_{s,s-t} \right)\mathbf{q}_1. \tag{A-4}$$

A.2 Discussion on the asymptotic bias of the CCEP estimator $\widehat{\rho}$

In order to gain a better understanding of the driving forces behind the asymptotic bias of the CCEP estimator for the autoregressive parameter ρ in eq.(1) of the main text, we first derive the following corollary result to Theorem 1 (with notation introduced in (C-37)).

Corollary 1. *Under the conditions of Theorem 1 and conditional on $\mathcal{C}$, the asymptotic bias of $\widehat{\rho}$ is*

$$\operatorname*{plim}_{N\to\infty}(\widehat{\rho}-\rho) = -\frac{\left[A(\rho)+D(\rho,\widetilde{\mathbf{H}})\right]}{\left[B(\rho)-E(\rho,\widetilde{\mathbf{H}})+TC\right]} = -\psi(\rho,\widetilde{\mathbf{H}},C), \tag{A-5}$$

with

- $A(\rho)=\frac{1}{1-\rho}\left(1-\frac{1}{T}\frac{1-\rho^T}{1-\rho}\right)$, $\quad D(\rho,\widetilde{\mathbf{H}})=\sum_{t=1}^{T-1}\rho^{t-1}\sum_{s=t+1}^{T}\widetilde{h}_{s,s-t}$,
- $B(\rho)=\frac{T}{1-\rho^2}\left(1-\frac{1}{T}\frac{1+\rho}{1-\rho}-\frac{2\rho}{T^2}\frac{1-\rho^T}{(1-\rho)^2}\right)$, $\quad E(\rho,\widetilde{\mathbf{H}})=\frac{1}{1-\rho^2}\left[c-1+2\rho D(\rho,\widetilde{\mathbf{H}})\right]$,

where $\widetilde{h}_{s,s-t}$ denotes the element on row s and column $s-t$ of $\widetilde{\mathbf{H}}=\widetilde{\mathbf{Q}}(\widetilde{\mathbf{Q}}'\widetilde{\mathbf{Q}})^{\dagger}\widetilde{\mathbf{Q}}'$, with $\mathbf{B}=\mathbf{I}_T-\boldsymbol{\iota}_T\boldsymbol{\iota}_T'/T$ and $\widetilde{\mathbf{Q}}=\mathbf{BQ}$ the matrix of CSA in deviation of its column means, and

$$C=\operatorname*{plim}_{N\to\infty}\frac{\boldsymbol{\beta}'\boldsymbol{\Omega}_{\check{\mathrm{x}}}\boldsymbol{\beta}+\boldsymbol{\Lambda}'\boldsymbol{\Omega}_{\check{\mathrm{f}}}\boldsymbol{\Lambda}+2\boldsymbol{\beta}'\boldsymbol{\Omega}_{\check{\mathrm{x}},\check{\mathrm{f}}}\boldsymbol{\Lambda}}{\sigma_\varepsilon^2}, \tag{A-6}$$

with $\boldsymbol{\Omega}_{\check{\mathrm{x}}}=(\mathbf{X}_{-1}^{+})'\mathbb{M}_{\mathrm{X}}\mathbf{X}_{-1}^{+}/NT$, $\boldsymbol{\Omega}_{\check{\mathrm{f}}}=(\mathbb{F}_{-1}^{+})'\mathbb{M}_{\mathrm{X}}\mathbb{F}_{-1}^{+}/NT$ and $\boldsymbol{\Omega}_{\check{\mathrm{x}},\check{\mathrm{f}}}=(\mathbf{X}_{-1}^{+})'\mathbb{M}_{\mathrm{X}}\mathbb{F}_{-1}^{+}/NT$. Variables with a $+$ superscript are defined as $\mathbf{X}^{+}=(1-\rho L)^{-1}\mathbf{X}$.

The expression in eq.(A-5) shows that the inconsistency of the CCEP estimator $\widehat{\rho}$ is determined by the interplay of (i) the numerator, which is the covariance between the defactored lagged dependent variable $\check{\mathbf{y}}_{-1}=\mathbb{M}_{\mathrm{X}}\mathbf{y}_{-1}$ and the error term $\boldsymbol{\varepsilon}$, and (ii) the denominator, which is the signal that remains in the lagged dependent variable after orthogonalizing the data on the CSA $\mathbf{Q}$ through the $\mathbf{M}$ matrix. We elaborate below.

First consider the covariance terms in the numerator. The correlation between $\check{\mathbf{y}}_{-1}$ and $\boldsymbol{\varepsilon}$ originates from projecting out the nuisance parameters using the orthogonalization matrix $\mathbf{M}$. The term $A(\rho)$ is induced by the within transformation (time-demeaning implied by including $\boldsymbol{\iota}_T$ in $\mathbf{Q}$) and also appears in bias expressions for the FE estimator in dynamic models without common factors (see Nickell, 1981), whereas the additional orthogonalization on the CSA induces the CCEP-specific term $D(\rho,\widetilde{\mathbf{H}})$. The latter is stochastic as in fixed T settings the matrix $\widetilde{\mathbf{H}}$ depends, through the CSA, on the particular realization of the factors. This is the reason for why we need to condition on the σ-algebra $\mathcal{C}$ to derive Theorem 1. We expect D to be negative and smaller in magnitude[1] than A, which is positive. Hence, the asymptotic bias is expected to be negative, with the orthogonalization on

[1]This is because D is a reweighing of the sum $\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}\widetilde{h}_{s,s-t}=-(c-1)/2<0$ in function of ρ. With positive weights ($\rho>0$) it is therefore likely that $D<0$. Similarly, $A+D$ is a reweighing of $\sum_{t=1}^{T-1}\sum_{s=t+1}^{T}h_{s,s-t}=(T-c)/2>0$ such that we can expect this sum to be positive when $\rho>0$.

the CSA counteracting the A term in the numerator of (A-5) and therefore reducing the bias in absolute terms.

The second determinant of the bias is the denominator, which denotes the variation that remains in the lagged dependent variable after multiplying the model through with $\mathbf{M}$. The C-term represents the remaining variation due to the presence of exogenous regressors and factors, expressed relative to σ_ε^2, whereas B and E relate to the variation due to $\boldsymbol{\varepsilon}$. The positive B term is again a shared term with the FE estimator due to the within transformation, whereas the $-E$ term (which is negative) indicates that additional variation is lost compared to the FE estimator by orthogonalizing on the CSA. Including CSA will similarly reduce C. Hence, when the set of CSA cut out a relatively large amount of variation, the denominator of eq.(A-5) may decrease faster than the induced reduction in the numerator and hence result in a larger bias. For a given number of factors and regressors, increasing the number of CSA used by the CCEP estimator is therefore likely to increase its asymptotic bias. This is confirmed by the Monte Carlo simulations in Section 5 of the main paper. Finally, since we can show that $\mathbb{M}\mathbb{F}_{-1}^{+} \rightarrow^{p} \mathbf{0}_{T\times 1}$ for $m = 1$ (see Lemma 4) the second and last term in the numerator of C drop out in single factor settings. As such, an increase in the importance of the factors will, ceteris paribus, increase the signal in the model and reduce the asymptotic bias of the CCEP estimator, but only when more than one factor is present.

A.3 Restricted bias corrections for models with a single factor

The procedure outlined in Section 4 of the main paper is a generally applicable method in the sense that it does not require the number of factors to be known. In the single factor setting, eq.(A-5) of Corollary 1 can be used to develop more efficient restricted bias corrections, denoted CCEPbcr. Below we outline two alternative CCEPbcr estimators, depending on whether the dynamic model includes additional covariates or not.

Firstly, in a model with a single common factor ($m = 1$) and no covariates ($\boldsymbol{\beta} = \mathbf{0}$), the bias expression (A-5) simplifies considerably as $C = 0$ for $N \rightarrow \infty$. This is convenient as it is the presence of the C-term that makes bias correction from eq.(A-5) infeasible due to its dependence on the unobservable sums $\mathbf{X}_{-1}^{+}$ and $\mathbf{F}_{-1}^{+}$. Furthermore, the bias expression for $\widehat{\rho}$ no longer depends on σ_ε^2, such that ρ is the only unknown parameter in eq.(A-5). In this setting, the CCEPbcr estimator $\widehat{\boldsymbol{\delta}}_{bcr1}$ can be obtained as

$$\widehat{\boldsymbol{\delta}}_{bcr1} = \underset{|\rho_0|<1}{\arg\min} \frac{1}{2} \left\| \widehat{\rho} - \rho_0 + \psi(\rho_0, \widetilde{\mathbf{H}}, 0) \right\|^2. \tag{A-7}$$

Secondly, adding exogenous regressors implies that $C \neq 0$ but if the single factor assumption is maintained we get the relatively simple form

$$C = \underset{N\rightarrow\infty}{\text{plim}} \frac{\boldsymbol{\beta}'\boldsymbol{\Omega}_{\check{\mathbf{x}}}\boldsymbol{\beta}}{\sigma_\varepsilon^2}, \tag{A-8}$$

which through $\boldsymbol{\Omega}_{\check{\mathbf{x}}}$ also depends on the unknown parameter ρ and on the infinite sum of explanatory variables $\mathbf{X}_{-1}^{+} = \sum_{l=0}^{\infty} \rho^l \mathbf{X}_{-1-l}$. In a finite sample, the latter can be approximated by the truncated sum $\widehat{\mathbf{X}_{-1}^{+}} = \left[\widehat{\mathbf{X}^{+}}'_{1,-1}, \ldots, \widehat{\mathbf{X}^{+}}'_{N,-1}\right]'$ where $\widehat{\mathbf{X}^{+}}_{i,-1} = \mathbf{J}^{-1}\mathbf{X}_{i,-1}$, and

$\mathbf{J}$ is a $T \times T$ matrix with ones on the main diagonal and $-\rho$ on the first sub-diagonal. The variance-covariance matrix is then estimated as $\widehat{\boldsymbol{\Omega}}_{\check{\mathrm{x}}}(\rho) = \widehat{\mathbf{X}^{+}_{-1}}' \mathbb{M}_{\mathrm{X}} \widehat{\mathbf{X}^{+}_{-1}}/NT$. Further substituting $\widehat{\sigma}^2_{\varepsilon}(\cdot)$ as defined in (20) for σ^2_{ε}, the estimator for C is

$$\widehat{C}(\boldsymbol{\delta}) = \frac{\boldsymbol{\beta}' \widehat{\boldsymbol{\Omega}}_{\check{\mathrm{x}}}(\rho) \boldsymbol{\beta}}{\widehat{\sigma}^2_{\varepsilon}(\boldsymbol{\delta})}, \tag{A-9}$$

which is, conditional on the unknown parameters ρ and $\boldsymbol{\beta}$, a function of the observed data only. Hence, in this setting the CCEPbcr estimator $\widehat{\boldsymbol{\delta}}_{bcr2}$ is

$$\widehat{\boldsymbol{\delta}}_{bcr2} = \underset{\boldsymbol{\delta}_0 \in \chi, |\rho_0|<1}{\arg\min} \frac{1}{2} \left\| \widehat{\boldsymbol{\delta}} - \boldsymbol{\delta}_0 + \widehat{\boldsymbol{\nu}} \psi(\rho_0, \widetilde{\mathbf{H}}, \widehat{C}(\boldsymbol{\delta}_0)) \right\|^2, \tag{A-10}$$

where $\widehat{\boldsymbol{\nu}} = \left[1, -\widehat{\boldsymbol{\zeta}}'\right]'$ and $\widehat{\boldsymbol{\zeta}} = (\mathbf{S}'_{\mathrm{x}} \widehat{\boldsymbol{\Sigma}} \mathbf{S}_{\mathrm{x}})^{-1} \mathbf{S}'_{\mathrm{x}} \widehat{\boldsymbol{\Sigma}} \mathbf{q}_1$. This bias correction should perform well when the single factor assumption is true and the approximation of $\mathbf{X}^{+}_{-1}$ is not too inaccurate. Note that the truncation implies that $\widehat{\boldsymbol{\delta}}_{bcr2}$ is inconsistent for finite T, but in practice the bias may be negligible (depending on the size of ρ). In case more than one factor is present, eq.($A-9$) can be a poor approximation of C and lead to additional bias, especially when the factors have a large overall influence on the model (relative to σ^2_{ε}).

A.4 Finite sample properties of CCEPbc versus CCEPbcr

In this section we compare the performance of the unrestricted bias correction CCEPbc to that of the restricted version CCEPbcr $\widehat{\boldsymbol{\delta}}_{bcr2}$ derived in Section A.3 for a model with covariates and a single factor. As in the Monte Carlo simulation experiment presented in the main text, we also report results for variants that add the additional CSA $\bar{\mathrm{g}}_t$ to the orthogonalization matrix.

Table A-1 compares the performance of the CCEPbc estimator to that of CCEPbcr in settings with one and two common factors. The distinction between these scenarios is of interest since CCEPbcr is derived under the assumption that only one factor is present whereas CCEPbc is applicable irrespectively of the number of factors (provided that the rank condition is satisfied). In general, we find that CCEPbcr is a fairly accurate bias-correction method, even in the case of two factors. Comparing the unrestricted and restricted version shows some trade-off between bias and variance, though. CCEPbc dominates in terms of bias correction but has a downside that the estimator $\widehat{\boldsymbol{\Sigma}}$ used in eq.(18) introduces uncertainty in small samples. CCEPbcr has a smaller variance as it imposes a specific form for the denominator in (A-10) but is less effective as a bias correction method because of the truncation error made in the estimation of C and the resulting finite T inconsistency. Because this bias is offset by the lower variance (in rmse terms) in small samples (also see Table A-2 for $N = 25$), CCEPbcr may still be an interesting alternative to CCEPbc. As N grows large, however, this relative efficiency only compensates for bias when the single factor assumption is true (see upper panel of Table A-1) or when the factors are not too strong in case $m > 1$ (see lower left panel of Table A-1). Moreover, as a result of the inconsistency for finite T, CCEPbcr displays a size distortion especially when N is large. For the unrestricted version, inference is reliable in all settings (although this may require adding $\bar{\mathrm{g}}_t$), but at the cost of a higher variance.

Table A-1: Monte Carlo results for ρ : number and strength of factors ($N = 500$)

	bias	*rmse*	*size*	*bias*	*rmse*	*size*	*bias*	*rmse*	*size*	*bias*	*rmse*	*size*	*bias*	*rmse*	*size*	*bias*	*rmse*	*size*
	one factor																	
	$RI = 1$									$RI = 3$								
	$T = 10$			$T = 20$			$T = 30$			$T = 10$			$T = 20$			$T = 30$		
CCEPbc	0.000	0.057	0.06	0.001	0.014	0.04	0.000	0.008	0.05	0.001	0.055	0.06	0.000	0.014	0.04	0.000	0.008	0.04
CCEPbcr	−0.022	0.034	0.20	−0.009	0.014	0.16	−0.005	0.008	0.10	−0.023	0.034	0.20	−0.009	0.014	0.17	−0.005	0.008	0.10
CCEPbc(+g)	0.000	0.066	0.06	0.000	0.014	0.02	0.000	0.008	0.04	0.000	0.064	0.06	0.000	0.014	0.03	0.000	0.008	0.04
CCEPbcr(+g)	−0.023	0.036	0.16	−0.009	0.014	0.16	−0.005	0.009	0.10	−0.024	0.036	0.15	−0.009	0.014	0.15	−0.005	0.009	0.10
	two factors																	
	$RI = 1$									$RI = 3$								
	$T = 10$			$T = 20$			$T = 30$			$T = 10$			$T = 20$			$T = 30$		
CCEPbc	0.000	0.058	0.06	0.000	0.014	0.04	0.000	0.008	0.04	0.006	0.070	0.10	0.006	0.020	0.08	0.005	0.012	0.12
CCEPbcr	−0.022	0.033	0.19	−0.009	0.014	0.17	−0.005	0.008	0.10	0.002	0.045	0.21	0.007	0.022	0.28	0.008	0.016	0.32
CCEPbc(+g)	0.000	0.070	0.08	0.000	0.016	0.03	0.000	0.008	0.05	0.000	0.067	0.06	0.001	0.014	0.04	0.000	0.008	0.05
CCEPbcr(+g)	−0.023	0.036	0.16	−0.009	0.015	0.17	−0.005	0.009	0.10	0.003	0.047	0.18	0.007	0.020	0.25	0.007	0.014	0.28

Notes: (*i*) see notes to Table 2 in the main paper. (*ii*) CCEPbcr is the restricted CCEPbc estimator derived under the assumption of a single factor ($m = 1$).

Table A-2: Monte Carlo results for ρ : number and strength of factors ($N = 25$)

	bias	*rmse*	*size*	*bias*	*rmse*	*size*	*bias*	*rmse*	*size*	*bias*	*rmse*	*size*	*bias*	*rmse*	*size*	*bias*	*rmse*	*size*
	one factor																	
	$RI = 1$									$RI = 3$								
	$T = 10$			$T = 20$			$T = 30$			$T = 10$			$T = 20$			$T = 30$		
CCEPbc	−0.004	0.151	0.06	0.000	0.064	0.08	0.000	0.038	0.06	0.002	0.153	0.08	0.000	0.063	0.08	0.000	0.038	0.06
CCEPbcr	−0.029	0.108	0.06	−0.010	0.046	0.07	−0.005	0.032	0.07	−0.026	0.110	0.06	−0.010	0.046	0.06	−0.006	0.033	0.08
CCEPbc(+g)	−0.003	0.169	0.04	0.001	0.065	0.07	0.000	0.039	0.06	0.002	0.166	0.05	0.001	0.066	0.07	0.000	0.039	0.06
CCEPbcr(+g)	−0.026	0.116	0.04	−0.010	0.047	0.06	−0.005	0.033	0.07	−0.024	0.119	0.05	−0.010	0.048	0.06	−0.006	0.033	0.08
	two factors																	
	$RI = 1$									$RI = 3$								
	$T = 10$			$T = 20$			$T = 30$			$T = 10$			$T = 20$			$T = 30$		
CCEPbc	0.002	0.155	0.06	−0.001	0.063	0.08	−0.001	0.038	0.05	0.014	0.161	0.08	0.019	0.073	0.12	0.017	0.046	0.09
CCEPbcr	−0.027	0.106	0.06	−0.010	0.045	0.05	−0.005	0.032	0.07	−0.004	0.109	0.07	0.010	0.050	0.10	0.013	0.037	0.13
CCEPbc(+g)	−0.006	0.173	0.05	−0.001	0.067	0.07	−0.001	0.041	0.06	0.006	0.169	0.05	0.008	0.068	0.08	0.006	0.039	0.05
CCEPbcr(+g)	−0.028	0.120	0.05	−0.010	0.047	0.06	−0.006	0.033	0.07	0.001	0.118	0.05	0.007	0.047	0.08	0.008	0.034	0.09

Notes: see Table A-1.

B Notation, definitions and preliminary results

B.1 Notation

We first introduce some notation that will be used later on. In what follows we define $K = 1 + k$, $k_w = 1 + k_x$ and we set $p = 1$ for convenience but note that generalizations follow straightforwardly. With $p = 1$ model (1)-(3) can be written in VAR(1) form

$$\begin{bmatrix} 1 & -(\boldsymbol{\beta}^*)' \\ \mathbf{0}_{k\times 1} & \mathbf{I}_k \end{bmatrix} \begin{bmatrix} \mathrm{y}_{it} \\ \mathbf{z}_{it} \end{bmatrix} = \begin{bmatrix} \alpha_i \\ \mathbf{c}_{\mathrm{z},i} \end{bmatrix} + \begin{bmatrix} \rho & \mathbf{0}_{1\times k} \\ \mathbf{0}_{k\times 1} & \boldsymbol{\lambda} \end{bmatrix} \begin{bmatrix} \mathrm{y}_{it-1} \\ \mathbf{z}_{it-1} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\gamma}_i' \\ \boldsymbol{\Gamma}_i' \end{bmatrix} \mathbf{f}_t + \begin{bmatrix} \varepsilon_{it} \\ \mathbf{v}_{it} \end{bmatrix}, \tag{B-1}$$

with $\boldsymbol{\beta}^* = [\boldsymbol{\beta}', \mathbf{0}_{1\times k_g}]'$ and the associated more compact form

$$\mathbf{A}_0 \mathbf{d}_{it} = \mathbf{c}_{\mathrm{d},i} + \boldsymbol{\Theta} L \mathbf{d}_{it} + \mathbf{C}_i \mathbf{f}_t + \mathbf{u}_{it},$$

where $\mathbf{c}_{\mathrm{d},i} = [\alpha_i, \mathbf{c}_{\mathrm{z},i}']'$, $\mathbf{d}_{it} = [\mathrm{y}_{it}, \mathbf{z}_{it}']'$, $\mathbf{u}_{it} = [\varepsilon_{it}, \mathbf{v}_{it}']'$ are $K \times 1$ vectors and

$$\underset{(K\times K)}{\mathbf{A}_0} = \begin{bmatrix} 1 & -(\boldsymbol{\beta}^*)' \\ \mathbf{0}_{k\times 1} & \mathbf{I}_k \end{bmatrix}, \qquad \underset{(K\times K)}{\boldsymbol{\Theta}} = \begin{bmatrix} \rho & \mathbf{0}_{1\times k} \\ \mathbf{0}_{k\times 1} & \boldsymbol{\lambda} \end{bmatrix}, \qquad \underset{(K\times m)}{\mathbf{C}_i} = \begin{bmatrix} \boldsymbol{\gamma}_i' \\ \boldsymbol{\Gamma}_i' \end{bmatrix}.$$

Since $\mathbf{A}_0$ is invertible,

$$\mathbf{d}_{it} = \mathbf{A}_0^{-1} \mathbf{c}_{\mathrm{d},i} + \mathbf{A}_0^{-1} \boldsymbol{\Theta} L \mathbf{d}_{it} + \mathbf{A}_0^{-1} \mathbf{C}_i \mathbf{f}_t + \mathbf{A}_0^{-1} \mathbf{u}_{it},$$

which can be rewritten further as

$$\begin{aligned} (\mathbf{I}_K - \boldsymbol{\Theta}^* L)\mathbf{d}_{it} &= \mathbf{c}_{\mathrm{d},i}^* + \mathbf{C}_i^* \mathbf{f}_t + \mathbf{u}_{it}^*, \\ \boldsymbol{\Theta}(L)\mathbf{d}_{it} &= \mathbf{c}_{\mathrm{d},i}^* + \mathbf{C}_i^* \mathbf{f}_t + \mathbf{u}_{it}^*, \end{aligned}$$

where the terms with an asterisk are defined as $\boldsymbol{\Theta}^* = \mathbf{A}_0^{-1}\boldsymbol{\Theta}$ and with $\boldsymbol{\Theta}(L) = \mathbf{I}_K - \boldsymbol{\Theta}^* L$. Then, as $\boldsymbol{\Theta}(L)$ is invertible by Assumption 5 we obtain the reduced form

$$\begin{aligned} \mathbf{d}_{it} &= \boldsymbol{\Theta}^{-1}(L)\mathbf{c}_{\mathrm{d},i}^* + \boldsymbol{\Theta}^{-1}(L)\mathbf{C}_i^* \mathbf{f}_t + \boldsymbol{\Theta}^{-1}(L)\mathbf{u}_{it}^*, \\ &= \ddot{\mathbf{c}}_{\mathrm{d},i} + (\ddot{\mathbf{C}}_i \otimes \mathbf{I}_K)' \check{\mathbf{f}}_t + \ddot{\mathbf{u}}_{it}, \end{aligned} \tag{B-2}$$

with $\ddot{\mathbf{u}}_{it} = \boldsymbol{\Theta}^{-1}(L)\mathbf{u}_{it}^*$, $\ddot{\mathbf{c}}_{\mathrm{d},i} = \boldsymbol{\Theta}^{-1}(L)\mathbf{c}_{\mathrm{d},i}^*$, $\check{\mathbf{f}}_t = vec(\mathbf{f}_t' \otimes \boldsymbol{\Theta}^{-1}(L))$ is $K^2 m \times 1$ and $\ddot{\mathbf{C}}_i = vec(\mathbf{C}_i^*)$ is $Km \times 1$. Its cross-section average is

$$\bar{\mathbf{d}}_t = \ddot{\mathbf{c}}_{\mathrm{d}} + (\ddot{\mathbf{C}} \otimes \mathbf{I}_K)' \check{\mathbf{f}}_t + \ddot{\mathbf{u}}_t, \tag{B-3}$$

where $\ddot{\mathbf{u}}_t = \boldsymbol{\Theta}^{-1}(L)\bar{\mathbf{u}}_t^*$, $\bar{\mathbf{u}}_t^* = \frac{1}{N}\sum_{i=1}^N \mathbf{u}_{it}^*$, $\ddot{\mathbf{c}}_{\mathrm{d}} = \boldsymbol{\Theta}^{-1}(L)\bar{\mathbf{c}}_{\mathrm{d}}^*$, $\ddot{\mathbf{C}} = vec(\bar{\mathbf{C}}^*)$ and $\bar{\mathbf{C}}^* = \frac{1}{N}\sum_{i=1}^N \mathbf{C}_i^*$. Stack the observations over time into the $T \times K$ matrix $\mathbf{D}_i = [\mathbf{d}_{i1}, \ldots, \mathbf{d}_{iT}]'$ and let $\mathbf{D} = [\bar{\mathbf{d}}_1, \ldots, \bar{\mathbf{d}}_T]'$ be its cross-section average. Next, define

$$\underset{(T\times c)}{\mathbf{Q}_i} = [\boldsymbol{\iota}_T, \mathbf{D}_i, \ldots, \mathbf{D}_{i,-p^*}], \qquad \underset{(T\times c)}{\mathbf{Q}} = \frac{1}{N}\sum_{i=1}^N \mathbf{Q}_i = [\boldsymbol{\iota}_T, \mathbf{D}, \ldots, \mathbf{D}_{-p^*}], \tag{B-4}$$

with $c = 1 + K(1 + p^*)$ the number of columns of $\mathbf{Q}_i$ and $\mathbf{Q}$. Also, defining

$$\underset{(T\times 1+K^2m(1+p^*))}{\check{\mathbf{F}}} = [\boldsymbol{\iota}_T, \check{\mathbf{F}}_0, \check{\mathbf{F}}_{-1}, \ldots, \check{\mathbf{F}}_{-p^*}] \tag{B-5}$$

with $\check{\mathbf{F}}_0 = [\check{\mathbf{f}}_1, \ldots, \check{\mathbf{f}}_T]'$, $\check{\mathbf{F}}_{-1} = [\check{\mathbf{f}}_0, \ldots, \check{\mathbf{f}}_{T-1}]'$, $\check{\mathbf{F}}_{-p^*} = [\check{\mathbf{f}}_{1-p^*}, \ldots, \check{\mathbf{f}}_{T-p^*}]'$ and so on, and

$$\underset{(1+K^2m(1+p^*)\times 1+K(1+p^*))}{\ddot{\mathbf{P}}_i} = \begin{bmatrix} 1 & (\boldsymbol{\iota}'_{1+p^*} \otimes \ddot{\mathbf{c}}'_{\mathrm{d},i}) \\ \mathbf{0}_{K^2m(1+p^*)\times 1} & \mathbf{I}_{1+p^*} \otimes (\ddot{\mathbf{C}}_i \otimes \mathbf{I}_K) \end{bmatrix}, \quad \ddot{\mathbf{P}} = \frac{1}{N}\sum_{i=1}^{N} \ddot{\mathbf{P}}_i. \tag{B-6}$$

Given that $\mathbf{C}_i^* = \mathbf{A}_0^{-1}(\mathbf{C} + [\boldsymbol{\eta}_i, \boldsymbol{\nu}_i]')$ by Ass.3 we have also

$$\ddot{\mathbf{P}}_i = \mathbf{P} + \check{\mathbf{P}}_i, \tag{B-7}$$

with

$$\mathbf{P} = \begin{bmatrix} 1 & (\boldsymbol{\iota}'_{1+p^*} \otimes \mathbf{0}'_{K\times 1}) \\ \mathbf{0}_{K^2m(1+p^*)\times 1} & \mathbf{I}_{1+p^*} \otimes (\dot{\mathbf{C}} \otimes \mathbf{I}_K) \end{bmatrix}, \quad \check{\mathbf{P}}_i = \begin{bmatrix} 1 & (\boldsymbol{\iota}'_{1+p^*} \otimes \mathbf{0}'_{K\times 1}) \\ \mathbf{0}_{K^2m(1+p^*)\times 1} & \mathbf{I}_{1+p^*} \otimes (\widetilde{\mathbf{C}}_i \otimes \mathbf{I}_K) \end{bmatrix},$$

and where $\dot{\mathbf{C}} = vec(\mathbf{A}_0^{-1}\mathbf{C})$ and $\widetilde{\mathbf{C}}_i = vec(\mathbf{A}_0^{-1}[\boldsymbol{\eta}_i, \boldsymbol{\nu}_i]')$.

With the definitions above, the $T \times c$ matrix of observations is

$$\mathbf{Q}_i = \check{\mathbf{F}}\ddot{\mathbf{P}}_i + \ddot{\mathbf{U}}_i, \tag{B-8}$$

such that the observed matrix of cross-section averages can similarly be decomposed into

$$\mathbf{Q} = \check{\mathbf{F}}\ddot{\mathbf{P}} + \ddot{\mathbf{U}}, \tag{B-9}$$

where

$$\underset{(T\times c)}{\ddot{\mathbf{U}}_i} = \begin{bmatrix} 0 & \ddot{\mathbf{u}}'_{i1} & \ldots & \ddot{\mathbf{u}}'_{i,1-p^*} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \ddot{\mathbf{u}}'_{iT} & \ldots & \ddot{\mathbf{u}}'_{i,T-p^*} \end{bmatrix}, \quad \underset{(T\times c)}{\ddot{\mathbf{U}}} = \frac{1}{N}\sum_{i=1}^{N} \ddot{\mathbf{U}}_i = \begin{bmatrix} 0 & \ddot{\mathbf{u}}'_{1} & \ldots & \ddot{\mathbf{u}}'_{1-p^*} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \ddot{\mathbf{u}}'_{T} & \ldots & \ddot{\mathbf{u}}'_{T-p^*} \end{bmatrix}. \tag{B-10}$$

Next, we express all the regression variables in the model in terms of $\mathbf{Q}_i$ by defining the $k \times k_x$, $c \times 1$ and $c \times k_w$ selector matrices

$$\underset{(k\times k_x)}{\mathbf{S}_x} = \begin{bmatrix} \mathbf{I}_{k_x} \\ \mathbf{0}_{k_g\times k_x} \end{bmatrix}, \quad \underset{(c\times 1)}{\mathbf{S}_y} = \begin{bmatrix} 0 \\ 1 \\ \mathbf{0}_{(c-2)\times 1} \end{bmatrix}, \quad \underset{(c\times k_w)}{\mathbf{S}_w} = \begin{bmatrix} \mathbf{0}_{2\times 1} & \mathbf{0}_{2\times k_x} \\ \mathbf{0}_{k\times 1} & \mathbf{S}_x \\ 1 & \mathbf{0}_{1\times k_x} \\ \mathbf{0}_{c-(3+k)\times 1} & \mathbf{0}_{c-(3+k)\times k_x} \end{bmatrix}, \tag{B-11}$$

such that

$$\mathbf{y}_i = \mathbf{Q}_i\mathbf{S}_y = \check{\mathbf{F}}\ddot{\mathbf{P}}_i\mathbf{S}_y + \ddot{\mathbf{U}}_i\mathbf{S}_y, \tag{B-12}$$

$$\mathbf{w}_i = \mathbf{Q}_i\mathbf{S}_w = \check{\mathbf{F}}\ddot{\mathbf{P}}_i\mathbf{S}_w + \boldsymbol{\epsilon}_i, \tag{B-13}$$

where notably $\boldsymbol{\epsilon}_i$ is a $T \times k_w$ matrix given by

$$\boldsymbol{\epsilon}_i = \ddot{\mathbf{U}}_i\mathbf{S}_w = [\boldsymbol{\varrho}^+_{i,-1}, \dot{\mathbf{V}}_i\mathbf{S}_{\mathrm{x}}], \tag{B-14}$$

with $\dot{\mathbf{V}}_i = [\dot{\mathbf{v}}_{i,1}, \ldots, \dot{\mathbf{v}}_{i,T}]'$ and $\dot{\mathbf{v}}_{i,t} = \boldsymbol{\lambda}(L)^{-1}\mathbf{v}_{i,t}$. Also, with $\boldsymbol{\varrho}_i = \boldsymbol{\varepsilon}_i + \dot{\mathbf{V}}_i\boldsymbol{\beta}^*$ we have $\boldsymbol{\varrho}^+_{i,-1} = \boldsymbol{\varepsilon}^+_{i,-1} + \dot{\mathbf{V}}^+_{i,-1}\boldsymbol{\beta}^*$, a $T \times 1$ vector.

B.2 Rotating the projection matrix

To proceed with the terms involving the projector $\mathbf{H}$, we extend the approach of Karabiyik et al. (2017) to dynamic settings. To that end, let $\mathbf{T}$ be a $K \times K$ orthogonal matrix such that $(\bar{\mathbf{C}}^*)'\mathbf{T} = [\bar{\mathbf{C}}_m, \bar{\mathbf{C}}_{-m}]$, with $\bar{\mathbf{C}}_m$ the full rank $m \times m$ partitioning of $(\bar{\mathbf{C}}^*)'$, and $\bar{\mathbf{C}}_{-m}$ is the $m \times (K-m)$ matrix containing the remaining $K-m$ columns. Let $\bar{\mathbf{U}}^* = [\bar{\mathbf{u}}_1^*, \dots, \bar{\mathbf{u}}_T^*]'$ such that $\bar{\mathbf{U}}_m^*$ and $\bar{\mathbf{U}}_{-m}^*$ are the corresponding partitioning that follows from $\bar{\mathbf{U}}^*\mathbf{T} = [\bar{\mathbf{U}}_m^*, \bar{\mathbf{U}}_{-m}^*]$. Next, we introduce the $k(1+p^*) \times k(1+p^*)$ rotation matrix $\mathbf{R}$. First, let

$$\mathbf{B} = \begin{bmatrix} \bar{\mathbf{C}}_m^{-1} & -\bar{\mathbf{C}}_m^{-1}\bar{\mathbf{C}}_{-m} \\ \mathbf{0}_{(K-m)\times m} & \mathbf{I}_{K-m} \end{bmatrix} = [\mathbf{B}_m, \mathbf{B}_{-m}]. \tag{B-15}$$

In what follows it is convenient to set $p^* = 1$ in order to save on notation. However, we note that the results generalize directly. The matrices $\widetilde{\mathbf{R}}$, $\widetilde{\mathbf{T}}$ and $\mathbf{R}$, defined next, are in general[2] of dimension $K(1+p^*) \times K(1+p^*)$, $K(1+p^*) \times K(1+p^*)$ and $1+K(1+p^*) \times 1+K(1+p^*)$ respectively. In the $p^* = 1$ case we then have

$$\underset{(2K\times K)}{\mathbf{R}^*} = \begin{bmatrix} \mathbf{I}_K \\ -(\boldsymbol{\Theta}^*)' \end{bmatrix}, \qquad \underset{(2K\times 2K)}{\widetilde{\mathbf{R}}} = \begin{bmatrix} \mathbf{R}^*, & \mathbf{0}_{K\times K} \\ & \mathbf{I}_K \end{bmatrix}, \qquad \underset{(2K\times 2K)}{\widetilde{\mathbf{T}}} = \begin{bmatrix} \mathbf{TB} & \mathbf{0}_{K\times K} \\ \mathbf{0}_{K\times K} & \mathbf{I}_K \end{bmatrix},$$

and, accounting for the row of constants in $\mathbf{Q}$,

$$\mathbf{R} = \begin{bmatrix} 1 & \mathbf{0}_{1\times 2K} \\ \mathbf{0}_{2K\times 1} & \widetilde{\mathbf{R}}\widetilde{\mathbf{T}} \end{bmatrix}.$$

Next, since by Lemma 1 the distribution of the CCEP estimator, or all its components, is invariant to the presence of the fixed effects, we can, without loss of generality, simplify notation by setting $\mathbf{c}_{\mathrm{d},i} = \mathbf{0}_{K\times 1}$ for all i such that $\ddot{\mathbf{c}}_{\mathrm{d}} = \mathbf{0}_{K\times 1}$. Making use of (B-9) we then get the following restructuring of $\mathbf{Q}$

$$\mathbf{QR} = \check{\mathbf{F}}\ddot{\mathbf{P}}\mathbf{R} + \ddot{\mathbf{U}}\mathbf{R} = [\boldsymbol{\iota}_T, \mathbf{F}, \mathbf{0}_{T\times(K-m)}, \check{\mathbf{F}}_{-1}(\ddot{\mathbf{C}} \otimes \mathbf{I}_K)] + [\mathbf{0}_{T\times 1}, \bar{\mathbf{U}}_m, \bar{\mathbf{U}}_{-m}, \ddot{\mathbf{U}}_{-1}],$$

where $\bar{\mathbf{U}}_m = \bar{\mathbf{U}}_m^*\bar{\mathbf{C}}_m^{-1}$, $\bar{\mathbf{U}}_{-m} = \bar{\mathbf{U}}_{-m}^* - \bar{\mathbf{U}}_m^*\bar{\mathbf{C}}_m^{-1}\bar{\mathbf{C}}_{-m}$ and $\ddot{\mathbf{U}}_{-1} = [\ddot{\mathbf{u}}_0, \dots, \ddot{\mathbf{u}}_{T-1}]'$. The matrix $\mathbf{N}$ rearranges the columns conveniently as follows

$$\mathbf{QRN} = [\boldsymbol{\iota}_T, \mathbf{F}, \check{\mathbf{F}}_{-1}(\ddot{\mathbf{C}} \otimes \mathbf{I}_K), \mathbf{0}_{T\times(K-m)}] + [\mathbf{0}_{T\times 1}, \bar{\mathbf{U}}_m, \ddot{\mathbf{U}}_{-1}, \bar{\mathbf{U}}_{-m}].$$

[2] To illustrate: for any p^* we have, with $\mathbf{L}_{(1+p^*)}$ denoting a $(1+p^*) \times (1+p^*)$ matrix of zeros with ones on the first lower sub-diagonal

$$\underset{(K(1+p^*)\times K(1+p^*))}{\widetilde{\mathbf{R}}} = \mathbf{I}_{K(1+p^*)} - (\mathbf{L}_{(1+p^*)} \otimes (\boldsymbol{\Theta}^*)'), \qquad \underset{(K(1+p^*)\times K(1+p^*))}{\widetilde{\mathbf{T}}} = \begin{bmatrix} \mathbf{I}_{p^*} \otimes \mathbf{TB} & \mathbf{0}_{Kp^*\times K} \\ \mathbf{0}_{K\times Kp^*} & \mathbf{I}_K \end{bmatrix},$$

and

$$\mathbf{R} = \begin{bmatrix} 1 & \mathbf{0}_{1\times K(1+p^*)} \\ \mathbf{0}_{K(1+p^*)\times 1} & \widetilde{\mathbf{R}}\widetilde{\mathbf{T}} \end{bmatrix}.$$

Note that $\check{\mathbf{F}}_{-1}(\ddot{\mathbf{C}} \otimes \mathbf{I}_K)$ is a full column rank matrix ($rk(\check{\mathbf{F}}_{-1}(\ddot{\mathbf{C}} \otimes \mathbf{I}_K)) = K$) such that $rk([\mathbf{F}, \check{\mathbf{F}}_{-1}(\ddot{\mathbf{C}} \otimes \mathbf{I}_K), \mathbf{0}_{T\times(K-m)}]) = K + m \leq c - 1 = 2K$.[3] When $m < K$, the final $K - m$ columns of $\mathbf{QRN}$ are degenerate as $\left\|[\bar{\mathbf{U}}_m, \ddot{\mathbf{U}}_{-1}, \bar{\mathbf{U}}_{-m}]\right\| = O_p(N^{-1/2})$ by Lemma 2. Hence, post-multiplying by $\mathbf{D}_N = diag(\boldsymbol{\iota}'_{(1+K+m)}, \sqrt{N}\boldsymbol{\iota}'_{(K-m)})$

$$\mathbf{Q}_0 = \mathbf{QRND}_N = [\boldsymbol{\iota}_T, \mathbf{F}, \check{\mathbf{F}}_{-1}(\ddot{\mathbf{C}} \otimes \mathbf{I}_K), \mathbf{0}_{T\times(K-m)}] + [\mathbf{0}_{T\times 1}, \bar{\mathbf{U}}_m, \ddot{\mathbf{U}}_{-1}, \sqrt{N}\bar{\mathbf{U}}_{-m}] = \mathbf{F}^0 + \bar{\mathbf{U}}^0,$$

with $\mathbf{F}^0 = [\mathbf{F}^*, \mathbf{0}_{T\times(K-m)}]$ and $\mathbf{F}^* = [\boldsymbol{\iota}_T, \mathbf{F}, \check{\mathbf{F}}_{-1}(\ddot{\mathbf{C}} \otimes \mathbf{I}_K)]$ is a $T \times (1 + K + m)$ full rank matrix. Additionally, $\bar{\mathbf{U}}^0 = [\bar{\mathbf{U}}^0_m, \bar{\mathbf{U}}^0_{-m}]$, $\bar{\mathbf{U}}^0_m = [\mathbf{0}_{T\times 1}, \bar{\mathbf{U}}_m, \ddot{\mathbf{U}}_{-1}]$ and $\bar{\mathbf{U}}^0_{-m} = \sqrt{N}\bar{\mathbf{U}}_{-m}$. Therefore, we obtain for the rotated $\mathbf{Q}$ matrix with $\mathbf{F}^+_u = [\mathbf{F}^*, \bar{\mathbf{U}}^0_{-m}]$

$$\mathbf{Q}_0 = \mathbf{F}^0 + \bar{\mathbf{U}}^0 = [\mathbf{F}^*, \bar{\mathbf{U}}^0_{-m}] + [\bar{\mathbf{U}}^0_m, \mathbf{0}_{T\times(K-m)}] = \mathbf{F}^+_u + O_p(N^{-1/2}), \tag{B-16}$$

since $\left\|\bar{\mathbf{U}}^0_m\right\| = O_p(N^{-1/2})$ and $\left\|\bar{\mathbf{U}}^0_{-m}\right\| = O_p(1)$ by Lemma 2. Hence, in contrast to $\mathbf{Q}$, the columns of $\mathbf{Q}_0$ are non-degenerate even in case $m < K$, which, given that $\mathbf{H} = \mathbf{Q}(\mathbf{Q}'\mathbf{Q})^\dagger\mathbf{Q} = \mathbf{Q}_0(\mathbf{Q}_0'\mathbf{Q}_0)^{-1}\mathbf{Q}_0$ will now allow us to evaluate the limit of $\mathbf{H}$.

Finally, it is convenient to define the selector matrices

$$\mathbf{S}_m = \begin{bmatrix} \mathbf{I}_{1+K+m} \\ \mathbf{0}_{(K-m)\times(1+K+m)} \end{bmatrix}, \qquad \mathbf{S}_{-m} = \begin{bmatrix} \mathbf{0}_{(1+K+m)\times(K-m)} \\ \mathbf{I}_{K-m} \end{bmatrix}, \tag{B-17}$$

such that we obtain the following key identities that will be used throughout the appendix

$$\mathbf{F}^* = \check{\mathbf{F}}\ddot{\mathbf{P}}\mathbf{RNS}_m, \tag{B-18}$$
$$\bar{\mathbf{U}}^0_m = \ddot{\mathbf{U}}\mathbf{RNS}_m, \tag{B-19}$$
$$\bar{\mathbf{U}}^0_{-m} = \sqrt{N}\ddot{\mathbf{U}}\mathbf{RNS}_{-m}. \tag{B-20}$$

B.3 Preliminary results

Assume that Ass.4 holds and $p^* \geq p$. Define next $\mathbf{R}_0$ as follows

$$\mathbf{R}_0 = \begin{bmatrix} \mathbf{0}_{1\times K} \\ \mathbf{R}^* \\ \mathbf{0}_{K(p^*-p)\times K} \end{bmatrix},$$

[3] In general, for any p^* we have

$$\begin{aligned}\mathbf{QR} = \check{\mathbf{F}}\ddot{\mathbf{P}}\mathbf{R} + \ddot{\mathbf{U}}\mathbf{R} &= [\boldsymbol{\iota}_T, \mathbf{F}, \mathbf{0}_{T\times(K-m)}, \mathbf{F}_{-1}, \mathbf{0}_{T\times(K-m)}, \dots, \mathbf{F}_{-(p^*-1)}, \mathbf{0}_{T\times(K-m)}, \check{\mathbf{F}}_{-p^*}(\ddot{\mathbf{C}} \otimes \mathbf{I}_K)] \\ &+ [\mathbf{0}_{T\times 1}, \bar{\mathbf{U}}_m, \bar{\mathbf{U}}_{-m}, \bar{\mathbf{U}}_{m,-1}, \bar{\mathbf{U}}_{-m,-1}, \dots, \bar{\mathbf{U}}_{m,-(p^*-1)}, \bar{\mathbf{U}}_{-m,-(p^*-1)}, \ddot{\mathbf{U}}_{-p^*}],\end{aligned}$$

and

$$\begin{aligned}\mathbf{QRN} &= [\boldsymbol{\iota}_T, \mathbf{F}, \dots, \mathbf{F}_{-(p^*-1)}, \check{\mathbf{F}}_{-1}(\ddot{\mathbf{C}} \otimes \mathbf{I}_K), \mathbf{0}_{T\times((K-m)(p^*-1))}] \\ &+ [\mathbf{0}_{T\times 1}, \bar{\mathbf{U}}_m, \dots, \bar{\mathbf{U}}_{m,-(p^*-1)}, \ddot{\mathbf{U}}_{-p^*}, \bar{\mathbf{U}}_{-m}, \dots, \bar{\mathbf{U}}_{-m,-(p^*-1)}],\end{aligned}$$

with $rk([\mathbf{F}, \dots, \mathbf{F}_{-(p^*-1)}, \check{\mathbf{F}}_{-p^*}(\ddot{\mathbf{C}} \otimes \mathbf{I}_K), \mathbf{0}_{T\times(K-m)(p^*-1)}]) = K + mp^* \leq c - 1 = K(1 + p^*)$.

such that we can write

$$\mathbf{Q}\mathbf{R}_0\mathbf{T} = \mathbf{F}[\bar{\mathbf{C}}_m, \bar{\mathbf{C}}_{-m}] + [\bar{\mathbf{U}}^*_m, \bar{\mathbf{U}}^*_{-m}].$$

This gives, multiplied by $\mathbf{B}_m$ defined in (B-15),

$$\mathbf{Q}\mathbf{R}_0\mathbf{T}\mathbf{B}_m = \mathbf{F} + \bar{\mathbf{U}}_m, \tag{B-21}$$

such that we also have the following important relation

$$\bar{\mathbf{U}}_m = \ddot{\mathbf{U}}\mathbf{R}_0\mathbf{T}\mathbf{B}_m. \tag{B-22}$$

Solving (B-21) for $\mathbf{F}$ and multiplying by $\mathbf{M}$ gives

$$\mathbf{M}\mathbf{F} = \mathbf{M}(\mathbf{Q}\mathbf{R}_0\mathbf{T}\mathbf{B}_m - \bar{\mathbf{U}}_m),$$

which in turn, given that by definition $\mathbf{M}\mathbf{Q} = \mathbf{0}_{T\times c}$, leads to the following key result

$$\mathbf{M}\mathbf{F} = -\mathbf{M}\bar{\mathbf{U}}_m. \tag{B-23}$$

C Analysis for $N \to \infty$ and T fixed

C.1 Statement of lemmas

Lemma 1. *Suppose that Ass.5 holds and a vector of constants $\boldsymbol{\iota}_T$ is included in $\mathbf{Q}$. Then, the CCEP estimator in eq.(14), or its components $\mathbf{w}_i'\mathbf{M}\mathbf{w}_i$ and $\mathbf{w}_i'\mathbf{M}\mathbf{y}_i$ are invariant to α_i and $\mathbf{c}_{z,i}$ for all sample sizes. If additionally Ass.2 holds then it is equivalent to evaluate (14) with $E(\check{\mathbf{F}}) = \mathbf{0}$ for all N and T.*

Lemma 2. *Let Ass.1 and 5 hold. Then, as $N \to \infty$ and T fixed,*

$$\left\|\ddot{\mathbf{U}}\right\| = O_p(N^{-1/2}), \qquad \left\|\bar{\mathbf{U}}^*\right\| = O_p(N^{-1/2}), \tag{C-1}$$

$$\left\|\bar{\mathbf{U}}_m\right\| = O_p(N^{-1/2}), \qquad \left\|\bar{\mathbf{U}}_m^0\right\| = O_p(N^{-1/2}), \qquad \left\|\bar{\mathbf{U}}_{-m}^0\right\| = O_p(1). \tag{C-2}$$

Lemma 3. *Let c be the number of columns in $\mathbf{Q}$. For any $N \to \infty$ and $c < \infty$,*

$$\|\mathbf{H}\| \leq M, \tag{C-3}$$

irrespective of m, with M a finite constant.

Lemma 4. *Let Ass.1-5 hold and suppose that $m = 1$ and $p = 0$, then,*

$$\mathbf{M}\mathbf{F}_{-1}^{+} \longrightarrow^{p} \mathbf{0}_{T\times 1} \qquad \text{as} \qquad N \to \infty. \tag{C-4}$$

Lemma 5. *Let Ass.1-5 hold and suppose that $p^* \geq p$, then, as $N \to \infty$*

$$\mathbf{A}^{\mathbf{F}} = \frac{1}{N}\sum_{i=1}^{N}\frac{\mathbf{w}_i'\mathbf{M}\mathbf{F}}{T}\boldsymbol{\gamma}_i = O_p\left(N^{\omega}\right), \tag{C-5}$$

with $\omega = -1$ in case $m = 1$, $p = 0$ and $\omega = -1/2$ otherwise.

Lemma 6. *Let Ass.1-3 and 5 hold, then,*

$$\widehat{\boldsymbol{\Sigma}} = \frac{1}{N}\sum_{i=1}^{N}\frac{\mathbf{w}_i'\mathbf{M}\mathbf{w}_i}{T} = O_p(1), \tag{C-6}$$

for all N and T.

Lemma 7. *Let Ass.1-3 and 5 hold. Then, as $N \to \infty$,*

$$\widehat{\boldsymbol{\Sigma}} = \boldsymbol{\Sigma} + O_p(N^{-1/2}),$$

with

$$\boldsymbol{\Sigma} = \left(vec(\mathbf{I}_{k_w})' \otimes \mathbf{I}_{k_w}\right)\left(\mathbf{I}_{k_w} \otimes \left[\boldsymbol{\Sigma}_{\epsilon} + \left(\mathbf{S}_w' \otimes \mathbf{S}_w'\right)\boldsymbol{\Sigma}_{\ddot{\mathbf{P}}}(\check{\mathbf{F}}' \otimes \check{\mathbf{F}}')\right]T^{-1}vec\left(\mathbf{M}\right)\right), \tag{C-7}$$

and where $\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}} = E(\boldsymbol{\epsilon}_i' \otimes \boldsymbol{\epsilon}_i')$ and $\boldsymbol{\Sigma}_{\ddot{\mathbf{P}}} = E(\ddot{\mathbf{P}}_i' \otimes \ddot{\mathbf{P}}_i')$.

Lemma 8. *Let Ass.1-5 hold and suppose that $p^* \geq p$. Then, for $\widehat{\sigma}_\varepsilon^2$ evaluated at $\boldsymbol{\delta}_0 \neq \boldsymbol{\delta}$ with $\|\boldsymbol{\delta} - \boldsymbol{\delta}_0\| < \infty$, as $N \to \infty$,*

$$\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta}_0) = \sigma_\varepsilon^2 - \sigma_\varepsilon^2 c_1 \boldsymbol{\upsilon}(\rho, \mathbf{H})'(\boldsymbol{\delta} - \boldsymbol{\delta}_0) + c_2(\boldsymbol{\delta} - \boldsymbol{\delta}_0)'\boldsymbol{\Sigma}(\boldsymbol{\delta} - \boldsymbol{\delta}_0) + O_p(N^{-1/2}), \tag{C-8}$$

with $c_1 = 2/(T-c)$ and $c_2 = T/(T-c)$. When evaluated at $\boldsymbol{\delta}_0 = \boldsymbol{\delta}$,

$$\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta}) = \frac{1}{N(T-c)} \sum_{i=1}^N \boldsymbol{\varepsilon}_i' \mathbf{M} \boldsymbol{\varepsilon}_i + O_p(N^{-1}), \tag{C-9}$$

and also

$$\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta}) \longrightarrow^p \sigma_\varepsilon^2, \tag{C-10}$$

for $\widehat{\sigma}_\varepsilon^2(\cdot)$ defined in eq.(20).

C.2 Proof of lemmas

C.2.1 Proof of Lemma 1

Let $\mathbf{D}_0 = \mathbf{I}_T - \boldsymbol{\iota}_T \boldsymbol{\iota}_T'/T$ and consider that $\mathbf{D}_0 = \mathbf{D}_0'$ and $\mathbf{D}_0\mathbf{D}_0 = \mathbf{D}_0$. Then, with (B-12)-(B-13) we can write the components of the CCEP estimator in (14) as

$$\mathbf{w}_i'\mathbf{M}\mathbf{y}_i = \mathbf{S}_w'\mathbf{Q}_i'\mathbf{M}\mathbf{Q}_i\mathbf{S}_y = \mathbf{S}_w'\mathbf{Q}_i'\mathbf{D}_0\mathbf{Q}_i\mathbf{S}_y - \mathbf{S}_w'\mathbf{Q}_i'\mathbf{D}_0\mathbf{Q}(\mathbf{Q}'\mathbf{D}_0\mathbf{Q})^\dagger\mathbf{Q}'\mathbf{D}_0\mathbf{Q}_i\mathbf{S}_y, \tag{C-11}$$
$$\mathbf{w}_i'\mathbf{M}\mathbf{w}_i = \mathbf{S}_w'\mathbf{Q}_i'\mathbf{M}\mathbf{Q}_i\mathbf{S}_w = \mathbf{S}_w'\mathbf{Q}_i\mathbf{D}_0\mathbf{Q}_i\mathbf{S}_w - \mathbf{S}_w'\mathbf{Q}_i'\mathbf{D}_0\mathbf{Q}(\mathbf{Q}'\mathbf{D}_0\mathbf{Q})^\dagger\mathbf{Q}'\mathbf{D}_0\mathbf{Q}_i\mathbf{S}_w. \tag{C-12}$$

Next, making use of (B-8) and (B-5)

$$\begin{aligned}\mathbf{D}_0\mathbf{Q}_i &= \mathbf{D}_0(\check{\mathbf{F}}\ddot{\mathbf{P}}_i + \ddot{\mathbf{U}}_i) = [\mathbf{D}_0\boldsymbol{\iota}_T, \mathbf{D}_0\check{\mathbf{F}}_0, \ldots, \mathbf{D}_0\check{\mathbf{F}}_{-p^*}]\ddot{\mathbf{P}}_i + \mathbf{D}_0\ddot{\mathbf{U}}_i, \\ &= \left[\mathbf{0}_{T\times 1}, \mathbf{D}_0[\check{\mathbf{F}}_0, \ldots, \check{\mathbf{F}}_{-p^*}][\mathbf{I}_{1+p^*} \otimes (\ddot{\mathbf{C}}_i \otimes \mathbf{I}_K)]\right] + \mathbf{D}_0\ddot{\mathbf{U}}_i\end{aligned} \tag{C-13}$$

because $\mathbf{D}_0\boldsymbol{\iota}_T = \mathbf{0}_{T\times 1}$, and therefore also for the CSA

$$\mathbf{D}_0\mathbf{Q} = \mathbf{D}_0(\check{\mathbf{F}}\ddot{\mathbf{P}} + \ddot{\mathbf{U}}) = \left[\mathbf{0}_{T\times 1}, \mathbf{D}_0[\check{\mathbf{F}}_0, \ldots, \check{\mathbf{F}}_{-p^*}][\mathbf{I}_{1+p^*} \otimes (\ddot{\mathbf{C}} \otimes \mathbf{I}_K)]\right] + \mathbf{D}_0\ddot{\mathbf{U}}. \tag{C-14}$$

By consequence of (C-13) and (C-14), the right hand side of (C-11)-(C-12) is devoid of the fixed effects such that both $\mathbf{w}_i'\mathbf{M}\mathbf{y}_i$ and $\mathbf{w}_i'\mathbf{M}\mathbf{w}_i$ are invariant to their presence for all sample sizes. Additionally, since from Ass.2 and 5 follows $E(\mathbf{D}_0\check{\mathbf{F}}) = \mathbf{0}$, by (C-13) and (C-14) we can without loss of generality evaluate (C-11)-(C-12) assuming $E(\check{\mathbf{F}}) = \mathbf{0}$.

C.2.2 Proof of Lemma 2

From the definition we have $\bar{\mathbf{U}}^* = [\bar{\mathbf{u}}_1^*, \ldots, \bar{\mathbf{u}}_T^*]'$ such that its t-th row can be written as $\bar{\mathbf{u}}_t^* = N^{-1}\sum_{i=1}^N \mathbf{u}_{it}^* = N^{-1}\sum_{i=1}^N \mathbf{A}_0^{-1}\mathbf{u}_{it}$, where $\mathbf{A}_0^{-1}$ always exists and has fixed and finite entries. From Ass.1 follows $E(\mathbf{u}_{it}) = \mathbf{0}$ and therefore $E(\bar{\mathbf{u}}_t^*) = \mathbf{0}$. Consider now the variance

$$Var\left(\bar{\mathbf{u}}_t^*\right) = E\left(\frac{1}{N}\sum_{i=1}^N \mathbf{u}_{it}^*\right)\left(\frac{1}{N}\sum_{j=1}^N \mathbf{u}_{jt}^*\right)' = E\left(\frac{1}{N^2}\sum_{i=1}^N \mathbf{u}_{it}^*\mathbf{u}_{it}^{*\prime}\right),$$

$$= \mathbf{A}_0^{-1}\left(\frac{1}{N^2}\sum_{i=1}^{N} E(\mathbf{u}_{it}\mathbf{u}_{it}')\right)(\mathbf{A}_0^{-1})' = \mathbf{A}_0^{-1}\left(\frac{1}{N^2}\sum_{i=1}^{N} \boldsymbol{\Omega}_{\mathbf{u}}\right)(\mathbf{A}_0^{-1})' = O\left(\frac{1}{N}\right),$$

because by Ass.1 the $\mathbf{u}_{it}$ are independent over i and the entries of $\boldsymbol{\Omega}_{\mathbf{u}} = \begin{bmatrix} \sigma_\varepsilon^2 & \mathbf{0}_{k\times 1}' \\ \mathbf{0}_{k\times 1} & \boldsymbol{\Omega}_{\mathbf{v}} \end{bmatrix}$ are bounded for all i. Consequently, $\|\bar{\mathbf{u}}_t^*\| = O_p(N^{-1/2})$ and $\left\|\bar{\mathbf{U}}^*\right\| = O_p(N^{-1/2})$. Consider next $\ddot{\mathbf{U}}$ defined in (B-10) and let $\boldsymbol{\xi}_q = [0, \ddot{\mathbf{u}}_q', \ddot{\mathbf{u}}_{q-1}', \ldots, \ddot{\mathbf{u}}_{q-p^*}']'$ be its q-th row. Since its entries are defined as $\ddot{\mathbf{u}}_t = \boldsymbol{\Theta}^{-1}(L)\bar{\mathbf{u}}_t^*$, with $\boldsymbol{\Theta}^{-1}(L)$ a fixed and stable lag polynomial by Ass.5 such that $\ddot{\mathbf{u}}_t$ is stationary, it follows from the above that $\|\ddot{\mathbf{u}}_t\| = O_p(N^{-1/2})$ and $E(\boldsymbol{\xi}_q) = \mathbf{0}$. This in turn implies that $E\left\|\boldsymbol{\xi}_q\right\|^2 = \sum_{l=0}^{p^*} E(\ddot{\mathbf{u}}_{q-l}'\ddot{\mathbf{u}}_{q-l}) \leq O(N^{-1})$, which establishes that $\left\|\boldsymbol{\xi}_q\right\| = O_p(N^{-1/2})$ and $\left\|\ddot{\mathbf{U}}\right\| = O_p(N^{-1/2})$. Combining this result with eqs.(B-19), (B-20) and (B-22) gives

$$\begin{aligned}
\left\|\bar{\mathbf{U}}_m\right\| &\leq \left\|\ddot{\mathbf{U}}\right\| \|\mathbf{R}_0\| \|\mathbf{T}\| \|\mathbf{B}_m\| = O_p(N^{-1/2}), \\
\left\|\bar{\mathbf{U}}_m^0\right\| &\leq \left\|\ddot{\mathbf{U}}\right\| \|\mathbf{R}\| \|\mathbf{N}\| \|\mathbf{S}_m\| = O_p(N^{-1/2}), \\
\left\|\bar{\mathbf{U}}_{-m}^0\right\| &\leq \sqrt{N}\left\|\ddot{\mathbf{U}}\right\| \|\mathbf{R}\| \|\mathbf{N}\| \|\mathbf{S}_{-m}\| = O_p(1),
\end{aligned}$$

which ends the proof.

C.2.3 Proof of Lemma 3

Recall that $\mathbf{Q} = \check{\mathbf{F}}\ddot{\mathbf{P}} + \ddot{\mathbf{U}}$ is a $T \times c$ real stochastic matrix with $T \geq c$ and $\ddot{\mathbf{U}} = O_p(N^{-1/2})$ by Lemma 2. Let r be the rank of $\mathbf{Q}$ and note that $r_0 = rk(\check{\mathbf{F}}\ddot{\mathbf{P}}) \leq r$ depending on m and k. Despite that $r_0 \leq r$, Feng and Zhang (2007) show that $r \overset{a.s.}{\to} c$ as $N \to \infty$ irrespective of r_0 (also see Karabiyik et al., 2017). Accordingly, $rk(\mathbf{H}) \overset{a.s.}{\to} c$ with $N \to \infty$ such that, by the property $rk(\mathbf{H}) = tr(\mathbf{H})$ of idempotent matrices, also $tr(\mathbf{H}) \overset{a.s.}{\to} c$. Consider next the matrix norm of $\mathbf{H}$. Given the above

$$\|\mathbf{H}\| = \sqrt{tr\left(\mathbf{H}\mathbf{H}'\right)} = \sqrt{tr\left(\mathbf{H}\right)} = \sqrt{c}, \tag{C-15}$$

and therefore $\mathbf{H}$ is bounded for any N irrespective of r_0 since c does not depend on N.

C.2.4 Proof of Lemma 4

Suppose that $p = 0$, $m = 1$ and write the one period lag of (1) as

$$\begin{aligned}
(1-\rho L)\mathrm{y}_{i,t-1} &= \alpha_i + \mathbf{x}_{i,t-1}'\boldsymbol{\beta} + \gamma_i \mathrm{f}_{t-1} + \varepsilon_{i,t-1}, \\
&= (\alpha_i + \mathbf{c}_{\mathrm{z,i}}'\boldsymbol{\beta}^*) + (\gamma_i + \boldsymbol{\beta}^{*\prime}\boldsymbol{\Gamma}_i')\mathrm{f}_{t-1} + (\varepsilon_{i,t-1} + \mathbf{v}_{i,t-1}'\boldsymbol{\beta}^*), \\
&= \alpha_i^* + \gamma_i^*\mathrm{f}_{t-1} + \varepsilon_{i,t-1}^*,
\end{aligned}$$

where $\mathbf{x}_{i,t-1}'\boldsymbol{\beta} = \mathbf{z}_{i,t-1}'\boldsymbol{\beta}^* = \mathbf{c}_{\mathrm{z,i}}'\boldsymbol{\beta}^* + \mathrm{f}_{t-1}\boldsymbol{\Gamma}_i\boldsymbol{\beta}^* + \mathbf{v}_{i,t-1}'\boldsymbol{\beta}^*$ was substituted in. Solve for f_t

$$\mathrm{f}_{t-1} = \frac{1}{\gamma_i^*}\left((1-\rho L)\mathrm{y}_{i,t-1} - \alpha_i^* - \varepsilon_{i,t-1}^*\right),$$

with $\gamma_i^* = \gamma_i + \boldsymbol{\beta}^{*\prime}\boldsymbol{\Gamma}_i'$ and multiply both sides with $(1-\rho L)^{-1}$

$$(1-\rho L)^{-1}\mathrm{f}_{t-1} = \frac{(1-\rho L)^{-1}}{\gamma_i^*}\left((1-\rho L)\mathrm{y}_{i,t-1} - \alpha_i^* - \varepsilon_{i,t-1}^*\right),$$
$$\mathrm{f}_{t-1}^+ = \frac{1}{\gamma_i^*}\left((\mathrm{y}_{i,t-1} - (1-\rho L)^{-1}\alpha_i^* - (1-\rho L)^{-1}\varepsilon_{i,t-1}^*\right),$$

where $\mathrm{f}_{t-1}^+ = (1-\rho L)^{-1}\mathrm{f}_{t-1}$. Next, averaging over i gives

$$\mathrm{f}_{t-1}^+ = \frac{1}{\bar{\gamma}^*}\left(\bar{\mathrm{y}}_{t-1} - \bar{\alpha}^*/(1-\rho) - (1-\rho L)^{-1}\bar{\varepsilon}_{t-1}^*\right),$$

where barred variables are averages and it follows from Lemma 2 that $(1-\rho L)^{-1}\bar{\varepsilon}_{t-1}^* = O_p(N^{-1/2})$. Given the above we can write $\mathbf{F}_{-1}^+ = (1-\rho L)^{-1}\mathbf{F}_{-1} = [\mathrm{f}_0^+, \ldots, \mathrm{f}_{T-1}^+]'$ using $\bar{\boldsymbol{\varepsilon}}_{-1}^{*+} = (1-\rho L)^{-1}[\bar{\varepsilon}_0^*, \ldots, \bar{\varepsilon}_{T-1}^*]'$ as

$$\mathbf{F}_{-1}^+ = \mathbf{Q}^*\begin{bmatrix}-\bar{\alpha}^* \\ 1-\rho\end{bmatrix}\frac{1}{\bar{\gamma}^*(1-\rho)} - \frac{\bar{\boldsymbol{\varepsilon}}_{-1}^{*+}}{\bar{\gamma}^*} = \mathbf{Q}^*\mathbf{P}^* + O_p\left(\frac{1}{\sqrt{N}}\right), \tag{C-16}$$

with $\mathbf{Q}^* = [\boldsymbol{\iota}_T, \bar{\mathbf{y}}_{-1}]$ and obvious definition for $\mathbf{P}^*$. Provided a constant and $\bar{\mathbf{y}}_{-1}$ are included in $\mathbf{Q}$, we have

$$\mathbf{M}\mathbf{F}_{-1}^+ = O_p(N^{-1/2}), \tag{C-17}$$

because in this case $\mathbf{M}\mathbf{Q}^* = \mathbf{0}$ by definition and $\mathbf{M}$ is bounded in norm by Lemma 3. Note that (C-17) does not go through in the multiple factor case or with $p > 0$ since, lagging (9) and multiplying both sides with $\rho(L)^{-1} = (1-\rho L)^{-1}$ yields

$$\mathbf{f}_{t-1}^+ = \left(\mathbf{C}'\mathbf{C}\right)^{-1}\mathbf{C}'\left(\begin{bmatrix}1 & -\rho(L)^{-1}(\boldsymbol{\beta}^*)' \\ 0 & \rho(L)^{-1}\boldsymbol{\lambda}(L)\end{bmatrix}\begin{bmatrix}\bar{\mathrm{y}}_{t-1} \\ \bar{\mathbf{z}}_{t-1}\end{bmatrix} - \rho(L)^{-1}\begin{bmatrix}\bar{\alpha} \\ \bar{\mathbf{c}}_{\mathrm{z}}\end{bmatrix}\right) + O_p(N^{-1/2}),$$

which shows that an infinite number of lags of $\bar{\mathbf{z}}_{t-1}$ are required to approximate $\mathbf{f}_t^+$.

C.2.5 Proof of Lemma 5

Let $\mathbf{A}^{\mathbf{F}} = \frac{1}{NT}\sum_{i=1}^N \mathbf{w}_i'\mathbf{M}\mathbf{F}\boldsymbol{\gamma}_i$. Since the rank condition holds by Ass.4 we have substituting in (B-23) and using $\boldsymbol{\gamma}_i = \boldsymbol{\gamma} + \boldsymbol{\eta}_i$ from Ass.3

$$\mathbf{A}^{\mathbf{F}} = -\frac{1}{NT}\sum_{i=1}^N \mathbf{w}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\gamma}_i = -\frac{1}{T}\bar{\mathbf{w}}'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\gamma} - \frac{1}{NT}\sum_{i=1}^N \mathbf{w}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i = -\frac{1}{NT}\sum_{i=1}^N \mathbf{w}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i,$$

since $\bar{\mathbf{w}} \subset \mathbf{Q}$ such that $\mathbf{M}\bar{\mathbf{w}} = \mathbf{0}_{T\times k_w}$. We next make use of $\mathbf{M} = \mathbf{I}_T - \mathbf{H}$ to write the matrix norm of $\mathbf{A}^{\mathbf{F}}$ as

$$\left\|\mathbf{A}^{\mathbf{F}}\right\| \leq \left\|\frac{1}{NT}\sum_{i=1}^N \mathbf{w}_i'\bar{\mathbf{U}}_m\boldsymbol{\eta}_i\right\| + \left\|\frac{1}{NT}\sum_{i=1}^N \mathbf{w}_i'\mathbf{H}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i\right\|. \tag{C-18}$$

Turning to the first term gives

$$\left\|\frac{1}{NT}\sum_{i=1}^{N}\mathbf{w}_i'\bar{\mathbf{U}}_m\boldsymbol{\eta}_i\right\| \leq \left\|\frac{1}{NT}\sum_{i=1}^{N}(\boldsymbol{\eta}_i'\otimes\mathbf{w}_i')\right\|\left\|\bar{\mathbf{U}}_m\right\| = O_p\left(\frac{1}{\sqrt{N}}\right),$$

since $\left\|\bar{\mathbf{U}}_m\right\| = O_p(N^{-1/2})$ by Lemma 2 and since substituting in $\mathbf{w}_i = \check{\mathbf{F}}\ddot{\mathbf{P}}_i\mathbf{S}_w + \boldsymbol{\epsilon}_i$ by (B-13) leads to

$$\left\|\frac{1}{NT}\sum_{i=1}^{N}(\boldsymbol{\eta}_i'\otimes\mathbf{w}_i')\right\| \leq \left\|\frac{1}{NT}\sum_{i=1}^{N}\left(\boldsymbol{\eta}_i'\otimes\mathbf{S}_w'\ddot{\mathbf{P}}_i'\check{\mathbf{F}}'\right)\right\| + \left\|\frac{1}{NT}\sum_{i=1}^{N}(\boldsymbol{\eta}_i'\otimes\boldsymbol{\epsilon}_i')\right\|,$$

$$= \left\|\frac{1}{NT}\sum_{i=1}^{N}\left(\boldsymbol{\eta}_i'\otimes\mathbf{S}_w'\ddot{\mathbf{P}}_i'\check{\mathbf{F}}'\right)\right\| + O_p\left(\frac{1}{\sqrt{N}}\right) = O_p(1), \tag{C-19}$$

because $\boldsymbol{\epsilon}_i$ and $\boldsymbol{\eta}_i$ are independent and loadings are i.i.d. with bounded fourth moments by Ass.3. For the second term, we find with (C-19),

$$\left\|\frac{1}{NT}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{H}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i\right\| \leq \left\|\frac{1}{NT}\sum_{i=1}^{N}(\boldsymbol{\eta}_i'\otimes\mathbf{w}_i')\right\|\|\mathbf{H}\|\left\|\bar{\mathbf{U}}_m\right\| = O_p\left(\frac{1}{\sqrt{N}}\right), \tag{C-20}$$

since $\|\mathbf{H}\|$ is bounded by Lemma 3. Combining results in (C-18) gives

$$\left\|\mathbf{A}^{\mathbf{F}}\right\| = O_p(N^{-1/2}),$$

which proves that in general $\left\|\mathbf{A}^{\mathbf{F}}\right\| = O_p(N^{\omega})$ with $\omega = -1/2$.

It remains to show that $\omega = -1$ when $m = 1$ and $p = 0$. Write $\mathbf{A}^{\mathbf{F}}$ explicitly as

$$\mathbf{A}^{\mathbf{F}} = -\frac{1}{N}\sum_{i=1}^{N}\frac{\mathbf{w}_i'\mathbf{M}\bar{\mathbf{U}}_m}{T}\boldsymbol{\eta}_i = -\frac{1}{NT}\sum_{i=1}^{N}\begin{bmatrix}\mathbf{y}_{i,-1}'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i\\ \mathbf{X}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i\end{bmatrix}. \tag{C-21}$$

Suppose that $m = 1$, $p = 0$. We can then write $\mathbf{M}\mathbf{y}_{i,-1}$ more explicitly by inverting eq.(6) and employing (C-16) of Lemma 4

$$\mathbf{M}\mathbf{y}_{i,-1} = \mathbf{M}\left(\mathbf{F}_{-1}^{+}\gamma_i + \mathbf{X}_{i,-1}^{+}\boldsymbol{\beta} + \boldsymbol{\varepsilon}_{i,-1}^{+}\right) = \mathbf{M}\left(\mathbf{X}_{i,-1}^{+}\boldsymbol{\beta} + \boldsymbol{\varepsilon}_{i,-1}^{+} - \frac{\gamma_i}{\bar{\gamma}^*}\bar{\boldsymbol{\varepsilon}}_{-1}^{*+}\right), \tag{C-22}$$

and since $p = 0$ (no dynamics in $\mathbf{z}_{it}$) we can also write (3) in matrix notation as

$$\mathbf{Z}_i = [\mathbf{X}_i, \mathbf{G}_i] = \boldsymbol{\iota}_T\mathbf{c}_{z,i}' + \mathbf{F}\boldsymbol{\Gamma}_i + \mathbf{V}_i,$$

where $\mathbf{V}_i = [\mathbf{v}_{i1}, \ldots, \mathbf{v}_{iT}]'$. Defining $\mathbf{S}_x = [\mathbf{I}_{k_x}, \mathbf{0}_{k_x\times k_g}]'$ as the matrix selecting $\mathbf{X}_i$ from $\mathbf{Z}_i$ and substituting in (B-23) gives

$$\mathbf{M}\mathbf{X}_i = \mathbf{M}\mathbf{Z}_i\mathbf{S}_x = \mathbf{M}(\mathbf{F}\boldsymbol{\Gamma}_i + \mathbf{V}_i)\mathbf{S}_x = \mathbf{M}(\mathbf{V}_i - \bar{\mathbf{U}}_m\boldsymbol{\Gamma}_i)\mathbf{S}_x. \tag{C-23}$$

Similarly, from (C-16) in Lemma 4

$$\mathbf{M}\mathbf{X}_{i,-1}^{+} = \mathbf{M}\rho(L)^{-1}\mathbf{Z}_{i,-1}\mathbf{S}_x = \mathbf{M}(\mathbf{F}_{-1}^{+}\boldsymbol{\Gamma}_i + \mathbf{V}_{i,-1}^{+})\mathbf{S}_x = \mathbf{M}(\mathbf{V}_{i,-1}^{+} - \bar{\gamma}^{*-1}\bar{\boldsymbol{\varepsilon}}_{-1}^{*+}\boldsymbol{\Gamma}_i)\mathbf{S}_x. \tag{C-24}$$

Consider the first row of (C-21), substituting in (C-22) gives

$$\frac{1}{NT}\sum_{i=1}^{N}\mathbf{y}_{i,-1}'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i = \frac{1}{NT}\sum_{i=1}^{N}\left(\boldsymbol{\beta}'\mathbf{X}_{i,-1}^{+\prime} + \boldsymbol{\varepsilon}_{i,-1}^{+\prime} - \frac{\gamma_i}{\bar{\gamma}^*}\bar{\boldsymbol{\varepsilon}}_{-1}^{*+\prime}\right)\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i, \tag{C-25}$$

where since $\bar{\mathbf{U}}_m$ and $\bar{\boldsymbol{\varepsilon}}_{-1}^{+*}$ are $O_p(N^{-1/2})$ and loadings and errors are independent

$$\left\|\frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\varepsilon}_{i,-1}^{+\prime}\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i\right\| \leq \frac{1}{T}\left\|\frac{1}{N}\sum_{i=1}^{N}\left(\boldsymbol{\eta}_i'\otimes\boldsymbol{\varepsilon}_{i,-1}^{+\prime}\right)\right\|\|\mathbf{M}\|\left\|\bar{\mathbf{U}}_m\right\| = O_p(N^{-1}),$$

$$\left\|\frac{1}{NT}\sum_{i=1}^{N}\frac{\gamma_i}{\bar{\gamma}^*}\bar{\boldsymbol{\varepsilon}}_{-1}^{+*\prime}\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i\right\| \leq \frac{1}{T}\left\|\frac{1}{N}\sum_{i=1}^{N}\left(\boldsymbol{\eta}_i'\otimes\frac{\gamma_i}{\bar{\gamma}^*}\right)\right\|\left\|\bar{\boldsymbol{\varepsilon}}_{-1}^{+*}\right\|\|\mathbf{M}\|\left\|\bar{\mathbf{U}}_m\right\| = O_p(N^{-1}),$$

and we find for first term of (C-25), after substituting in (C-24),

$$\frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\beta}'\mathbf{S}_x'\left(\mathbf{V}_{i,-1}^{+\prime} - \bar{\gamma}^{*-1}\boldsymbol{\Gamma}_i'\bar{\boldsymbol{\varepsilon}}_{-1}^{*+\prime}\right)\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i = O_p(N^{-1}),$$

because

$$\left\|\frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\beta}'\mathbf{S}_x'\mathbf{V}_{i,-1}^{+\prime}\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i\right\| \leq \frac{1}{T}\left\|\frac{1}{N}\sum_{i=1}^{N}\left(\boldsymbol{\eta}_i'\otimes\boldsymbol{\beta}'\mathbf{S}_x'\mathbf{V}_{i,-1}^{+\prime}\right)\right\|\|\mathbf{M}\|\left\|\bar{\mathbf{U}}_m\right\| = O_p\left(\frac{1}{N}\right),$$

$$\left\|\frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\beta}'\mathbf{S}_x'\frac{\boldsymbol{\Gamma}_i'}{\bar{\gamma}^*}\bar{\boldsymbol{\varepsilon}}_{-1}^{+*\prime}\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i\right\| \leq \frac{1}{T}\left\|\frac{1}{N}\sum_{i=1}^{N}\left(\boldsymbol{\eta}_i'\otimes\boldsymbol{\beta}'\mathbf{S}_x'\frac{\boldsymbol{\Gamma}_i'}{\bar{\gamma}^*}\right)\right\|\left\|\bar{\boldsymbol{\varepsilon}}_{-1}^{+*}\right\|\|\mathbf{M}\|\left\|\bar{\mathbf{U}}_m\right\| = O_p\left(\frac{1}{N}\right),$$

where we note that the last bound can be sharpened to $O_p(N^{-3/2})$ when $\boldsymbol{\gamma}_i$ and $\boldsymbol{\Gamma}_i$ are independent. Regardless, combining results in (C-25) gives

$$\frac{1}{NT}\sum_{i=1}^{N}\mathbf{y}_{i,-1}'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i = O_p(N^{-1}). \tag{C-26}$$

For rows 2 to k_w of (C-21) we find, after substituting in (C-23) and using similar arguments as before

$$\frac{1}{NT}\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i = \frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}_x'(\mathbf{V}_i' - \boldsymbol{\Gamma}_i'\bar{\mathbf{U}}_m')\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i = O_p(N^{-1}). \tag{C-27}$$

Combining (C-26)-(C-27) in (C-21) leads to $\mathbf{A}^{\mathbf{F}} = O_p(N^{\omega})$ with $\omega = -1$, as required.

C.2.6 Proof of Lemma 6

Recall from eq.(B-13) that $\mathbf{w}_i = \check{\mathbf{F}}\ddot{\mathbf{P}}_i\mathbf{S}_w + \boldsymbol{\epsilon}_i$ with $\mathbf{S}_w$ the selector matrix defined in (B-11) and $\check{\mathbf{F}}$, $\ddot{\mathbf{P}}_i$ and $\boldsymbol{\epsilon}_i$ are defined in eq.(B-5), (B-6) and (B-14) respectively. Let $\boldsymbol{\vartheta}_{i,s}$ be the s-th column of $\mathbf{w}_i$ and note that by Ass.1-3 and 5 the $\ddot{\mathbf{P}}_i$, $\boldsymbol{\epsilon}_i$ and $\check{\mathbf{F}}$ are independent and stationary with finite variance such that $\boldsymbol{\vartheta}_{i,s} = O_p(1)$ for every i and s and $\|\boldsymbol{\vartheta}_{i,s}\| = O_p(\sqrt{T})$. Consider the matrix $\widehat{\boldsymbol{\Sigma}} = \sum_{i=1}^{N}\mathbf{w}_i'\mathbf{M}\mathbf{w}_i/NT$ and note that element s on its diagonal is $\frac{1}{NT}\sum_{i=1}^{N}\|\mathbf{M}\boldsymbol{\vartheta}_{i,s}\|^2 = O_p(1)$, since $\|\mathbf{M}\boldsymbol{\vartheta}_{i,s}\| \leq \|\boldsymbol{\vartheta}_{i,s}\| = O_p(\sqrt{T})$ for all i and s. Using the same argument we have for the off-diagonal element on row s and column $s' \neq s$

$$\left\|\frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\vartheta}'_{i,s}\mathbf{M}\boldsymbol{\vartheta}_{i,s'}\right\| \leq \frac{1}{NT}\sum_{i=1}^{N}\left\|\boldsymbol{\vartheta}'_{i,s}\mathbf{M}\boldsymbol{\vartheta}_{i,s'}\right\| \leq \frac{1}{NT}\sum_{i=1}^{N}\|\mathbf{M}\boldsymbol{\vartheta}_{i,s}\|\,\|\mathbf{M}\boldsymbol{\vartheta}_{i,s'}\| = O_p(1),$$

such that $\widehat{\boldsymbol{\Sigma}} = O_p(1)$ and the lemma is proved.

C.2.7 Proof of Lemma 7

Consider the following decomposition of $\widehat{\boldsymbol{\Sigma}}$ obtained by substituting in eq.(B-13)

$$\begin{aligned}\widehat{\boldsymbol{\Sigma}} = \frac{1}{NT}\sum_{i=1}^{N}\mathbf{w}'_i\mathbf{M}\mathbf{w}_i =& \frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}'_w\ddot{\mathbf{P}}'_i\check{\mathbf{F}}'\mathbf{M}\check{\mathbf{F}}\ddot{\mathbf{P}}_i\mathbf{S}_w + \frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\epsilon}'_i\mathbf{M}\boldsymbol{\epsilon}_i \\ &+ \frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}'_w\ddot{\mathbf{P}}'_i\check{\mathbf{F}}'\mathbf{M}\boldsymbol{\epsilon}_i + \frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\epsilon}'_i\mathbf{M}\check{\mathbf{F}}\ddot{\mathbf{P}}_i\mathbf{S}_w.\end{aligned}$$

By Ass.1 and 3, the $\boldsymbol{\epsilon}_i$ and $\ddot{\mathbf{P}}_i$ are independent of each other and over i such that

$$\left\|\frac{1}{N}\sum_{i=1}^{N}\mathbf{S}'_w\ddot{\mathbf{P}}'_i\check{\mathbf{F}}'\mathbf{M}\boldsymbol{\epsilon}_i\right\| \leq \left\|\frac{1}{N}\sum_{i=1}^{N}\left(\boldsymbol{\epsilon}'_i\otimes\mathbf{S}'_w\ddot{\mathbf{P}}'_i\right)\right\|\left\|\check{\mathbf{F}}\right\|\|\mathbf{M}\| = O_p(N^{-1/2}),$$

which also uses Lemma 3. Substituting in this result and noting that the summation operates only on $\boldsymbol{\epsilon}_i$ and $\ddot{\mathbf{P}}_i$ we can write

$$vec(\widehat{\boldsymbol{\Sigma}}) = T^{-1}\left[\widehat{\boldsymbol{\Sigma}}_\epsilon + (\mathbf{S}'_w\otimes\mathbf{S}'_w)\,\widehat{\boldsymbol{\Sigma}}_{\ddot{\mathbf{P}}}(\check{\mathbf{F}}'\otimes\check{\mathbf{F}}')\right]vec\,(\mathbf{M}) + O_p(N^{-1/2}),$$

where $\widehat{\boldsymbol{\Sigma}}_\epsilon = \frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{\epsilon}'_i\otimes\boldsymbol{\epsilon}'_i)$ and $\widehat{\boldsymbol{\Sigma}}_{\ddot{\mathbf{P}}} = \frac{1}{N}\sum_{i=1}^{N}(\ddot{\mathbf{P}}'_i\otimes\ddot{\mathbf{P}}'_i)$. For the latter, since by Ass.1 and 3 the $\boldsymbol{\epsilon}_i$ and $\ddot{\mathbf{P}}_i$ are independent over i with bounded moments up to the fourth order

$$\begin{aligned}\widehat{\boldsymbol{\Sigma}}_\epsilon &= E\left(\boldsymbol{\epsilon}'_i\otimes\boldsymbol{\epsilon}'_i\right) + O_p(N^{-1/2}) = \boldsymbol{\Sigma}_\epsilon + O_p(N^{-1/2}),\\ \widehat{\boldsymbol{\Sigma}}_{\ddot{\mathbf{P}}} &= E(\ddot{\mathbf{P}}'_i\otimes\ddot{\mathbf{P}}'_i) + O_p(N^{-1/2}) = \boldsymbol{\Sigma}_{\ddot{\mathbf{P}}} + O_p(N^{-1/2}),\end{aligned}$$

with $\boldsymbol{\Sigma}_{\ddot{\mathbf{P}}} = E(\ddot{\mathbf{P}}'_i\otimes\ddot{\mathbf{P}}'_i)$ and $\boldsymbol{\Sigma}_\epsilon = E(\boldsymbol{\epsilon}'_i\otimes\boldsymbol{\epsilon}'_i)$. Therefore, matricising $vec(\widehat{\boldsymbol{\Sigma}})$ yields

$$\widehat{\boldsymbol{\Sigma}} = (vec(\mathbf{I}_{k_w})'\otimes\mathbf{I}_{k_w})\left(\mathbf{I}_{k_w}\otimes\left[\boldsymbol{\Sigma}_\epsilon + (\mathbf{S}'_w\otimes\mathbf{S}'_w)\,\boldsymbol{\Sigma}_{\ddot{\mathbf{P}}}(\check{\mathbf{F}}'\otimes\check{\mathbf{F}}')\right]T^{-1}vec\,(\mathbf{M})\right) + O_p(N^{-1/2}),$$

which is the result stated in the lemma.

C.2.8 Proof of Lemma 8

Consider the estimator $\widehat{\sigma}^2_\varepsilon(\cdot)$ defined in equation (20) evaluated at $\boldsymbol{\delta}_0 \neq \boldsymbol{\delta}$, with $\boldsymbol{\delta} = [\rho, \boldsymbol{\beta}']'$ the true parameter vector. Suppose that $p^* \geq p$ and Ass.1-5 hold. We can then make use of eqs.(6) and (B-23) to obtain

$$\widehat{\sigma}^2_\varepsilon(\boldsymbol{\delta}_0) = \frac{1}{N(T-c)}\sum_{i=1}^{N}\|\mathbf{M}(\mathbf{y}_i - \mathbf{w}_i\boldsymbol{\delta}_0)\|^2 = \frac{1}{N(T-c)}\sum_{i=1}^{N}\|\mathbf{M}(\mathbf{w}_i(\boldsymbol{\delta}-\boldsymbol{\delta}_0) + \mathbf{F}\boldsymbol{\gamma}_i + \boldsymbol{\varepsilon}_i)\|^2,$$

$$
\begin{aligned}
=&\frac{1}{N(T-c)}\sum_{i=1}^{N}\left\|\mathbf{M}\left(\mathbf{w}_i(\boldsymbol{\delta}-\boldsymbol{\delta}_0)-\bar{\mathbf{U}}_m\boldsymbol{\gamma}_i+\boldsymbol{\varepsilon}_i\right)\right\|^2,\\
=&\frac{1}{N(T-c)}\sum_{i=1}^{N}\left\|\mathbf{M}\left(\mathbf{w}_i(\boldsymbol{\delta}-\boldsymbol{\delta}_0)+\boldsymbol{\varepsilon}_i\right)\right\|^2+O_p(N^{-1/2}),\\
=&\frac{T}{T-c}(\boldsymbol{\delta}-\boldsymbol{\delta}_0)'\widehat{\boldsymbol{\Sigma}}(\boldsymbol{\delta}-\boldsymbol{\delta}_0)+\frac{2}{N(T-c)}\sum_{i=1}^{N}(\boldsymbol{\delta}-\boldsymbol{\delta}_0)'\mathbf{w}_i'\mathbf{M}\boldsymbol{\varepsilon}_i\\
&+\frac{1}{N(T-c)}\sum_{i=1}^{N}\boldsymbol{\varepsilon}_i'\mathbf{M}\boldsymbol{\varepsilon}_i+O_p(N^{-1/2}), \qquad \text{(C-28)}
\end{aligned}
$$

since we have $\left\|\frac{1}{N}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\gamma}_i\right\| = \left\|\frac{1}{N}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i\right\| = O_p(N^{-1/2})$ as proved in Lemma 5 such that for any $\|\boldsymbol{\delta}-\boldsymbol{\delta}_0\|<\infty$,

$$
\left\|\frac{1}{N}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i(\boldsymbol{\delta}-\boldsymbol{\delta}_0)\right\| \le \left\|\frac{1}{N}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i\right\|\|\boldsymbol{\delta}-\boldsymbol{\delta}_0\| = O_p\left(\frac{1}{\sqrt{N}}\right),
$$

and because

$$
\left\|\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\varepsilon}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\gamma}_i\right\| \le \left\|\frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{\gamma}_i'\otimes\boldsymbol{\varepsilon}_i')\right\|\left\|\bar{\mathbf{U}}_m\right\|\|\mathbf{M}\| = O_p\left(\frac{1}{N}\right), \qquad \text{(C-29)}
$$

$$
\left\|\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\gamma}_i'\bar{\mathbf{U}}_m'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\gamma}_i\right\| \le \left\|\frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{\gamma}_i'\otimes\boldsymbol{\gamma}_i')\right\|\left\|\bar{\mathbf{U}}_m\right\|^2\|\mathbf{M}\| = O_p\left(\frac{1}{N}\right), \qquad \text{(C-30)}
$$

due to $\left\|\frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{\gamma}_i'\otimes\boldsymbol{\gamma}_i')\right\| = O_p(1)$ and $\left\|\frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{\gamma}_i'\otimes\boldsymbol{\varepsilon}_i')\right\| = O_p(N^{-1/2})$ by Ass.1 and 3, $\left\|\bar{\mathbf{U}}_m\right\| = O_p(N^{-1/2})$ by Lemma 2 and $\|\mathbf{M}\| = O(1)$ by Lemma 3. Next, we take the first two remaining terms in (C-28) individually as $N\to\infty$,

$$
\frac{T}{T-c}(\boldsymbol{\delta}-\boldsymbol{\delta}_0)'\widehat{\boldsymbol{\Sigma}}(\boldsymbol{\delta}-\boldsymbol{\delta}_0) = c_2(\boldsymbol{\delta}-\boldsymbol{\delta}_0)'\boldsymbol{\Sigma}(\boldsymbol{\delta}-\boldsymbol{\delta}_0)+O_p(N^{-1/2}), \qquad \text{(C-31)}
$$

$$
\frac{2}{T-c}\frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{\delta}-\boldsymbol{\delta}_0)'\mathbf{w}_i'\mathbf{M}\boldsymbol{\varepsilon}_i = -c_1\sigma_\varepsilon^2\boldsymbol{\upsilon}(\rho,\mathbf{H})'(\boldsymbol{\delta}-\boldsymbol{\delta}_0)+O_p(N^{-1/2}), \qquad \text{(C-32)}
$$

where $c_1 = \frac{2}{T-c}$ and $c_2 = \frac{T}{T-c}$. The first result follows from Lemma 7 and the second from Theorem 1. Also, letting $h_{t,s}$ denote the element on row t and column s of $\mathbf{H}$, $\widetilde{c} = T-c$ and with $\bar{\varepsilon}_{t,s} = \frac{1}{N}\sum_{i=1}^{N}\varepsilon_{i,t}\varepsilon_{i,s}$,

$$
\begin{aligned}
\frac{1}{T-c}\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\varepsilon}_i'\mathbf{M}\boldsymbol{\varepsilon}_i &= \widetilde{c}^{-1}\frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}\varepsilon_{i,t}^2-\widetilde{c}^{-1}\frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}h_{t,s}\varepsilon_{i,t}\varepsilon_{i,s},\\
&= \widetilde{c}^{-1}\sum_{t=1}^{T}\bar{\varepsilon}_{t,t}-\widetilde{c}^{-1}\sum_{t=1}^{T}\sum_{s=1}^{T}h_{t,s}\bar{\varepsilon}_{t,s},\\
&= \widetilde{c}^{-1}\sum_{t=1}^{T}\bar{\varepsilon}_{t,t}-\widetilde{c}^{-1}\sum_{t=1}^{T}h_{t,t}\bar{\varepsilon}_{t,t}+O_p(N^{-1/2}),
\end{aligned}
$$

$$= \widetilde{c}^{-1}\left[T\sigma_\varepsilon^2 + \sum_{t=1}^{T}\left(\bar{\varepsilon}_{t,t} - \sigma_\varepsilon^2\right)\right] - \widetilde{c}^{-1}\left[\sigma_\varepsilon^2 \sum_{t=1}^{T} h_{t,t} + \sum_{t=1}^{T} h_{t,t}\left(\bar{\varepsilon}_{t,t} - \sigma_\varepsilon^2\right)\right] + O_p(N^{-1/2}),$$

$$= \widetilde{c}^{-1}\sigma_\varepsilon^2\left(T - \sum_{t=1}^{T} h_{t,t}\right) + O_p(N^{-1/2}) = \sigma_\varepsilon^2 \widetilde{c}^{-1}\widetilde{c} + O_p(N^{-1/2}),$$

$$= \sigma_\varepsilon^2 + O_p(N^{-1/2}), \tag{C-33}$$

since $\sum_{t=1}^{T} h_{t,t} = tr(\mathbf{H}) = c$ and by Ass.1 $\bar{\varepsilon}_{t,s} = O_p(N^{-1/2})$ for $t \neq s$ and $\bar{\varepsilon}_{t,t} = \sigma_\varepsilon^2 + O_p(N^{-1/2})$. Combining results gives

$$\widehat{\sigma}_\varepsilon^2\left(\boldsymbol{\delta}_0\right) = \sigma_\varepsilon^2 - \sigma_\varepsilon^2 c_1 \boldsymbol{\upsilon}(\rho, \mathbf{H})'(\boldsymbol{\delta} - \boldsymbol{\delta}_0) + c_2(\boldsymbol{\delta} - \boldsymbol{\delta}_0)'\boldsymbol{\Sigma}(\boldsymbol{\delta} - \boldsymbol{\delta}_0) + O_p(N^{-1/2}). \tag{C-34}$$

This proves (C-8).

Finally, evaluating eq.(20) at $\boldsymbol{\delta}_0 = \boldsymbol{\delta}$ we have, employing again (B-23) in $\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta})$,

$$\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta}) = \frac{1}{N(T-c)}\sum_{i=1}^{N}\left\|\mathbf{M}(\mathbf{w}_i(\boldsymbol{\delta} - \boldsymbol{\delta}) + \mathbf{F}\boldsymbol{\gamma}_i + \boldsymbol{\varepsilon}_i)\right\|^2 = \frac{1}{N(T-c)}\sum_{i=1}^{N}\left\|\mathbf{M}(\boldsymbol{\varepsilon}_i - \bar{\mathbf{U}}_m\boldsymbol{\gamma}_i)\right\|^2,$$

$$= \frac{1}{N(T-c)}\sum_{i=1}^{N}\boldsymbol{\varepsilon}_i'\mathbf{M}\boldsymbol{\varepsilon}_i - 2\frac{1}{N(T-c)}\sum_{i=1}^{N}\boldsymbol{\varepsilon}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\gamma}_i + \frac{1}{N(T-c)}\sum_{i=1}^{N}\boldsymbol{\gamma}_i'\bar{\mathbf{U}}_m'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\gamma}_i,$$

which makes, by (C-29) and (C-30),

$$\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta}) = \frac{1}{N(T-c)}\sum_{i=1}^{N}\boldsymbol{\varepsilon}_i'\mathbf{M}\boldsymbol{\varepsilon}_i + O_p(N^{-1}),$$

and proves (C-9) of the lemma. Finally, eq.(C-10) in the lemma follows directly from (C-34) evaluated at $\boldsymbol{\delta}_0 = \boldsymbol{\delta}$ and letting $N \to \infty$.

C.3 Statement of theorems

Below we state the theorems that are not presented in the main text or in section A.2.

Theorem 3. *Let* $\boldsymbol{\phi}(\cdot) = \widehat{\boldsymbol{\delta}} - \widehat{\mathbf{m}}(\cdot)$, $\widetilde{\boldsymbol{\phi}}(\cdot) = \lim_{N\to\infty}\boldsymbol{\phi}(\cdot)$ *and suppose that* $p^* \geq p$ *and Ass.1-5 hold. Assuming that* $\widetilde{\boldsymbol{\phi}}(\boldsymbol{\delta}_0) = \mathbf{0}$ *implies* $\boldsymbol{\delta}_0 = \boldsymbol{\delta}$, *and that* $\chi \subseteq \mathbb{R}^{k_w}$ *is compact with* $\boldsymbol{\delta} \in \chi$,

$$\widehat{\boldsymbol{\delta}}_{bc} \longrightarrow^p \boldsymbol{\delta} \qquad as \qquad N \to \infty,$$

with $\widehat{\boldsymbol{\delta}}_{bc}$ *defined in eq.(21).*

C.4 Proof of theorems and corollaries

C.4.1 Proof of Theorem 1

The CCEP estimator for $\boldsymbol{\delta}$ defined in (14) is

$$\widehat{\boldsymbol{\delta}} = \left(\frac{1}{N}\sum_{i=1}^{N}\frac{\mathbf{w}_i'\mathbf{M}\mathbf{w}_i}{T}\right)^{-1}\frac{1}{N}\sum_{i=1}^{N}\frac{\mathbf{w}_i'\mathbf{M}\mathbf{y}_i}{T}.$$

Substituting in eq.(6) gives

$$\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta} = \widehat{\boldsymbol{\Sigma}}^{-1}\left(\mathbf{A}^{\varepsilon} + \mathbf{A}^{\mathbf{F}}\right) = \widehat{\boldsymbol{\Sigma}}^{-1}\mathbf{A}^{\varepsilon} + O_p(N^{-1/2}), \tag{C-35}$$

where $\mathbf{A}^{\varepsilon} = \frac{1}{NT}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{M}\boldsymbol{\varepsilon}_i$ and because $\widehat{\boldsymbol{\Sigma}} = \frac{1}{NT}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{M}\mathbf{w}_i = O_p(1)$ by Lemma 6 and $\mathbf{A}^{\mathbf{F}} = \frac{1}{NT}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{M}\mathbf{F}\boldsymbol{\gamma}_i = O_p(N^{-1/2})$ by Lemma 5. Substituting in (B-13), we can decompose $\mathbf{A}^{\varepsilon}$ as

$$\mathbf{A}^{\varepsilon} = \frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}_w'\ddot{\mathbf{P}}_i'\check{\mathbf{F}}'\mathbf{M}\boldsymbol{\varepsilon}_i + \frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\epsilon}_i'\mathbf{M}\boldsymbol{\varepsilon}_i,$$

where by the independence of $\boldsymbol{\varepsilon}_i$ and $\ddot{\mathbf{P}}_i$ by Ass.1 and 3, and Lemma 3

$$\left\|\frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}_w'\ddot{\mathbf{P}}_i'\check{\mathbf{F}}'\mathbf{M}\boldsymbol{\varepsilon}_i\right\| \leq \frac{1}{T}\left\|\frac{1}{N}\sum_{i=1}^{N}\left(\boldsymbol{\varepsilon}_i'\otimes\mathbf{S}_w'\ddot{\mathbf{P}}_i'\right)\right\|\left\|\check{\mathbf{F}}\right\|\|\mathbf{M}\| = O_p(N^{-1/2}).$$

Next, note that we can write, with $h_{t,s}$ denoting the element on row t and column s of $\mathbf{H}$, and with $\bar{\boldsymbol{\epsilon}}_{t,s} = \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\epsilon}_{it}\varepsilon_{is}$,

$$\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\epsilon}_i'\mathbf{M}\boldsymbol{\varepsilon}_i = \sum_{t=1}^{T}\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\epsilon}_{it}\varepsilon_{it} - \sum_{t=1}^{T}\sum_{s=1}^{T}h_{t,s}\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\epsilon}_{it}\varepsilon_{is} = \sum_{t=1}^{T}\bar{\boldsymbol{\epsilon}}_{t,t} - \sum_{t=1}^{T}\sum_{s=1}^{T}h_{t,s}\bar{\boldsymbol{\epsilon}}_{t,s},$$

where making use of (B-14) and Ass.1 and 5, for all t and s

$$\left[\bar{\boldsymbol{\epsilon}}_{t,s} - \sigma_{\varepsilon}^2\mathbf{q}_1\rho^{t-1-s}\mathbb{1}_{(t-1\geq s)}\right] = O_p(N^{-1/2}),$$

with $\mathbf{q}_1 = [1, \mathbf{0}_{k_x\times 1}']'$ and $\mathbb{1}_a$ is the indicator function returning 1 if the condition a is true, and zero otherwise. This gives, since by Lemma 3 all $h_{t,s}$ are bounded and $\bar{\boldsymbol{\epsilon}}_{t,t} = O_p(N^{-1/2})$,

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\epsilon}_i'\mathbf{M}\boldsymbol{\varepsilon}_i &= \sum_{t=1}^{T}\bar{\boldsymbol{\epsilon}}_{t,t} - \sum_{t=1}^{T}\sum_{s=1}^{T}h_{t,s}\left[\bar{\boldsymbol{\epsilon}}_{t,s} - \sigma_{\varepsilon}^2\mathbf{q}_1\rho^{t-1-s}\mathbb{1}_{(t-1\geq s)}\right] - \sigma_{\varepsilon}^2\mathbf{q}_1\sum_{t=1}^{T}\sum_{s=1}^{T}h_{t,s}\rho^{t-1-s}\mathbb{1}_{(t-1\geq s)}, \\ &= -\sigma_{\varepsilon}^2\mathbf{q}_1\sum_{t=1}^{T}\sum_{s=1}^{T}h_{t,s}\rho^{t-1-s}\mathbb{1}_{(t-1\geq s)} + O_p(N^{-1/2}),\end{aligned}$$

and in turn leads to the conclusion

$$\operatorname*{plim}_{N\to\infty}\mathbf{A}^{\varepsilon} = -T^{-1}\sigma_{\varepsilon}^2\mathbf{q}_1\sum_{t=1}^{T}\sum_{s=1}^{T}h_{t,s}\rho^{t-1-s}\mathbb{1}_{(t-1\geq s)} = -T^{-1}\sigma_{\varepsilon}^2\boldsymbol{\upsilon}(\rho, \mathbf{H}),$$

with $\boldsymbol{\upsilon}(\rho, \mathbf{H}) = \upsilon(\rho, \mathbf{H})\mathbf{q}_1$ and $\upsilon(\rho, \mathbf{H}) = \sum_{t=1}^{T-1}\rho^{t-1}\sum_{s=t+1}^{T}h_{s,s-t}$. Next up is the denominator. From Lemma 7,

$$\begin{aligned}\widehat{\boldsymbol{\Sigma}} &= (vec(\mathbf{I}_{k_w})'\otimes\mathbf{I}_{k_w})\left(\mathbf{I}_{k_w}\otimes\left[\boldsymbol{\Sigma}_{\epsilon} + (\mathbf{S}_w'\otimes\mathbf{S}_w')\,\boldsymbol{\Sigma}_{\ddot{\mathbf{P}}}(\check{\mathbf{F}}'\otimes\check{\mathbf{F}}')\right]T^{-1}vec\,(\mathbf{M})\right) + O_p(N^{-1/2}), \\ &= \boldsymbol{\Sigma} + O_p(N^{-1/2}),\end{aligned}$$

with $\boldsymbol{\Sigma}_{\ddot{\mathbf{P}}} = E(\ddot{\mathbf{P}}_i' \otimes \ddot{\mathbf{P}}_i')$ and $\boldsymbol{\Sigma}_{\epsilon} = E(\boldsymbol{\epsilon}_i' \otimes \boldsymbol{\epsilon}_i')$, which are all $O(1)$ terms by Ass.1, 3 and 5. Hence, combining results gives

$$\underset{N\to\infty}{\text{plim}}(\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta}) = -\frac{\sigma_\varepsilon^2}{T}\boldsymbol{\Sigma}^{-1}\boldsymbol{\upsilon}(\rho, \mathbf{H}), \tag{C-36}$$

which is the result stated in eq.(15). Equations (16)-(17) in Theorem 1 are a reformulation of (C-36) obtained by application of the Frisch-Waugh-Lovell theorem and defining $\boldsymbol{\zeta} = \text{plim}_{N\to\infty}(\breve{\mathbf{X}}'\breve{\mathbf{X}})^{-1}\breve{\mathbf{X}}'\mathbf{y}_{-1} = (\mathbf{S}_{\text{x}}'\boldsymbol{\Sigma}\mathbf{S}_{\text{x}})^{-1}\mathbf{S}_{\text{x}}'\boldsymbol{\Sigma}\mathbf{q}_1$ and $\sigma^2_{\breve{\mathbf{y}}_{-1}} = \text{plim}_{N\to\infty}\frac{\breve{\mathbf{y}}_{-1}'\breve{\mathbf{y}}_{-1}}{NT}$, with $\breve{\mathbf{y}}_{-1} = \mathbb{M}_x[\mathbf{y}_{1,-1}', \ldots, \mathbf{y}_{N,-1}']'$, $\mathbb{M}_x = \mathbb{M} - \breve{\mathbf{X}}(\breve{\mathbf{X}}'\breve{\mathbf{X}})^{-1}\breve{\mathbf{X}}'$, $\breve{\mathbf{X}} = \mathbb{M}[\mathbf{X}_1', \ldots, \mathbf{X}_N']'$ *and* $\mathbb{M} = \mathbf{I}_N \otimes \mathbf{M}$.

C.4.2 Proof for Corollary 1

It will be useful for the derivation of the explicit bias expression in eq.(A-5) to stack eq.(6) over individuals as

$$\mathbf{y} \;=\; (\mathbf{I}_N \otimes \boldsymbol{\iota}_T)\,\boldsymbol{\alpha} + \rho\mathbf{y}_{-1} + \mathbf{X}\boldsymbol{\beta} + \mathbb{F}\boldsymbol{\Lambda} + \boldsymbol{\varepsilon}, \tag{C-37}$$

with $\mathbb{F} = (\mathbf{I}_N \otimes \mathbf{F})$, $\mathbf{y} = [\mathbf{y}_1', \ldots, \mathbf{y}_N']'$, $\mathbf{X} = [\mathbf{X}_1', \ldots, \mathbf{X}_N']'$, $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_N]'$, $\boldsymbol{\Lambda} = [\boldsymbol{\gamma}_1', \ldots, \boldsymbol{\gamma}_N']'$ and $\boldsymbol{\varepsilon} = [\boldsymbol{\varepsilon}_1', \ldots, \boldsymbol{\varepsilon}_N']'$. With Ass.5 expression (C-37) can be inverted to get

$$\mathbf{y} \;=\; (\mathbf{I}_N \otimes \boldsymbol{\iota}_T)\,\boldsymbol{\alpha}^+ + \mathbf{X}^+\boldsymbol{\beta} + \mathbb{F}^+\boldsymbol{\Lambda} + \boldsymbol{\varepsilon}^+, \tag{C-38}$$

with $\mathbb{F}^+ = (\mathbf{I}_N \otimes \mathbf{F}^+)$ and variables with a + superscript defined as $\mathbf{X}^+ = (1-\rho L)^{-1}\mathbf{X}$. Using eq.(C-37) and the Frisch-Waugh-Lovell theorem, write the CCEP estimator as

$$\widehat{\rho} \;=\; (\mathbf{y}_{-1}'\mathbb{M}_{\text{X}}\mathbf{y}_{-1})^{-1}\mathbf{y}_{-1}'\mathbb{M}_{\text{X}}\mathbf{y}, \tag{C-39}$$

$$\widehat{\boldsymbol{\beta}} \;=\; (\mathbf{X}'\mathbb{M}\mathbf{X})^{-1}\,\mathbf{X}'\mathbb{M}\,(\mathbf{y} - \widehat{\rho}\mathbf{y}_{-1})\,, \tag{C-40}$$

with $\mathbb{M}_{\text{X}} = \mathbb{M}_x\mathbb{M}$, $\mathbb{M} = \mathbf{I}_N \otimes \mathbf{M}$ and $\mathbb{M}_x = \mathbf{I}_{NT} - \mathbb{M}\mathbf{X}\,(\mathbf{X}'\mathbb{M}\mathbf{X})^{-1}\,\mathbf{X}'\mathbb{M}$. Eq.(C-35) implies

$$\underset{N\to\infty}{\text{plim}}(\widehat{\rho} - \rho) = \underset{N\to\infty}{\text{plim}}\,\frac{\mathbf{y}_{-1}'\mathbb{M}_{\text{X}}\boldsymbol{\varepsilon}}{\mathbf{y}_{-1}'\mathbb{M}_{\text{X}}\mathbf{y}_{-1}}, \tag{C-41}$$

$$\underset{N\to\infty}{\text{plim}}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}) = \underset{N\to\infty}{\text{plim}}\,(\mathbf{X}'\mathbb{M}\mathbf{X})^{-1}\,\mathbf{X}'\mathbb{M}\mathbf{y}_{-1}\,(\rho - \widehat{\rho})\,, \tag{C-42}$$

such that, defining $\boldsymbol{\zeta} = \text{plim}_{N\to\infty}\,(\mathbf{X}'\mathbb{M}\mathbf{X})^{-1}\,\mathbf{X}'\mathbb{M}\mathbf{y}_{-1}$ we obtain for (C-42)

$$\underset{N\to\infty}{\text{plim}}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}) = -\boldsymbol{\zeta}\,\underset{N\to\infty}{\text{plim}}(\rho - \widehat{\rho}), \tag{C-43}$$

which is the expression in eq.(17).

Next, consider that lagging eq.(C-38) one period gives the following expression for $\mathbf{y}_{-1}$

$$\mathbf{y}_{-1} \;=\; (\mathbf{I}_N \otimes \boldsymbol{\iota}_T)\,\boldsymbol{\alpha}^+ + \mathbf{X}_{-1}^+\boldsymbol{\beta} + (\mathbf{I}_N \otimes \mathbf{F}_{-1}^+)\boldsymbol{\Lambda} + \boldsymbol{\varepsilon}_{-1}^+.$$

This leads to

$$\mathbb{M}\mathbf{y}_{-1} \;=\; \mathbb{M}\mathbf{X}_{-1}^+\boldsymbol{\beta} + (\mathbf{I}_N \otimes \mathbf{M}\mathbf{F}_{-1}^+)\boldsymbol{\Lambda} + \mathbb{M}\boldsymbol{\varepsilon}_{-1}^+. \tag{C-44}$$

We will use this result to evaluate (C-41) conditional on $\mathcal{C} = \sigma\{\check{\mathbf{F}}, \mathbf{Q}\}$. From the strict exogeneity of $\mathbf{X}$ (Ass.1) and the independence of $\boldsymbol{\Lambda}$ and $\boldsymbol{\varepsilon}$ (Ass.3) follows

$$\begin{aligned}
\operatorname*{plim}_{N\to\infty} \frac{1}{NT}\mathbf{y}_{-1}'\mathbb{M}_{\mathrm{X}}\boldsymbol{\varepsilon} &= \operatorname*{plim}_{N\to\infty} \frac{1}{NT}\mathbf{y}_{-1}'\mathbb{M}\boldsymbol{\varepsilon} = \operatorname*{plim}_{N\to\infty} \frac{1}{NT}(\boldsymbol{\varepsilon}_{-1}^{+})'\mathbb{M}\boldsymbol{\varepsilon}, \\
&= \operatorname*{plim}_{N\to\infty} \frac{1}{NT}\sum_{i=1}^{N}(\boldsymbol{\varepsilon}_{i,-1}^{+})'\mathbf{M}\boldsymbol{\varepsilon}_i. \qquad \text{(C-45)}
\end{aligned}$$

Defining $\widetilde{\mathbf{Q}} = \mathbf{B}\mathbf{Q}$, with $\mathbf{Q}$ a fixed matrix conditional on $\mathcal{C}$, and $\mathbf{B} = \mathbf{I}_T - \boldsymbol{\iota}_T\boldsymbol{\iota}_T'/T$, the numerator of (C-41) is

$$\begin{aligned}
\operatorname*{plim}_{N\to\infty} \frac{1}{NT}\sum_{i=1}^{N}(\boldsymbol{\varepsilon}_{i,-1}^{+})'\mathbf{M}\boldsymbol{\varepsilon}_i &= \operatorname*{plim}_{N\to\infty} \frac{1}{NT}\sum_{i=1}^{N}(\boldsymbol{\varepsilon}_{i,-1}^{+})'\left[(\boldsymbol{\varepsilon}_i - \bar{\boldsymbol{\varepsilon}}_i) - \widetilde{\mathbf{Q}}(\widetilde{\mathbf{Q}}'\widetilde{\mathbf{Q}})^{\dagger}\widetilde{\mathbf{Q}}'\left(\boldsymbol{\varepsilon}_i - \bar{\boldsymbol{\varepsilon}}_i\right)\right], \\
&= -\operatorname*{plim}_{N\to\infty} \frac{1}{NT}\sum_{i=1}^{N}(\boldsymbol{\varepsilon}_{i,-1}^{+})'\bar{\boldsymbol{\varepsilon}}_i - \operatorname*{plim}_{N\to\infty} \frac{1}{NT}\sum_{i=1}^{N}(\boldsymbol{\varepsilon}_{i,-1}^{+})'\widetilde{\mathbf{Q}}(\widetilde{\mathbf{Q}}'\widetilde{\mathbf{Q}})^{\dagger}\widetilde{\mathbf{Q}}'\boldsymbol{\varepsilon}_i, \\
&= -\frac{\sigma_\varepsilon^2}{T}A\left(\rho\right) - \frac{\sigma_\varepsilon^2}{T}\sum_{t=1}^{T-1}\rho^{t-1}\sum_{s=t+1}^{T}\widetilde{h}_{s,s-t}, \\
&= -\frac{\sigma_\varepsilon^2}{T}A\left(\rho\right) - \frac{\sigma_\varepsilon^2}{T}D(\rho, \widetilde{\mathbf{H}}), \qquad \text{(C-46)}
\end{aligned}$$

with $A(\rho) = \frac{1}{1-\rho}\left(1 - \frac{1}{T}\frac{1-\rho^T}{1-\rho}\right)$, $D(\rho, \widetilde{\mathbf{H}}) = \sum_{t=1}^{T-1}\rho^{t-1}\sum_{s=t+1}^{T}\widetilde{h}_{s,s-t}$ and $\widetilde{\mathbf{H}} = \widetilde{\mathbf{Q}}(\widetilde{\mathbf{Q}}'\widetilde{\mathbf{Q}})^{\dagger}\widetilde{\mathbf{Q}}'$.

Turning to the denominator of equation (C-41), using (C-44) we get

$$\begin{aligned}
\mathbf{y}_{-1}'\mathbb{M}_{\mathrm{X}}\mathbf{y}_{-1} &= \left\|\mathbb{M}_x\left(\mathbb{M}\mathbf{X}_{-1}^{+}\boldsymbol{\beta} + (\mathbf{I}_N \otimes \mathbf{M}\mathbf{F}_{-1}^{+})\boldsymbol{\Lambda} + \mathbb{M}\boldsymbol{\varepsilon}_{-1}^{+}\right)\right\|^2, \\
&= \left\|\mathbb{M}_{\mathrm{X}}\mathbf{X}_{-1}^{+}\boldsymbol{\beta}\right\|^2 + \left\|\mathbb{M}_{\mathrm{X}}(\mathbf{I}_N \otimes \mathbf{F}_{-1}^{+})\boldsymbol{\Lambda}\right\|^2 + \left\|\mathbb{M}_{\mathrm{X}}\boldsymbol{\varepsilon}_{-1}^{+}\right\|^2 \\
&\quad + 2\boldsymbol{\beta}'(\mathbf{X}_{-1}^{+})'\mathbb{M}_{\mathrm{X}}\boldsymbol{\varepsilon}_{-1}^{+} + 2\boldsymbol{\beta}'(\mathbf{X}_{-1}^{+})'\mathbb{M}_{\mathrm{X}}(\mathbf{I}_N \otimes \mathbf{F}_{-1}^{+})\boldsymbol{\Lambda} \\
&\quad + 2\boldsymbol{\Lambda}'(\mathbf{I}_N \otimes (\mathbf{F}_{-1}^{+})')\mathbb{M}_{\mathrm{X}}\boldsymbol{\varepsilon}_{-1}^{+}.
\end{aligned}$$

Defining first

$$\begin{aligned}
C^{+} &= \left\|\mathbb{M}_{\mathrm{X}}\mathbf{X}_{-1}^{+}\boldsymbol{\beta}\right\|^2 + \left\|\mathbb{M}_{\mathrm{X}}(\mathbf{I}_N \otimes \mathbf{F}_{-1}^{+})\boldsymbol{\Lambda}\right\|^2 + 2\boldsymbol{\beta}'(\mathbf{X}_{-1}^{+})'\mathbb{M}_{\mathrm{X}}\boldsymbol{\varepsilon}_{-1}^{+} \\
&\quad + 2\boldsymbol{\beta}'(\mathbf{X}_{-1}^{+})'\mathbb{M}_{\mathrm{X}}(\mathbf{I}_N \otimes \mathbf{F}_{-1}^{+})\boldsymbol{\Lambda} + 2\boldsymbol{\Lambda}'(\mathbf{I}_N \otimes (\mathbf{F}_{-1}^{+})')\mathbb{M}_{\mathrm{X}}\boldsymbol{\varepsilon}_{-1}^{+},
\end{aligned}$$

and taking the limit (conditional on $\mathcal{C}$) gives

$$\begin{aligned}
\operatorname*{plim}_{N\to\infty} \frac{1}{NT}C^{+} &= \operatorname*{plim}_{N\to\infty} \frac{1}{NT}\left\|\mathbb{M}_{\mathrm{X}}\mathbf{X}_{-1}^{+}\boldsymbol{\beta}\right\|^2 + \operatorname*{plim}_{N\to\infty} \frac{1}{NT}\left\|\mathbb{M}_{\mathrm{X}}(\mathbf{I}_N \otimes \mathbf{F}_{-1}^{+})\boldsymbol{\Lambda}\right\|^2 \\
&\quad + \operatorname*{plim}_{N\to\infty} \frac{1}{NT}2\boldsymbol{\beta}'(\mathbf{X}_{-1}^{+})'\mathbb{M}_{\mathrm{X}}(\mathbf{I}_N \otimes \mathbf{F}_{-1}^{+})\boldsymbol{\Lambda}, \qquad \text{(C-47)}
\end{aligned}$$

because by Ass.1 and 3

$$\operatorname*{plim}_{N\to\infty} \frac{1}{NT}2\boldsymbol{\Lambda}'(\mathbf{I}_N \otimes (\mathbf{F}_{-1}^{+})')\mathbb{M}_{\mathrm{X}}\boldsymbol{\varepsilon}_{-1}^{+} = 0,$$

$$\underset{N\to\infty}{\text{plim}}\frac{1}{NT}2\boldsymbol{\beta}'(\mathbf{X}_{-1}^{+})'\mathbb{M}_{\mathrm{X}}\boldsymbol{\varepsilon}_{-1}^{+} \;=\; 0.$$

Hence

$$\begin{aligned}
\underset{N\to\infty}{\text{plim}}\frac{1}{NT}\mathbf{y}_{-1}'\mathbb{M}_{\mathrm{X}}\mathbf{y}_{-1} &= \underset{N\to\infty}{\text{plim}}\frac{1}{NT}\left\|\mathbb{M}_{\mathrm{X}}\boldsymbol{\varepsilon}_{-1}^{+}\right\|^{2}+\underset{N\to\infty}{\text{plim}}\frac{1}{NT}C^{+},\\
&= \underset{N\to\infty}{\text{plim}}\frac{1}{NT}\left\|\mathbb{M}\boldsymbol{\varepsilon}_{-1}^{+}\right\|^{2}-\underset{N\to\infty}{\text{plim}}\frac{1}{NT}(\boldsymbol{\varepsilon}_{-1}^{+})'\mathbb{M}\mathbf{X}\left(\mathbf{X}'\mathbb{M}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbb{M}\boldsymbol{\varepsilon}_{-1}^{+}\\
&\qquad +\underset{N\to\infty}{\text{plim}}\frac{1}{NT}C^{+},\\
&= \underset{N\to\infty}{\text{plim}}\frac{1}{NT}\sum_{i=1}^{N}(\boldsymbol{\varepsilon}_{i,-1}^{+})'\mathbf{M}\boldsymbol{\varepsilon}_{i,-1}^{+}+\underset{N\to\infty}{\text{plim}}\frac{1}{NT}C^{+}. \qquad \text{(C-48)}
\end{aligned}$$

Focusing on the first term of (C-48) and using earlier definitions gives

$$\begin{aligned}
\underset{N\to\infty}{\text{plim}}\frac{1}{NT}\sum_{i=1}^{N}(\boldsymbol{\varepsilon}_{i,-1}^{+})'\mathbf{M}\boldsymbol{\varepsilon}_{i,-1}^{+} &= \underset{N\to\infty}{\text{plim}}\frac{1}{NT}\sum_{i=1}^{N}(\boldsymbol{\varepsilon}_{i,-1}^{+})'\Big[\left(\boldsymbol{\varepsilon}_{i,-1}^{+}-\bar{\boldsymbol{\varepsilon}}_{i,-1}^{+}\right)\\
&\qquad -\widetilde{\mathbf{Q}}(\widetilde{\mathbf{Q}}'\widetilde{\mathbf{Q}})^{\dagger}\widetilde{\mathbf{Q}}'\left(\boldsymbol{\varepsilon}_{i,-1}^{+}-\bar{\boldsymbol{\varepsilon}}_{i,-1}^{+}\right)\Big],\\
&= \underset{N\to\infty}{\text{plim}}\frac{1}{NT}\sum_{t=1}^{T}\sum_{i=1}^{N}\left(\varepsilon_{i,t-1}^{+}-\bar{\varepsilon}_{i,-1}^{+}\right)^{2}\\
&\qquad -\underset{N\to\infty}{\text{plim}}\frac{1}{NT}\sum_{i=1}^{N}(\boldsymbol{\varepsilon}_{i,-1}^{+})'\widetilde{\mathbf{H}}\boldsymbol{\varepsilon}_{i,-1}^{+},\\
&= \frac{\sigma_{\varepsilon}^{2}}{T}B\left(\rho\right)-\frac{\sigma_{\varepsilon}^{2}}{1-\rho^{2}}\frac{1}{T}\left[tr(\widetilde{\mathbf{H}})+2\rho\sum_{t=1}^{T-1}\rho^{t-1}\sum_{s=t+1}^{T}\widetilde{h}_{s,s-t}\right],\\
&= \frac{\sigma_{\varepsilon}^{2}}{T}\left(B\left(\rho\right)-\frac{1}{1-\rho^{2}}\left[c-1+2\rho D(\rho,\widetilde{\mathbf{H}})\right]\right), \qquad \text{(C-49)}
\end{aligned}$$

where $B(\rho)=\frac{T}{1-\rho^{2}}\left(1-\frac{1}{T}\frac{1+\rho}{1-\rho}-\frac{2\rho}{T^{2}}\frac{1-\rho^{T}}{(1-\rho)^{2}}\right)$.

Combining (C-46), (C-47) and (C-49)

$$\begin{aligned}
\underset{N\to\infty}{\text{plim}}\left(\widehat{\rho}-\rho\right) &= \frac{-\frac{\sigma_{\varepsilon}^{2}}{T}\left(A\left(\rho\right)+D(\rho,\widetilde{\mathbf{H}})\right)}{\frac{\sigma_{\varepsilon}^{2}}{T}\left(B\left(\rho\right)-\frac{1}{1-\rho^{2}}\left[c-1+2\rho D(\rho,\widetilde{\mathbf{H}})\right]\right)+\text{plim}_{N\to\infty}\frac{1}{NT}C^{+}},\\
&= \frac{-A\left(\rho\right)-D(\rho,\widetilde{\mathbf{H}})}{B\left(\rho\right)-\frac{1}{1-\rho^{2}}\left[c-1+2\rho D(\rho,\widetilde{\mathbf{H}})\right]+\text{plim}_{N\to\infty}\frac{1}{N\sigma_{\varepsilon}^{2}}C^{+}}, \qquad \text{(C-50)}
\end{aligned}$$

which we reformulate to

$$\underset{N\to\infty}{\text{plim}}\left(\widehat{\rho}-\rho\right) \;=\; -\frac{\left[A\left(\rho\right)+D(\rho,\widetilde{\mathbf{H}})\right]}{\left[B\left(\rho\right)-E(\rho,\widetilde{\mathbf{H}})+TC\right]}, \qquad \text{(C-51)}$$

where $E(\rho,\widetilde{\mathbf{H}}) = \frac{1}{1-\rho^2}\left[c-1+2\rho D(\rho,\widetilde{\mathbf{H}})\right]$ and

$$\begin{aligned}
C &= \operatorname*{plim}_{N\to\infty} \frac{1}{NT\sigma_\varepsilon^2}\left(\left\|\mathbb{M}_{\mathrm{X}}\mathbf{X}_{-1}^{+}\boldsymbol{\beta}\right\|^2 + \left\|\mathbb{M}_{\mathrm{X}}(\mathbf{I}_N\otimes\mathbf{F}_{-1}^{+})\boldsymbol{\Lambda}\right\|^2 + 2\boldsymbol{\beta}'(\mathbf{X}_{-1}^{+})'\mathbb{M}_{\mathrm{X}}(\mathbf{I}_N\otimes\mathbf{F}_{-1}^{+})\boldsymbol{\Lambda}\right),\\
&= \operatorname*{plim}_{N\to\infty} \frac{1}{NT\sigma_\varepsilon^2}\left(\boldsymbol{\beta}'(\mathbf{X}_{-1}^{+})'\mathbb{M}_{\mathrm{X}}\mathbf{X}_{-1}^{+}\boldsymbol{\beta} + \boldsymbol{\Lambda}'(\mathbb{F}_{-1}^{+})'\mathbb{M}_{\mathrm{X}}\mathbb{F}_{-1}^{+}\boldsymbol{\Lambda} + 2\boldsymbol{\beta}'(\mathbf{X}_{-1}^{+})'\mathbb{M}_{\mathrm{X}}\mathbb{F}_{-1}^{+}\boldsymbol{\Lambda}\right),\\
&= \operatorname*{plim}_{N\to\infty} \frac{\boldsymbol{\beta}'\boldsymbol{\Omega}_{\breve{\mathrm{x}}}\boldsymbol{\beta} + \boldsymbol{\Lambda}'\boldsymbol{\Omega}_{\breve{\mathrm{f}}}\boldsymbol{\Lambda} + 2\boldsymbol{\beta}'\boldsymbol{\Omega}_{\breve{\mathrm{x}},\breve{\mathrm{f}}}\boldsymbol{\Lambda}}{\sigma_\varepsilon^2},
\end{aligned}$$

with $\boldsymbol{\Omega}_{\breve{\mathrm{x}}} = (\mathbf{X}_{-1}^{+})'\mathbb{M}_{\mathrm{X}}\mathbf{X}_{-1}^{+}/NT$, $\boldsymbol{\Omega}_{\breve{\mathrm{f}}} = (\mathbb{F}_{-1}^{+})'\mathbb{M}_{\mathrm{X}}\mathbb{F}_{-1}^{+}/NT$, $\boldsymbol{\Omega}_{\breve{\mathrm{x}},\breve{\mathrm{f}}} = (\mathbf{X}_{-1}^{+})'\mathbb{M}_{\mathrm{X}}\mathbb{F}_{-1}^{+}/NT$ and $\mathbb{F}_{-1}^{+} = (\mathbf{I}_N\otimes\mathbf{F}_{-1}^{+})$.

C.4.3 Proof of Theorem 3

Let $\boldsymbol{\phi}(\boldsymbol{\delta}_0)$ be the vector of moment conditions employed by CCEPbc in (21) evaluated at $\boldsymbol{\delta}_0 \neq \boldsymbol{\delta}$, with $\boldsymbol{\delta}$ the population parameter vector $\boldsymbol{\delta} = [\rho, \boldsymbol{\beta}']'$. Multiplying by $\widehat{\boldsymbol{\Sigma}}$ and solving in eq.(6) gives

$$\begin{aligned}
\widehat{\boldsymbol{\Sigma}}\boldsymbol{\phi}(\boldsymbol{\delta}_0) &= \frac{1}{NT}\sum_{i=1}^N \mathbf{w}_i'\mathbf{M}\mathbf{y}_i - \frac{1}{NT}\sum_{i=1}^N \mathbf{w}_i'\mathbf{M}\mathbf{w}_i\boldsymbol{\delta}_0 + \frac{1}{T}\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta}_0)\boldsymbol{\upsilon}(\rho_0),\\
&= \widehat{\boldsymbol{\Sigma}}\left(\boldsymbol{\delta}-\boldsymbol{\delta}_0\right) + \frac{1}{NT}\sum_{i=1}^N \mathbf{w}_i'\mathbf{M}\boldsymbol{\varepsilon}_i + \frac{1}{NT}\sum_{i=1}^N \mathbf{w}_i'\mathbf{M}\mathbf{F}\boldsymbol{\gamma}_i + \frac{1}{T}\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta}_0)\boldsymbol{\upsilon}(\rho_0),\\
&= \widehat{\boldsymbol{\Sigma}}(\boldsymbol{\delta}-\boldsymbol{\delta}_0) + \frac{1}{T}\left(\frac{1}{N}\sum_{i=1}^N \mathbf{w}_i'\mathbf{M}\boldsymbol{\varepsilon}_i + \widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta}_0)\boldsymbol{\upsilon}(\rho)\right) + O_p(N^{-1/2}), \qquad \text{(C-52)}
\end{aligned}$$

because $\frac{1}{NT}\sum_{i=1}^N \mathbf{w}_i'\mathbf{M}\mathbf{F}\boldsymbol{\gamma}_i = O_p(N^{-1/2})$ by Lemma 5. Note that we have dropped the dependence of $\boldsymbol{\upsilon}(\cdot)$ on $\mathbf{H}$ for simplicity.

Consider the middle term. From Lemma 8 with $\|\boldsymbol{\delta}-\boldsymbol{\delta}_0\| < \infty$ given compactness of χ,

$$\widehat{\sigma}_\varepsilon^2\left(\boldsymbol{\delta}_0\right) = \sigma_\varepsilon^2 - \sigma_\varepsilon^2 c_1\boldsymbol{\upsilon}(\rho)'(\boldsymbol{\delta}-\boldsymbol{\delta}_0) + c_2(\boldsymbol{\delta}-\boldsymbol{\delta}_0)'\boldsymbol{\Sigma}(\boldsymbol{\delta}-\boldsymbol{\delta}_0) + O_p(N^{-1/2}),$$

with $\boldsymbol{\Sigma}$ defined in eq.(C-7) of Lemma 7, and where $c_1 = \frac{2}{T-c}$ and $c_2 = \frac{T}{T-c}$. We also have

$$\frac{1}{N}\sum_{i=1}^N \mathbf{w}_i'\mathbf{M}\boldsymbol{\varepsilon}_i = -\sigma_\varepsilon^2\boldsymbol{\upsilon}(\rho) + O_p(N^{-1/2}),$$

by Theorem 1. As such, by combining results we can write as $N\to\infty$ that

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^N \mathbf{w}_i'\mathbf{M}\boldsymbol{\varepsilon}_i + \widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta}_0)\boldsymbol{\upsilon}(\rho_0)\\
&= -\sigma_\varepsilon^2\boldsymbol{\upsilon}(\rho) + \boldsymbol{\upsilon}(\rho_0)\left[\sigma_\varepsilon^2 - \sigma_\varepsilon^2 c_1\boldsymbol{\upsilon}(\rho)'(\boldsymbol{\delta}-\boldsymbol{\delta}_0) + c_2(\boldsymbol{\delta}-\boldsymbol{\delta}_0)'\boldsymbol{\Sigma}(\boldsymbol{\delta}-\boldsymbol{\delta}_0)\right] + o_p(1),\\
&= -\sigma_\varepsilon^2\left[\boldsymbol{\upsilon}(\rho)-\boldsymbol{\upsilon}(\rho_0)\right] - \sigma_\varepsilon^2 c_1\boldsymbol{\upsilon}(\rho_0)\boldsymbol{\upsilon}(\rho)'(\boldsymbol{\delta}-\boldsymbol{\delta}_0) + c_2\boldsymbol{\upsilon}(\rho_0)(\boldsymbol{\delta}-\boldsymbol{\delta}_0)'\boldsymbol{\Sigma}(\boldsymbol{\delta}-\boldsymbol{\delta}_0) + o_p(1),
\end{aligned}$$

Substituting this result in (C-52) gives $\left\|\boldsymbol{\phi}(\boldsymbol{\delta}_0) - \widetilde{\boldsymbol{\phi}}(\boldsymbol{\delta}_0)\right\| = o_p(1)$ for $\|\boldsymbol{\delta} - \boldsymbol{\delta}_0\| < \infty$, with

$$\widetilde{\boldsymbol{\phi}}(\boldsymbol{\delta}_0) = (\boldsymbol{\delta} - \boldsymbol{\delta}_0) - \frac{1}{T}\Bigg[\sigma_\varepsilon^2 \boldsymbol{\Sigma}^{-1}\left[\boldsymbol{\upsilon}(\rho) - \boldsymbol{\upsilon}(\rho_0)\right] + \sigma_\varepsilon^2 c_1 \boldsymbol{\Sigma}^{-1}\boldsymbol{\upsilon}(\rho_0)\boldsymbol{\upsilon}(\rho)'(\boldsymbol{\delta} - \boldsymbol{\delta}_0)$$
$$- c_2 \boldsymbol{\Sigma}^{-1}\boldsymbol{\upsilon}(\rho_0)(\boldsymbol{\delta} - \boldsymbol{\delta}_0)'\boldsymbol{\Sigma}(\boldsymbol{\delta} - \boldsymbol{\delta}_0)\Bigg]. \quad \text{(C-53)}$$

In (C-53) we note that $\|\boldsymbol{\upsilon}(\rho_0)\| < \infty$ since $\|\mathbf{H}\| = \sqrt{c}$ from Lemma 3 such that $\upsilon(\rho_0, \mathbf{H}) < \infty$ for any finite ρ_0 (and where $|\rho| < 1$ by Ass.5 ensures $\|\boldsymbol{\upsilon}(\rho)\| < \infty$ also as $T \to \infty$). Also, $c_2 = O(1)$ and since $T > c$, $c_1 = O(1)$. $\|\boldsymbol{\Sigma}\| = O(1)$ is shown in Lemma 7 and $\sigma_\varepsilon^2 < \infty$ by Ass.1. This implies that $\left\|\widetilde{\boldsymbol{\phi}}(\boldsymbol{\delta}_0)\right\| < \infty$ provided $\|\boldsymbol{\delta} - \boldsymbol{\delta}_0\| < \infty$. Also, clearly from (C-53),

$$\widetilde{\boldsymbol{\phi}}(\boldsymbol{\delta}_0) = \mathbf{0}_{k_w \times 1}, \qquad \text{for} \qquad \boldsymbol{\delta}_0 = \boldsymbol{\delta}.$$

Finally, since $[\boldsymbol{\upsilon}(\rho) - \boldsymbol{\upsilon}(\rho_0)]$ is determined only by $\rho_0 - \rho$ and is zero only for $\rho_0 = \rho$ we take that $\widetilde{\boldsymbol{\phi}}(\boldsymbol{\delta}_0) = \mathbf{0}_{k_w \times 1}$ implies $\boldsymbol{\delta}_0 = \boldsymbol{\delta}$ such that, assuming that the admissible parameter space $\chi \subseteq \mathbb{R}^{k_w}$ in (21) is compact with $\boldsymbol{\delta}$ contained in its interior, we have as in Newey and McFadden (1994) (Section 2.5) that

$$\widehat{\boldsymbol{\delta}}_{bc} \longrightarrow^p \boldsymbol{\delta},$$

as $N \to \infty$.

D Analysis for $(N,T) \to \infty$

D.1 Preliminary results

Consider the decomposition

$$\begin{aligned}\mathbf{M_F} - \mathbf{M} &= \bar{\mathbf{U}}^0[\mathbf{Q}_0'\mathbf{Q}_0]^{-1}(\bar{\mathbf{U}}^0)' + \bar{\mathbf{U}}^0[\mathbf{Q}_0'\mathbf{Q}_0]^{-1}(\mathbf{F}^0)' + \mathbf{F}^0[\mathbf{Q}_0'\mathbf{Q}_0]^{-1}(\bar{\mathbf{U}}^0)' \\ &\quad + \mathbf{F}^0\left([\mathbf{Q}_0'\mathbf{Q}_0]^{-1} - [(\mathbf{F}^0)'\mathbf{F}^0]^{-1}\right)(\mathbf{F}^0)',\end{aligned}$$

and note that, since $\mathbf{F}^0 = [\mathbf{F}^*, \mathbf{0}_{T\times(K-m)}]$ and $\bar{\mathbf{U}}^0 = [\bar{\mathbf{U}}^0_m, \bar{\mathbf{U}}^0_{-m}]$, we have using similar steps as in the proof of Lemma S.1 of Karabiyik et al. (2017)

$$\begin{aligned}\mathbf{M_F} - \mathbf{M} &= T^{-1}\bar{\mathbf{U}}^0_{-m}[T^{-1}(\bar{\mathbf{U}}^0_{-m})'\bar{\mathbf{U}}^0_{-m}]^{-1}(\bar{\mathbf{U}}^0_{-m})' + T^{-1}\bar{\mathbf{U}}^0_m[T^{-1}(\mathbf{F}^*)'\mathbf{F}^*]^{-1}(\bar{\mathbf{U}}^0_m)' \\ &\quad + T^{-1}\mathbf{F}^*[T^{-1}(\mathbf{F}^*)'\mathbf{F}^*]^{-1}(\bar{\mathbf{U}}^0_m)' + T^{-1}\bar{\mathbf{U}}^0_m[T^{-1}(\mathbf{F}^*)'\mathbf{F}^*]^{-1}(\mathbf{F}^*)' \\ &\quad + T^{-1}\mathbf{Q}_0\left(\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1} - \widehat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+}^{-1}\right)\mathbf{Q}_0', \end{aligned} \tag{D-1}$$

with $\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}} = T^{-1}\mathbf{Q}_0'\mathbf{Q}_0$ and

$$\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+} = \frac{1}{T}\begin{bmatrix} (\mathbf{F}^*)'\mathbf{F}^* & \mathbf{0}_{(1+K+m)\times(K-m)} \\ \mathbf{0}_{(K-m)\times(1+K+m)} & (\bar{\mathbf{U}}^0_{-m})'\bar{\mathbf{U}}^0_{-m}\end{bmatrix} = \begin{bmatrix} \widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*} & \mathbf{0}_{(1+K+m)\times(K-m)} \\ \mathbf{0}_{(K-m)\times(1+K+m)} & \widehat{\boldsymbol{\Sigma}}_{\mathbf{u}^0_{-m}}\end{bmatrix}, \tag{D-2}$$

where $\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*} = T^{-1}(\mathbf{F}^*)'\mathbf{F}^*$ and $\widehat{\boldsymbol{\Sigma}}_{\mathbf{u}^0_{-m}} = T^{-1}(\bar{\mathbf{U}}^0_{-m})'\bar{\mathbf{U}}^0_{-m}$.

D.2 Statement of lemmas

Lemma 9. *Suppose Assumptions 1-3 and 5 hold, then, as $(N,T)\to\infty$,*

$$\frac{\ddot{\mathbf{U}}'\ddot{\mathbf{U}}}{T} = O_p\left(\frac{1}{N}\right), \tag{D-3}$$

$$\frac{\ddot{\mathbf{U}}'\check{\mathbf{F}}}{T} = O_p\left(\frac{1}{\sqrt{NT}}\right), \qquad \frac{\check{\mathbf{F}}'\check{\mathbf{F}}}{T} = O_p(1), \tag{D-4}$$

$$\frac{\ddot{\mathbf{U}}_i'\check{\mathbf{F}}}{T} = O_p\left(\frac{1}{\sqrt{T}}\right), \qquad \frac{\boldsymbol{\varepsilon}_i'\check{\mathbf{F}}}{T} = O_p\left(\frac{1}{\sqrt{T}}\right), \qquad \frac{\boldsymbol{\epsilon}_i'\check{\mathbf{F}}}{T} = O_p\left(\frac{1}{\sqrt{T}}\right), \tag{D-5}$$

$$\frac{\boldsymbol{\varepsilon}_i'\ddot{\mathbf{U}}}{T} = O_p\left(\frac{1}{N}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right), \qquad \frac{\boldsymbol{\epsilon}_i'\ddot{\mathbf{U}}}{T} = O_p\left(\frac{1}{N}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right). \tag{D-6}$$

Lemma 10. *Suppose Assumptions 1-5 hold, then, as $(N,T)\to\infty$,*

$$\frac{(\bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}^0_m}{T} = O_p\left(\frac{1}{N}\right), \qquad \frac{(\bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}^0_{-m}}{T} = O_p\left(\frac{1}{\sqrt{N}}\right), \qquad \frac{(\bar{\mathbf{U}}^0_{-m})'\bar{\mathbf{U}}^0_{-m}}{T} = O_p(1), \tag{D-7}$$

$$\frac{(\mathbf{F}^*)'\mathbf{F}^*}{T} = O_p(1), \qquad \frac{(\bar{\mathbf{U}}^0_m)'\mathbf{F}^*}{T} = O_p\left(\frac{1}{\sqrt{NT}}\right), \qquad \frac{(\bar{\mathbf{U}}^0_{-m})'\mathbf{F}^*}{T} = O_p\left(\frac{1}{\sqrt{T}}\right), \tag{D-8}$$

$$\frac{(\bar{\mathbf{U}}^0)'\mathbf{F}^*}{T} = O_p\left(\frac{1}{\sqrt{T}}\right), \qquad \frac{\boldsymbol{\varepsilon}_i'\mathbf{F}^*}{T} = O_p\left(\frac{1}{\sqrt{T}}\right), \qquad \frac{\boldsymbol{\epsilon}_i'\mathbf{F}^*}{T} = O_p\left(\frac{1}{\sqrt{T}}\right), \tag{D-9}$$

$$\frac{\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}^0_m}{T} = O_p\left(\frac{1}{N}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right), \qquad \frac{\boldsymbol{\varepsilon}_i'\bar{\mathbf{U}}^0_m}{T} = O_p\left(\frac{1}{N}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right), \tag{D-10}$$

$$\frac{\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}^0_{-m}}{T} = O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right), \qquad \frac{\boldsymbol{\varepsilon}_i'\bar{\mathbf{U}}^0_{-m}}{T} = O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right), \tag{D-11}$$

$$\frac{\boldsymbol{\epsilon}_i'\mathbf{Q}_0}{T} = O_p\left(\frac{1}{\sqrt{T}}\right), \qquad \frac{\boldsymbol{\varepsilon}_i'\mathbf{Q}_0}{T} = O_p\left(\frac{1}{\sqrt{T}}\right), \qquad \frac{\mathbf{Q}_0'\bar{\mathbf{U}}^0_m}{T} = O_p\left(\frac{1}{\sqrt{N}}\right). \tag{D-12}$$

Lemma 11. *Suppose Assumptions 1-5 hold and let* $\widetilde{\mathbf{P}}_i = \ddot{\mathbf{P}}_i - N^{-1}\sum_{i=1}^N \ddot{\mathbf{P}}_i$. *Then, as* $(N,T) \to \infty$,

$$N^{-1}\sum_{i=1}^N \left(T^{-1}\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}^0_m \otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right) = O_p\left(\frac{1}{N^{3/2}}\right) + O_p\left(\frac{1}{N\sqrt{T}}\right), \tag{D-13}$$

$$N^{-1}\sum_{i=1}^N \left(T^{-1}\boldsymbol{\varepsilon}_i'\bar{\mathbf{U}}^0_m \otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right) = O_p\left(\frac{1}{N^{3/2}}\right) + O_p\left(\frac{1}{N\sqrt{T}}\right), \tag{D-14}$$

$$N^{-1}\sum_{i=1}^N \left(T^{-1}\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}^0_{-m} \otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right) = O_p\left(\frac{1}{N}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right), \tag{D-15}$$

$$N^{-1}\sum_{i=1}^N \left(T^{-1}\boldsymbol{\varepsilon}_i'\bar{\mathbf{U}}^0_{-m} \otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right) = O_p\left(\frac{1}{N}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right), \tag{D-16}$$

$$N^{-1}\sum_{i=1}^N \left(T^{-1}\boldsymbol{\epsilon}_i'\check{\mathbf{F}} \otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right) = O_p\left(\frac{1}{\sqrt{NT}}\right), \tag{D-17}$$

$$N^{-1}\sum_{i=1}^N \left(T^{-1}\boldsymbol{\varepsilon}_i'\check{\mathbf{F}} \otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right) = O_p\left(\frac{1}{\sqrt{NT}}\right), \tag{D-18}$$

$$N^{-1}\sum_{i=1}^N \left(T^{-1}\boldsymbol{\varepsilon}_i'\mathbf{Q}_0 \otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right) = O_p\left(\frac{1}{N}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right), \tag{D-19}$$

where the results hold similarly if $\mathbf{S}_w'\widetilde{\mathbf{P}}_i'$ *is substituted for* $\boldsymbol{\eta}_i'$.

Lemma 12. *Let Assumptions 1-5 hold. Then, as* $(N,T) \to \infty$,

$$\begin{aligned}
\widehat{\boldsymbol{\Sigma}}_{\check{\mathbf{F}}} &= T^{-1}\check{\mathbf{F}}'\check{\mathbf{F}} = \boldsymbol{\Sigma}_{\check{\mathbf{F}}} + O_p(T^{-1/2}),\\
\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*} &= T^{-1}(\mathbf{F}^*)'\mathbf{F}^* = \boldsymbol{\Sigma}_{\mathbf{F}^*} + O_p(N^{-1/2}) + O_p(T^{-1/2}),\\
\widehat{\boldsymbol{\Sigma}}_{\check{\mathbf{F}}\mathbf{F}^*} &= T^{-1}(\mathbf{F}^*)'\mathbf{F}^* = \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*} + O_p(N^{-1/2}) + O_p(T^{-1/2}),\\
\widehat{\boldsymbol{\Sigma}}_{\mathbf{u}^0_{-m}} &= T^{-1}(\bar{\mathbf{U}}^0_{-m})'\bar{\mathbf{U}}^0_{-m} = \boldsymbol{\Sigma}_{\mathbf{u}^0_{-m}} + O_p(T^{-1/2}),
\end{aligned}$$

$$\widehat{\mathbf{\Sigma}}_{\mathbf{u}^0_{-m}\mathbf{u}^0_m} = T^{-1}(\bar{\mathbf{U}}^0_{-m})'\sqrt{N}\bar{\mathbf{U}}^0_m = \mathbf{\Sigma}_{\mathbf{u}^0_{-m}\mathbf{u}^0_m} + O_p(T^{-1/2}),$$

and also

$$\left\|\widehat{\mathbf{\Sigma}}^{-1}_{\mathbf{F}^*} - \mathbf{\Sigma}^{-1}_{\mathbf{F}^*}\right\| = O_p(N^{-1/2}) + O_p(T^{-1/2}),$$
$$\left\|\widehat{\mathbf{\Sigma}}^{-1}_{\mathbf{u}^0_{-m}} - \mathbf{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}\right\| = O_p(T^{-1/2}),$$

where

$$\mathbf{\Sigma}_{\check{\mathbf{F}}} = E(\check{\mathbf{F}}'\check{\mathbf{F}}/T), \qquad \mathbf{\Sigma}_{\ddot{\mathbf{U}}} = E(\ddot{\mathbf{U}}_i'\ddot{\mathbf{U}}_i/T), \qquad \mathbf{\Sigma}_{\mathbf{F}^*} = \mathbf{S}_m'\mathbf{N}'\mathbf{R}'\mathbf{P}'\mathbf{\Sigma}_{\check{\mathbf{F}}}\mathbf{PRNS}_m,$$
$$\mathbf{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*} = \mathbf{\Sigma}_{\check{\mathbf{F}}}\mathbf{PRNS}_m, \qquad \mathbf{\Sigma}_{\mathbf{u}^0_{-m}} = \mathbf{B}_{-m}'\mathbf{T}'\mathbf{A}_0^{-1}\mathbf{\Omega}_{\mathbf{u}}(\mathbf{A}_0^{-1})'\mathbf{TB}_{-m},$$
$$\mathbf{\Sigma}_{\mathbf{u}^0_{-m}\mathbf{u}^0_m} = \mathbf{S}_{-m}'\mathbf{N}'\mathbf{R}'\mathbf{\Sigma}_{\ddot{\mathbf{U}}}\mathbf{RNS}_m, \qquad \mathbf{\Omega}_{\mathbf{u}} = \begin{bmatrix} \sigma^2_{\varepsilon} & \mathbf{0}'_{k\times 1} \\ \mathbf{0}_{k\times 1} & \mathbf{\Omega}_{\mathbf{v}} \end{bmatrix}.$$

Lemma 13. *Let $\widehat{\mathbf{\Sigma}}_{\mathbf{Q}} = T^{-1}\mathbf{Q}_0'\mathbf{Q}_0$ and suppose Assumptions 1-5 hold. Then, as $(N,T) \to \infty$, with $\widehat{\mathbf{\Sigma}}_{\mathbf{F}^+_u}$ defined in eq.(D-2),*

$$\left\|\widehat{\mathbf{\Sigma}}^{-1}_{\mathbf{Q}} - \widehat{\mathbf{\Sigma}}^{-1}_{\mathbf{F}^+_u}\right\| = O_p(N^{-1/2}) + O_p(T^{-1/2}), \tag{D-20}$$

and also

$$\sqrt{T}\left[\widehat{\mathbf{\Sigma}}^{-1}_{\mathbf{Q}} - \widehat{\mathbf{\Sigma}}^{-1}_{\mathbf{F}^+_u}\right] = -\begin{bmatrix} \mathbf{0}_{(1+K+m)\times(1+K+m)} & T^{-1/2}\widehat{\mathbf{\Sigma}}_{\mathbf{Fu}} \\ T^{-1/2}\widehat{\mathbf{\Sigma}}'_{\mathbf{Fu}} & \mathbf{0}_{(K-m)\times(K-m)} \end{bmatrix} + O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\frac{\sqrt{T}}{N}\right), \tag{D-21}$$

where $\widehat{\mathbf{\Sigma}}_{\mathbf{Fu}} = \mathbf{\Sigma}^{-1}_{\mathbf{F}^*}(\mathbf{F}^* + \bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}^0_{-m}\mathbf{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}$.

Lemma 14. *Suppose Assumptions 1-5 hold and let $p^* \geq p$. Then, as $(N,T) \to \infty$,*

$$\mathbf{A}^{\mathbf{F}} = \frac{1}{NT}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{MF}\boldsymbol{\gamma}_i = O_p\left(\frac{1}{N}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right), \tag{D-22}$$

and letting $\mathbf{A}^{\mathbf{F}}_{NT} = \sqrt{NT}\mathbf{A}^{\mathbf{F}}$, provided that $T/N \to M < \infty$,

$$\mathbf{A}^{\mathbf{F}}_{NT} = \mathbf{\Psi}_{\mathbf{F}} vec\left(\frac{\check{\mathbf{F}}'\sqrt{N}\ddot{\mathbf{U}}}{\sqrt{T}}\right) + \sqrt{\frac{T}{N}}\left(\mathbf{b}^{\mathbf{F}}_0 - \mathbf{b}^{\mathbf{F}}_1\right) + O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right), \tag{D-23}$$

with

$$\mathbf{b}^{\mathbf{F}}_0 = \mathbf{\Sigma}_{\boldsymbol{\eta}}\left[\mathbf{B}_m'\mathbf{T}'\mathbf{R}_0' \otimes \mathbf{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*}\mathbf{\Sigma}^{-1}_{\mathbf{F}^*}\mathbf{S}_m'\mathbf{N}'\mathbf{R}'\right] vec\left(\mathbf{\Sigma}_{\ddot{\mathbf{U}}}\right),$$

$$\mathbf{b}_1^{\mathbf{F}} = \boldsymbol{\Sigma}_{\boldsymbol{\eta}}\left[\boldsymbol{\Sigma}'_{\mathbf{u}^0_{-m}\mathbf{u}_m}\boldsymbol{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}\mathbf{S}'_{-m}\mathbf{N}'\mathbf{R}' \otimes \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*}\boldsymbol{\Sigma}^{-1}_{\mathbf{F}^*}\mathbf{S}'_m\mathbf{N}'\mathbf{R}'\right] vec\left(\boldsymbol{\Sigma}_{\ddot{\mathbf{U}}}\right),$$
$$\boldsymbol{\Psi}_{\mathbf{F}} = -\mathbf{V}_{\mathbf{F},1} + \mathbf{V}_{\mathbf{F},2} + \mathbf{V}_{\mathbf{F},3} - \mathbf{V}_{\mathbf{F},4},$$
$$\mathbf{V}_{\mathbf{F},1} = \boldsymbol{\Sigma}_{\boldsymbol{\eta}}\left[\mathbf{B}'_m\mathbf{T}'\mathbf{R}'_0 \otimes \mathbf{I}_{1+K^2m(1+p^*)}\right],$$
$$\mathbf{V}_{\mathbf{F},2} = \boldsymbol{\Sigma}_{\boldsymbol{\eta}}\left[\mathbf{B}'_m\mathbf{T}'\mathbf{R}'_0 \otimes \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*}\boldsymbol{\Sigma}^{-1}_{\mathbf{F}^*}\mathbf{S}'_m\mathbf{N}'\mathbf{R}'\mathbf{P}'\right],$$
$$\mathbf{V}_{\mathbf{F},3} = \boldsymbol{\Sigma}_{\boldsymbol{\eta}}\left[\mathbf{B}'_m\mathbf{T}'\mathbf{R}'_0\boldsymbol{\Sigma}_{\ddot{\mathbf{U}}}\mathbf{R}\mathbf{N}\mathbf{S}_{-m}\boldsymbol{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}\mathbf{S}'_{-m}\mathbf{N}'\mathbf{R}' \otimes \mathbf{I}_{1+k^2m(1+p^*)}\right],$$
$$\mathbf{V}_{\mathbf{F},4} = \boldsymbol{\Sigma}_{\boldsymbol{\eta}}\left[\boldsymbol{\Sigma}'_{\mathbf{u}^0_{-m}\mathbf{u}_m}\boldsymbol{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}\mathbf{S}'_{-m}\mathbf{N}'\mathbf{R}' \otimes \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*}\boldsymbol{\Sigma}^{-1}_{\mathbf{F}^*}\mathbf{S}'_m\mathbf{N}'\mathbf{R}'\mathbf{P}'\right].$$

and where $\boldsymbol{\Sigma}_{\boldsymbol{\eta}} = E(\boldsymbol{\eta}'_i \otimes \mathbf{S}'_w\ddot{\mathbf{P}}'_i)$.

Lemma 15. *Suppose Assumptions 1-5 hold. Then, as* $(N,T) \to \infty$,

$$\mathbf{A}^{\varepsilon} = \frac{1}{NT}\sum_{i=1}^{N}\mathbf{w}'_i\mathbf{M}\boldsymbol{\varepsilon}_i = O_p\left(\frac{1}{N}\right) + O_p\left(\frac{1}{T}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right), \tag{D-24}$$

and letting $\mathbf{A}^{\varepsilon}_{NT} = \sqrt{NT}\mathbf{A}^{\varepsilon}$,

$$\mathbf{A}^{\varepsilon}_{NT} = O_p(1) + O_p(\sqrt{T}N^{-1/2}) + O_p(\sqrt{N}T^{-1/2}). \tag{D-25}$$

Lemma 16. *Suppose Assumptions 1-5 hold and let* $p^* \geq p$. *Then, for any* $\boldsymbol{\delta}_0 \neq \boldsymbol{\delta}$ *such that* $\|\boldsymbol{\delta} - \boldsymbol{\delta}_0\| < \infty$ *we have as* $(N,T) \to \infty$

$$\widehat{\sigma}^2_{\varepsilon}\left(\boldsymbol{\delta}_0\right) = \sigma^2_{\varepsilon} + (\boldsymbol{\delta} - \boldsymbol{\delta}_0)'\widehat{\boldsymbol{\Sigma}}(\boldsymbol{\delta} - \boldsymbol{\delta}_0) + O_p(N^{-1}) + O_p(T^{-1}) + O_p((NT)^{-1/2}), \tag{D-26}$$

whereas if $\boldsymbol{\delta}_0 = \boldsymbol{\delta}$ *then*

$$\widehat{\sigma}^2_{\varepsilon}\left(\boldsymbol{\delta}\right) = \sigma^2_{\varepsilon} + O_p(N^{-1}) + O_p((NT)^{-1/2}), \tag{D-27}$$

with $\widehat{\sigma}^2(\cdot)$ *defined in (20).*

Lemma 17. *Suppose Assumptions 1-5 hold and let* $p^* \geq p$. *Then, as* $(N,T) \to \infty$ *with* $\boldsymbol{\upsilon} = \upsilon(\rho, \mathbf{H})\mathbf{q}_1$ *and* $\widehat{\sigma}^2(\cdot)$ *defined in (20)*

$$\mathbf{A}^{c} = \frac{1}{NT}\sum_{i=1}^{N}\mathbf{w}'_i\mathbf{M}\boldsymbol{\varepsilon}_i + \frac{1}{T}\widehat{\sigma}^2_{\varepsilon}(\boldsymbol{\delta})\boldsymbol{\upsilon} = O_p(N^{-1}) + O_p((NT)^{-1/2}). \tag{D-28}$$

Letting $\mathbf{A}^{c}_{NT} = \sqrt{NT}\mathbf{A}^{c}$ *and* $T/N \to M < \infty$,

$$\mathbf{A}^{c}_{NT} = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{\boldsymbol{\epsilon}'_i\boldsymbol{\varepsilon}_i}{\sqrt{T}} + \boldsymbol{\Psi}_{\boldsymbol{\epsilon}} vec\left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\frac{\boldsymbol{\varepsilon}'_i\check{\mathbf{F}}}{\sqrt{T}} \otimes \mathbf{S}'_w\widetilde{\mathbf{P}}'_i\right)\right] - \sqrt{T}N^{-1/2}\mathbf{b}^{\mathbf{U}} \tag{D-29}$$

$$+ O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right),$$

with $\mathbf{b^U} = \mathbf{\Sigma}_{\boldsymbol{\epsilon}\mathbf{U}_{-m}} \mathbf{\Sigma}^{-1}_{\mathbf{u}^0_{-m}} \mathbf{\Sigma}'_{\boldsymbol{\varepsilon}\mathbf{U}_{-m}}$, $\mathbf{\Sigma}_{\boldsymbol{\epsilon}\mathbf{U}_{-m}} = \mathbf{S}'_w \mathbf{\Sigma}_{\ddot{\mathbf{U}}} \mathbf{RNS}_{-m}$, $\mathbf{\Sigma}_{\boldsymbol{\varepsilon}\mathbf{U}_{-m}} = E(\boldsymbol{\varepsilon}'_i \ddot{\mathbf{U}}_i/T)\mathbf{RNS}_{-m}$, $\mathbf{B^F} = \mathbf{I}_{1+K^2 m(1+p^*)} - \mathbf{\Sigma}_{\check{\mathbf{F}}} \mathbf{PRNS}_m \mathbf{\Sigma}^{-1}_{\mathbf{F}^*} \mathbf{S}'_m \mathbf{N}'\mathbf{R}'\mathbf{P}'$ *and* $\mathbf{\Psi}_{\boldsymbol{\epsilon}} = \left[vec\left(\mathbf{B^F}\right)' \otimes \mathbf{I}_{k_w}\right]$.
Finally, for $\widetilde{\mathbf{A}}^c(\boldsymbol{\delta}_0)$ *the vector* $\mathbf{A}^c$ *evaluated at* $\boldsymbol{\delta}_0 \neq \boldsymbol{\delta}$

$$\widetilde{\mathbf{A}}^c(\boldsymbol{\delta}_0) = \frac{1}{T}(\boldsymbol{\delta} - \boldsymbol{\delta}_0)'\widehat{\mathbf{\Sigma}}(\boldsymbol{\delta} - \boldsymbol{\delta}_0)\boldsymbol{\upsilon}(\rho_0) + \frac{1}{T}\sigma^2_\varepsilon \left[\boldsymbol{\upsilon}(\rho_0) - \boldsymbol{\upsilon}\right] + O_p\left(\frac{1}{N}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right),$$

where $\boldsymbol{\upsilon}(\rho_0) = \upsilon(\rho_0, \mathbf{H})\mathbf{q}_1$.

Lemma 18. *Suppose Assumptions 1-5 hold. Then, as* $(N,T) \to \infty$,

$$\widehat{\mathbf{\Sigma}} \longrightarrow^p \dot{\mathbf{\Sigma}} = \mathbf{\Sigma}_{\check{\mathbf{F}}\mathbf{P}} + \mathbf{\Sigma}_{\boldsymbol{\epsilon}}, \tag{D-30}$$

where $\mathbf{\Sigma}_{\check{\mathbf{F}}\mathbf{P}} = (vec(\mathbf{I}_{k_w})' \otimes \mathbf{I}_{k_w})\left(\mathbf{I}_{k_w} \otimes \mathbf{\Sigma}_{\widetilde{\mathbf{P}}} vec(\mathbf{V^F})\right)$, $\mathbf{\Sigma}_{\widetilde{\mathbf{P}}} = E(\mathbf{S}'_w \widetilde{\mathbf{P}}'_i \otimes \mathbf{S}'_w \widetilde{\mathbf{P}}'_i)$, $\mathbf{V^F} = \mathbf{\Sigma}_{\check{\mathbf{F}}} - \mathbf{\Sigma}_{\check{\mathbf{F}}} \mathbf{PRNS}_m \mathbf{\Sigma}^{-1}_{\mathbf{F}^*} \mathbf{S}'_m \mathbf{N}'\mathbf{R}'\mathbf{P}'\mathbf{\Sigma}_{\check{\mathbf{F}}}$ *and* $\mathbf{\Sigma}_{\boldsymbol{\epsilon}} = E(\boldsymbol{\epsilon}'_i \boldsymbol{\epsilon}_i/T)$.

D.3 Proof of lemmas

Proof of Lemma 9

The proof for this Lemma is, under Ass.1-3 and 5, identical to that of Lemmas 1 and 2 in Pesaran (2006). The proof is therefore omitted.

Proof of Lemma 10

To prove this lemma, recall from eqs.(B-18)-(B-20) that $\mathbf{F}^* = \check{\mathbf{F}}\ddot{\mathbf{P}}\mathbf{RNS}_m$, $\bar{\mathbf{U}}^0_m = \ddot{\mathbf{U}}\mathbf{RNS}_m$, and $\bar{\mathbf{U}}^0_{-m} = \sqrt{N}\ddot{\mathbf{U}}\mathbf{RNS}_{-m}$. Hence, we have

$$\left\|T^{-1}(\bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}^0_m\right\| = \left\|\mathbf{S}'_m \mathbf{N}'\mathbf{R}' T^{-1}\ddot{\mathbf{U}}'\ddot{\mathbf{U}}\mathbf{RNS}_m\right\| \leq \|\mathbf{RNS}_m\|^2 \left\|T^{-1}\ddot{\mathbf{U}}'\ddot{\mathbf{U}}\right\| = O_p(N^{-1}),$$

since $\left\|T^{-1}\ddot{\mathbf{U}}'\ddot{\mathbf{U}}\right\| = O_p(N^{-1})$ by (D-3) of Lemma 9 and we have by definition that $\|\mathbf{R}\| = O_p(1)$ and $\|\mathbf{N}\|$ and $\|\mathbf{S}_m\|$ are $O(1)$. Similarly we obtain

$$\begin{aligned}
\left\|T^{-1}(\bar{\mathbf{U}}^0_{-m})'\bar{\mathbf{U}}^0_{-m}\right\| &= N\left\|\mathbf{S}'_{-m}\mathbf{N}'\mathbf{R}'T^{-1}\ddot{\mathbf{U}}'\ddot{\mathbf{U}}\mathbf{RNS}_{-m}\right\| \leq \|\mathbf{RNS}_{-m}\|^2 N \left\|T^{-1}\ddot{\mathbf{U}}'\ddot{\mathbf{U}}\right\| = O_p(1),\\
\left\|T^{-1}(\bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}^0_{-m}\right\| &= \sqrt{N}\left\|\mathbf{S}'_m\mathbf{N}'\mathbf{R}'T^{-1}\ddot{\mathbf{U}}'\ddot{\mathbf{U}}\mathbf{RNS}_{-m}\right\|,\\
&\leq \|\mathbf{RN}\|^2 \|\mathbf{S}_m\| \|\mathbf{S}_{-m}\| \sqrt{N}\left\|T^{-1}\ddot{\mathbf{U}}'\ddot{\mathbf{U}}\right\| = O_p(N^{-1/2}),
\end{aligned}$$

which proves (D-7). Moving on to (D-8), we have, noting that $\left\|\ddot{\mathbf{P}}\right\| = O_p(1)$,

$$\left\|T^{-1}(\mathbf{F}^*)'\mathbf{F}^*\right\| = \left\|\mathbf{S}'_m\mathbf{N}'\mathbf{R}'\ddot{\mathbf{P}}'T^{-1}\check{\mathbf{F}}'\check{\mathbf{F}}\ddot{\mathbf{P}}\mathbf{RNS}_m\right\| \leq \left\|\ddot{\mathbf{P}}\mathbf{RNS}_m\right\|^2 \left\|T^{-1}\check{\mathbf{F}}'\check{\mathbf{F}}\right\| = O_p(1),$$

$$\left\|T^{-1}(\bar{\mathbf{U}}_m^0)'\mathbf{F}^*\right\| = \left\|\mathbf{S}_m'\mathbf{N}'\mathbf{R}'T^{-1}\ddot{\mathbf{U}}'\check{\mathbf{F}}\ddot{\mathbf{P}}\mathbf{RNS}_m\right\| \leq \|\mathbf{RNS}_m\|^2 \left\|\ddot{\mathbf{P}}\right\| \left\|T^{-1}\ddot{\mathbf{U}}'\check{\mathbf{F}}\right\| = O_p((NT)^{-1/2}),$$
$$\left\|T^{-1}(\bar{\mathbf{U}}_{-m}^0)'\mathbf{F}^*\right\| = \sqrt{N}\left\|\mathbf{S}_{-m}'\mathbf{N}'\mathbf{R}'T^{-1}\ddot{\mathbf{U}}'\check{\mathbf{F}}\ddot{\mathbf{P}}\mathbf{RNS}_m\right\|,$$
$$\leq \|\mathbf{RN}\|^2 \|\mathbf{S}_m\| \|\mathbf{S}_{-m}\| \left\|\ddot{\mathbf{P}}\right\| \sqrt{N}\left\|T^{-1}\ddot{\mathbf{U}}'\check{\mathbf{F}}\right\| = O_p(T^{-1/2}),$$

where we have made use of (D-4) of Lemma 9. The second and third result in (D-9) follow analogously from (D-5) of Lemma 9 and, given (D-8), the first result in (D-9) follows from the definition $\bar{\mathbf{U}}^0 = [\bar{\mathbf{U}}_m^0, \bar{\mathbf{U}}_{-m}^0]$. Next up, making use of (D-6) gives

$$\left\|T^{-1}\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}_m^0\right\| = \left\|T^{-1}\boldsymbol{\epsilon}_i'\ddot{\mathbf{U}}\mathbf{RNS}_m\right\| \leq \|\mathbf{RNS}_m\| \left\|T^{-1}\boldsymbol{\epsilon}_i'\ddot{\mathbf{U}}\right\| = O_p(N^{-1}) + O_p((NT)^{-1/2}),$$

and similarly for $\left\|T^{-1}\boldsymbol{\varepsilon}_i'\bar{\mathbf{U}}_m^0\right\|$. Also

$$\left\|T^{-1}\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}_{-m}^0\right\| = \left\|T^{-1}\boldsymbol{\epsilon}_i'\ddot{\mathbf{U}}\mathbf{RNS}_{-m}\right\| \leq \|\mathbf{RNS}_{-m}\| \sqrt{N}\left\|T^{-1}\boldsymbol{\epsilon}_i'\ddot{\mathbf{U}}\right\| = O_p(N^{-1/2}) + O_p(T^{-1/2}),$$

with the argument being identical for $\left\|T^{-1}\boldsymbol{\varepsilon}_i'\bar{\mathbf{U}}_{-m}^0\right\|$. This establishes (D-10) and (D-11). Turning next to $T^{-1}\boldsymbol{\epsilon}_i'\mathbf{Q}_0$ of (D-12) we find making use of the definition in (B-16)

$$\left\|T^{-1}\boldsymbol{\epsilon}_i'\mathbf{Q}_0\right\| = \left\|T^{-1}\boldsymbol{\epsilon}_i'(\mathbf{F}^0 + \bar{\mathbf{U}}^0)\right\| \leq \left\|T^{-1}\boldsymbol{\epsilon}_i'\mathbf{F}^0\right\| + \left\|T^{-1}\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}^0\right\| = O_p(T^{-1/2}),$$

because $\|T^{-1}\boldsymbol{\epsilon}_i'\mathbf{F}^0\| = \left\|T^{-1}\boldsymbol{\epsilon}_i'[\mathbf{F}^*, \mathbf{0}_{T\times(K-m)}]\right\| = \left\|T^{-1}\boldsymbol{\epsilon}_i'\check{\mathbf{F}}\ddot{\mathbf{P}}\mathbf{RNS}_m\right\| \leq \left\|T^{-1}\boldsymbol{\epsilon}_i'\check{\mathbf{F}}\right\| \left\|\ddot{\mathbf{P}}\mathbf{RNS}_m\right\| = O_p(T^{-1/2})$ by (D-5) of Lemma 9 and because $\left\|T^{-1}\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}^0\right\| = \left\|T^{-1}\boldsymbol{\epsilon}_i'[\bar{\mathbf{U}}_m^0, \bar{\mathbf{U}}_{-m}^0]\right\| = O_p(N^{-1/2}) + O_p(T^{-1/2})$ by (D-10) and (D-11). $\|T^{-1}\boldsymbol{\varepsilon}_i'\mathbf{Q}_0\| = O_p(T^{-1/2})$ of (D-12) can be established in the same way. Finally, for $\left\|T^{-1}\mathbf{Q}_0'\bar{\mathbf{U}}_m^0\right\|$, making use of (D-7) and (D-8)

$$\left\|T^{-1}\mathbf{Q}_0'\bar{\mathbf{U}}_m^0\right\| = \left\|T^{-1}(\mathbf{F}^0 + \bar{\mathbf{U}}^0)'\bar{\mathbf{U}}_m^0\right\| \leq \left\|T^{-1}[\mathbf{F}^*, \mathbf{0}_{T\times(K-m)}]'\bar{\mathbf{U}}_m^0\right\| + \left\|T^{-1}[\bar{\mathbf{U}}_m^0, \bar{\mathbf{U}}_{-m}^0]'\bar{\mathbf{U}}_m^0\right\|,$$
$$\leq \left\|T^{-1}(\mathbf{F}^*)'\bar{\mathbf{U}}_m^0\right\| + \left\|T^{-1}[\bar{\mathbf{U}}_m^0, \bar{\mathbf{U}}_{-m}^0]'\bar{\mathbf{U}}_m^0\right\| = O_p(N^{-1/2}).$$

which then proves the final statement in (D-12), and therefore the lemma. □

Proof of Lemma 11

Note that substituting in $\ddot{\mathbf{P}}_i = \mathbf{P} + \check{\mathbf{P}}_i$ from eq.(B-7) gives by Ass.3 that $\widetilde{\mathbf{P}}_i = \check{\mathbf{P}}_i + O_p(N^{-1/2})$. Then, since the following matrix norms are identical

$$\left\|\frac{1}{N}\sum_{i=1}^{N}\left(T^{-1}\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}_m^0 \otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)\right\| = \left\|\frac{1}{N}\sum_{i=1}^{N}\left(\mathbf{S}_w'\widetilde{\mathbf{P}}_i' \otimes T^{-1}\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}_m^0\right)\right\|,$$

we will evaluate the second. Let $\widetilde{p}_{i,r,d}$ denote the element on row $r = 1, \ldots, k_w$ and column $d = 1, \ldots, 1 + K^2 m(1 + p^*)$ of $\mathbf{S}_w'\widetilde{\mathbf{P}}_i'$. Then the elements on rows $k_w(r-1)+1$ to $k_w r$ and columns $k_w(d-1)+1$ to $k_w d$ of the second Kronecker product are given by $\frac{1}{N}\sum_{i=1}^{N}\widetilde{p}_{i,r,d}\frac{\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}_m^0}{T}$. To evaluate these terms, consider that we can write, making use of (B-19) and (B-10),

$$\bar{\mathbf{U}}_m^0 = \bar{\mathbf{U}}_{m,-i}^0 + \frac{1}{N}\mathbf{U}_{m,i}^0,$$

where $\bar{\mathbf{U}}^0_{m,-i} = N^{-1}\sum_{j=1,j\neq i}^{N} \ddot{\mathbf{U}}_j\mathbf{RNS}_m$ and $\mathbf{U}^0_{m,i} = \ddot{\mathbf{U}}_i\mathbf{RNS}_m$. Hence

$$\left\|\frac{1}{N}\sum_{i=1}^{N}\widetilde{p}_{i,r,d}\frac{\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}^0_m}{T}\right\| \leq \left\|\frac{1}{N}\sum_{i=1}^{N}\widetilde{p}_{i,r,d}\left(\frac{\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}^0_{m,-i}}{T}\right)\right\| + \left\|\frac{1}{N^2}\sum_{i=1}^{N}\widetilde{p}_{i,r,d}\left(\frac{\boldsymbol{\epsilon}_i'\mathbf{U}^0_{m,i}}{T}\right)\right\|,$$
$$= O_p\left(\frac{1}{N\sqrt{T}}\right) + O_p\left(\frac{1}{N^{3/2}}\right),$$

because $T^{-1}\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}^0_{m,-i} = O_p((NT)^{-1/2})$, $T^{-1}\boldsymbol{\epsilon}_i'\mathbf{U}^0_{m,i} = O_p(1)$ and by Ass.3 $\frac{1}{N}\sum_{i=1}^{N}\widetilde{p}_{i,r,d} = O_p(N^{-1/2})$ with $\widetilde{p}_{i,r,d}$ independent of the other variables. Since this applies for all $r = 1,\ldots,k_w$ and $d = 1,\ldots,1+K^2m(1+p^*)$ we have

$$\left\|N^{-1}\sum_{i=1}^{N}\left(T^{-1}\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}^0_m \otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)\right\| = O_p\left(\frac{1}{N^{3/2}}\right) + O_p\left(\frac{1}{N\sqrt{T}}\right),$$

which is the result in (D-13), and (D-14) follows in similar fashion. In turn, to prove (D-15) we note that

$$\bar{\mathbf{U}}^0_{-m} = \sqrt{N}\bar{\mathbf{U}}^0_{-m,-i} + \frac{1}{\sqrt{N}}\mathbf{U}^0_{-m,i},$$

with $\bar{\mathbf{U}}^0_{-m,-i} = N^{-1}\sum_{j=1,j\neq i}^{N}\ddot{\mathbf{U}}_j\mathbf{RNS}_{-m}$ and $\mathbf{U}^0_{-m,i} = \ddot{\mathbf{U}}_i\mathbf{RNS}_{-m}$ such that for $r = 1,\ldots,k_w$ and column $d = 1,\ldots,1+K^2m(1+p^*)$ we have for the corresponding elements in the Kronecker product

$$\left\|\frac{1}{N}\sum_{i=1}^{N}\widetilde{p}_{i,r,d}\frac{\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}^0_{-m}}{T}\right\| \leq \sqrt{N}\left\|\frac{1}{N}\sum_{i=1}^{N}\widetilde{p}_{i,r,d}\left(\frac{\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}^0_{-m,-i}}{T}\right)\right\| + \sqrt{N}\left\|\frac{1}{N^2}\sum_{i=1}^{N}\widetilde{p}_{i,r,d}\left(\frac{\boldsymbol{\epsilon}_i'\mathbf{U}^0_{-m,i}}{T}\right)\right\|,$$
$$= O_p\left(\frac{1}{N}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right),$$

since also $T^{-1}\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}^0_{-m,-i} = O_p((NT)^{-1/2})$ and $T^{-1}\boldsymbol{\epsilon}_i'\mathbf{U}^0_{-m,i} = O_p(1)$. This implies (D-15) and the result in (D-16) can be established in the same way. Next up is (D-17). The elements on rows $k_w(r-1)+1$ to k_wr and columns $(1+K^2m(1+p^*))(d-1)+1$ to $(1+K^2m(1+p^*))d$ of $\frac{1}{N}\sum_{i=1}^{N}\left(\mathbf{S}_w'\widetilde{\mathbf{P}}_i' \otimes T^{-1}\boldsymbol{\epsilon}_i'\check{\mathbf{F}}\right)$ are given by

$$\frac{1}{N}\sum_{i=1}^{N}\widetilde{p}_{i,r,d}\frac{\boldsymbol{\epsilon}_i'\check{\mathbf{F}}}{T} = \frac{\bar{\mathbf{a}}_{r,d}'\check{\mathbf{F}}}{T} = O_p\left(\frac{1}{\sqrt{NT}}\right), \tag{D-31}$$

with $\bar{\mathbf{a}}_{r,d} = \frac{1}{N}\sum_{i=1}^{N}\widetilde{p}_{i,r,d}\boldsymbol{\epsilon}_i$ and $\|\bar{\mathbf{a}}_{r,d}\| = O_p(N^{-1/2})$ by the independence of $\widetilde{p}_{i,r,d}$ and $\boldsymbol{\epsilon}_i$ from Ass.1 and 3. The result then follows because also $\bar{\mathbf{a}}_{r,d}$ and $\check{\mathbf{F}}$ are independent stationary variables. Since (D-31) holds for every sub-matrix

$$\left\|\frac{1}{N}\sum_{i=1}^{N}\left(T^{-1}\boldsymbol{\epsilon}_i'\check{\mathbf{F}} \otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)\right\| = O_p\left(\frac{1}{\sqrt{NT}}\right),$$

with again an analogous argument for (D-18). The final result is found by noting that

$$\frac{1}{N}\sum_{i=1}^{N}\left(T^{-1}\boldsymbol{\varepsilon}_i'\mathbf{Q}_0 \otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right) = \frac{1}{N}\sum_{i=1}^{N}\left(T^{-1}\boldsymbol{\varepsilon}_i'\left([\mathbf{F}^*, \mathbf{0}_{T\times(K-m)}] + [\bar{\mathbf{U}}_m^0, \bar{\mathbf{U}}_{-m}^0]\right)\otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right),$$

such that since $\mathbf{F}^* = \check{\mathbf{F}}\ddot{\mathbf{P}}\mathbf{RNS}_m$ from (B-18), inserting the preceding results gives

$$\frac{1}{N}\sum_{i=1}^{N}\left(T^{-1}\boldsymbol{\varepsilon}_i'\mathbf{Q}_0 \otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right) = O_p\left(\frac{1}{N}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right).$$

Finally, given the independence of $\boldsymbol{\eta}_i$ from $\boldsymbol{\epsilon}_j, \boldsymbol{\varepsilon}_j$ and $\check{\mathbf{F}}$ for all i, j, t by Ass.3 all the stated results also hold true when $\mathbf{S}_w'\widetilde{\mathbf{P}}_i'$ is substituted for $\boldsymbol{\eta}_i'$. This establishes the lemma.

Proof of Lemma 12

Consider that by Assumptions 2 and 5, $\check{\mathbf{F}}$ is a matrix of covariance stationary variables with finite fourth moments. As such, the first result $\widehat{\boldsymbol{\Sigma}}_{\check{\mathbf{F}}} = T^{-1}\check{\mathbf{F}}'\check{\mathbf{F}} = \boldsymbol{\Sigma}_{\check{\mathbf{F}}} + O_p(T^{-1/2})$, with $\boldsymbol{\Sigma}_{\check{\mathbf{F}}} = E(T^{-1}\check{\mathbf{F}}'\check{\mathbf{F}})$ follows directly. Similarly, from Ass.1 and 5 follows that $\widehat{\boldsymbol{\Sigma}}_{\ddot{\mathbf{U}}} = T^{-1}N\ddot{\mathbf{U}}'\ddot{\mathbf{U}} = \boldsymbol{\Sigma}_{\ddot{\mathbf{U}}} + O_p(T^{-1/2})$, with $\boldsymbol{\Sigma}_{\ddot{\mathbf{U}}} = E(\ddot{\mathbf{U}}_i'\ddot{\mathbf{U}}_i/T)$ since error terms are independent over i. The second and third statements of the lemma are obtained by substituting in (B-18) and by making use of the first statement and $\ddot{\mathbf{P}} = \mathbf{P} + O_p(N^{-1/2})$ by Ass.3

$$\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*} = \frac{(\mathbf{F}^*)'\mathbf{F}^*}{T} = \mathbf{S}_m'\mathbf{N}'\mathbf{R}'\ddot{\mathbf{P}}'\widehat{\boldsymbol{\Sigma}}_{\check{\mathbf{F}}}\ddot{\mathbf{P}}\mathbf{RNS}_m = \mathbf{S}_m'\mathbf{N}'\mathbf{R}'\mathbf{P}'\boldsymbol{\Sigma}_{\check{\mathbf{F}}}\mathbf{PRNS}_m + O_p(N^{-1/2}) + O_p(T^{-1/2}),$$

$$\widehat{\boldsymbol{\Sigma}}_{\check{\mathbf{F}}\mathbf{F}^*} = \frac{\check{\mathbf{F}}'\mathbf{F}^*}{T} = \widehat{\boldsymbol{\Sigma}}_{\check{\mathbf{F}}}\ddot{\mathbf{P}}\mathbf{RNS}_m = \boldsymbol{\Sigma}_{\check{\mathbf{F}}}\mathbf{PRNS}_m + O_p(N^{-1/2}) + O_p(T^{-1/2}).$$

Since $\mathbf{F}^{*\prime}\mathbf{F}^*/T$ is by construction a $1+K+m \times 1+K+m$ full rank matrix we also have $\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*}^{-1} = \boldsymbol{\Sigma}_{\mathbf{F}^*}^{-1} + O_p(N^{-1/2}) + O_p(T^{-1/2})$. For the next result, consider that $\bar{\mathbf{U}}_{-m}^0 = \sqrt{N}\bar{\mathbf{U}}^*\mathbf{TB}_{-m}$, with $\bar{\mathbf{U}}^* = [\bar{\mathbf{u}}_1^*, \ldots, \bar{\mathbf{u}}_T^*]'$ and $\bar{\mathbf{u}}_t^* = \mathbf{A}_0^{-1}\bar{\mathbf{u}}_t$. Therefore, by Ass.1

$$\begin{aligned}
\widehat{\boldsymbol{\Sigma}}_{\mathbf{u}_{-m}^0} = T^{-1}(\bar{\mathbf{U}}_{-m}^0)'\bar{\mathbf{U}}_{-m}^0 &= \mathbf{B}_{-m}'\mathbf{T}'\mathbf{A}_0^{-1}\left(NT^{-1}\sum_{t=1}^{T}\bar{\mathbf{u}}_t\bar{\mathbf{u}}_t'\right)(\mathbf{A}_0^{-1})'\mathbf{TB}_{-m},\\
&= \mathbf{B}_{-m}'\mathbf{T}'\mathbf{A}_0^{-1}\boldsymbol{\Omega}_{\mathbf{u}}(\mathbf{A}_0^{-1})'\mathbf{TB}_{-m} + O_p(T^{-1/2}),\\
&= \boldsymbol{\Sigma}_{\mathbf{u}_{-m}^0} + O_p(T^{-1/2}),
\end{aligned}$$

where $\boldsymbol{\Sigma}_{\mathbf{u}_{-m}^0} = \mathbf{B}_{-m}'\mathbf{T}'\mathbf{A}_0^{-1}\boldsymbol{\Omega}_{\mathbf{u}}(\mathbf{A}_0^{-1})'\mathbf{TB}_{-m}$ is a $(K-m)\times(K-m)$ positive definite matrix because Ass.1 implies $\boldsymbol{\Omega}_{\mathbf{u}} = E(\mathbf{u}_{i,t}\mathbf{u}_{i,t}') = \begin{bmatrix}\sigma_\varepsilon^2 & \mathbf{0}_{k\times 1}' \\ \mathbf{0}_{k\times 1} & \boldsymbol{\Omega}_{\mathbf{v}}\end{bmatrix}$. Consequently also $\widehat{\boldsymbol{\Sigma}}_{\mathbf{u}_{-m}^0}^{-1} = \boldsymbol{\Sigma}_{\mathbf{u}_{-m}^0}^{-1} + O_p(T^{-1/2})$. Finally, the last result can be obtained by substituting in (B-19)-(B-20) and $\widehat{\boldsymbol{\Sigma}}_{\ddot{\mathbf{U}}} = \boldsymbol{\Sigma}_{\ddot{\mathbf{U}}} + O_p(T^{-1/2})$ as follows

$$\widehat{\boldsymbol{\Sigma}}_{\mathbf{u}_{-m}^0\mathbf{u}_m^0} = T^{-1}(\bar{\mathbf{U}}_{-m}^0)'\sqrt{N}\bar{\mathbf{U}}_m^0 = \mathbf{S}_{-m}'\mathbf{N}'\mathbf{R}'\widehat{\boldsymbol{\Sigma}}_{\ddot{\mathbf{U}}}\mathbf{RNS}_m = \mathbf{S}_{-m}'\mathbf{N}'\mathbf{R}'\boldsymbol{\Sigma}_{\ddot{\mathbf{U}}}\mathbf{RNS}_m + O_p(T^{-1/2}).$$

Proof of Lemma 13

Consider that by definition

$$\widehat{\mathbf{\Sigma}}_{\mathbf{Q}} = T^{-1}\mathbf{Q}_0'\mathbf{Q}_0 = T^{-1}(\mathbf{F}^0)'\mathbf{F}^0 + T^{-1}(\bar{\mathbf{U}}^0)'\mathbf{F}^0 + T^{-1}(\mathbf{F}^0)'\bar{\mathbf{U}}^0 + T^{-1}(\bar{\mathbf{U}}^0)'\bar{\mathbf{U}}^0,$$

with, since $\mathbf{F}^0 = [\mathbf{F}^*, \mathbf{0}_{T\times(K-m)}]$,

$$T^{-1}(\mathbf{F}^0)'\mathbf{F}^0 = \begin{bmatrix} \widehat{\mathbf{\Sigma}}_{\mathbf{F}^*} & \mathbf{0}_{(1+K+m)\times(K-m)} \\ \mathbf{0}_{(K-m)\times(1+K+m)} & \mathbf{0}_{(K-m)\times(K-m)} \end{bmatrix},$$

and also because by Lemma 10 we have $\left\|T^{-1}(\bar{\mathbf{U}}^0_{-m})'\mathbf{F}^*\right\| = O_p(T^{-1/2})$ and $\left\|T^{-1}(\bar{\mathbf{U}}^0_{m})'\mathbf{F}^*\right\| = O_p((NT)^{-1/2})$, it follows that

$$T^{-1}(\bar{\mathbf{U}}^0)'\mathbf{F}^0 + T^{-1}(\mathbf{F}^0)'\bar{\mathbf{U}}^0 = \begin{bmatrix} T^{-1}(\mathbf{F}^*)'\bar{\mathbf{U}}^0_m + T^{-1}(\bar{\mathbf{U}}^0_m)'\mathbf{F}^* & T^{-1}(\mathbf{F}^*)'\bar{\mathbf{U}}^0_{-m} \\ T^{-1}(\bar{\mathbf{U}}^0_{-m})'\mathbf{F}^* & \mathbf{0}_{(K-m)\times(K-m)} \end{bmatrix} = O_p\left(\frac{1}{\sqrt{T}}\right).$$

Next, making use of Lemma 10

$$\begin{aligned} T^{-1}(\bar{\mathbf{U}}^0)'\bar{\mathbf{U}}^0 &= \frac{1}{T}\begin{bmatrix} (\bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}^0_m & (\bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}^0_{-m} \\ (\bar{\mathbf{U}}^0_{-m})'\bar{\mathbf{U}}^0_m & (\bar{\mathbf{U}}^0_{-m})'\bar{\mathbf{U}}^0_{-m} \end{bmatrix}, \\ &= \begin{bmatrix} \mathbf{0}_{(1+K+m)\times(1+K+m)} & \mathbf{0}_{(1+K+m)\times(K-m)} \\ \mathbf{0}_{(K-m)\times(1+K+m)} & \widehat{\mathbf{\Sigma}}_{\mathbf{u}^0_{-m}} \end{bmatrix} + O_p(N^{-1/2}), \end{aligned}$$

and recalling from (D-2) that

$$\widehat{\mathbf{\Sigma}}_{\mathbf{F}^+_u} = \begin{bmatrix} \widehat{\mathbf{\Sigma}}_{\mathbf{F}^*} & \mathbf{0}_{(1+K+m)\times(K-m)} \\ \mathbf{0}_{(K-m)\times(1+K+m)} & \widehat{\mathbf{\Sigma}}_{\mathbf{u}^0_{-m}} \end{bmatrix}, \tag{D-32}$$

we have, given the results above

$$\begin{aligned} \widehat{\mathbf{\Sigma}}_{\mathbf{Q}} - \widehat{\mathbf{\Sigma}}_{\mathbf{F}^+_u} &= \begin{bmatrix} T^{-1}(\mathbf{F}^*)'\bar{\mathbf{U}}^0_m + T^{-1}(\bar{\mathbf{U}}^0_m)'\mathbf{F}^* & T^{-1}(\mathbf{F}^*)'\bar{\mathbf{U}}^0_{-m} \\ T^{-1}(\bar{\mathbf{U}}^0_{-m})'\mathbf{F}^* & \mathbf{0}_{(K-m)\times(K-m)} \end{bmatrix} \\ &\quad + T^{-1}\begin{bmatrix} (\bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}^0_m & (\bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}^0_{-m} \\ (\bar{\mathbf{U}}^0_{-m})'\bar{\mathbf{U}}^0_m & \mathbf{0}_{(K-m)\times(K-m)} \end{bmatrix}, \\ &= O_p(N^{-1/2}) + O_p(T^{-1/2}). \end{aligned} \tag{D-33}$$

Then, since $p^* = 1$ we have $rk(\widehat{\mathbf{\Sigma}}_{\mathbf{Q}}) = 1 + K(1+p^*) = 1+2K$, $rk(\widehat{\mathbf{\Sigma}}_{\mathbf{F}^*}) = 1+K+m$ and $rk(\widehat{\mathbf{\Sigma}}_{\mathbf{u}^0_{-m}}) = K-m$, such that for the block diagonal matrix $rk(\widehat{\mathbf{\Sigma}}_{\mathbf{F}^+_u}) = rk(\widehat{\mathbf{\Sigma}}_{\mathbf{F}^*}) + rk(\widehat{\mathbf{\Sigma}}_{\mathbf{u}^0_{-m}}) = 1+2K$. Therefore, by Theorem 1 of Karabiyik et al. (2017)

$$\widehat{\mathbf{\Sigma}}_{\mathbf{Q}}^{-1} = \widehat{\mathbf{\Sigma}}_{\mathbf{F}^+_u}^{-1} + O_p(N^{-1/2}) + O_p(T^{-1/2}).$$

This proves (D-20) of the lemma.

Moving on to the second statement, consider that from Lemma 12

$$\left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{u}^0_{-m}} - \boldsymbol{\Sigma}_{\mathbf{u}^0_{-m}}\right\| = O_p(T^{-1/2}),$$

where $\boldsymbol{\Sigma}_{\mathbf{u}^0_{-m}} = \mathbf{B}'_{-m}\mathbf{T}'\mathbf{A}_0^{-1}\boldsymbol{\Omega}_{\mathbf{u}}(\mathbf{A}_0^{-1})'\mathbf{T}\mathbf{B}_{-m}$ is a $(K-m)\times(K-m)$ positive definite matrix. Consider also from Lemma 12 that $\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*} = \boldsymbol{\Sigma}_{\mathbf{F}^*} + O_p(N^{-1/2}) + O_p(T^{-1/2})$, with $\boldsymbol{\Sigma}_{\mathbf{F}^*}$ a $(1+K+m)\times(1+K+m)$ full rank matrix. Accordingly, we have denoting

$$\boldsymbol{\Sigma}_{\mathbf{F}_u^+} = \begin{bmatrix} \boldsymbol{\Sigma}_{\mathbf{F}^*} & \mathbf{0}_{(1+K+m)\times(K-m)} \\ \mathbf{0}_{(K-m)\times(1+K+m)} & \boldsymbol{\Sigma}_{\mathbf{u}^0_{-m}} \end{bmatrix},$$

that

$$\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+} = \boldsymbol{\Sigma}_{\mathbf{F}_u^+} + O_p(N^{-1/2}) + O_p(T^{-1/2}),$$

and since $rk(\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+}) = rk(\boldsymbol{\Sigma}_{\mathbf{F}_u^+})$ also

$$\widehat{\boldsymbol{\Sigma}}^{-1}_{\mathbf{F}_u^+} = \boldsymbol{\Sigma}^{-1}_{\mathbf{F}_u^+} + O_p(N^{-1/2}) + O_p(T^{-1/2}),$$

with

$$\boldsymbol{\Sigma}^{-1}_{\mathbf{F}_u^+} = \begin{bmatrix} \boldsymbol{\Sigma}^{-1}_{\mathbf{F}^*} & \mathbf{0}_{(1+K+m)\times(K-m)} \\ \mathbf{0}_{(K-m)\times(1+K+m)} & \boldsymbol{\Sigma}^{-1}_{\mathbf{u}^0_{-m}} \end{bmatrix}, \tag{D-34}$$

which implies in turn, making use of (D-20) that

$$\widehat{\boldsymbol{\Sigma}}^{-1}_{\mathbf{Q}} = \boldsymbol{\Sigma}^{-1}_{\mathbf{F}_u^+} + O_p(N^{-1/2}) + O_p(T^{-1/2}).$$

Consider then the following identity

$$\sqrt{T}\left[\widehat{\boldsymbol{\Sigma}}^{-1}_{\mathbf{Q}} - \widehat{\boldsymbol{\Sigma}}^{-1}_{\mathbf{F}_u^+}\right] = -\widehat{\boldsymbol{\Sigma}}^{-1}_{\mathbf{Q}}\sqrt{T}\left[\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}} - \widehat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+}\right]\widehat{\boldsymbol{\Sigma}}^{-1}_{\mathbf{F}_u^+}, \tag{D-35}$$

such that by the results above

$$\sqrt{T}\left[\widehat{\boldsymbol{\Sigma}}^{-1}_{\mathbf{Q}} - \widehat{\boldsymbol{\Sigma}}^{-1}_{\mathbf{F}_u^+}\right] = -\boldsymbol{\Sigma}^{-1}_{\mathbf{F}_u^+}\sqrt{T}\left[\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}} - \widehat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+}\right]\boldsymbol{\Sigma}^{-1}_{\mathbf{F}_u^+} + O_p(N^{-1/2}) + O_p(T^{-1/2}).$$

Using (D-33) we find for the middle term, also making use of Lemma 10,

$$\begin{aligned}
\sqrt{T}\left[\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}} - \widehat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+}\right] &= \frac{1}{\sqrt{T}}\begin{bmatrix} (\mathbf{F}^*)'\bar{\mathbf{U}}^0_m + (\bar{\mathbf{U}}^0_m)'\mathbf{F}^* & (\mathbf{F}^*)'\bar{\mathbf{U}}^0_{-m} \\ (\bar{\mathbf{U}}^0_{-m})'\mathbf{F}^* & \mathbf{0}_{(K-m)\times(K-m)} \end{bmatrix} \\
&\quad + \frac{1}{\sqrt{T}}\begin{bmatrix} (\bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}^0_m & (\bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}^0_{-m} \\ (\bar{\mathbf{U}}^0_{-m})'\bar{\mathbf{U}}^0_m & \mathbf{0}_{(K-m)\times(K-m)} \end{bmatrix}, \\
&= \frac{1}{\sqrt{T}}\begin{bmatrix} \mathbf{0}_{(1+K+m)\times(1+K+m)} & (\mathbf{F}^*)'\bar{\mathbf{U}}^0_{-m} \\ (\bar{\mathbf{U}}^0_{-m})'\mathbf{F}^* & \mathbf{0}_{(K-m)\times(K-m)} \end{bmatrix} + O_p\left(\frac{1}{\sqrt{N}}\right) \\
&\quad + \frac{1}{\sqrt{T}}\begin{bmatrix} \mathbf{0}_{(1+K+m)\times(1+K+m)} & (\bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}^0_{-m} \\ (\bar{\mathbf{U}}^0_{-m})'\bar{\mathbf{U}}^0_m & \mathbf{0}_{(K-m)\times(K-m)} \end{bmatrix} + O_p\left(\frac{\sqrt{T}}{N}\right),
\end{aligned}$$

$$= \frac{1}{\sqrt{T}} \begin{bmatrix} \mathbf{0}_{(1+K+m)\times(1+K+m)} & (\mathbf{F}^* + \bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}^0_{-m} \\ (\bar{\mathbf{U}}^0_{-m})'(\mathbf{F}^* + \bar{\mathbf{U}}^0_m) & \mathbf{0}_{(K-m)\times(K-m)} \end{bmatrix} + O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{\sqrt{T}}{N}\right).$$

Hence

$$\sqrt{T}\left[\widehat{\boldsymbol{\Sigma}}^{-1}_{\mathbf{Q}} - \widehat{\boldsymbol{\Sigma}}^{-1}_{\mathbf{F}^+_u}\right] = -\boldsymbol{\Sigma}^{-1}_{\mathbf{F}^+_u} \frac{1}{\sqrt{T}} \begin{bmatrix} \mathbf{0}_{(1+K+m)\times(1+K+m)} & (\mathbf{F}^* + \bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}^0_{-m} \\ (\bar{\mathbf{U}}^0_{-m})'(\mathbf{F}^* + \bar{\mathbf{U}}^0_m) & \mathbf{0}_{(K-m)\times(K-m)} \end{bmatrix} \boldsymbol{\Sigma}^{-1}_{\mathbf{F}^+_u} + O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\frac{\sqrt{T}}{N}\right),$$

and making use of (D-34)

$$\sqrt{T}\left[\widehat{\boldsymbol{\Sigma}}^{-1}_{\mathbf{Q}} - \widehat{\boldsymbol{\Sigma}}^{-1}_{\mathbf{F}^+_u}\right] = -\frac{1}{\sqrt{T}} \begin{bmatrix} \mathbf{0}_{(1+K+m)\times(1+K+m)} & \boldsymbol{\Sigma}^{-1}_{\mathbf{F}^*}(\mathbf{F}^* + \bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}^0_{-m} \\ \boldsymbol{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}(\bar{\mathbf{U}}^0_{-m})'(\mathbf{F}^* + \bar{\mathbf{U}}^0_m) & \mathbf{0}_{(K-m)\times(K-m)} \end{bmatrix} \boldsymbol{\Sigma}^{-1}_{\mathbf{F}^+_u} + O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\frac{\sqrt{T}}{N}\right),$$

$$= -\frac{1}{\sqrt{T}} \begin{bmatrix} \mathbf{0}_{(1+K+m)\times(1+K+m)} & \boldsymbol{\Sigma}^{-1}_{\mathbf{F}^*}(\mathbf{F}^* + \bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}^0_{-m}\boldsymbol{\Sigma}^{-1}_{\mathbf{u}^0_{-m}} \\ \boldsymbol{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}(\bar{\mathbf{U}}^0_{-m})'(\mathbf{F}^* + \bar{\mathbf{U}}^0_m)\boldsymbol{\Sigma}^{-1}_{\mathbf{F}^*} & \mathbf{0}_{(K-m)\times(K-m)} \end{bmatrix} + O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\frac{\sqrt{T}}{N}\right),$$

which, by defining $\widehat{\boldsymbol{\Sigma}}_{\mathbf{Fu}} = \boldsymbol{\Sigma}^{-1}_{\mathbf{F}^*}(\mathbf{F}^* + \bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}^0_{-m}\boldsymbol{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}$, can be written more compactly as

$$\sqrt{T}\left[\widehat{\boldsymbol{\Sigma}}^{-1}_{\mathbf{Q}} - \widehat{\boldsymbol{\Sigma}}^{-1}_{\mathbf{F}^+_u}\right] = -\begin{bmatrix} \mathbf{0}_{(1+K+m)\times(1+K+m)} & T^{-1/2}\widehat{\boldsymbol{\Sigma}}_{\mathbf{Fu}} \\ T^{-1/2}\widehat{\boldsymbol{\Sigma}}'_{\mathbf{Fu}} & \mathbf{0}_{(K-m)\times(K-m)} \end{bmatrix} + O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\frac{\sqrt{T}}{N}\right).$$

This is the result in (D-21).

Proof of Lemma 14

Consider $\mathbf{A}^{\mathbf{F}} = \frac{1}{NT}\sum_{i=1}^N \mathbf{w}'_i\mathbf{MF}\boldsymbol{\gamma}_i$. Under Ass.4 and assuming that $p^* \geq p$ we can substitute in (B-23) to obtain

$$\mathbf{A}^{\mathbf{F}} = \frac{1}{NT}\sum_{i=1}^N \mathbf{w}'_i\mathbf{MF}\boldsymbol{\gamma}_i = -\frac{1}{NT}\sum_{i=1}^N \mathbf{w}'_i\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\gamma}_i,$$

and also, by eq.(5) of Ass.3,

$$\mathbf{A}^{\mathbf{F}} = -\frac{1}{NT}\sum_{i=1}^N \mathbf{w}'_i\mathbf{M}\bar{\mathbf{U}}_m(\boldsymbol{\gamma} + \boldsymbol{\eta}_i) = -\frac{1}{T}\bar{\mathbf{w}}'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\gamma} - \frac{1}{NT}\sum_{i=1}^N \mathbf{w}'_i\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i,$$

$$= -\frac{1}{NT}\sum_{i=1}^N \mathbf{w}'_i\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i,$$

because $\mathbf{M}\bar{\mathbf{w}} = \mathbf{0}_{T\times k_w}$ since $\bar{\mathbf{w}} \subseteq \mathbf{Q}$. Substituting in (B-13) gives

$$\mathbf{A}^{\mathbf{F}} = -(\mathbf{A}_1^{\mathbf{F}} + \mathbf{A}_2^{\mathbf{F}}), \tag{D-36}$$

with $\mathbf{A}_1^{\mathbf{F}} = \frac{1}{NT}\sum_{i=1}^N \mathbf{S}_w'\ddot{\mathbf{P}}_i'\check{\mathbf{F}}'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i$ and $\mathbf{A}_2^{\mathbf{F}} = \frac{1}{NT}\sum_{i=1}^N \boldsymbol{\epsilon}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i$.
We start by evaluating $\mathbf{A}_2^{\mathbf{F}}$, and make use of $\mathbf{M} = \mathbf{I}_T - \mathbf{Q}(\mathbf{Q}'\mathbf{Q})^{\dagger}\mathbf{Q}' = \mathbf{I}_T - \mathbf{Q}_0(\mathbf{Q}_0'\mathbf{Q}_0)^{-1}\mathbf{Q}_0'$

$$\left\|\mathbf{A}_2^{\mathbf{F}}\right\| \le \left\|\frac{1}{N}\sum_{i=1}^N \frac{\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}_m}{T}\boldsymbol{\eta}_i\right\| + \left\|\frac{1}{N}\sum_{i=1}^N \frac{\boldsymbol{\epsilon}_i'\mathbf{Q}_0}{T}\left(\frac{\mathbf{Q}_0'\mathbf{Q}_0}{T}\right)^{-1}\frac{\mathbf{Q}_0'\bar{\mathbf{U}}_m}{T}\boldsymbol{\eta}_i\right\| = \left\|\mathbf{A}_{21}^{\mathbf{F}}\right\| + \left\|\mathbf{A}_{22}^{\mathbf{F}}\right\|$$

with obvious definitions for $\mathbf{A}_{21}^{\mathbf{F}}$ and $\mathbf{A}_{22}^{\mathbf{F}}$. Taking on first $\mathbf{A}_{21}^{\mathbf{F}}$, note that

$$\mathbf{A}_{21}^{\mathbf{F}} = \left[\frac{1}{N}\sum_{i=1}^N \left(\boldsymbol{\eta}_i' \otimes \frac{\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}_m}{T}\right)\right] vec(\mathbf{I}_m),$$

and therefore, by eq.(D-13) of Lemma 11,

$$\left\|\mathbf{A}_{21}^{\mathbf{F}}\right\| \le \left\|\frac{1}{N}\sum_{i=1}^N \left(\boldsymbol{\eta}_i' \otimes T^{-1}\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}_m\right)\right\| \|\mathbf{I}_m\| = O_p(N^{3/2}) + O_p(N^{-1}T^{-1/2}).$$

Next up is, $\mathbf{A}_{22}^{\mathbf{F}}$. We find

$$\begin{aligned}
\left\|\mathbf{A}_{22}^{\mathbf{F}}\right\| &\le \left\|\frac{1}{N}\sum_{i=1}^N \left(\boldsymbol{\eta}_i' \otimes \frac{\boldsymbol{\epsilon}_i'\mathbf{Q}_0}{T}\right)\right\| \left\|\left(\frac{\mathbf{Q}_0'\mathbf{Q}_0}{T}\right)^{-1}\right\| \left\|\frac{\mathbf{Q}_0'\bar{\mathbf{U}}_m}{T}\right\|, \\
&\le \left\|\frac{1}{N}\sum_{i=1}^N \left(\boldsymbol{\eta}_i' \otimes \frac{\boldsymbol{\epsilon}_i'(\mathbf{F}^0 + \bar{\mathbf{U}}^0)}{T}\right)\right\| \left\|\left(\frac{\mathbf{Q}_0'\mathbf{Q}_0}{T}\right)^{-1}\right\| \left\|\frac{\mathbf{Q}_0'\bar{\mathbf{U}}_m}{T}\right\|, \\
&= O_p\left(\frac{1}{N^{3/2}}\right) + O_p\left(\frac{1}{N\sqrt{T}}\right),
\end{aligned}$$

because $\|(T^{-1}\mathbf{Q}_0'\mathbf{Q}_0)^{-1}\| = O_p(1)$, $\left\|T^{-1}\mathbf{Q}_0'\bar{\mathbf{U}}_m^0\right\| = O_p(N^{-1/2})$ by (D-12) of Lemma 10 and

$$\begin{aligned}
&\left\|\frac{1}{N}\sum_{i=1}^N \left(\boldsymbol{\eta}_i' \otimes T^{-1}\boldsymbol{\epsilon}_i'(\mathbf{F}^0 + \bar{\mathbf{U}}^0)\right)\right\| \\
&\quad \le \left\|\frac{1}{N}\sum_{i=1}^N \left(\boldsymbol{\eta}_i' \otimes T^{-1}\boldsymbol{\epsilon}_i'\mathbf{F}^0\right)\right\| + \left\|\frac{1}{N}\sum_{i=1}^N \left(\boldsymbol{\eta}_i' \otimes T^{-1}\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}^0\right)\right\|, \\
&\quad \le \left\|\frac{1}{N}\sum_{i=1}^N \left(\boldsymbol{\eta}_i' \otimes T^{-1}\boldsymbol{\epsilon}_i'[\mathbf{F}^*, \mathbf{0}_{T\times(K-m)}]\right)\right\| + \left\|\frac{1}{N}\sum_{i=1}^N \left(\boldsymbol{\eta}_i' \otimes T^{-1}\boldsymbol{\epsilon}_i'[\bar{\mathbf{U}}_m^0, \bar{\mathbf{U}}_{-m}^0]\right)\right\|, \\
&\quad = O_p\left(\frac{1}{N}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right),
\end{aligned}$$

by (D-13), (D-15) and (D-17) of Lemma 11. It follows that

$$\left\|\mathbf{A}_2^{\mathbf{F}}\right\| = O_p\left(\frac{1}{N^{3/2}}\right) + O_p\left(\frac{1}{N\sqrt{T}}\right). \tag{D-37}$$

Next up is $\mathbf{A}_1^{\mathbf{F}}$. Recalling that $\mathbf{M}_{\mathbf{F}} = \mathbf{I}_T - \mathbf{H}_{\mathbf{F}}$ and $\mathbf{H}_{\mathbf{F}} = \mathbf{F}^*((\mathbf{F}^*)'\mathbf{F}^*)^{-1}(\mathbf{F}^*)'$ we can decompose it as

$$\begin{aligned}\mathbf{A}_1^{\mathbf{F}} &= \frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}_w'\ddot{\mathbf{P}}_i'\check{\mathbf{F}}'\bar{\mathbf{U}}_m\boldsymbol{\eta}_i - \frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}_w'\ddot{\mathbf{P}}_i'\check{\mathbf{F}}'\mathbf{H}_{\mathbf{F}}\bar{\mathbf{U}}_m\boldsymbol{\eta}_i - \frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}_w'\ddot{\mathbf{P}}_i'\check{\mathbf{F}}'(\mathbf{M}_{\mathbf{F}}-\mathbf{M})\bar{\mathbf{U}}_m\boldsymbol{\eta}_i,\\ &= \mathbf{A}_{11}^{\mathbf{F}} - \mathbf{A}_{12}^{\mathbf{F}} - \mathbf{A}_{13}^{\mathbf{F}},\end{aligned}$$

with obvious definitions for $\mathbf{A}_{11}^{\mathbf{F}}$, $\mathbf{A}_{12}^{\mathbf{F}}$ and $\mathbf{A}_{13}^{\mathbf{F}}$. For the first two terms we find

$$\begin{aligned}\left\|\mathbf{A}_{11}^{\mathbf{F}}\right\| &\le \left\|\frac{1}{N}\sum_{i=1}^{N}\left(\boldsymbol{\eta}_i'\otimes\mathbf{S}_w'\ddot{\mathbf{P}}_i'\right)\right\|\left\|\frac{\check{\mathbf{F}}'\bar{\mathbf{U}}_m}{T}\right\| = O_p\left(\frac{1}{\sqrt{NT}}\right),\\ \left\|\mathbf{A}_{12}^{\mathbf{F}}\right\| &\le \left\|\frac{1}{N}\sum_{i=1}^{N}\left(\boldsymbol{\eta}_i'\otimes\mathbf{S}_w'\ddot{\mathbf{P}}_i'\right)\right\|\left\|\frac{\check{\mathbf{F}}'\mathbf{F}^*}{T}\right\|\left\|\left(\frac{(\mathbf{F}^*)'\mathbf{F}^*}{T}\right)^{-1}\right\|\left\|\frac{(\mathbf{F}^*)'\bar{\mathbf{U}}_m}{T}\right\| = O_p\left(\frac{1}{\sqrt{NT}}\right),\end{aligned}$$

since $\left\|\frac{1}{N}\sum_{i=1}^{N}\left(\boldsymbol{\eta}_i'\otimes\mathbf{S}_w'\ddot{\mathbf{P}}_i'\right)\right\| = O_p(1)$, and Lemmas 9 and 10 show that $\left\|T^{-1}(\mathbf{F}^*)'\bar{\mathbf{U}}_m\right\| = O_p((NT)^{-1/2})$, $\left\|T^{-1}\check{\mathbf{F}}'\mathbf{F}^*\right\| = O_p(1)$ and $\|(T^{-1}(\mathbf{F}^*)'\mathbf{F}^*)^{-1}\| = O_p(1)$.

Next is $\mathbf{A}_{13}^{\mathbf{F}}$. Making use of (D-1) gives the following decomposition

$$\mathbf{A}_{13}^{\mathbf{F}} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{S}_w'\ddot{\mathbf{P}}_i'\left[\mathbf{A}_{131}^{\mathbf{F}} + \mathbf{A}_{132}^{\mathbf{F}} + \mathbf{A}_{133}^{\mathbf{F}} + \mathbf{A}_{134}^{\mathbf{F}} + \mathbf{A}_{135}^{\mathbf{F}}\right]\boldsymbol{\eta}_i,$$

with, defining $\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}} = [T^{-1}\mathbf{Q}_0'\mathbf{Q}_0]$,

$$\begin{aligned}\mathbf{A}_{131}^{\mathbf{F}} &= T^{-1}\check{\mathbf{F}}'\bar{\mathbf{U}}_{-m}^0[T^{-1}(\bar{\mathbf{U}}_{-m}^0)'\bar{\mathbf{U}}_{-m}^0]^{-1}T^{-1}(\bar{\mathbf{U}}_{-m}^0)'\bar{\mathbf{U}}_m,\\ \mathbf{A}_{132}^{\mathbf{F}} &= T^{-1}\check{\mathbf{F}}'\bar{\mathbf{U}}_m^0[T^{-1}(\mathbf{F}^*)'\mathbf{F}^*]^{-1}T^{-1}(\bar{\mathbf{U}}_m^0)'\bar{\mathbf{U}}_m,\\ \mathbf{A}_{133}^{\mathbf{F}} &= T^{-1}\check{\mathbf{F}}'\mathbf{F}^*[T^{-1}(\mathbf{F}^*)'\mathbf{F}^*]^{-1}T^{-1}(\bar{\mathbf{U}}_m^0)'\bar{\mathbf{U}}_m,\\ \mathbf{A}_{134}^{\mathbf{F}} &= T^{-1}\check{\mathbf{F}}'\bar{\mathbf{U}}_m^0[T^{-1}(\mathbf{F}^*)'\mathbf{F}^*]^{-1}T^{-1}(\mathbf{F}^*)'\bar{\mathbf{U}}_m,\\ \mathbf{A}_{135}^{\mathbf{F}} &= T^{-1}\check{\mathbf{F}}'\mathbf{Q}_0\left(\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1} - \widehat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+}^{-1}\right)T^{-1}\mathbf{Q}_0'\bar{\mathbf{U}}_m,\end{aligned}$$

which yields, by Lemma 10,

$$\begin{aligned}\left\|\mathbf{A}_{131}^{\mathbf{F}}\right\| &\le \left\|T^{-1}\check{\mathbf{F}}'\bar{\mathbf{U}}_{-m}^0\right\|\left\|[T^{-1}(\bar{\mathbf{U}}_{-m}^0)'\bar{\mathbf{U}}_{-m}^0]^{-1}\right\|\left\|T^{-1}(\bar{\mathbf{U}}_{-m}^0)'\bar{\mathbf{U}}_m\right\| = O_p((NT)^{-1/2}),\\ \left\|\mathbf{A}_{132}^{\mathbf{F}}\right\| &\le \left\|T^{-1}\check{\mathbf{F}}'\bar{\mathbf{U}}_m^0\right\|\left\|[T^{-1}(\mathbf{F}^*)'\mathbf{F}^*]^{-1}\right\|\left\|T^{-1}(\bar{\mathbf{U}}_m^0)'\bar{\mathbf{U}}_m\right\| = O_p(N^{-3/2}T^{-1/2}),\\ \left\|\mathbf{A}_{133}^{\mathbf{F}}\right\| &\le \left\|T^{-1}\check{\mathbf{F}}'\mathbf{F}^*\right\|\left\|[T^{-1}(\mathbf{F}^*)'\mathbf{F}^*]^{-1}\right\|\left\|T^{-1}(\bar{\mathbf{U}}_m^0)'\bar{\mathbf{U}}_m\right\| = O_p(N^{-1}),\\ \left\|\mathbf{A}_{134}^{\mathbf{F}}\right\| &\le \left\|T^{-1}\check{\mathbf{F}}'\bar{\mathbf{U}}_m^0\right\|\left\|[T^{-1}(\mathbf{F}^*)'\mathbf{F}^*]^{-1}\right\|\left\|T^{-1}(\mathbf{F}^*)'\bar{\mathbf{U}}_m\right\| = O_p((NT)^{-1}),\\ \left\|\mathbf{A}_{135}^{\mathbf{F}}\right\| &\le \left\|T^{-1}\check{\mathbf{F}}'\mathbf{Q}_0\right\|\left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1} - \widehat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+}^{-1}\right\|\left\|T^{-1}\mathbf{Q}_0'\bar{\mathbf{U}}_m\right\| = O_p(N^{-1}) + O_p((NT)^{-1/2}),\end{aligned}$$

because also $\left\|T^{-1}\check{\mathbf{F}}'\mathbf{Q}_0\right\| = \left\|T^{-1}\check{\mathbf{F}}'([\mathbf{F}^*, \mathbf{0}_{T\times(K-m)}] + \bar{\mathbf{U}}^0)\right\| = O_p(1)$ and $\left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1} - \widehat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+}^{-1}\right\| = O_p(N^{-1/2}) + O_p(T^{-1/2})$ from (D-20) of Lemma 13. Hence,

$$\left\|\mathbf{A}_{13}^{\mathbf{F}}\right\| = O_p(N^{-1}) + O_p((NT)^{-1/2}),$$

which implies, in turn $\left\|\mathbf{A}_1^{\mathbf{F}}\right\| = O_p(N^{-1}) + O_p((NT)^{-1/2})$, and therefore, combining results for $\left\|\mathbf{A}_1^{\mathbf{F}}\right\|$ and $\left\|\mathbf{A}_2^{\mathbf{F}}\right\|$ in (D-36)

$$\left\|\mathbf{A}^{\mathbf{F}}\right\| \leq \left\|\mathbf{A}_1^{\mathbf{F}}\right\| + \left\|\mathbf{A}_2^{\mathbf{F}}\right\| = O_p(N^{-1}) + O_p((NT)^{-1/2}),$$

which is the result stated in eq.(D-22).

Next, let $\mathbf{A}_{NT}^{\mathbf{F}} = \sqrt{NT}\mathbf{A}^{\mathbf{F}}$ such that by the results above

$$\mathbf{A}_{NT}^{\mathbf{F}} = \mathbf{A}_{NT,1}^{\mathbf{F}} + \mathbf{A}_{NT,2}^{\mathbf{F}} + \mathbf{A}_{NT,3}^{\mathbf{F}} + \mathbf{A}_{NT,4}^{\mathbf{F}} + \mathbf{A}_{NT,5}^{\mathbf{F}} + O_p(N^{-1/2}) + O_p(\sqrt{T}N^{-1}),$$

with

$$\begin{aligned}
\mathbf{A}_{NT,1}^{\mathbf{F}} &= -\frac{1}{N}\sum_{i=1}^{N}\mathbf{S}_w'\ddot{\mathbf{P}}_i'\frac{\check{\mathbf{F}}'\sqrt{N}\bar{\mathbf{U}}_m}{\sqrt{T}}\boldsymbol{\eta}_i,\\
\mathbf{A}_{NT,2}^{\mathbf{F}} &= \frac{1}{N}\sum_{i=1}^{N}\mathbf{S}_w'\ddot{\mathbf{P}}_i'\frac{\check{\mathbf{F}}'\mathbf{F}^*}{T}\left(\frac{(\mathbf{F}^*)'\mathbf{F}^*}{T}\right)^{-1}\frac{(\mathbf{F}^*)'\sqrt{N}\bar{\mathbf{U}}_m}{\sqrt{T}}\boldsymbol{\eta}_i,\\
\mathbf{A}_{NT,3}^{\mathbf{F}} &= \frac{1}{N}\sum_{i=1}^{N}\mathbf{S}_w'\ddot{\mathbf{P}}_i'\frac{\check{\mathbf{F}}'\bar{\mathbf{U}}_{-m}^0}{\sqrt{T}}\left(\frac{(\bar{\mathbf{U}}_{-m}^0)'\bar{\mathbf{U}}_{-m}^0}{T}\right)^{-1}\frac{(\bar{\mathbf{U}}_{-m}^0)'\sqrt{N}\bar{\mathbf{U}}_m}{T}\boldsymbol{\eta}_i,\\
\mathbf{A}_{NT,4}^{\mathbf{F}} &= \frac{1}{N}\sum_{i=1}^{N}\mathbf{S}_w'\ddot{\mathbf{P}}_i'\frac{\check{\mathbf{F}}'\mathbf{F}^*}{T}\left(\frac{(\mathbf{F}^*)'\mathbf{F}^*}{T}\right)^{-1}\frac{(\bar{\mathbf{U}}_m^0)'\sqrt{N}\bar{\mathbf{U}}_m}{\sqrt{T}}\boldsymbol{\eta}_i,\\
\mathbf{A}_{NT,5}^{\mathbf{F}} &= \frac{1}{N}\sum_{i=1}^{N}\mathbf{S}_w'\ddot{\mathbf{P}}_i'\frac{\check{\mathbf{F}}'\mathbf{Q}_0}{T}\sqrt{T}\left[\hat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1} - \hat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+}^{-1}\right]\frac{\mathbf{Q}_0'\sqrt{N}\bar{\mathbf{U}}_m}{T}\boldsymbol{\eta}_i.
\end{aligned}$$

Taking on first $\mathbf{A}_{NT,1}^{\mathbf{F}}$ we substitute in $\bar{\mathbf{U}}_m = \ddot{\mathbf{U}}\mathbf{R}_0\mathbf{T}\mathbf{B}_m$ by eq.(B-22) and write

$$\begin{aligned}
\mathbf{A}_{NT,1}^{\mathbf{F}} &= -\left[\frac{1}{N}\sum_{i=1}^{N}\left(\boldsymbol{\eta}_i'\otimes\mathbf{S}_w'\ddot{\mathbf{P}}_i'\right)\right] vec\left(\frac{\check{\mathbf{F}}'\sqrt{N}\bar{\mathbf{U}}_m}{\sqrt{T}}\right)\\
&= -\left[\frac{1}{N}\sum_{i=1}^{N}\left(\boldsymbol{\eta}_i'\otimes\mathbf{S}_w'\ddot{\mathbf{P}}_i'\right)\right]\left[\mathbf{B}_m'\mathbf{T}'\mathbf{R}_0'\otimes\mathbf{I}_{1+K^2m(1+p^*)}\right] vec\left(\frac{\check{\mathbf{F}}'\sqrt{N}\ddot{\mathbf{U}}}{\sqrt{T}}\right),
\end{aligned}$$

such that denoting

$$\boldsymbol{\Sigma}_{\boldsymbol{\eta}} = E\left(\boldsymbol{\eta}_i'\otimes\mathbf{S}_w'\ddot{\mathbf{P}}_i'\right),$$

which we note exists and is bounded by Ass.3, we have $\left[\frac{1}{N}\sum_{i=1}^{N}\left(\boldsymbol{\eta}_i'\otimes\mathbf{S}_w'\ddot{\mathbf{P}}_i'\right)\right] = \boldsymbol{\Sigma}_{\boldsymbol{\eta}} + O_p(N^{-1/2})$, and therefore

$$\mathbf{A}_{NT,1}^{\mathbf{F}} = -\boldsymbol{\Sigma}_{\boldsymbol{\eta}}\left[\mathbf{B}_m'\mathbf{T}'\mathbf{R}_0'\otimes\mathbf{I}_{1+K^2m(1+p^*)}\right] vec\left(T^{-1/2}\check{\mathbf{F}}'\sqrt{N}\ddot{\mathbf{U}}\right) + O_p(N^{-1/2}).$$

Next, for $\mathbf{A}_{NT,2}^{\mathbf{F}}$ we can write using (B-22) and (B-18) that $(\mathbf{F}^*)'\bar{\mathbf{U}}_m = \mathbf{S}_m'\mathbf{N}'\mathbf{R}'\ddot{\mathbf{P}}'\check{\mathbf{F}}'\ddot{\mathbf{U}}\mathbf{R}_0\mathbf{T}\mathbf{B}_m$ and substitute it into the expression to give

$$\mathbf{A}_{NT,2}^{\mathbf{F}} = \left[\frac{1}{N}\sum_{i=1}^{N}\left(\boldsymbol{\eta}_i'\otimes\mathbf{S}_w'\ddot{\mathbf{P}}_i'\right)\right]\left[\mathbf{I}_m\otimes\frac{\check{\mathbf{F}}'\mathbf{F}^*}{T}\left(\frac{(\mathbf{F}^*)'\mathbf{F}^*}{T}\right)^{-1}\right]\left[\mathbf{B}_m'\mathbf{T}'\mathbf{R}_0'\otimes\mathbf{S}_m'\mathbf{N}'\mathbf{R}'\ddot{\mathbf{P}}'\right]$$

$$\times\, vec\left(\frac{\check{\mathbf{F}}'\sqrt{N}\ddot{\mathbf{U}}}{\sqrt{T}}\right).$$

Since by Lemma 12

$$T^{-1}\check{\mathbf{F}}'\mathbf{F}^* = \mathbf{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*} + O_p(T^{-1/2}) + O_p(N^{-1/2}),$$
$$T^{-1}\mathbf{F}^{*\prime}\mathbf{F}^* = \mathbf{\Sigma}_{\mathbf{F}^*} + O_p(T^{-1/2}) + O_p(N^{-1/2}),$$

with $\mathbf{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*} = \mathbf{\Sigma}_{\check{\mathbf{F}}}\mathbf{PRNS}_m$ and $\mathbf{\Sigma}_{\mathbf{F}^*} = \mathbf{S}_m'\mathbf{N}'\mathbf{R}'\mathbf{P}'\mathbf{\Sigma}_{\check{\mathbf{F}}}\mathbf{PRNS}_m$, we get

$$\mathbf{A}^{\mathbf{F}}_{NT,2} = \mathbf{\Sigma}_{\boldsymbol{\eta}}\left[\mathbf{B}_m'\mathbf{T}'\mathbf{R}_0' \otimes \mathbf{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*}\mathbf{\Sigma}_{\mathbf{F}^*}^{-1}\mathbf{S}_m'\mathbf{N}'\mathbf{R}'\mathbf{P}'\right] vec\left(\frac{\check{\mathbf{F}}'\sqrt{N}\ddot{\mathbf{U}}}{\sqrt{T}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\frac{1}{\sqrt{N}}\right).$$

Continuing on to the next term, substituting in (B-20) and (B-22) gives

$$\mathbf{A}^{\mathbf{F}}_{NT,3} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{S}_w'\ddot{\mathbf{P}}_i'\left(\frac{\check{\mathbf{F}}'\sqrt{N}\ddot{\mathbf{U}}}{\sqrt{T}}\right)\mathbf{RNS}_{-m}\widehat{\mathbf{\Sigma}}^{-1}_{\mathbf{u}^0_{-m}}\mathbf{S}_{-m}'\mathbf{N}'\mathbf{R}'\left(\frac{N\ddot{\mathbf{U}}'\ddot{\mathbf{U}}}{T}\right)\mathbf{R}_0\mathbf{TB}_m\boldsymbol{\eta}_i,$$

where by Lemma 12

$$\left\|\widehat{\mathbf{\Sigma}}^{-1}_{\mathbf{u}^0_{-m}} - \mathbf{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}\right\| = O_p(T^{-1/2}),$$

with $\mathbf{\Sigma}_{\mathbf{u}^0_{-m}} = \mathbf{B}_{-m}'\mathbf{T}'\mathbf{A}_0^{-1}\mathbf{\Omega}_{\mathbf{u}}(\mathbf{A}_0^{-1})'\mathbf{TB}_{-m}$ and $\mathbf{\Omega}_{\mathbf{u}} = \begin{bmatrix}\sigma^2_\varepsilon & \mathbf{0}'_{k\times 1}\\ \mathbf{0}_{k\times 1} & \mathbf{\Omega}_{\mathbf{v}}\end{bmatrix}$. Also, from the proof of Lemma 12 we have $N\ddot{\mathbf{U}}'\ddot{\mathbf{U}}/T = \mathbf{\Sigma}_{\ddot{\mathbf{U}}} + O_p(T^{-1/2})$, with $\mathbf{\Sigma}_{\ddot{\mathbf{U}}} = E(\ddot{\mathbf{U}}_i'\ddot{\mathbf{U}}_i/T)$. As such,

$$\mathbf{A}^{\mathbf{F}}_{NT,3} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{S}_w'\ddot{\mathbf{P}}_i'\left(\frac{\check{\mathbf{F}}'\sqrt{N}\ddot{\mathbf{U}}}{\sqrt{T}}\right)\mathbf{RNS}_{-m}\mathbf{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}\mathbf{S}_{-m}'\mathbf{N}'\mathbf{R}'\mathbf{\Sigma}_{\ddot{\mathbf{U}}}\mathbf{R}_0\mathbf{TB}_m\boldsymbol{\eta}_i + O_p\left(\frac{1}{\sqrt{T}}\right),$$

and, as before

$$\mathbf{A}^{\mathbf{F}}_{NT,3} = \mathbf{\Sigma}_{\boldsymbol{\eta}}\left[\mathbf{B}_m'\mathbf{T}'\mathbf{R}_0'\mathbf{\Sigma}_{\ddot{\mathbf{U}}}\mathbf{RNS}_{-m}\mathbf{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}\mathbf{S}_{-m}'\mathbf{N}'\mathbf{R}' \otimes \mathbf{I}_{1+k^2m(1+p^*)}\right] vec\left(\frac{\check{\mathbf{F}}'\sqrt{N}\ddot{\mathbf{U}}}{\sqrt{T}}\right)$$
$$+ O_p(N^{-1/2}) + O_p(T^{-1/2}).$$

For the next term, since using earlier results $\mathbf{A}^{\mathbf{F}}_{NT,4} = O_p(\sqrt{T}N^{-1/2})$, we define first

$$\mathbf{B}^{\mathbf{F}}_{NT,4} = \sum_{i=1}^{N}\mathbf{S}_w'\ddot{\mathbf{P}}_i'\frac{\check{\mathbf{F}}'\mathbf{F}^*}{T}\left(\frac{(\mathbf{F}^*)'\mathbf{F}^*}{T}\right)^{-1}\frac{(\bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}_m}{T}\boldsymbol{\eta}_i,$$

and note that $\mathbf{A}^{\mathbf{F}}_{NT,4} = \sqrt{\frac{T}{N}}\mathbf{B}^{\mathbf{F}}_{NT,4}$. Substituting in (B-19) and (B-22) gives $(\bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}_m = \mathbf{S}_m'\mathbf{N}'\mathbf{R}'\ddot{\mathbf{U}}'\ddot{\mathbf{U}}\mathbf{R}_0\mathbf{TB}_m$ and therefore

$$\mathbf{B}^{\mathbf{F}}_{NT,4} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{S}_w'\ddot{\mathbf{P}}_i'\frac{\check{\mathbf{F}}'\mathbf{F}^*}{T}\left(\frac{(\mathbf{F}^*)'\mathbf{F}^*}{T}\right)^{-1}\frac{N(\bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}_m}{T}\boldsymbol{\eta}_i,$$

$$= \boldsymbol{\Sigma}_{\boldsymbol{\eta}} \left[\mathbf{I}_m \otimes \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*} \boldsymbol{\Sigma}_{\mathbf{F}^*}^{-1} \right] \left[\mathbf{B}_m' \mathbf{T}' \mathbf{R}_0' \otimes \mathbf{S}_m' \mathbf{N}' \mathbf{R}' \right] vec\left(\boldsymbol{\Sigma}_{\ddot{\mathbf{U}}} \right) + O_p(T^{-1/2}) + O_p(N^{-1/2}),$$
$$= \boldsymbol{\Sigma}_{\boldsymbol{\eta}} \left[\mathbf{B}_m' \mathbf{T}' \mathbf{R}_0' \otimes \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*} \boldsymbol{\Sigma}_{\mathbf{F}^*}^{-1} \mathbf{S}_m' \mathbf{N}' \mathbf{R}' \right] vec\left(\boldsymbol{\Sigma}_{\ddot{\mathbf{U}}} \right) + O_p(T^{-1/2}) + O_p(N^{-1/2}).$$

Hence, we have for $T/N \to M < \infty$ (implying $\sqrt{T}N^{-1} \to 0$)

$$\mathbf{A}_{NT,4}^{\mathbf{F}} = \sqrt{T}N^{-1/2}\mathbf{B}_{NT,4}^{\mathbf{F}} = \sqrt{T}N^{-1/2}\mathbf{b}_0^{\mathbf{F}} + O_p(N^{-1/2}), \tag{D-38}$$

with $\mathbf{b}_0^{\mathbf{F}} = \boldsymbol{\Sigma}_{\boldsymbol{\eta}} \left[\mathbf{B}_m' \mathbf{T}' \mathbf{R}_0' \otimes \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*} \boldsymbol{\Sigma}_{\mathbf{F}^*}^{-1} \mathbf{S}_m' \mathbf{N}' \mathbf{R}' \right] vec\left(\boldsymbol{\Sigma}_{\ddot{\mathbf{U}}} \right)$.

Last up is $\mathbf{A}_{NT,5}^{\mathbf{F}}$, given by

$$\mathbf{A}_{NT,5}^{\mathbf{F}} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{S}_w' \ddot{\mathbf{P}}_i' \frac{\check{\mathbf{F}}'\mathbf{Q}_0}{T} \sqrt{T} \left[\boldsymbol{\Sigma}_{\mathbf{Q}}^{-1} - \widehat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+}^{-1} \right] \frac{\mathbf{Q}_0' \sqrt{N} \bar{\mathbf{U}}_m}{T} \boldsymbol{\eta}_i.$$

First we decompose it into 4 parts using $\mathbf{Q}_0 = \mathbf{F}^0 + \bar{\mathbf{U}}^0$,

$$\mathbf{A}_{NT,5}^{\mathbf{F}} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{S}_w' \ddot{\mathbf{P}}_i' \left(\sum_{l=1}^{4} \mathbf{A}_{NT,5,l}^{\mathbf{F}} \right) \boldsymbol{\eta}_i,$$

with

$$\left\| \mathbf{A}_{NT,5,1}^{\mathbf{F}} \right\| \le \sqrt{NT} \left\| T^{-1}\check{\mathbf{F}}'\mathbf{F}^0 \right\| \left\| \widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1} - \widehat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+}^{-1} \right\| \left\| T^{-1}(\mathbf{F}^0)'\bar{\mathbf{U}}_m \right\| = O_p(N^{-1/2}) + O_p(T^{-1/2}),$$
$$\left\| \mathbf{A}_{NT,5,2}^{\mathbf{F}} \right\| \le \sqrt{NT} \left\| T^{-1}\check{\mathbf{F}}'\mathbf{F}^0 \right\| \left\| \widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1} - \widehat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+}^{-1} \right\| \left\| T^{-1}(\bar{\mathbf{U}}^0)'\bar{\mathbf{U}}_m \right\| = O_p(\sqrt{T}N^{-1/2}) + O_p(1),$$
$$\left\| \mathbf{A}_{NT,5,3}^{\mathbf{F}} \right\| \le \sqrt{NT} \left\| T^{-1}\check{\mathbf{F}}'\bar{\mathbf{U}}^0 \right\| \left\| \widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1} - \widehat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+}^{-1} \right\| \left\| T^{-1}(\mathbf{F}^0)'\bar{\mathbf{U}}_m \right\| = O_p((NT)^{-1/2}) + O_p(T^{-1}),$$
$$\left\| \mathbf{A}_{NT,5,4}^{\mathbf{F}} \right\| \le \sqrt{NT} \left\| T^{-1}\check{\mathbf{F}}'\bar{\mathbf{U}}^0 \right\| \left\| \widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1} - \widehat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+}^{-1} \right\| \left\| T^{-1}(\bar{\mathbf{U}}^0)'\bar{\mathbf{U}}_m \right\| = O_p(N^{-1/2}) + O_p(T^{-1/2}),$$

in which case the leading term is $\mathbf{A}_{NT,5,2}^{\mathbf{F}}$. Hence, imposing $T/N \to M < \infty$,

$$\mathbf{A}_{NT,5}^{\mathbf{F}} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{S}_w' \ddot{\mathbf{P}}_i' T^{-1}\check{\mathbf{F}}'\mathbf{F}^0 \sqrt{T} \left[\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1} - \widehat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+}^{-1} \right] T^{-1}(\bar{\mathbf{U}}^0)'\sqrt{N}\bar{\mathbf{U}}_m \boldsymbol{\eta}_i + O_p(N^{-1/2}) + O_p(T^{-1/2}),$$
$$= \left[\frac{1}{N}\sum_{i=1}^{N} \left(\boldsymbol{\eta}_i' \otimes \mathbf{S}_w' \ddot{\mathbf{P}}_i' \right) \right] vec\left(T^{-1}\check{\mathbf{F}}'\mathbf{F}^0 \sqrt{T} \left[\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1} - \widehat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+}^{-1} \right] T^{-1}(\bar{\mathbf{U}}^0)'\sqrt{N}\bar{\mathbf{U}}_m \right)$$
$$+ O_p(N^{-1/2}) + O_p(T^{-1/2}),$$
$$= \boldsymbol{\Sigma}_{\boldsymbol{\eta}} vec\left(T^{-1}\check{\mathbf{F}}'\mathbf{F}^0 \sqrt{T} \left[\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1} - \widehat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+}^{-1} \right] T^{-1}(\bar{\mathbf{U}}^0)'\sqrt{N}\bar{\mathbf{U}}_m \right) + O_p(N^{-1/2}) + O_p(T^{-1/2}).$$

Next, consider the term in the $vec(\cdot)$ operator. Substituting in eq.(D-21) of Lemma 13, which is

$$\sqrt{T} \left[\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1} - \widehat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+}^{-1} \right] = - \begin{bmatrix} \mathbf{0}_{(1+K+m)\times(1+K+m)} & T^{-1/2}\widehat{\boldsymbol{\Sigma}}_{\mathbf{Fu}} \\ T^{-1/2}\widehat{\boldsymbol{\Sigma}}_{\mathbf{Fu}}' & \mathbf{0}_{(K-m)\times(K-m)} \end{bmatrix}$$
$$+ O_p\left(\frac{1}{\sqrt{N}} \right) + O_p\left(\frac{1}{\sqrt{T}} \right) + O_p\left(\frac{\sqrt{T}}{N} \right),$$

and noting that $\mathbf{F}^0 = [\mathbf{F}^*, \mathbf{0}_{T\times(K-m)}]$ and $\bar{\mathbf{U}}^0 = [\bar{\mathbf{U}}^0_m, \bar{\mathbf{U}}^0_{-m}]$ gives (since $T/N \to M < \infty$)

$$\begin{aligned}\mathbf{A}^{\mathbf{F}}_{NT,5} &= -\boldsymbol{\Sigma}_{\boldsymbol{\eta}} vec\left((T^{-1}\check{\mathbf{F}}'\mathbf{F}^*)T^{-1/2}\widehat{\boldsymbol{\Sigma}}_{\mathbf{Fu}}T^{-1}(\bar{\mathbf{U}}^0_{-m})'\sqrt{N}\bar{\mathbf{U}}_m\right) + O_p(N^{-1/2}) + O_p(T^{-1/2}),\\ &= -\boldsymbol{\Sigma}_{\boldsymbol{\eta}} vec\left(\boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*}T^{-1/2}\widehat{\boldsymbol{\Sigma}}_{\mathbf{Fu}}\boldsymbol{\Sigma}_{\mathbf{u}^0_{-m}\mathbf{u}_m}\right) + O_p(N^{-1/2}) + O_p(T^{-1/2}),\end{aligned}$$

where $\boldsymbol{\Sigma}_{\mathbf{u}^0_{-m}\mathbf{u}_m} = \mathbf{S}'_{-m}\mathbf{N}'\mathbf{R}'\boldsymbol{\Sigma}_{\ddot{\mathbf{U}}}\mathbf{R}_0\mathbf{T}\mathbf{B}_m$ from Lemma 12 and we recall from Lemma 13 that $\widehat{\boldsymbol{\Sigma}}_{\mathbf{Fu}} = \boldsymbol{\Sigma}^{-1}_{\mathbf{F}^*}(\mathbf{F}^* + \bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}^0_{-m}\boldsymbol{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}$. By definition then

$$\begin{aligned}\boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*}T^{-1/2}\widehat{\boldsymbol{\Sigma}}_{\mathbf{Fu}}\boldsymbol{\Sigma}_{\mathbf{u}^0_{-m}\mathbf{u}_m} &= \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*}\boldsymbol{\Sigma}^{-1}_{\mathbf{F}^*}\left(T^{-1/2}(\mathbf{F}^*)'\bar{\mathbf{U}}^0_{-m}\right)\boldsymbol{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}\boldsymbol{\Sigma}_{\mathbf{u}^0_{-m}\mathbf{u}_m},\\ &\quad + \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*}\boldsymbol{\Sigma}^{-1}_{\mathbf{F}^*}\left(T^{-1/2}(\bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}^0_{-m}\right)\boldsymbol{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}\boldsymbol{\Sigma}_{\mathbf{u}^0_{-m}\mathbf{u}_m},\end{aligned}$$

where employing again (B-18)-(B-20) we have $(\mathbf{F}^*)'\bar{\mathbf{U}}^0_{-m} = \mathbf{S}'_m\mathbf{N}'\mathbf{R}'\ddot{\mathbf{P}}'\check{\mathbf{F}}'\sqrt{N}\ddot{\mathbf{U}}\mathbf{R}\mathbf{N}\mathbf{S}_{-m}$ and $(\bar{\mathbf{U}}^0_m)'\bar{\mathbf{U}}^0_{-m} = \mathbf{S}'_m\mathbf{N}'\mathbf{R}'\ddot{\mathbf{U}}'\sqrt{N}\ddot{\mathbf{U}}\mathbf{R}\mathbf{N}\mathbf{S}_{-m}$. This gives

$$\begin{aligned}\boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*}T^{-1/2}\widehat{\boldsymbol{\Sigma}}_{\mathbf{Fu}}\boldsymbol{\Sigma}_{\mathbf{u}^0_{-m}\mathbf{u}_m} &= \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*}\boldsymbol{\Sigma}^{-1}_{\mathbf{F}^*}\mathbf{S}'_m\mathbf{N}'\mathbf{R}'\ddot{\mathbf{P}}'\left(T^{-1/2}\check{\mathbf{F}}'\sqrt{N}\ddot{\mathbf{U}}\right)\mathbf{R}\mathbf{N}\mathbf{S}_{-m}\boldsymbol{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}\boldsymbol{\Sigma}_{\mathbf{u}^0_{-m}\mathbf{u}_m},\\ &\quad + \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*}\boldsymbol{\Sigma}^{-1}_{\mathbf{F}^*}\mathbf{S}'_m\mathbf{N}'\mathbf{R}'\left(T^{-1/2}\sqrt{N}\ddot{\mathbf{U}}'\ddot{\mathbf{U}}\right)\mathbf{R}\mathbf{N}\mathbf{S}_{-m}\boldsymbol{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}\boldsymbol{\Sigma}_{\mathbf{u}^0_{-m}\mathbf{u}_m},\end{aligned}$$

and in turn once substituted in $\mathbf{A}^{\mathbf{F}}_{NT,5}$

$$\begin{aligned}\mathbf{A}^{\mathbf{F}}_{NT,5} &= -\boldsymbol{\Sigma}_{\boldsymbol{\eta}}\left[\boldsymbol{\Sigma}'_{\mathbf{u}^0_{-m}\mathbf{u}_m}\boldsymbol{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}\mathbf{S}'_{-m}\mathbf{N}'\mathbf{R}' \otimes \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*}\boldsymbol{\Sigma}^{-1}_{\mathbf{F}^*}\mathbf{S}'_m\mathbf{N}'\mathbf{R}'\mathbf{P}'\right] vec\left(T^{-1/2}\check{\mathbf{F}}'\sqrt{N}\ddot{\mathbf{U}}\right)\\ &\quad - \boldsymbol{\Sigma}_{\boldsymbol{\eta}}\left[\boldsymbol{\Sigma}'_{\mathbf{u}^0_{-m}\mathbf{u}_m}\boldsymbol{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}\mathbf{S}'_{-m}\mathbf{N}'\mathbf{R}' \otimes \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*}\boldsymbol{\Sigma}^{-1}_{\mathbf{F}^*}\mathbf{S}'_m\mathbf{N}'\mathbf{R}'\right] vec\left(T^{-1/2}\sqrt{N}\ddot{\mathbf{U}}'\ddot{\mathbf{U}}\right)\\ &\quad + O_p(N^{-1/2}) + O_p(T^{-1/2}).\end{aligned}$$

Finally, since from Lemma 9 the second term in this expression is of order $O_p(\sqrt{T}N^{-1/2})$ it is clear that

$$\begin{aligned}\mathbf{A}^{\mathbf{F}}_{NT,5} &= -\boldsymbol{\Sigma}_{\boldsymbol{\eta}}\left[\boldsymbol{\Sigma}'_{\mathbf{u}^0_{-m}\mathbf{u}_m}\boldsymbol{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}\mathbf{S}'_{-m}\mathbf{N}'\mathbf{R}' \otimes \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*}\boldsymbol{\Sigma}^{-1}_{\mathbf{F}^*}\mathbf{S}'_m\mathbf{N}'\mathbf{R}'\mathbf{P}'\right] vec\left(T^{-1/2}\check{\mathbf{F}}'\sqrt{N}\ddot{\mathbf{U}}\right)\\ &\quad - \sqrt{T}N^{-1/2}\mathbf{b}^{\mathbf{F}}_1 + O_p(N^{-1/2}) + O_p(T^{-1/2}),\end{aligned}$$

with $\mathbf{b}^{\mathbf{F}}_1 = \boldsymbol{\Sigma}_{\boldsymbol{\eta}}\left[\boldsymbol{\Sigma}'_{\mathbf{u}^0_{-m}\mathbf{u}_m}\boldsymbol{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}\mathbf{S}'_{-m}\mathbf{N}'\mathbf{R}' \otimes \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*}\boldsymbol{\Sigma}^{-1}_{\mathbf{F}^*}\mathbf{S}'_m\mathbf{N}'\mathbf{R}'\right] vec\left(\boldsymbol{\Sigma}_{\ddot{\mathbf{U}}}\right)$.
In conclusion, combining the results gives, provided $T/N \to M < \infty$,

$$\mathbf{A}^{\mathbf{F}}_{NT} = \boldsymbol{\Psi}_{\mathbf{F}} vec\left(\frac{\check{\mathbf{F}}'\sqrt{N}\ddot{\mathbf{U}}}{\sqrt{T}}\right) + \sqrt{\frac{T}{N}}\left(\mathbf{b}^{\mathbf{F}}_0 - \mathbf{b}^{\mathbf{F}}_1\right) + O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right),$$

with

$$\begin{aligned}\mathbf{b}^{\mathbf{F}}_0 &= \boldsymbol{\Sigma}_{\boldsymbol{\eta}}\left[\mathbf{B}'_m\mathbf{T}'\mathbf{R}'_0 \otimes \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*}\boldsymbol{\Sigma}^{-1}_{\mathbf{F}^*}\mathbf{S}'_m\mathbf{N}'\mathbf{R}'\right] vec\left(\boldsymbol{\Sigma}_{\ddot{\mathbf{U}}}\right),\\ \mathbf{b}^{\mathbf{F}}_1 &= \boldsymbol{\Sigma}_{\boldsymbol{\eta}}\left[\boldsymbol{\Sigma}'_{\mathbf{u}^0_{-m}\mathbf{u}_m}\boldsymbol{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}\mathbf{S}'_{-m}\mathbf{N}'\mathbf{R}' \otimes \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*}\boldsymbol{\Sigma}^{-1}_{\mathbf{F}^*}\mathbf{S}'_m\mathbf{N}'\mathbf{R}'\right] vec\left(\boldsymbol{\Sigma}_{\ddot{\mathbf{U}}}\right),\end{aligned}$$

$$
\begin{aligned}
\mathbf{\Psi_F} &= -\mathbf{V}_{\mathbf{F},1} + \mathbf{V}_{\mathbf{F},2} + \mathbf{V}_{\mathbf{F},3} - \mathbf{V}_{\mathbf{F},4},\\
\mathbf{V}_{\mathbf{F},1} &= \mathbf{\Sigma}_{\boldsymbol{\eta}}\left[\mathbf{B}'_m\mathbf{T}'\mathbf{R}'_0 \otimes \mathbf{I}_{1+K^2m(1+p^*)}\right],\\
\mathbf{V}_{\mathbf{F},2} &= \mathbf{\Sigma}_{\boldsymbol{\eta}}\left[\mathbf{B}'_m\mathbf{T}'\mathbf{R}'_0 \otimes \mathbf{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*}\mathbf{\Sigma}^{-1}_{\mathbf{F}^*}\mathbf{S}'_m\mathbf{N}'\mathbf{R}'\mathbf{P}'\right],\\
\mathbf{V}_{\mathbf{F},3} &= \mathbf{\Sigma}_{\boldsymbol{\eta}}\left[\mathbf{B}'_m\mathbf{T}'\mathbf{R}'_0\mathbf{\Sigma}_{\ddot{\mathbf{U}}}\mathbf{R}\mathbf{N}\mathbf{S}_{-m}\mathbf{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}\mathbf{S}'_{-m}\mathbf{N}'\mathbf{R}' \otimes \mathbf{I}_{1+k^2m(1+p^*)}\right],\\
\mathbf{V}_{\mathbf{F},4} &= \mathbf{\Sigma}_{\boldsymbol{\eta}}\left[\mathbf{\Sigma}'_{\mathbf{u}^0_{-m}\mathbf{u}_m}\mathbf{\Sigma}^{-1}_{\mathbf{u}^0_{-m}}\mathbf{S}'_{-m}\mathbf{N}'\mathbf{R}' \otimes \mathbf{\Sigma}_{\check{\mathbf{F}}\mathbf{F}^*}\mathbf{\Sigma}^{-1}_{\mathbf{F}^*}\mathbf{S}'_m\mathbf{N}'\mathbf{R}'\mathbf{P}'\right],
\end{aligned}
$$

which is the result stated in eq.(D-23) of the lemma.

Proof of Lemma 15

Let $\mathbf{A}^{\varepsilon} = \frac{1}{NT}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{M}\boldsymbol{\varepsilon}_i$ and note that given $\bar{\mathbf{w}} \subseteq \mathbf{Q}$ then $\mathbf{M}\bar{\mathbf{w}} = \mathbf{0}_{T\times k_w}$. Therefore, substituting in (B-13)

$$
\begin{aligned}
\mathbf{A}^{\varepsilon} &= \frac{1}{NT}\sum_{i=1}^{N}\left(\mathbf{w}_i' - \bar{\mathbf{w}}\right)\mathbf{M}\boldsymbol{\varepsilon}_i = \frac{1}{NT}\sum_{i=1}^{N}\left(\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\check{\mathbf{F}}' + \boldsymbol{\epsilon}_i' - \bar{\boldsymbol{\epsilon}}'\right)\mathbf{M}\boldsymbol{\varepsilon}_i,\\
&= \frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\check{\mathbf{F}}'\mathbf{M}\boldsymbol{\varepsilon}_i + \frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\epsilon}_i'\mathbf{M}\boldsymbol{\varepsilon}_i - \frac{1}{T}\bar{\boldsymbol{\epsilon}}'\mathbf{M}\bar{\boldsymbol{\varepsilon}},\\
&= \mathbf{A}_1^{\varepsilon} + \mathbf{A}_2^{\varepsilon} - \mathbf{A}_3^{\varepsilon}, \qquad \text{(D-39)}
\end{aligned}
$$

with $\widetilde{\mathbf{P}}_i = \ddot{\mathbf{P}}_i - \ddot{\mathbf{P}}$ and obvious definitions for $\mathbf{A}_1^{\varepsilon}, \mathbf{A}_2^{\varepsilon}$ and $\mathbf{A}_3^{\varepsilon}$. We start with the first term and decompose it as

$$
\begin{aligned}
\mathbf{A}_1^{\varepsilon} &= \frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\check{\mathbf{F}}'\mathbf{M}_{\mathbf{F}}\boldsymbol{\varepsilon}_i + \frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\check{\mathbf{F}}'(\mathbf{M}_{\mathbf{F}} - \mathbf{M})\boldsymbol{\varepsilon}_i,\\
&= \mathbf{A}_{11}^{\varepsilon} + \mathbf{A}_{12}^{\varepsilon}.
\end{aligned}
$$

For the first term we find, writing it in full and substituting in (B-18), $\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*} = T^{-1}(\mathbf{F}^*)'\mathbf{F}^*$ and $\widehat{\boldsymbol{\Sigma}}_{\check{\mathbf{F}}} = T^{-1}\check{\mathbf{F}}'\check{\mathbf{F}}$,

$$
\begin{aligned}
\mathbf{A}_{11}^{\varepsilon} &= \frac{1}{N}\sum_{i=1}^{N}\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\frac{\check{\mathbf{F}}'\boldsymbol{\varepsilon}_i}{T} - \frac{1}{N}\sum_{i=1}^{N}\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\frac{\check{\mathbf{F}}'\mathbf{F}^*}{T}\left(\frac{\mathbf{F}^{*\prime}\mathbf{F}^*}{T}\right)^{-1}\frac{\mathbf{F}^{*\prime}\boldsymbol{\varepsilon}_i}{T},\\
&= \frac{1}{N}\sum_{i=1}^{N}\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\left[\mathbf{I}_{1+K^2m(1+p^*)} - \widehat{\boldsymbol{\Sigma}}_{\check{\mathbf{F}}}\ddot{\mathbf{P}}\mathbf{R}\mathbf{N}\mathbf{S}_m\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*}^{-1}\mathbf{S}_m'\mathbf{N}'\mathbf{R}'\ddot{\mathbf{P}}'\right]\frac{\check{\mathbf{F}}'\boldsymbol{\varepsilon}_i}{T},\\
&= \left[vec\left(\widehat{\mathbf{B}}^{\mathbf{F}}\right)' \otimes \mathbf{I}_{k_w}\right] vec\left[\frac{1}{N}\sum_{i=1}^{N}\left(\frac{\boldsymbol{\varepsilon}_i'\check{\mathbf{F}}}{T} \otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)\right],\\
&= \widehat{\boldsymbol{\Psi}}_{\boldsymbol{\epsilon}} vec\left[\frac{1}{N}\sum_{i=1}^{N}\left(\frac{\boldsymbol{\varepsilon}_i'\check{\mathbf{F}}}{T} \otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)\right],
\end{aligned}
$$

where $\widehat{\boldsymbol{\Psi}}_{\boldsymbol{\epsilon}} = \left[vec\left(\widehat{\mathbf{B}}^{\mathbf{F}}\right)' \otimes \mathbf{I}_{k_w}\right]$ and $\widehat{\mathbf{B}}^{\mathbf{F}} = \mathbf{I}_{1+K^2m(1+p^*)} - \widehat{\boldsymbol{\Sigma}}_{\check{\mathbf{F}}}\ddot{\mathbf{P}}\mathbf{R}\mathbf{N}\mathbf{S}_m\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*}^{-1}\mathbf{S}_m'\mathbf{N}'\mathbf{R}'\ddot{\mathbf{P}}'$. From $\left\|\widehat{\mathbf{B}}^{\mathbf{F}}\right\| = O_p(1)$ by results in Lemma 10 and eq.(D-17) of Lemma 11 follows

$$
\|\mathbf{A}_{11}^{\varepsilon}\| \le \left\|\widehat{\boldsymbol{\Psi}}_{\boldsymbol{\epsilon}}\right\|\left\|\frac{1}{N}\sum_{i=1}^{N}\left(T^{-1}\boldsymbol{\varepsilon}_i'\check{\mathbf{F}} \otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)\right\| = O_p\left(\frac{1}{\sqrt{NT}}\right).
$$

Next, for $\mathbf{A}_{12}^{\varepsilon}$ we use the decomposition in (D-1) and obtain

$$
\|\mathbf{A}_{12}^{\varepsilon}\| \le \|\mathbf{A}_{121}^{\varepsilon}\| + \|\mathbf{A}_{122}^{\varepsilon}\| + \|\mathbf{A}_{123}^{\varepsilon}\| + \|\mathbf{A}_{124}^{\varepsilon}\| + \|\mathbf{A}_{125}^{\varepsilon}\| = O_p(N^{-1}T^{-1/2}) + O_p(N^{-1/2}T^{-1}),
$$

because, denoting $\widehat{\boldsymbol{\Sigma}}_{\mathbf{u}_{-m}^0} = T^{-1}(\bar{\mathbf{U}}_{-m}^0)'\bar{\mathbf{U}}_{-m}^0$ we have

$$
\|\mathbf{A}_{121}^{\varepsilon}\| \le \left\|\frac{1}{N}\sum_{i=1}^{N}\left(\frac{\boldsymbol{\varepsilon}_i'\bar{\mathbf{U}}_{-m}^0}{T} \otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)\right\|\left\|\frac{\check{\mathbf{F}}'\bar{\mathbf{U}}_{-m}^0}{T}\right\|\left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{u}_{-m}^0}^{-1}\right\| = O_p\left(\frac{1}{N\sqrt{T}}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right),
$$

$$\|\mathbf{A}_{122}^{\varepsilon}\| \leq \left\|\frac{1}{N}\sum_{i=1}^{N}\left(\frac{\boldsymbol{\varepsilon}_i'\bar{\mathbf{U}}_m^0}{T}\otimes\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)\right\|\left\|\frac{\check{\mathbf{F}}'\bar{\mathbf{U}}_m^0}{T}\right\|\left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*}^{-1}\right\| = O_p\left(\frac{1}{N^2\sqrt{T}}\right)+O_p\left(\frac{1}{N^{3/2}T}\right),$$

$$\|\mathbf{A}_{123}^{\varepsilon}\| \leq \left\|\frac{1}{N}\sum_{i=1}^{N}\left(\frac{\boldsymbol{\varepsilon}_i'\mathbf{F}^*}{T}\otimes\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)\right\|\left\|\frac{\check{\mathbf{F}}'\bar{\mathbf{U}}_m^0}{T}\right\|\left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*}^{-1}\right\| = O_p\left(\frac{1}{NT}\right),$$

$$\|\mathbf{A}_{124}^{\varepsilon}\| \leq \left\|\frac{1}{N}\sum_{i=1}^{N}\left(\frac{\boldsymbol{\varepsilon}_i'\bar{\mathbf{U}}_m^0}{T}\otimes\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)\right\|\left\|\frac{\check{\mathbf{F}}'\mathbf{F}^*}{T}\right\|\left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*}^{-1}\right\| = O_p\left(\frac{1}{N\sqrt{T}}\right)+O_p\left(\frac{1}{N^{3/2}}\right),$$

$$\|\mathbf{A}_{125}^{\varepsilon}\| \leq \left\|\frac{1}{N}\sum_{i=1}^{N}\left(\frac{\boldsymbol{\varepsilon}_i'\mathbf{Q}_0}{T}\otimes\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)\right\|\left\|\frac{\check{\mathbf{F}}'\mathbf{Q}_0}{T}\right\|\left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1}-\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+}^{-1}\right\| = O_p\left(\frac{1}{N\sqrt{T}}\right)+O_p\left(\frac{1}{\sqrt{N}T}\right),$$

by the results in Lemmas 10, 11 and 13. Hence, we conclude that

$$\|\mathbf{A}_1^{\varepsilon}\| = O_p((NT)^{-1/2}), \tag{D-40}$$

and, defining $\mathbf{A}_{NT,1}^{\varepsilon}=\sqrt{NT}\mathbf{A}_1^{\varepsilon}$ also, since $\widehat{\mathbf{B}}^{\mathbf{F}}=\mathbf{B}^{\mathbf{F}}+O_p(N^{-1/2})+O_p(T^{-1/2})$ by Lemma 12, with $\mathbf{B}^{\mathbf{F}}=\mathbf{I}_{1+K^2m(1+p^*)}-\boldsymbol{\Sigma}_{\check{\mathbf{F}}}\mathbf{PRNS}_m\boldsymbol{\Sigma}_{\mathbf{F}^*}^{-1}\mathbf{S}_m'\mathbf{N}'\mathbf{R}'\mathbf{P}'$ and $\boldsymbol{\Psi}_{\boldsymbol{\epsilon}}=\left[vec\left(\mathbf{B}^{\mathbf{F}}\right)'\otimes\mathbf{I}_{k_w}\right]$

$$\mathbf{A}_{NT,1}^{\varepsilon}=\boldsymbol{\Psi}_{\boldsymbol{\epsilon}}vec\left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\frac{\boldsymbol{\varepsilon}_i'\check{\mathbf{F}}}{\sqrt{T}}\otimes\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)\right]+O_p\left(\frac{1}{\sqrt{N}}\right)+O_p\left(\frac{1}{\sqrt{T}}\right)+O_p\left(\frac{\sqrt{T}}{N}\right). \tag{D-41}$$

We take on $\mathbf{A}_2^{\varepsilon}$ next. Decomposing it as before returns

$$\begin{aligned}\mathbf{A}_2^{\varepsilon} &= \frac{1}{N}\sum_{i=1}^{N}\frac{\boldsymbol{\epsilon}_i'\boldsymbol{\varepsilon}_i}{T}-\frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\epsilon}_i'\mathbf{H}_{\mathbf{F}}\boldsymbol{\varepsilon}_i-\frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\epsilon}_i'(\mathbf{M}_{\mathbf{F}}-\mathbf{M})\boldsymbol{\varepsilon}_i,\\ &= \mathbf{A}_{21}^{\varepsilon}-\mathbf{A}_{22}^{\varepsilon}-\mathbf{A}_{23}^{\varepsilon}.\end{aligned} \tag{D-42}$$

Clearly, since by Ass.1 the elements of $\boldsymbol{\epsilon}_i$ and $\boldsymbol{\varepsilon}_i$ are contemporaneously uncorrelated

$$\|\mathbf{A}_{21}^{\varepsilon}\| = \left\|\frac{1}{N}\sum_{i=1}^{N}\frac{\boldsymbol{\epsilon}_i'\boldsymbol{\varepsilon}_i}{T}\right\| = O_p\left(\frac{1}{\sqrt{NT}}\right), \tag{D-43}$$

whereas for the second term, by (D-8)-(D-9) of Lemma 10,

$$\left\|T^{-1}\boldsymbol{\epsilon}_i'\mathbf{H}_{\mathbf{F}}\boldsymbol{\varepsilon}_i\right\| \leq \left\|\frac{\boldsymbol{\epsilon}_i'\mathbf{F}^*}{T}\right\|\left\|\left(\frac{(\mathbf{F}^*)'\mathbf{F}^*}{T}\right)^{-1}\right\|\left\|\frac{(\mathbf{F}^*)'\boldsymbol{\varepsilon}_i}{T}\right\| = O_p\left(\frac{1}{T}\right),$$

and therefore

$$\|\mathbf{A}_{22}^{\varepsilon}\| = O_p\left(T^{-1}\right).$$

Letting again $\mathbf{A}_{NT,22}^{\varepsilon}=\sqrt{NT}\mathbf{A}_{22}^{\varepsilon}$ it is clear that

$$\left\|\mathbf{A}_{NT,22}^{\varepsilon}\right\| = O_p\left(\sqrt{N}T^{-1/2}\right).$$

To evaluate $\mathbf{A}_{23}^{\varepsilon}$ we again split it into 5 key components

$$\|\mathbf{A}_{231}^{\varepsilon}\| \leq \left\|T^{-1}\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}_{-m}^0\right\| \left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{u}_{-m}^0}^{-1}\right\| \left\|T^{-1}(\bar{\mathbf{U}}_{-m}^0)'\boldsymbol{\varepsilon}_i\right\| = O_p(N^{-1}) + O_p(T^{-1}) + O_p((NT)^{-1/2}),$$
$$\|\mathbf{A}_{232}^{\varepsilon}\| \leq \left\|T^{-1}\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}_{m}^0\right\| \left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*}^{-1}\right\| \left\|T^{-1}(\bar{\mathbf{U}}_{m}^0)'\boldsymbol{\varepsilon}_i\right\| = O_p(N^{-2}) + O_p(N^{-3/2}T^{-1/2}) + O_p((NT)^{-1}),$$
$$\|\mathbf{A}_{233}^{\varepsilon}\| \leq \left\|T^{-1}\boldsymbol{\epsilon}_i'\mathbf{F}^*\right\| \left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*}^{-1}\right\| \left\|T^{-1}(\bar{\mathbf{U}}_{m}^0)'\boldsymbol{\varepsilon}_i\right\| = O_p(N^{-1}T^{-1/2}) + O_p(N^{-1/2}T^{-1}),$$
$$\|\mathbf{A}_{234}^{\varepsilon}\| \leq \left\|T^{-1}\boldsymbol{\epsilon}_i'\bar{\mathbf{U}}_{m}^0\right\| \left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*}^{-1}\right\| \left\|T^{-1}(\mathbf{F}^*)'\boldsymbol{\varepsilon}_i\right\| = O_p(N^{-1}T^{-1/2}) + O_p(N^{-1/2}T^{-1}),$$
$$\|\mathbf{A}_{235}^{\varepsilon}\| \leq \left\|T^{-1}\boldsymbol{\epsilon}_i'\mathbf{Q}_0\right\| \left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1} - \widehat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+}^{-1}\right\| \left\|T^{-1}\mathbf{Q}_0'\boldsymbol{\varepsilon}_i\right\| = O_p(T^{-1}) + O_p((NT)^{-1/2}),$$

which leads to

$$\|\mathbf{A}_{23}^{\varepsilon}\| = O_p(N^{-1}) + O_p(T^{-1}) + O_p((NT)^{-1/2}),$$

and therefore

$$\|\mathbf{A}_{2}^{\varepsilon}\| = O_p(N^{-1}) + O_p(T^{-1}) + O_p((NT)^{-1/2}). \tag{D-44}$$

Finally, for $\mathbf{A}_3^{\varepsilon}$ we find

$$\|\mathbf{A}_{3}^{\varepsilon}\| \leq \left\|T^{-1}\bar{\boldsymbol{\epsilon}}'\bar{\boldsymbol{\varepsilon}}\right\| + \left\|T^{-1}\bar{\boldsymbol{\epsilon}}'\mathbf{Q}_0(T^{-1}\mathbf{Q}_0'\mathbf{Q}_0)^{-1}\mathbf{Q}_0'\bar{\boldsymbol{\varepsilon}}\right\| = O_p(N^{-1}), \tag{D-45}$$

since $\|T^{-1}\bar{\boldsymbol{\epsilon}}'\bar{\boldsymbol{\varepsilon}}\| = O_p(T^{-1/2}N^{-1})$ due to $\bar{\boldsymbol{\epsilon}}$ and $\bar{\boldsymbol{\varepsilon}}$ being uncorrelated $O_p(N^{-1/2})$ variables, and because the norm of the final term can be decomposed in the following five components

$$\|\mathbf{A}_{31}^{\varepsilon}\| \leq \left\|T^{-1}\bar{\boldsymbol{\epsilon}}'\bar{\mathbf{U}}_{-m}^0\right\| \left\|[T^{-1}(\bar{\mathbf{U}}_{-m}^0)'\bar{\mathbf{U}}_{-m}^0]^{-1}\right\| \left\|T^{-1}(\bar{\mathbf{U}}_{-m}^0)'\bar{\boldsymbol{\varepsilon}}\right\| = O_p(N^{-1}),$$
$$\|\mathbf{A}_{32}^{\varepsilon}\| \leq \left\|T^{-1}\bar{\boldsymbol{\epsilon}}'\bar{\mathbf{U}}_{m}^0\right\| \left\|[T^{-1}(\mathbf{F}^*)'\mathbf{F}^*]^{-1}\right\| \left\|T^{-1}(\bar{\mathbf{U}}_{m}^0)'\bar{\boldsymbol{\varepsilon}}\right\| = O_p(N^{-2}),$$
$$\|\mathbf{A}_{33}^{\varepsilon}\| \leq \left\|T^{-1}\bar{\boldsymbol{\epsilon}}'\mathbf{F}^*\right\| \left\|[T^{-1}(\mathbf{F}^*)'\mathbf{F}^*]^{-1}\right\| \left\|T^{-1}(\bar{\mathbf{U}}_{m}^0)'\bar{\boldsymbol{\varepsilon}}\right\| = O_p(N^{-3/2}T^{-1/2}),$$
$$\|\mathbf{A}_{34}^{\varepsilon}\| \leq \left\|T^{-1}\bar{\boldsymbol{\epsilon}}'\bar{\mathbf{U}}_{m}^0\right\| \left\|[T^{-1}(\mathbf{F}^*)'\mathbf{F}^*]^{-1}\right\| \left\|T^{-1}(\mathbf{F}^*)'\bar{\boldsymbol{\varepsilon}}\right\| = O_p(N^{-3/2}T^{-1/2}),$$
$$\|\mathbf{A}_{35}^{\varepsilon}\| \leq \left\|T^{-1}\bar{\boldsymbol{\epsilon}}'\mathbf{Q}_0\right\| \left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1} - \widehat{\boldsymbol{\Sigma}}_{\mathbf{F}_u^+}^{-1}\right\| \left\|T^{-1}\mathbf{Q}_0'\bar{\boldsymbol{\varepsilon}}\right\| = O_p(N^{-3/2}) + O_p(N^{-1}T^{-1/2}),$$

where we used the fact that the terms involving $\bar{\boldsymbol{\epsilon}}$ and $\bar{\boldsymbol{\varepsilon}}$ have the same order as those involving $\bar{\mathbf{U}}_m^0$ in Lemma 10. It will be convenient to also define $\mathbf{A}_{NT,3}^{\varepsilon} = \sqrt{NT}\mathbf{A}_3^{\varepsilon}$

$$\mathbf{A}_{NT,3}^{\varepsilon} = \sqrt{\frac{T}{N}}\left(\frac{\sqrt{N}\bar{\boldsymbol{\epsilon}}'\bar{\mathbf{U}}_{-m}^0}{T}\right)\left(\frac{(\bar{\mathbf{U}}_{-m}^0)'\bar{\mathbf{U}}_{-m}^0}{T}\right)^{-1}\left(\frac{(\bar{\mathbf{U}}_{-m}^0)'\sqrt{N}\bar{\boldsymbol{\varepsilon}}}{T}\right) + O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{\sqrt{T}}{N}\right),$$

such that if $T/N \to M < \infty$, making use of (B-14),

$$\mathbf{A}_{NT,3}^{\varepsilon} = \sqrt{T}N^{-1/2}\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}\mathbf{U}_{-m}}\boldsymbol{\Sigma}_{\mathbf{u}_{-m}^0}^{-1}\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}\mathbf{U}_{-m}}' + O_p(T^{-1/2}) + O_p(N^{-1/2}), \tag{D-46}$$

with $\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}\mathbf{U}_{-m}} = \mathbf{S}_w'\boldsymbol{\Sigma}_{\ddot{\mathbf{U}}}\mathbf{RNS}_{-m}$, $\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}\mathbf{U}_{-m}} = E(\boldsymbol{\varepsilon}_i'\ddot{\mathbf{U}}_i/T)\mathbf{RNS}_{-m}$, $\boldsymbol{\Sigma}_{\mathbf{u}_{-m}^0} = \mathbf{S}_{-m}'\mathbf{N}'\mathbf{R}'\boldsymbol{\Sigma}_{\ddot{\mathbf{U}}}\mathbf{RNS}_{-m}$ and $\boldsymbol{\Sigma}_{\ddot{\mathbf{U}}} = E(\ddot{\mathbf{U}}_i'\ddot{\mathbf{U}}_i/T)$.

Combining (D-40)-(D-45) in (D-39) leads to the conclusion that

$$\|\mathbf{A}^{\varepsilon}\| = O_p(N^{-1}) + O_p(T^{-1}) + O_p((NT)^{-1/2}),$$

which is the result stated in the lemma. Letting $\mathbf{A}_{NT}^{\varepsilon} = \sqrt{NT}\mathbf{A}^{\varepsilon}$, the result above implies

$$\|\mathbf{A}_{NT}^{\varepsilon}\| = O_p(1) + O_p(\sqrt{T}N^{-1/2}) + O_p(\sqrt{N}T^{-1/2}).$$

Proof of Lemma 16

Consider $\widehat{\sigma}^2_\varepsilon(\cdot)$ defined in eq.(20) evaluated at $\boldsymbol{\delta}_0 \neq \boldsymbol{\delta}$, with $\boldsymbol{\delta} = [\rho, \boldsymbol{\beta}']'$ the true parameter vector. Suppose that $p^* \geq p$ and Ass.1-5 hold. We can then make use of (B-23) to get

$$\widehat{\sigma}^2_\varepsilon(\boldsymbol{\delta}_0) = \frac{1}{N(T-c)} \sum_{i=1}^N \|\mathbf{M}(\mathbf{y}_i - \mathbf{w}_i\boldsymbol{\delta}_0)\|^2 = \frac{1}{N(T-c)} \sum_{i=1}^N \|\mathbf{M}(\mathbf{w}_i(\boldsymbol{\delta}-\boldsymbol{\delta}_0) + \mathbf{F}\boldsymbol{\gamma}_i + \boldsymbol{\varepsilon}_i)\|^2,$$
$$= \frac{T}{T-c}\frac{1}{NT} \sum_{i=1}^N \left\|\mathbf{M}\left(\mathbf{w}_i(\boldsymbol{\delta}-\boldsymbol{\delta}_0) - \bar{\mathbf{U}}_m\boldsymbol{\gamma}_i + \boldsymbol{\varepsilon}_i\right)\right\|^2.$$

For its components we find, denoting first $\widehat{\boldsymbol{\Sigma}}_\gamma = \left[\frac{1}{N}\sum_{i=1}^N (\boldsymbol{\gamma}_i' \otimes \boldsymbol{\gamma}_i')\right]$, with $\left\|\widehat{\boldsymbol{\Sigma}}_\gamma\right\| = O_p(1)$ by Ass.3,

$$\left\|\frac{1}{NT}\sum_{i=1}^N \boldsymbol{\gamma}_i'\bar{\mathbf{U}}_m'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\gamma}_i\right\| \leq \left\|\widehat{\boldsymbol{\Sigma}}_\gamma\right\| \left\|\frac{\bar{\mathbf{U}}_m'\bar{\mathbf{U}}_m}{T}\right\| + \left\|\widehat{\boldsymbol{\Sigma}}_\gamma\right\| \left\|\frac{\bar{\mathbf{U}}_m'\mathbf{Q}_0}{T}\right\| \left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1}\right\| \left\|\frac{\mathbf{Q}_0'\bar{\mathbf{U}}_m}{T}\right\| = O_p\left(\frac{1}{N}\right),$$

where we made use of Lemma 10 by noting that $\bar{\mathbf{U}}_m$ is by the definition above (B-16) a subset of $\bar{\mathbf{U}}_m^0$. Also, since for any $\|\boldsymbol{\delta}-\boldsymbol{\delta}_0\| < \infty$, by (D-22) of Lemma 14

$$\left\|\frac{1}{N}\sum_{i=1}^N (\boldsymbol{\delta}-\boldsymbol{\delta}_0)'\frac{\mathbf{w}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\gamma}_i}{T}\right\| \leq \|\boldsymbol{\delta}-\boldsymbol{\delta}_0\| \left\|\frac{1}{N}\sum_{i=1}^N \frac{\mathbf{w}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\gamma}_i}{T}\right\| = O_p\left(\frac{1}{N}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right),$$

and similarly by (D-24) of Lemma 15

$$\left\|\frac{1}{N}\sum_{i=1}^N (\boldsymbol{\delta}-\boldsymbol{\delta}_0)'\frac{\mathbf{w}_i'\mathbf{M}\boldsymbol{\varepsilon}_i}{T}\right\| \leq \|\boldsymbol{\delta}-\boldsymbol{\delta}_0\| \left\|\frac{1}{N}\sum_{i=1}^N \frac{\mathbf{w}_i'\mathbf{M}\boldsymbol{\varepsilon}_i}{T}\right\| = O_p\left(\frac{1}{N}\right) + O_p\left(\frac{1}{T}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right).$$

Also

$$\left\|\frac{1}{NT}\sum_{i=1}^N \boldsymbol{\varepsilon}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\gamma}_i\right\| \leq \left\|\frac{1}{N}\sum_{i=1}^N \frac{\boldsymbol{\varepsilon}_i'\bar{\mathbf{U}}_m}{T}\boldsymbol{\gamma}_i\right\| + \left\|\frac{1}{N}\sum_{i=1}^N\left(\boldsymbol{\gamma}_i' \otimes \frac{\boldsymbol{\varepsilon}_i'\mathbf{Q}_0}{T}\right)\right\| \left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1}\right\| \left\|\frac{\mathbf{Q}_0'\bar{\mathbf{U}}_m}{T}\right\|,$$
$$\leq \left\|\frac{1}{N}\sum_{i=1}^N\left(\boldsymbol{\gamma}_i' \otimes \frac{\boldsymbol{\varepsilon}_i'\bar{\mathbf{U}}_m}{T}\right)\right\| \|\mathbf{I}_m\| + \left\|\frac{1}{N}\sum_{i=1}^N\left(\boldsymbol{\gamma}_i' \otimes \frac{\boldsymbol{\varepsilon}_i'\mathbf{Q}_0}{T}\right)\right\| \left\|\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1}\right\| \left\|\frac{\mathbf{Q}_0'\bar{\mathbf{U}}_m}{T}\right\|,$$

Letting $\gamma_{i,d}$ denote the element on row $d = 1, \ldots, m$ of $\boldsymbol{\gamma}_i$, the elements on columns $c(d-1)+1$ to cd of $\frac{1}{N}\sum_{i=1}^N (\boldsymbol{\gamma}_i' \otimes T^{-1}\boldsymbol{\varepsilon}_i'\mathbf{Q}_0)$ and columns $m(d-1)+1$ to md of $\frac{1}{N}\sum_{i=1}^N \left(\boldsymbol{\gamma}_i' \otimes T^{-1}\boldsymbol{\varepsilon}_i'\bar{\mathbf{U}}_m\right)$ are given by

$$\frac{1}{N}\sum_{i=1}^N \gamma_{i,d}\frac{\boldsymbol{\varepsilon}_i'\mathbf{F}^0}{T} + \frac{1}{N}\sum_{i=1}^N \gamma_{i,d}\frac{\boldsymbol{\varepsilon}_i'\bar{\mathbf{U}}^0}{T} = \left[\frac{\bar{\mathbf{a}}_d'\mathbf{F}^*}{T}, \mathbf{0}_{1\times(K-m)}\right] + \left[\frac{\bar{\mathbf{a}}_d'\bar{\mathbf{U}}_m^0}{T}, \frac{\bar{\mathbf{a}}_d'\bar{\mathbf{U}}_{-m}^0}{T}\right],$$
$$\frac{1}{N}\sum_{i=1}^N \gamma_{i,d}\frac{\boldsymbol{\varepsilon}_i'\bar{\mathbf{U}}_m}{T} = \frac{\bar{\mathbf{a}}_d'\bar{\mathbf{U}}_m}{T},$$

respectively, with $\bar{\mathbf{a}}_d = \frac{1}{N}\sum_{i=1}^N \gamma_{i,d}\boldsymbol{\varepsilon}_i$ and where we note that $\|\bar{\mathbf{a}}_d\| = O_p(\sqrt{T}N^{-1/2})$ by the independence of $\gamma_{i,d}$ and $\boldsymbol{\varepsilon}_i$ from Ass.1 and 3. As such, with (B-18)-(B-20) and (B-22)

$$\left\|\frac{\bar{\mathbf{a}}_d'\mathbf{F}^*}{T}\right\| \leq \left\|\frac{\bar{\mathbf{a}}_d'\check{\mathbf{F}}}{T}\right\| \left\|\ddot{\mathbf{P}}\right\| \|\mathbf{R}\| \|\mathbf{N}\| \|\mathbf{S}_m\| = O_p\left(\frac{1}{\sqrt{NT}}\right),$$

$$\left\|\frac{\bar{\mathbf{a}}_d'\bar{\mathbf{U}}_m^0}{T}\right\| \leq \left\|\frac{\bar{\mathbf{a}}_d'\ddot{\mathbf{U}}}{T}\right\| \|\mathbf{R}\| \|\mathbf{N}\| \|\mathbf{S}_m\| = O_p\left(\frac{1}{N}\right),$$
$$\left\|\frac{\bar{\mathbf{a}}_d'\bar{\mathbf{U}}_{-m}^0}{T}\right\| \leq \sqrt{N}\left\|\frac{\bar{\mathbf{a}}_d'\ddot{\mathbf{U}}}{T}\right\| \|\mathbf{R}\| \|\mathbf{N}\| \|\mathbf{S}_{-m}\| = O_p\left(\frac{1}{\sqrt{N}}\right),$$
$$\left\|\frac{\bar{\mathbf{a}}_d'\bar{\mathbf{U}}_m}{T}\right\| \leq \left\|\frac{\bar{\mathbf{a}}_d'\ddot{\mathbf{U}}}{T}\right\| \|\mathbf{R}_0\| \|\mathbf{T}\| \|\mathbf{B}_m\| = O_p\left(\frac{1}{N}\right),$$

since $\left\|\frac{\bar{\mathbf{a}}_d'\check{\mathbf{F}}}{T}\right\| = O_p((NT)^{-1/2})$ by independence of $\bar{\mathbf{a}}_d$ and $\check{\mathbf{F}}$, and because $\left\|\frac{\bar{\mathbf{a}}_d'\ddot{\mathbf{U}}}{T}\right\| \leq T^{-1}\|\bar{\mathbf{a}}_d\| \left\|\ddot{\mathbf{U}}\right\| = O_p(N^{-1})$ since $\left\|\ddot{\mathbf{U}}\right\| = O_p(\sqrt{T}N^{-1/2})$. As such, $\left\|\frac{1}{N}\sum_{i=1}^N(\boldsymbol{\gamma}_i' \otimes T^{-1}\boldsymbol{\varepsilon}_i'\mathbf{Q}_0)\right\| = O_p(N^{-1/2})$ and $\left\|\frac{1}{N}\sum_{i=1}^N\left(\boldsymbol{\gamma}_i' \otimes T^{-1}\boldsymbol{\varepsilon}_i'\bar{\mathbf{U}}_m\right)\right\| = O_p(N^{-1})$. Thus, inserting also $\left\|T^{-1}\mathbf{Q}_0'\bar{\mathbf{U}}_m\right\| = O_p(N^{-1/2})$ by Lemma 10 gives

$$\left\|\frac{1}{NT}\sum_{i=1}^N \boldsymbol{\varepsilon}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\gamma}_i\right\| = O_p\left(\frac{1}{N}\right), \tag{D-47}$$

and therefore,

$$\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta}_0) = \frac{T}{T-c}(\boldsymbol{\delta}-\boldsymbol{\delta}_0)'\widehat{\boldsymbol{\Sigma}}(\boldsymbol{\delta}-\boldsymbol{\delta}_0) + \frac{T}{T-c}\frac{1}{NT}\sum_{i=1}^N \boldsymbol{\varepsilon}_i'\mathbf{M}\boldsymbol{\varepsilon}_i$$
$$+ O_p\left(\frac{1}{N}\right) + O_p\left(\frac{1}{T}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right).$$

The final term in this expression we can decompose as

$$\frac{1}{NT}\sum_{i=1}^N \boldsymbol{\varepsilon}_i'\mathbf{M}\boldsymbol{\varepsilon}_i = \frac{1}{N}\sum_{i=1}^N \frac{\boldsymbol{\varepsilon}_i'\boldsymbol{\varepsilon}_i}{T} - \frac{1}{NT}\sum_{i=1}^N \boldsymbol{\varepsilon}_i'\mathbf{H}\boldsymbol{\varepsilon}_i.$$

Consider the last term and recall from Lemma 10 that $\|T^{-1}\boldsymbol{\varepsilon}_i'\mathbf{Q}_0\| = O_p(T^{-1/2})$. Note that we can write with $\mathbf{q}_{0,t}$ denoting the $t-th$ row of $\mathbf{Q}_0$ and $\bar{\varepsilon}_{t,s} = \frac{1}{N}\sum_{i=1}^N \varepsilon_{it}\varepsilon_{is}$, with notably $\bar{\varepsilon}_{t,s} = O_p(N^{-1/2})$ for $s \neq t$ and $\bar{\varepsilon}_{t,t} = \sigma_\varepsilon^2 + O_p(N^{-1/2})$,

$$\frac{1}{NT}\sum_{i=1}^N \boldsymbol{\varepsilon}_i'\mathbf{H}\boldsymbol{\varepsilon}_i = \frac{1}{N}\sum_{i=1}^N \frac{\boldsymbol{\varepsilon}_i'\mathbf{Q}_0}{T}\left(\frac{\mathbf{Q}_0'\mathbf{Q}_0}{T}\right)^{-1}\frac{\mathbf{Q}_0'\boldsymbol{\varepsilon}_i}{T} = \frac{1}{NT^2}\sum_{i=1}^N \boldsymbol{\varepsilon}_i'\mathbf{Q}_0\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1}\mathbf{Q}_0'\boldsymbol{\varepsilon}_i,$$
$$= \frac{1}{NT^2}\sum_{i=1}^N\sum_{t=1}^T\sum_{s=1}^T \varepsilon_{it}\varepsilon_{is}\mathbf{q}_{0,t}'\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1}\mathbf{q}_{0,s},$$
$$= \frac{1}{NT^2}\sum_{t=1}^T\sum_{s=1}^T \bar{\varepsilon}_{t,s}\mathbf{q}_{0,t}'\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1}\mathbf{q}_{0,s},$$
$$= \frac{1}{T^2}\sigma_\varepsilon^2 tr(\mathbf{Q}_0\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1}\mathbf{Q}_0') + \frac{1}{NT^2}\sum_{t=1}^T(\bar{\varepsilon}_{t,t}-\sigma_\varepsilon^2)\mathbf{q}_{0,t}'\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1}\mathbf{q}_{0,t}$$
$$+ \frac{1}{NT^2}\sum_{t=1}^T\sum_{s\neq t}^T \bar{\varepsilon}_{t,s}\mathbf{q}_{0,t}'\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1}\mathbf{q}_{0,s},$$

$$= \frac{1}{T^2}\sigma_\varepsilon^2 Tc + O_p(N^{-1/2}T^{-1}),$$
$$= \frac{c}{T}\sigma_\varepsilon^2 + O_p(N^{-1/2}T^{-1}),$$

and also for the first term

$$\frac{1}{N}\sum_{i=1}^{N}\frac{\boldsymbol{\varepsilon}_i'\boldsymbol{\varepsilon}_i}{T} = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\varepsilon_{it}^2 = \sigma_\varepsilon^2 + \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}(\varepsilon_{it}^2 - \sigma_\varepsilon^2) = \sigma_\varepsilon^2 + \frac{1}{T}\sum_{t=1}^{T}(\bar{\varepsilon}_{t,t} - \sigma_\varepsilon^2),$$
$$= \sigma_\varepsilon^2 + O_p((NT)^{-1/2}),$$

which gives, combined into the expression above,

$$\frac{T}{T-c}\left[\frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\varepsilon}_i'\mathbf{M}\boldsymbol{\varepsilon}_i\right] = \frac{T}{T-c}\left[\sigma_\varepsilon^2 - \frac{c}{T}\sigma_\varepsilon^2\right] + O_p\left(\frac{1}{\sqrt{NT}}\right) = \sigma_\varepsilon^2 + O_p\left(\frac{1}{\sqrt{NT}}\right). \quad \text{(D-48)}$$

Finally, since $\left\|\widehat{\boldsymbol{\Sigma}}\right\| = O_p(1)$ and making use of $\frac{T}{T-c} \to 1$ we conclude that

$$\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta}_0) = \sigma_\varepsilon^2 + (\boldsymbol{\delta} - \boldsymbol{\delta}_0)'\widehat{\boldsymbol{\Sigma}}(\boldsymbol{\delta} - \boldsymbol{\delta}_0) + O_p(N^{-1}) + O_p(T^{-1}) + O_p((NT)^{-1/2}),$$

which is the first result stated in the lemma.

It remains to consider $\boldsymbol{\delta}_0 = \boldsymbol{\delta}$. Clearly, in this case

$$\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta}) = \frac{T}{T-c}\frac{1}{NT}\sum_{i=1}^{N}\left\|\mathbf{M}(\boldsymbol{\varepsilon}_i - \bar{\mathbf{U}}_m\boldsymbol{\gamma}_i)\right\|^2,$$
$$= \frac{T}{T-c}\left[\frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\varepsilon}_i'\mathbf{M}\boldsymbol{\varepsilon}_i - 2\frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\varepsilon}_i'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\gamma}_i + \frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\gamma}_i'\bar{\mathbf{U}}_m'\mathbf{M}\bar{\mathbf{U}}_m\boldsymbol{\gamma}_i\right],$$

and therefore, substituting in earlier results such as (D-47) and (D-48)

$$\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta}) = \sigma_\varepsilon^2 + O_p(N^{-1}) + O_p((NT)^{-1/2}).$$

This proves the lemma.

Proof of Lemma 17

Consider $\mathbf{A}^c = \mathbf{A}^\varepsilon + \frac{1}{T}\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta})\boldsymbol{\upsilon}$ evaluated at $\boldsymbol{\delta}_0 = \boldsymbol{\delta}$, where $\boldsymbol{\upsilon}$ denotes $\boldsymbol{\upsilon}(\rho_0, \mathbf{H})$ evaluated at $\rho_0 = \rho$. Making use of the notation introduced in Lemma 15, specifically (D-39), we can decompose it as follows

$$\mathbf{A}^c = \mathbf{A}^\varepsilon + \frac{1}{T}\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta})\boldsymbol{\upsilon},$$
$$= \mathbf{A}_1^\varepsilon + \mathbf{A}_2^\varepsilon - \mathbf{A}_3^\varepsilon + \frac{1}{T}\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta})\boldsymbol{\upsilon},$$

$$= \mathbf{A}_1^{\varepsilon} + \mathbf{A}_{21}^{\varepsilon} - \mathbf{A}_3^{\varepsilon} - \frac{1}{NT}\sum_{i=1}^{N} \boldsymbol{\epsilon}_i' \mathbf{H} \boldsymbol{\varepsilon}_i + \frac{1}{T}\widehat{\sigma}_{\varepsilon}^2(\boldsymbol{\delta})\boldsymbol{\upsilon},$$
$$= \mathbf{A}_1^{\varepsilon} + \mathbf{A}_{21}^{\varepsilon} - \mathbf{A}_3^{\varepsilon} - \mathbf{A}_0^{c},$$

where in the final equality we substituted in $\mathbf{A}_{21}^{\varepsilon} = \frac{1}{NT}\sum_{i=1}^{N} \boldsymbol{\epsilon}_i' \boldsymbol{\varepsilon}_i$ of (D-42) and defined

$$\mathbf{A}_0^{c} = \frac{1}{NT}\sum_{i=1}^{N} \boldsymbol{\epsilon}_i' \mathbf{H} \boldsymbol{\varepsilon}_i - \frac{1}{T}\widehat{\sigma}_{\varepsilon}^2(\boldsymbol{\delta})\boldsymbol{\upsilon}.$$

For this term we can write

$$\mathbf{A}_0^{c} = \frac{1}{NT}\sum_{i=1}^{N} \left[\boldsymbol{\epsilon}_i' \mathbf{H} \boldsymbol{\varepsilon}_i - \sigma_{\varepsilon}^2 \boldsymbol{\upsilon}\right] - \frac{1}{T}\left[\widehat{\sigma}_{\varepsilon}^2(\boldsymbol{\delta}) - \sigma_{\varepsilon}^2\right]\boldsymbol{\upsilon}.$$

Recall that $T^{-1}\boldsymbol{\epsilon}_i'\mathbf{H}\boldsymbol{\varepsilon}_i = T^{-1}\boldsymbol{\epsilon}_i'\mathbf{Q}_0\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}}^{-1}T^{-1}\mathbf{Q}_0'\boldsymbol{\varepsilon}_i$ and that from Lemma 10 $\|T^{-1}\boldsymbol{\epsilon}_i'\mathbf{Q}_0\|$ and $\|T^{-1}\boldsymbol{\varepsilon}_i'\mathbf{Q}_0\|$ are $O_p(T^{-1/2})$. Also denote with $h_{t,s}$ the element on row t and column s of $\mathbf{H}$ and $\bar{\boldsymbol{\epsilon}}_{t,s} = \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{\epsilon}_{it}\varepsilon_{is}$ such that $\left[\bar{\boldsymbol{\epsilon}}_{t,s} - \sigma_{\varepsilon}^2\mathbf{q}_1\rho^{t-1-s}\mathbb{1}_{(t-1\geq s)}\right] = O_p(N^{-1/2})$ for all t and s, where $\mathbb{1}_a$ denotes the indicator function that returns one if the condition a is true, and zero otherwise. This gives

$$\frac{1}{NT}\sum_{i=1}^{N}\left[\boldsymbol{\epsilon}_i'\mathbf{H}\boldsymbol{\varepsilon}_i - \sigma_{\varepsilon}^2\boldsymbol{\upsilon}\right] = \frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T} h_{t,s}\frac{1}{N}\sum_{i=1}^{N}\left[\boldsymbol{\epsilon}_{it}\varepsilon_{is} - \sigma_{\varepsilon}^2\mathbf{q}_1\rho^{t-1-s}\mathbb{1}_{(t-1\geq s)}\right],$$
$$= \frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T} h_{t,s}\left[\bar{\boldsymbol{\epsilon}}_{t,s} - \sigma_{\varepsilon}^2\mathbf{q}_1\rho^{t-1-s}\mathbb{1}_{(t-1\geq s)}\right] = O_p(N^{-1/2}T^{-1}).$$

Second, note that the function $\boldsymbol{\upsilon} = \upsilon(\rho, \mathbf{H})\mathbf{q}_1 = tr(\mathbf{H}\mathbf{L}\mathbf{J}^{-1}(\rho))\mathbf{q}_1$ calculates the sum of the lower triangular elements of $\mathbf{H}$ weighted by the columns of $\mathbf{J}^{-1}(\rho)$, with $\mathbf{J}(\rho)$ a $T \times T$ matrix with ones on the main diagonal, $-\rho$ on the first lower sub-diagonal, and zeros on all other entries, and $\mathbf{L}$ the $T \times T$ lag operator with ones on the first lower sub-diagonal and zeros on all other entries. We then have that $\|\boldsymbol{\upsilon}\| = O_p(1)$ since $\rho < 1$ under Ass.5 such that each column of the weighting matrix $\mathbf{J}^{-1}(\rho)$ contains an exponentially decaying sequence and its row and column norms are bounded by a finite constant which is independent of T.

Therefore, also substituting in $\|\widehat{\sigma}_{\varepsilon}^2(\boldsymbol{\delta}) - \sigma_{\varepsilon}^2\| = O_p(N^{-1}) + O_p((NT)^{-1/2})$ by (D-27) of Lemma 16 gives

$$\|\mathbf{A}_0^{c}\| \leq \left\|\frac{1}{NT}\sum_{i=1}^{N}\left[\boldsymbol{\epsilon}_i'\mathbf{H}\boldsymbol{\varepsilon}_i - \sigma_{\varepsilon}^2\boldsymbol{\upsilon}\right]\right\| + \frac{1}{T}\left\|\widehat{\sigma}_{\varepsilon}^2(\boldsymbol{\delta}) - \sigma_{\varepsilon}^2\right\|\|\boldsymbol{\upsilon}\| = O_p(N^{-1/2}T^{-1}).$$

such that from the respective results in eqs.(D-40), (D-43) and (D-45) of the proof for Lemma 15 follows

$$\|\mathbf{A}^{c}\| \leq \|\mathbf{A}_1^{\varepsilon}\| + \|\mathbf{A}_{21}^{\varepsilon}\| + \|\mathbf{A}_3^{\varepsilon}\| + \|\mathbf{A}_0^{c}\| = O_p(N^{-1}) + O_p((NT)^{-1/2}).$$

Also, letting $\mathbf{A}_{NT}^{c} = \sqrt{NT}\mathbf{A}^{c}$ and imposing that $T/N \to M < \infty$ yields

$$\mathbf{A}_{NT}^{c} = \mathbf{A}_{NT,1}^{\varepsilon} + \mathbf{A}_{NT,21}^{\varepsilon} - \mathbf{A}_{NT,3}^{\varepsilon} + O_p(T^{-1/2}),$$

with $\mathbf{A}^{\varepsilon}_{NT,1}$ and $\mathbf{A}^{\varepsilon}_{NT,3}$ defined in (D-41) and (D-46), respectively, and where $\mathbf{A}^{\varepsilon}_{NT,21} = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{\boldsymbol{\epsilon}_i'\boldsymbol{\varepsilon}_i}{\sqrt{T}}$. Substituting in the respective definitions gives the result stated in the lemma.

Next, consider the moment vector evaluated at any $\boldsymbol{\delta}_0 \neq \boldsymbol{\delta}$ such that $\|\boldsymbol{\delta} - \boldsymbol{\delta}_0\| < \infty$,

$$\widetilde{\mathbf{A}}^c(\boldsymbol{\delta}_0) = \mathbf{A}^{\varepsilon} + T^{-1}\widehat{\sigma}^2_{\varepsilon}(\boldsymbol{\delta}_0)\boldsymbol{\upsilon}(\rho_0) = \mathbf{A}^{\varepsilon}_1 + \mathbf{A}^{\varepsilon}_{21} - \mathbf{A}^{\varepsilon}_3 - \widetilde{\mathbf{A}}^c_0(\boldsymbol{\delta}_0),$$

with $\widetilde{\mathbf{A}}^c_0(\boldsymbol{\delta}_0) = \frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\epsilon}_i'\mathbf{H}\boldsymbol{\varepsilon}_i - \frac{1}{T}\widehat{\sigma}^2_{\varepsilon}(\boldsymbol{\delta}_0)\boldsymbol{\upsilon}(\rho_0)$ and $\boldsymbol{\upsilon}(\rho_0) = \upsilon(\rho_0, \mathbf{H})\mathbf{q}_1$, and, as before by eqs.(D-40), (D-43) and (D-45)

$$\widetilde{\mathbf{A}}^c(\boldsymbol{\delta}_0) = -\widetilde{\mathbf{A}}^c_0(\boldsymbol{\delta}_0) + O_p(N^{-1}) + O_p((NT)^{-1/2}).$$

We get using the same steps as above and substituting in earlier results

$$\begin{aligned}\widetilde{\mathbf{A}}^c_0(\boldsymbol{\delta}_0) &= \frac{1}{NT}\sum_{i=1}^{N}\left[\boldsymbol{\epsilon}_i'\mathbf{H}\boldsymbol{\varepsilon}_i - \sigma^2_{\varepsilon}\boldsymbol{\upsilon}\right] - \frac{1}{T}\left[\widehat{\sigma}^2_{\varepsilon}(\boldsymbol{\delta}_0) - \sigma^2_{\varepsilon}\right]\boldsymbol{\upsilon}(\rho_0) - \frac{1}{T}\sigma^2_{\varepsilon}\left[\boldsymbol{\upsilon}(\rho_0) - \boldsymbol{\upsilon}\right], \\ &= -\frac{1}{T}\left[\widehat{\sigma}^2_{\varepsilon}(\boldsymbol{\delta}_0) - \sigma^2_{\varepsilon}\right]\boldsymbol{\upsilon}(\rho_0) - \frac{1}{T}\sigma^2_{\varepsilon}\left[\boldsymbol{\upsilon}(\rho_0) - \boldsymbol{\upsilon}\right] + O_p(N^{-1/2}T^{-1}).\end{aligned}$$

In turn, substituting in (D-26) of Lemma 16 returns

$$\widetilde{\mathbf{A}}^c_0(\boldsymbol{\delta}_0) = -\frac{1}{T}(\boldsymbol{\delta} - \boldsymbol{\delta}_0)'\widehat{\boldsymbol{\Sigma}}(\boldsymbol{\delta} - \boldsymbol{\delta}_0)\boldsymbol{\upsilon}(\rho_0) - \frac{1}{T}\sigma^2_{\varepsilon}\left[\boldsymbol{\upsilon}(\rho_0) - \boldsymbol{\upsilon}\right] + O_p(N^{-1/2}T^{-1}),$$

and therefore

$$\widetilde{\mathbf{A}}^c(\boldsymbol{\delta}_0) = \frac{1}{T}(\boldsymbol{\delta} - \boldsymbol{\delta}_0)'\widehat{\boldsymbol{\Sigma}}(\boldsymbol{\delta} - \boldsymbol{\delta}_0)\boldsymbol{\upsilon}(\rho_0) + \frac{1}{T}\sigma^2_{\varepsilon}\left[\boldsymbol{\upsilon}(\rho_0) - \boldsymbol{\upsilon}\right] + O_p(N^{-1}) + O_p((NT)^{-1/2}),$$

which ends the proof.

Proof of Lemma 18

Consider, since $\mathbf{M}\bar{\mathbf{w}} = \mathbf{0}$,

$$\begin{aligned}\widehat{\boldsymbol{\Sigma}} &= \frac{1}{NT}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{M}\mathbf{w}_i = \frac{1}{NT}\sum_{i=1}^{N}\left(\mathbf{w}_i - \bar{\mathbf{w}}\right)'\mathbf{M}\left(\mathbf{w}_i - \bar{\mathbf{w}}\right), \\ &= \frac{1}{NT}\sum_{i=1}^{N}\left(\check{\mathbf{F}}\widetilde{\mathbf{P}}_i\mathbf{S}_w + \boldsymbol{\epsilon}_i - \bar{\boldsymbol{\epsilon}}\right)'\mathbf{M}\left(\check{\mathbf{F}}\widetilde{\mathbf{P}}_i\mathbf{S}_w + \boldsymbol{\epsilon}_i - \bar{\boldsymbol{\epsilon}}\right),\end{aligned}$$

where noting that $\boldsymbol{\epsilon}_i = \ddot{\mathbf{U}}_i\mathbf{S}_w$ it is easily seen from Lemmas 10 and 11

$$\left\|\frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\check{\mathbf{F}}'\mathbf{M}\bar{\boldsymbol{\epsilon}}\right\| = O_p\left(\frac{1}{N}\right), \left\|\frac{1}{NT}\sum_{i=1}^{N}\bar{\boldsymbol{\epsilon}}'\mathbf{M}\bar{\boldsymbol{\epsilon}}\right\| = O_p\left(\frac{1}{N}\right), \left\|\frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\epsilon}_i'\mathbf{M}\bar{\boldsymbol{\epsilon}}\right\| = O_p\left(\frac{1}{N}\right).$$

Also, from (D-40)

$$\left\|\frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\check{\mathbf{F}}'\mathbf{M}\boldsymbol{\epsilon}_i\right\| = O_p\left(\frac{1}{\sqrt{NT}}\right),$$

and making use of Lemma 10 and Ass.1 and 5,

$$\frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\epsilon}_i'\mathbf{M}\boldsymbol{\epsilon}_i = \frac{1}{N}\sum_{i=1}^{N}\frac{\boldsymbol{\epsilon}_i'\boldsymbol{\epsilon}_i}{T} - \frac{1}{N}\sum_{i=1}^{N}\frac{\boldsymbol{\epsilon}_i'\mathbf{Q}_0}{T}\left(\frac{\mathbf{Q}_0'\mathbf{Q}_0}{T}\right)^{-1}\frac{\mathbf{Q}_0'\boldsymbol{\epsilon}_i}{T} = \frac{1}{N}\sum_{i=1}^{N}\frac{\boldsymbol{\epsilon}_i'\boldsymbol{\epsilon}_i}{T} + O_p\left(\frac{1}{T}\right),$$
$$= \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}} + O_p(T^{-1}) + O_p((NT)^{-1/2}),$$

with $\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}} = E(\boldsymbol{\epsilon}_i'\boldsymbol{\epsilon}_i/T)$. Next up is

$$\frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\check{\mathbf{F}}'\mathbf{M}\check{\mathbf{F}}\widetilde{\mathbf{P}}_i\mathbf{S}_w = \frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\check{\mathbf{F}}'\mathbf{M}_{\mathbf{F}}\check{\mathbf{F}}\widetilde{\mathbf{P}}_i\mathbf{S}_w - \frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\check{\mathbf{F}}'(\mathbf{M}_{\mathbf{F}}-\mathbf{M})\check{\mathbf{F}}\widetilde{\mathbf{P}}_i\mathbf{S}_w,$$

where for the second term, defining $\widehat{\boldsymbol{\Sigma}}_{\widetilde{\mathbf{P}}} = \left[\frac{1}{N}\sum_{i=1}^{N}\left(\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\otimes\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)\right]$,

$$\left\|\frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\check{\mathbf{F}}'(\mathbf{M}_{\mathbf{F}}-\mathbf{M})\check{\mathbf{F}}\widetilde{\mathbf{P}}_i\mathbf{S}_w\right\| \le \left\|\widehat{\boldsymbol{\Sigma}}_{\widetilde{\mathbf{P}}}\right\|\left\|\check{\mathbf{F}}'(\mathbf{M}_{\mathbf{F}}-\mathbf{M})\check{\mathbf{F}}\right\|,$$

for which $\left\|\widehat{\boldsymbol{\Sigma}}_{\widetilde{\mathbf{P}}}\right\| = O_p(1)$ by Ass.3 and the norm in the end can be decomposed into 5 parts by (D-1). Using the shorthand $\widehat{\boldsymbol{\Sigma}}_{\mathbf{Q}} = [T^{-1}\mathbf{Q}_0'\mathbf{Q}_0]$, $\widehat{\boldsymbol{\Sigma}}_{\mathbf{F}^*} = [T^{-1}(\mathbf{F}^*)'\mathbf{F}^*]$ and $\widehat{\boldsymbol{\Sigma}}_{\mathbf{u}^0_{-m}} = [T^{-1}(\bar{\mathbf{U}}^0_{-m})'\bar{\mathbf{U}}^0_{-m}]$ we get for each respective component

$$\|\mathbf{K}_1\| \le \left\|\frac{\check{\mathbf{F}}'\bar{\mathbf{U}}^0_{-m}}{T}\right\|\left\|\widehat{\boldsymbol{\Sigma}}^{-1}_{\mathbf{u}^0_{-m}}\right\|\left\|\frac{(\bar{\mathbf{U}}^0_{-m})'\check{\mathbf{F}}}{T}\right\| = O_p\left(\frac{1}{T}\right),$$
$$\|\mathbf{K}_2\| \le \left\|\frac{\check{\mathbf{F}}'\bar{\mathbf{U}}^0_{m}}{T}\right\|\left\|\widehat{\boldsymbol{\Sigma}}^{-1}_{\mathbf{F}^*}\right\|\left\|\frac{(\bar{\mathbf{U}}^0_{m})'\check{\mathbf{F}}}{T}\right\| = O_p\left(\frac{1}{NT}\right),$$
$$\|\mathbf{K}_3\| \le \left\|\frac{\check{\mathbf{F}}'\mathbf{F}^*}{T}\right\|\left\|\widehat{\boldsymbol{\Sigma}}^{-1}_{\mathbf{F}^*}\right\|\left\|\frac{(\bar{\mathbf{U}}^0_{m})'\check{\mathbf{F}}}{T}\right\| = O_p\left(\frac{1}{\sqrt{NT}}\right),$$
$$\|\mathbf{K}_4\| \le \left\|\frac{\check{\mathbf{F}}'\bar{\mathbf{U}}^0_{m}}{T}\right\|\left\|\widehat{\boldsymbol{\Sigma}}^{-1}_{\mathbf{F}^*}\right\|\left\|\frac{(\mathbf{F}^*)'\check{\mathbf{F}}}{T}\right\| = O_p\left(\frac{1}{\sqrt{NT}}\right),$$
$$\|\mathbf{K}_5\| \le \left\|\frac{\check{\mathbf{F}}'\mathbf{Q}_0}{T}\right\|\left\|\widehat{\boldsymbol{\Sigma}}^{-1}_{\mathbf{Q}} - \widehat{\boldsymbol{\Sigma}}^{-1}_{\mathbf{F}^+_u}\right\|\left\|\frac{\mathbf{Q}_0'\check{\mathbf{F}}}{T}\right\| = O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right),$$

which makes use of (B-18)-(B-20) and Lemmas 9, 10 and 13. As such, $\left\|\check{\mathbf{F}}'(\mathbf{M}_{\mathbf{F}}-\mathbf{M})\check{\mathbf{F}}\right\| = O_p(N^{-1/2}) + O_p(T^{-1/2})$ and

$$\frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\check{\mathbf{F}}'\mathbf{M}\check{\mathbf{F}}\widetilde{\mathbf{P}}_i\mathbf{S}_w = \frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\check{\mathbf{F}}'\mathbf{M}_{\mathbf{F}}\check{\mathbf{F}}\widetilde{\mathbf{P}}_i\mathbf{S}_w + O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right).$$

Here we have, recalling $\widehat{\boldsymbol{\Sigma}}_{\check{\mathbf{F}}} = T^{-1}\check{\mathbf{F}}'\check{\mathbf{F}}$ and using (B-18) and Lemma 12,

$$\frac{1}{NT}\sum_{i=1}^{N}\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\check{\mathbf{F}}'\mathbf{M}_{\mathbf{F}}\check{\mathbf{F}}\widetilde{\mathbf{P}}_i\mathbf{S}_w$$

$$= \frac{1}{N}\sum_{i=1}^{N} \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\widehat{\mathbf{\Sigma}}_{\check{\mathbf{F}}}\widetilde{\mathbf{P}}_i\mathbf{S}_w - \frac{1}{N}\sum_{i=1}^{N} \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\widehat{\mathbf{\Sigma}}_{\check{\mathbf{F}}}\ddot{\mathbf{P}}\mathbf{R}\mathbf{N}\mathbf{S}_m\widehat{\mathbf{\Sigma}}_{\mathbf{F}^*}^{-1}\mathbf{S}_m'\mathbf{N}'\mathbf{R}'\ddot{\mathbf{P}}'\widehat{\mathbf{\Sigma}}_{\check{\mathbf{F}}}\widetilde{\mathbf{P}}_i\mathbf{S}_w,$$
$$= \frac{1}{N}\sum_{i=1}^{N} \mathbf{S}_w'\widetilde{\mathbf{P}}_i'\mathbf{V}^{\mathbf{F}}\widetilde{\mathbf{P}}_i\mathbf{S}_w + O_p(N^{-1/2}) + O_p(T^{-1/2}),$$
$$= (vec(\mathbf{I}_{k_w})' \otimes \mathbf{I}_{k_w})\left(\mathbf{I}_{k_w} \otimes \mathbf{\Sigma}_{\widetilde{\mathbf{P}}} vec(\mathbf{V}^{\mathbf{F}})\right) + O_p(N^{-1/2}) + O_p(T^{-1/2}),$$

where $\mathbf{V}^{\mathbf{F}} = \mathbf{\Sigma}_{\check{\mathbf{F}}} - \mathbf{\Sigma}_{\check{\mathbf{F}}}\mathbf{P}\mathbf{R}\mathbf{N}\mathbf{S}_m\mathbf{\Sigma}_{\mathbf{F}^*}^{-1}\mathbf{S}_m'\mathbf{N}'\mathbf{R}'\mathbf{P}'\mathbf{\Sigma}_{\check{\mathbf{F}}}$ and $\mathbf{\Sigma}_{\widetilde{\mathbf{P}}} = E(\mathbf{S}_w'\widetilde{\mathbf{P}}_i' \otimes \mathbf{S}_w'\widetilde{\mathbf{P}}_i')$.

In conclusion, we have as $(N,T) \to \infty$ that

$$\widehat{\mathbf{\Sigma}} \longrightarrow^p \mathbf{\Sigma}_{\check{\mathbf{F}}\mathbf{P}} + \mathbf{\Sigma}_{\boldsymbol{\epsilon}},$$

with $\mathbf{\Sigma}_{\check{\mathbf{F}}\mathbf{P}} = (vec(\mathbf{I}_{k_w})' \otimes \mathbf{I}_{k_w})\left(\mathbf{I}_{k_w} \otimes \mathbf{\Sigma}_{\widetilde{\mathbf{P}}} vec(\mathbf{V}^{\mathbf{F}})\right)$.

D.4 Proof of theorems

D.4.1 Proof of Theorem 2

Consider that the CCEPbc estimator in eq.(21) is equivalent to

$$\widehat{\boldsymbol{\delta}}_{bc} = \arg\min_{\boldsymbol{\delta}_0 \in \chi} \frac{1}{2} \|\boldsymbol{\varphi}(\boldsymbol{\delta}_0)\|^2, \tag{D-49}$$

with $\boldsymbol{\varphi}(\boldsymbol{\delta}_0)$ given by

$$\boldsymbol{\varphi}(\boldsymbol{\delta}_0) = \frac{1}{NT}\sum_{i=1}^{N} \mathbf{w}_i'\mathbf{M}\mathbf{y}_i - \widehat{\boldsymbol{\Sigma}}\boldsymbol{\delta}_0 + \frac{1}{T}\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta}_0)\boldsymbol{\upsilon}(\rho_0),$$

and we will use $\boldsymbol{\upsilon}(\rho_0) = \upsilon(\rho_0, \mathbf{H})\mathbf{q}_1$. With eq.(6) the latter can be reformulated as

$$\begin{aligned}\boldsymbol{\varphi}(\boldsymbol{\delta}_0) &= \widehat{\boldsymbol{\Sigma}}\left(\boldsymbol{\delta} - \boldsymbol{\delta}_0\right) + \frac{1}{NT}\sum_{i=1}^{N} \mathbf{w}_i'\mathbf{M}(\mathbf{F}\boldsymbol{\gamma}_i + \boldsymbol{\varepsilon}_i) + \frac{1}{T}\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta}_0)\boldsymbol{\upsilon}(\rho_0),\\ &= \widehat{\boldsymbol{\Sigma}}\left(\boldsymbol{\delta} - \boldsymbol{\delta}_0\right) + \mathbf{A}^{\mathbf{F}} + \widetilde{\mathbf{A}}^c(\rho_0),\end{aligned}$$

where $\mathbf{A}^{\mathbf{F}} = \frac{1}{NT}\sum_{i=1}^{N} \mathbf{w}_i'\mathbf{M}\mathbf{F}\boldsymbol{\gamma}_i$ and $\widetilde{\mathbf{A}}^c(\rho_0) = \frac{1}{NT}\sum_{i=1}^{N} \mathbf{w}_i'\mathbf{M}\boldsymbol{\varepsilon}_i + \frac{1}{T}\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta}_0)\boldsymbol{\upsilon}(\rho_0)$. Under the assumption that χ is compact such that $\|\boldsymbol{\delta} - \boldsymbol{\delta}_0\| < \infty$ it follows from Lemmas 14 and 17

$$\boldsymbol{\varphi}(\boldsymbol{\delta}_0) = \widehat{\boldsymbol{\Sigma}}\left(\boldsymbol{\delta} - \boldsymbol{\delta}_0\right) + \frac{1}{T}(\boldsymbol{\delta} - \boldsymbol{\delta}_0)'\widehat{\boldsymbol{\Sigma}}(\boldsymbol{\delta} - \boldsymbol{\delta}_0)\boldsymbol{\upsilon}(\rho_0) + \frac{1}{T}\sigma_\varepsilon^2\left[\boldsymbol{\upsilon}(\rho_0) - \boldsymbol{\upsilon}\right] + O_p\left(\frac{1}{N}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right),$$

where from here onward we omit the functional dependence of $\boldsymbol{\upsilon}(\cdot)$ when it is evaluated at the population parameter ρ. Also inserting Lemma 18 gives

$$\boldsymbol{\varphi}(\boldsymbol{\delta}_0) = \dot{\boldsymbol{\Sigma}}\left(\boldsymbol{\delta} - \boldsymbol{\delta}_0\right) + \frac{1}{T}(\boldsymbol{\delta} - \boldsymbol{\delta}_0)'\dot{\boldsymbol{\Sigma}}(\boldsymbol{\delta} - \boldsymbol{\delta}_0)\boldsymbol{\upsilon}(\rho_0) + \frac{1}{T}\sigma_\varepsilon^2\left[\boldsymbol{\upsilon}(\rho_0) - \boldsymbol{\upsilon}\right] + o_p(1).$$

Note that $\|\boldsymbol{\upsilon}\| = O_p(1)$ since $|\rho| < 1$ by Ass.5, and if χ in eq.(D-49) is compact and accordingly restricted then $|\rho_0| < 1$ and therefore $\|\boldsymbol{\upsilon}(\rho_0)\| = O_p(1)$. Since also $\left\|\dot{\boldsymbol{\Sigma}}\right\| = O(1)$ it follows that as $(N,T) \to \infty$

$$\boldsymbol{\varphi}(\boldsymbol{\delta}_0) = \dot{\boldsymbol{\Sigma}}\left(\boldsymbol{\delta} - \boldsymbol{\delta}_0\right) + o_p(1),$$

for which the solution in (D-49) is clearly unique at $\boldsymbol{\delta}_0 = \boldsymbol{\delta}$ and therefore

$$\widehat{\boldsymbol{\delta}}_{bc} \longrightarrow^p \boldsymbol{\delta}, \tag{D-50}$$

as $(N,T) \to \infty$.

Define next the following vector evaluated at $\boldsymbol{\delta}_0 = \boldsymbol{\delta}$,

$$\boldsymbol{\psi}_{NT} = \sqrt{NT}\boldsymbol{\varphi}(\boldsymbol{\delta}) = \mathbf{A}_{NT}^{\mathbf{F}} + \mathbf{A}_{NT}^c,$$

with $\mathbf{A}_{NT}^{\mathbf{F}} = \sqrt{NT}\mathbf{A}^{\mathbf{F}}$, $\mathbf{A}_{NT}^{c} = \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{M}\boldsymbol{\varepsilon}_i + T^{-1/2}\sqrt{N}\widehat{\sigma}_{\varepsilon}^2(\boldsymbol{\delta})\boldsymbol{\upsilon}$. Assuming that $T/N \to M < \infty$ and combining in this expression Lemmas 14 and 17 gives

$$\boldsymbol{\psi}_{NT} = \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\boldsymbol{\epsilon}_i'\boldsymbol{\varepsilon}_i + \boldsymbol{\Psi}_{\mathbf{F}}vec\left(\frac{\check{\mathbf{F}}'\sqrt{N}\ddot{\mathbf{U}}}{\sqrt{T}}\right) + \boldsymbol{\Psi}_{\boldsymbol{\epsilon}}vec\left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\frac{\boldsymbol{\varepsilon}_i'\check{\mathbf{F}}}{\sqrt{T}}\otimes\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)\right]$$
$$+\sqrt{\frac{T}{N}}\left(\mathbf{b}_0^{\mathbf{F}} - \mathbf{b}_1^{\mathbf{F}} - \mathbf{b}^{\mathbf{U}}\right) + O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right),$$

where $\boldsymbol{\Psi}_{\mathbf{F}}, \mathbf{b}_0^{\mathbf{F}}$ and $\mathbf{b}_1^{\mathbf{F}}$ are fixed finite matrices defined below eq.(D-23) and similarly for $\boldsymbol{\Psi}_{\boldsymbol{\epsilon}}$ and $\mathbf{b}^{\mathbf{U}}$, which are stated below (D-29).
Then, recalling that the typical element of $\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\frac{\boldsymbol{\varepsilon}_i'\check{\mathbf{F}}}{\sqrt{T}}\otimes\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)$ is given by $\frac{\sqrt{N}\bar{\mathbf{a}}_{r,s}'\check{\mathbf{F}}}{\sqrt{T}}$, with $\bar{\mathbf{a}}_{r,s} = \frac{1}{N}\sum_{i=1}^{N}\widetilde{p}_{i,r,s}\boldsymbol{\varepsilon}_i$ and $\widetilde{p}_{i,r,s}$ denoting row r and column s of $\mathbf{S}_w'\widetilde{\mathbf{P}}_i'$, and that $\widetilde{p}_{i,r,s}$, $\boldsymbol{\varepsilon}_i$ and $\check{\mathbf{F}}$ are independent over all i and t, we have given the moment restrictions in Ass.1-3 by a CLT for independent stationary variables as $(N,T)\to\infty$

$$\boldsymbol{\xi}_1 = vec\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\frac{\boldsymbol{\varepsilon}_i'\check{\mathbf{F}}}{\sqrt{T}}\otimes\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)\right) \xrightarrow{d} \mathbf{n}^{\varepsilon\eta} \overset{d}{=} \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_{\check{\mathbf{F}}\boldsymbol{\varepsilon}}),$$

with $\boldsymbol{\Sigma}_{\check{\mathbf{F}}\boldsymbol{\varepsilon}} = \frac{1}{T}E\left[vec\left(\boldsymbol{\varepsilon}_i'\check{\mathbf{F}}\otimes\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)vec\left(\boldsymbol{\varepsilon}_i'\check{\mathbf{F}}\otimes\mathbf{S}_w'\widetilde{\mathbf{P}}_i'\right)'\right]$. Also

$$\boldsymbol{\xi}_2 = vec\left(\frac{\check{\mathbf{F}}'\sqrt{N}\ddot{\mathbf{U}}}{\sqrt{T}}\right) \xrightarrow{d} \mathbf{n}^{fu} \overset{d}{=} \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{u}}),$$

where $\boldsymbol{\Sigma}_{\check{\mathbf{F}}\mathbf{u}} = \frac{1}{T}E\left[vec\left(\check{\mathbf{F}}'\ddot{\mathbf{U}}_i\right)vec\left(\check{\mathbf{F}}'\ddot{\mathbf{U}}_i\right)'\right]$ and finally

$$\boldsymbol{\xi}_3 = \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\boldsymbol{\epsilon}_i'\boldsymbol{\varepsilon}_i \xrightarrow{d} \mathbf{n}^{\epsilon\varepsilon} \overset{d}{=} \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}\boldsymbol{\varepsilon}}),$$

with $\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}\boldsymbol{\varepsilon}} = \frac{1}{T}E\left[\boldsymbol{\epsilon}_i'\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i'\boldsymbol{\epsilon}_i'\right]$.

Let $\xi_{1,l}$ be the element on the $l-$th row of $\boldsymbol{\xi}_1$, and similarly for vectors $\boldsymbol{\xi}_2$ and $\boldsymbol{\xi}_3$. Then we have for any l and s

$$Cov(\xi_{1,l},\xi_{2,s}) = 0, \qquad Cov(\xi_{1,l},\xi_{3,s}) = 0, \qquad Cov(\xi_{2,l},\xi_{3,s}) = 0,$$

where the first two statements hold by $E(\widetilde{\mathbf{P}}_i) = \mathbf{0}$ and the independence of $\widetilde{\mathbf{P}}_i$ from $\check{\mathbf{F}}, \boldsymbol{\epsilon}_i$ and $\boldsymbol{\varepsilon}_i$ for all i and t by Ass.3, and the last result holds since $E(\check{\mathbf{F}}) = \mathbf{0}$ from Lemma 1 and the independence of $\check{\mathbf{F}}$ from $\boldsymbol{\epsilon}_i$ and $\boldsymbol{\varepsilon}_i$ by Ass.2. The three normals $\mathbf{n}^{\varepsilon\eta}$, $\mathbf{n}^{fu}$ and $\mathbf{n}^{\epsilon\varepsilon}$ are therefore independent, and as $(N,T)\to\infty$ such that $T/N \to M < \infty$ follows

$$\boldsymbol{\psi}_{NT} \xrightarrow{d} \mathcal{N}\left(\sqrt{T}N^{-1/2}\mathbf{b}_0, \boldsymbol{\Phi}\right), \tag{D-51}$$

where

$$\mathbf{b}_0 = \mathbf{b}_0^{\mathbf{F}} - \mathbf{b}_1^{\mathbf{F}} - \mathbf{b}^{\mathbf{U}}, \tag{D-52}$$

$$\mathbf{\Phi} = \mathbf{\Sigma}_{\boldsymbol{\epsilon\varepsilon}} + \mathbf{\Psi}_{\mathbf{F}}\mathbf{\Sigma}_{\check{\mathbf{F}}\mathbf{u}}\mathbf{\Psi}_{\mathbf{F}}' + \mathbf{\Psi}_{\boldsymbol{\epsilon}}\mathbf{\Sigma}_{\check{\mathbf{F}}\boldsymbol{\varepsilon}}\mathbf{\Psi}_{\boldsymbol{\epsilon}}'. \tag{D-53}$$

Next, recall from Section A.1 that the Jacobian for the CCEPbc estimator in (D-49) evaluated at $\boldsymbol{\delta}_0$ is given by

$$\mathbf{J}_{\mathrm{a}}(\boldsymbol{\delta}_0) = \frac{1}{T}\left[(\boldsymbol{\upsilon}(\rho_0) \otimes \dot{\boldsymbol{\sigma}}') + \left(\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta}_0)\mathbf{q}_1 \otimes \dot{\boldsymbol{\upsilon}}'\right)\right] - \widehat{\mathbf{\Sigma}},$$

with

$$\dot{\boldsymbol{\sigma}} = 2\frac{T}{T-c}\widehat{\mathbf{\Sigma}}\left(\boldsymbol{\delta}_0 - \widehat{\boldsymbol{\delta}}\right), \qquad \dot{\boldsymbol{\upsilon}} = \left(\sum_{t=1}^{T-1}(t-1)\rho_0^{t-2}\sum_{s=t+1}^{T} h_{s,s-t}\right)\mathbf{q}_1.$$

Consider then that as $(N,T) \to \infty$, $\widehat{\mathbf{\Sigma}} \to^p \dot{\mathbf{\Sigma}}$ by Lemma 18, $\widehat{\sigma}_\varepsilon^2(\boldsymbol{\delta}) \to^p \sigma_\varepsilon^2$ by Lemma 16 and $\|\dot{\boldsymbol{\upsilon}}\| = O_p(1)$. Also, from Lemmas 14, 15 and 18 follows $\widehat{\boldsymbol{\delta}} - \boldsymbol{\delta} = \widehat{\mathbf{\Sigma}}^{-1}(\mathbf{A}^{\varepsilon} + \mathbf{A}^{\mathbf{F}}) \to^p \mathbf{0}_{k_w \times 1}$. Hence, evaluated at $\boldsymbol{\delta}_0 = \boldsymbol{\delta}$

$$\mathbf{\Delta} = \underset{(N,T)\to\infty}{\text{plim}} \mathbf{J}_{\mathrm{a}}(\boldsymbol{\delta}) = -\dot{\mathbf{\Sigma}}. \tag{D-54}$$

As such, with (D-50) and (D-51) we have using standard arguments as in Newey and McFadden (1994), as $(N,T) \to \infty$ such that $T/N \to M < \infty$,

$$\sqrt{NT}(\widehat{\boldsymbol{\delta}}_{bc} - \boldsymbol{\delta}) \xrightarrow{d} -(\mathbf{\Delta}'\mathbf{\Delta})^{-1}\mathbf{\Delta}'\boldsymbol{\psi}_{NT},$$

which implies, given (D-51),

$$\sqrt{NT}\left(\widehat{\boldsymbol{\delta}}_{bc} - \boldsymbol{\delta}\right) \xrightarrow{d} \mathcal{N}\left(-\sqrt{T}N^{-1/2}(\mathbf{\Delta}'\mathbf{\Delta})^{-1}\mathbf{\Delta}'\mathbf{b}_0, (\mathbf{\Delta}'\mathbf{\Delta})^{-1}\mathbf{\Delta}'\mathbf{\Phi}\mathbf{\Delta}(\mathbf{\Delta}'\mathbf{\Delta})^{-1}\right),$$

and in turn, since $\mathbf{\Delta} = -\dot{\mathbf{\Sigma}}$ such that $(\mathbf{\Delta}'\mathbf{\Delta})^{-1}\mathbf{\Delta}' = -\dot{\mathbf{\Sigma}}^{-1}$,

$$\sqrt{NT}\left(\widehat{\boldsymbol{\delta}}_{bc} - \boldsymbol{\delta}\right) \xrightarrow{d} \mathcal{N}\left(\sqrt{\kappa}\mathbf{b}, \dot{\mathbf{\Sigma}}^{-1}\mathbf{\Phi}\dot{\mathbf{\Sigma}}^{-1}\right), \tag{D-55}$$

where $\mathbf{b} = \dot{\mathbf{\Sigma}}^{-1}\mathbf{b}_0$ and we denote $\kappa = T/N$. Letting next $\kappa \to 0$ gives

$$\sqrt{NT}\left(\widehat{\boldsymbol{\delta}}_{bc} - \boldsymbol{\delta}\right) \xrightarrow{d} \mathcal{N}\left(\mathbf{0}_{k_w \times 1}, \dot{\mathbf{\Sigma}}^{-1}\mathbf{\Phi}\dot{\mathbf{\Sigma}}^{-1}\right),$$

which is the result reported in the theorem.

E Additional simulation tables

Table E-1: Monte Carlo results for ρ and β : baseline design with $\rho = 0.4$

		Results for $\widehat{\rho}$											
		bias				*rmse*				$size_b$			
Estimator	(N,T)	10	20	30	50	10	20	30	50	10	20	30	50
CCEP	25	-0.198	-0.091	-0.058	-0.035	0.222	0.102	0.067	0.042	0.61	0.55	0.43	0.32
	100	-0.201	-0.093	-0.061	-0.036	0.216	0.098	0.064	0.038	0.92	0.97	0.94	0.84
	500	-0.199	-0.095	-0.061	-0.036	0.213	0.097	0.062	0.037	0.99	1.00	1.00	1.00
	5000	-0.200	-0.094	-0.062	-0.036	0.215	0.096	0.062	0.036	1.00	1.00	1.00	1.00
CCEPbc	25	-0.001	-0.001	0.001	0.000	0.093	0.045	0.033	0.024	0.04	0.06	0.07	0.06
	100	0.000	-0.001	0.000	0.000	0.043	0.022	0.016	0.012	0.04	0.05	0.06	0.05
	500	0.001	0.000	0.000	0.000	0.020	0.010	0.007	0.005	0.04	0.04	0.04	0.04
	5000	0.000	0.000	0.000	0.000	0.006	0.003	0.002	0.002	0.03	0.05	0.05	0.05
CCEPjk	25	0.070	0.015	0.008	0.003	0.267	0.069	0.042	0.028	0.41	0.20	0.13	0.09
	100	0.075	0.015	0.007	0.003	0.233	0.049	0.025	0.014	0.63	0.36	0.19	0.09
	500	0.077	0.017	0.008	0.003	0.228	0.043	0.019	0.008	0.75	0.64	0.44	0.18
	5000	0.079	0.016	0.009	0.003	0.220	0.040	0.017	0.006	0.82	0.80	0.75	0.58
FLSbc	25	-0.085	-0.018	-0.007	-0.003	0.130	0.052	0.037	0.026	0.10	0.03	0.02	0.02
	100	-0.105	-0.026	-0.012	-0.005	0.114	0.034	0.020	0.012	0.60	0.20	0.10	0.06
	500	-0.110	-0.026	-0.012	-0.005	0.112	0.029	0.014	0.007	0.96	0.72	0.39	0.14
	5000	-0.109	-0.026	-0.012	-0.005	0.110	0.028	0.012	0.005	1.00	1.00	0.99	0.77
		Results for $\widehat{\beta}$											
CCEP	25	-0.033	-0.010	-0.005	-0.002	0.086	0.048	0.036	0.028	0.07	0.06	0.06	0.06
	100	-0.033	-0.008	-0.004	-0.001	0.055	0.025	0.018	0.014	0.15	0.06	0.06	0.06
	500	-0.033	-0.008	-0.003	-0.001	0.042	0.014	0.009	0.006	0.40	0.13	0.08	0.06
	5000	-0.032	-0.008	-0.004	-0.001	0.040	0.009	0.004	0.002	0.77	0.60	0.27	0.11
CCEPbc	25	0.000	-0.002	-0.002	0.000	0.080	0.047	0.037	0.028	0.04	0.06	0.06	0.05
	100	-0.001	0.000	-0.001	0.000	0.038	0.023	0.018	0.014	0.04	0.05	0.05	0.06
	500	0.000	0.000	0.000	0.000	0.017	0.010	0.008	0.006	0.04	0.05	0.06	0.05
	5000	0.000	0.000	0.000	0.000	0.005	0.003	0.003	0.002	0.03	0.06	0.05	0.05
CCEPjk	25	0.087	0.016	0.006	0.002	0.185	0.060	0.041	0.030	0.35	0.11	0.09	0.07
	100	0.083	0.018	0.008	0.003	0.134	0.035	0.021	0.015	0.54	0.20	0.09	0.07
	500	0.081	0.017	0.008	0.003	0.123	0.025	0.013	0.007	0.74	0.42	0.20	0.09
	5000	0.081	0.018	0.008	0.003	0.119	0.022	0.009	0.003	0.88	0.85	0.76	0.31
FLSbc	25	-0.002	0.009	0.004	0.004	0.085	0.057	0.043	0.032	0.04	0.04	0.03	0.01
	100	-0.016	-0.001	0.000	0.001	0.044	0.024	0.018	0.014	0.08	0.04	0.04	0.04
	500	-0.022	-0.003	-0.001	0.000	0.029	0.011	0.008	0.006	0.32	0.06	0.06	0.05
	5000	-0.021	-0.003	-0.001	0.000	0.025	0.005	0.003	0.002	0.85	0.21	0.08	0.06

Note: See Table 1, but with $\rho = 0.4$ and $\beta = 0.6$

Table E-2: Monte Carlo results for ρ : number and strength of factors ($N = 25$)

	bias	*rmse*	$size_b$	*bias*	*rmse*	$size_b$	*bias*	*rmse*	$size_b$	*bias*	*rmse*	$size_b$	*bias*	*rmse*	$size_b$	*bias*	*rmse*	$size_b$
	one factor																	
	$RI = 1$									$RI = 3$								
	$T = 10$			$T = 20$			$T = 30$			$T = 10$			$T = 20$			$T = 30$		
CCEP	−0.385	0.417	0.90	−0.176	0.188	0.92	−0.109	0.118	0.90	−0.393	0.424	0.92	−0.179	0.191	0.94	−0.111	0.120	0.90
CCEPbc	−0.004	0.151	0.06	0.000	0.064	0.08	0.000	0.038	0.06	0.002	0.153	0.08	0.000	0.063	0.08	0.000	0.038	0.06
CCEPjk	0.027	0.358	0.40	0.037	0.124	0.31	0.028	0.074	0.30	0.017	0.349	0.40	0.039	0.124	0.32	0.030	0.076	0.31
CCEP(+g)	−0.391	0.424	0.83	−0.177	0.190	0.91	−0.110	0.119	0.89	−0.397	0.431	0.84	−0.180	0.193	0.92	−0.112	0.121	0.89
CCEPbc(+g)	−0.003	0.169	0.04	0.001	0.065	0.07	0.000	0.039	0.06	0.002	0.166	0.05	0.001	0.066	0.07	0.000	0.039	0.06
CCEPjk(+g)	-	-	-	0.041	0.133	0.13	0.027	0.077	0.14	-	-	-	0.045	0.133	0.15	0.029	0.078	0.16
FLSbc	−0.261	0.276	0.37	−0.067	0.089	0.04	−0.029	0.054	0.04	−0.233	0.260	0.32	−0.058	0.084	0.03	−0.028	0.054	0.05
	two factors																	
	$RI = 1$									$RI = 3$								
	$T = 10$			$T = 20$			$T = 30$			$T = 10$			$T = 20$			$T = 30$		
CCEP	−0.386	0.417	0.89	−0.176	0.188	0.93	−0.112	0.119	0.89	−0.380	0.412	0.89	−0.161	0.175	0.88	−0.095	0.105	0.78
CCEPbc	0.002	0.155	0.06	−0.001	0.063	0.08	−0.001	0.038	0.05	0.014	0.161	0.08	0.019	0.073	0.12	0.017	0.046	0.09
CCEPjk	0.031	0.356	0.41	0.035	0.123	0.31	0.027	0.074	0.29	0.063	0.359	0.41	0.052	0.139	0.37	0.045	0.086	0.38
CCEP(+g)	−0.424	0.459	0.87	−0.196	0.209	0.94	−0.124	0.132	0.91	−0.401	0.440	0.86	−0.178	0.191	0.91	−0.109	0.117	0.86
CCEPbc(+g)	−0.006	0.173	0.05	−0.001	0.067	0.07	−0.001	0.041	0.06	0.006	0.169	0.05	0.008	0.068	0.08	0.006	0.039	0.05
CCEPjk(+g)	-	-	-	0.033	0.136	0.13	0.026	0.079	0.15	-	-	-	0.042	0.141	0.16	0.039	0.083	0.18
FLSbc	−0.527	0.531	0.85	−0.169	0.184	0.22	−0.063	0.088	0.07	−0.506	0.510	0.81	−0.141	0.164	0.14	−0.053	0.079	0.04

Note: see Table 2 but with $N = 25$.

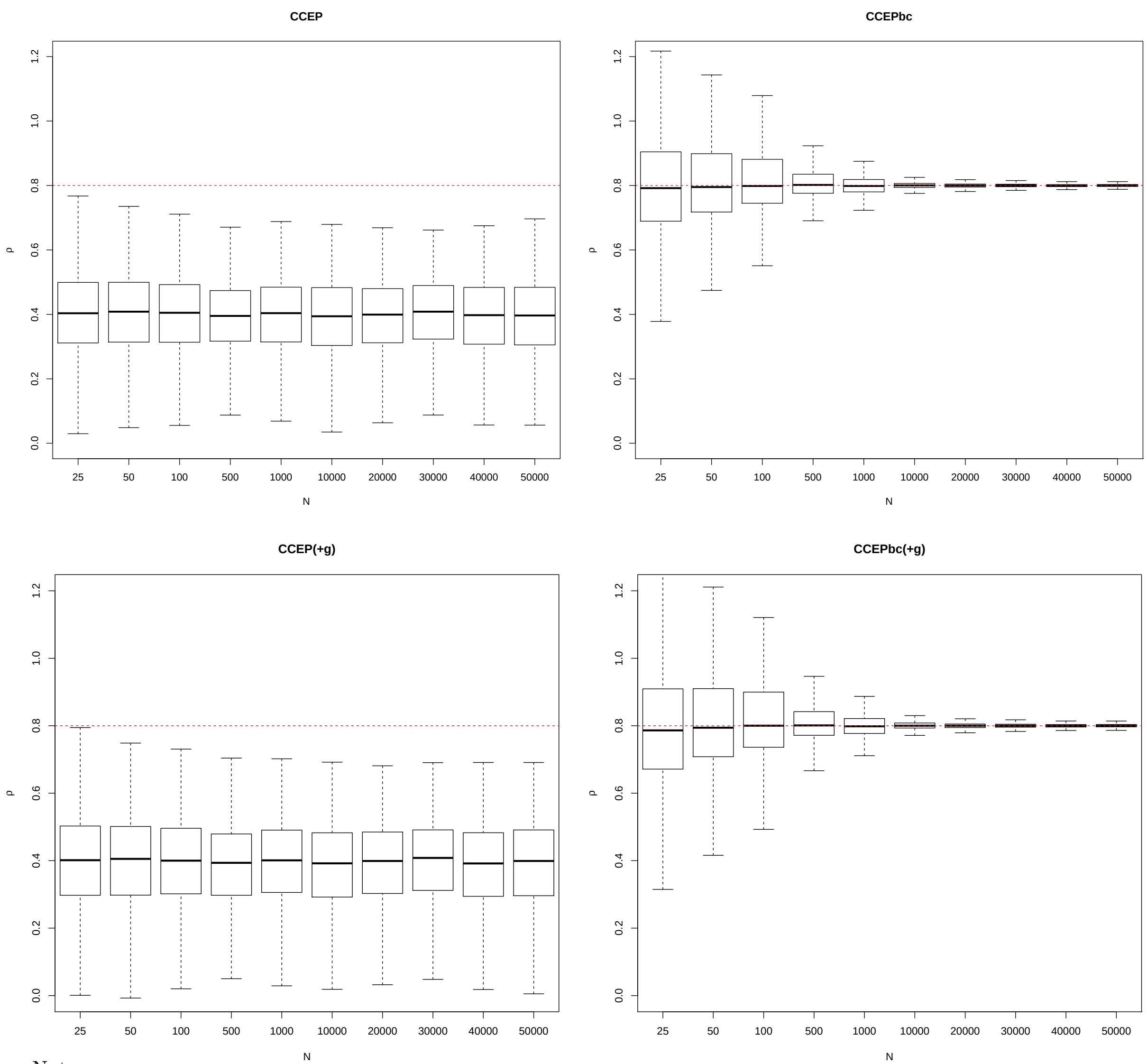
Table E-3: Monte Carlo results for ρ : dynamics in $\mathbf{z}_{it}$ with strong factors ($N = 25$)

	bias	*rmse*	$size_b$	*bias*	*rmse*	$size_b$	*bias*	*rmse*	$size_b$	*bias*	*rmse*	$size_b$
	one factor											
	$T = 10$			$T = 20$			$T = 30$			$T = 50$		
CCEP_p_0(+g)	−0.530	0.563	0.90	−0.236	0.251	0.95	−0.140	0.149	0.94	−0.076	0.081	0.90
CCEP_p_1(+g)	−0.666	0.702	0.81	−0.261	0.279	0.93	−0.148	0.159	0.93	−0.078	0.084	0.89
CCEP_p_T(+g)	-	-	-	−0.321	0.343	0.89	−0.196	0.210	0.88	−0.090	0.096	0.88
CCEPbc_p_0(+g)	−0.014	0.193	0.04	−0.006	0.073	0.06	−0.002	0.040	0.05	−0.002	0.023	0.06
CCEPbc_p_1(+g)	−0.033	0.265	0.03	−0.002	0.084	0.05	−0.001	0.044	0.04	0.000	0.023	0.05
CCEPbc_p_T(+g)	-	-	-	−0.006	0.106	0.03	−0.004	0.061	0.05	−0.001	0.026	0.04
CCEPjk_p_1(+g)	-	-	-	0.123	0.244	0.13	0.084	0.139	0.21	0.034	0.058	0.17
FLSbc	−0.254	0.270	0.47	−0.057	0.081	0.06	−0.026	0.048	0.04	−0.011	0.029	0.04
	two factors											
	$T = 10$			$T = 20$			$T = 30$			$T = 50$		
CCEP_p_0(+g)	−0.560	0.590	0.93	−0.252	0.268	0.97	−0.156	0.164	0.97	−0.085	0.090	0.95
CCEP_p_1(+g)	−0.720	0.750	0.86	−0.294	0.310	0.96	−0.169	0.177	0.96	−0.086	0.091	0.93
CCEP_p_T(+g)	-	-	-	−0.364	0.383	0.94	−0.225	0.239	0.94	−0.101	0.106	0.94
CCEPbc_p_0(+g)	−0.021	0.204	0.05	−0.015	0.075	0.07	−0.010	0.043	0.06	−0.007	0.024	0.06
CCEPbc_p_1(+g)	−0.023	0.275	0.03	−0.005	0.090	0.07	−0.002	0.045	0.05	−0.001	0.024	0.05
CCEPbc_p_T(+g)	-	-	-	−0.009	0.109	0.05	0.000	0.065	0.05	0.000	0.027	0.04
CCEPjk_p_1(+g)	-	-	-	0.120	0.240	0.14	0.090	0.149	0.24	0.044	0.064	0.22
FLSbc	−0.524	0.526	0.89	−0.138	0.162	0.19	−0.047	0.072	0.06	−0.012	0.033	0.03

Note: see Table 4 but with $N = 25$.

F Additional figures

Figure F-1: Monte Carlo results for ρ : Boxplots for CCEP and CCEPbc estimators over N for one normal factor ($m = 1$, $RI = 1$) with $T = 10$

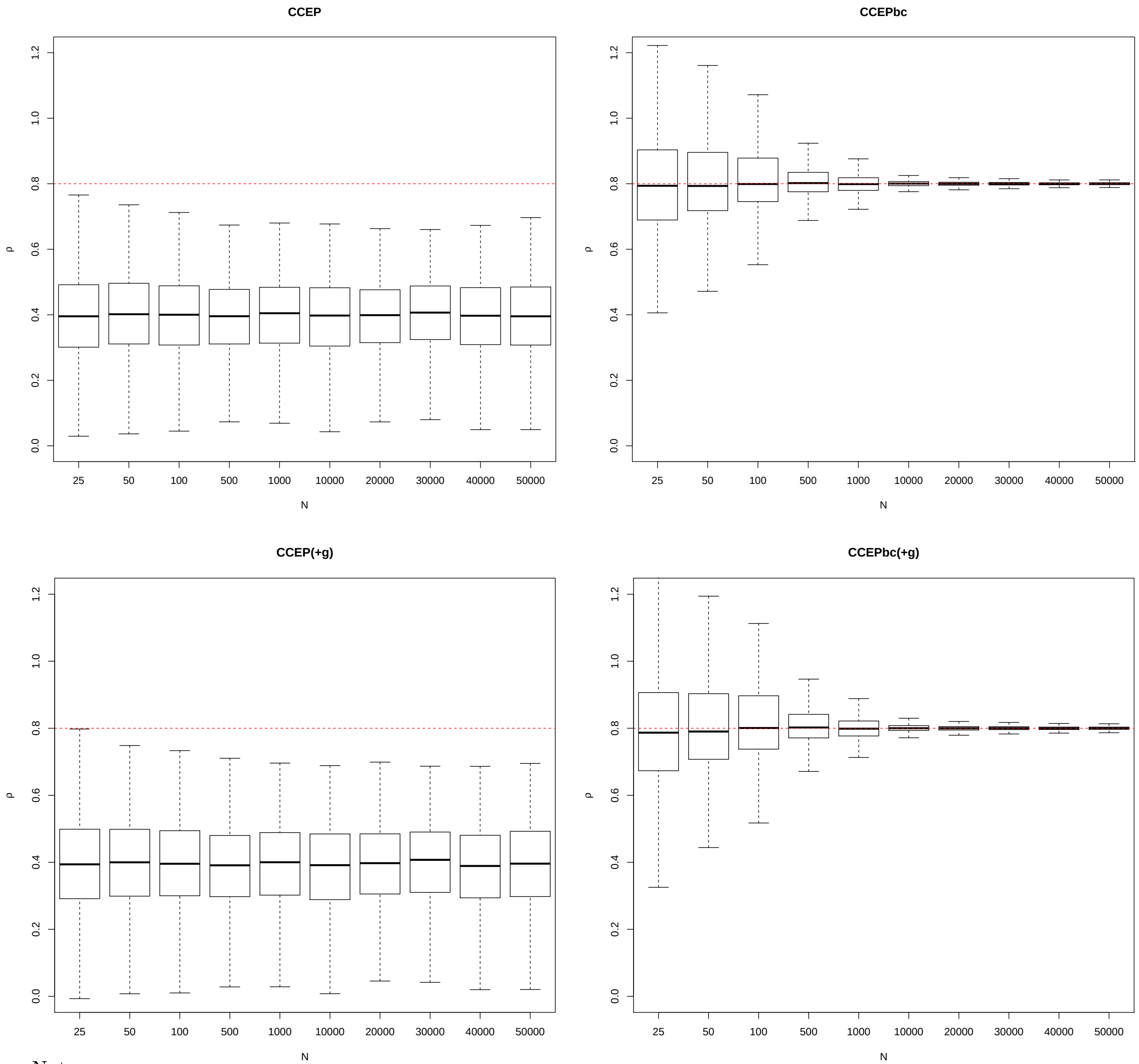


Notes:

(i) Reported are simulation results for estimating ρ in the baseline case for $T = 10$ and $N = 25, 50, 100, ..., 50.000$ (see notes Table 1). The CCEP estimators with a (+g) suffix (lower panel) make use of the $\overline{g}_t$ variable to project out the factors.

(ii) Dotted red lines indicate the population parameter value ($\rho = 0.8$). The boxplot 'whiskers' extend to the most extreme data point which is no more than 1.5 times the interquartile range from the box.

Figure F-2: Monte Carlo results for ρ : Boxplots for CCEP and CCEPbc estimators over N for one strong factor ($m = 1$, $RI = 3$) with $T = 10$.

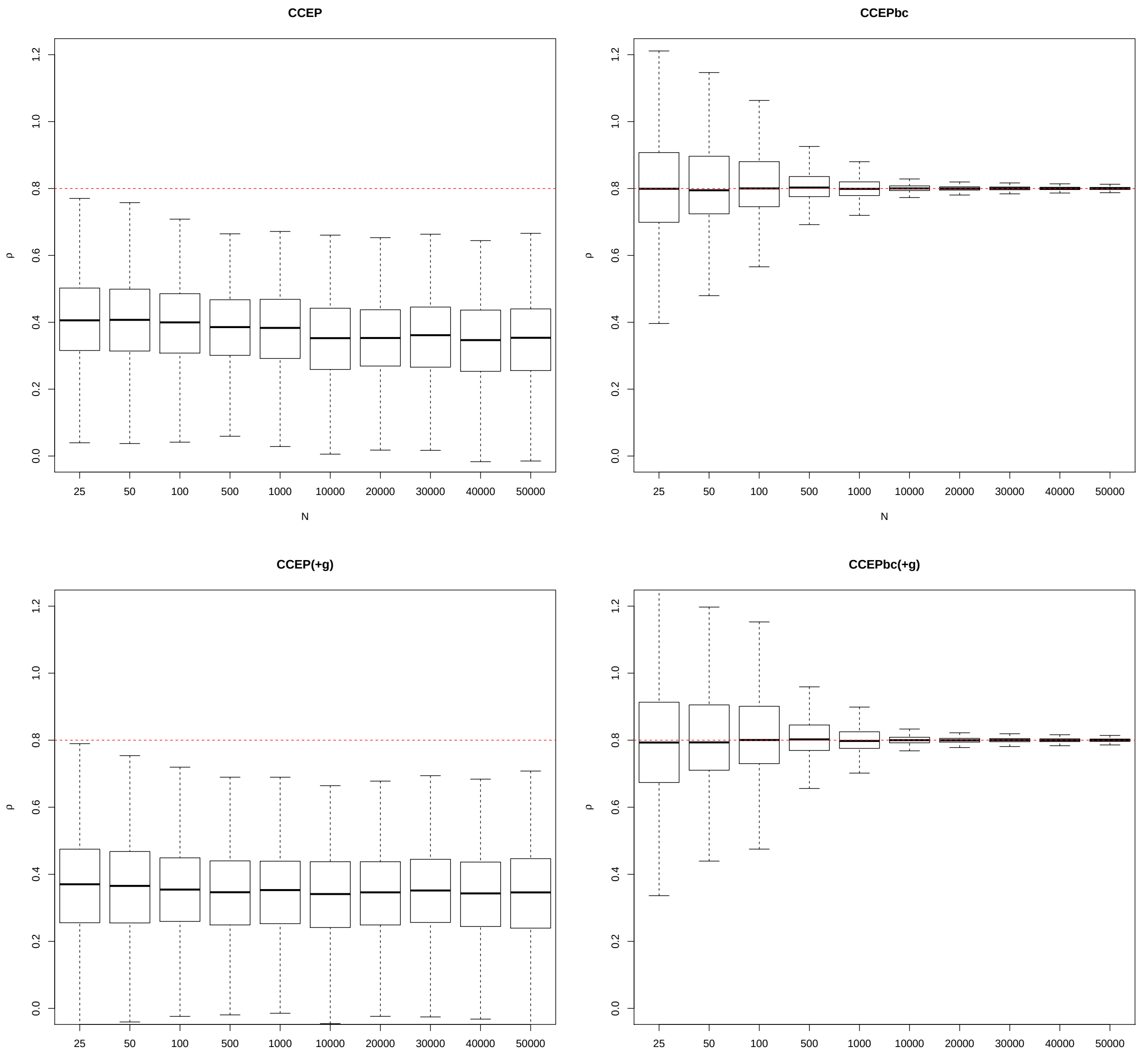


Notes:

(i) Reported are simulation results for estimating $\rho = 0.8$ with $m = 1$ and $RI = 3$ for $N = 25, 50, 100, ..., 50.000$. The CCEP estimators with a (+g) suffix (lower panel) make use of the $\overline{g}_t$ variable to project out the factors.

(ii) Dotted red lines indicate the population parameter value ($\rho = 0.8$). The boxplot 'whiskers' extend to the most extreme data point which is no more than 1.5 times the interquartile range from the box.

Figure F-3: Monte Carlo results for ρ : Boxplots for CCEP and CCEPbc estimators over N for two normal factors ($m = 2$, $RI = 1$) with $T = 10$.

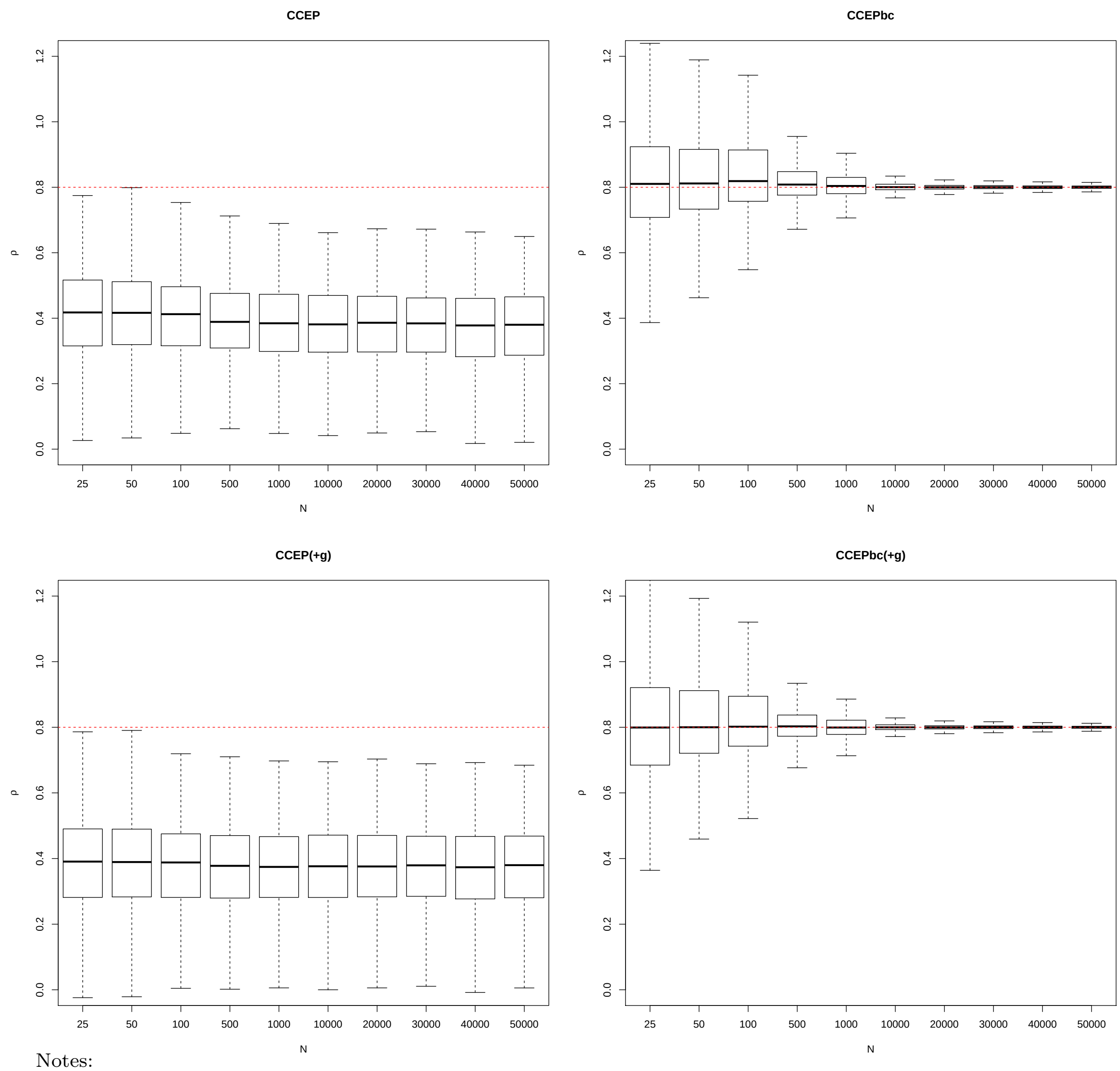


Notes:

(i) Reported are simulation results for estimating $\rho = 0.8$ with $m = 2$ and $RI = 1$ for $N = 25, 50, 100, ..., 50.000$. The CCEP estimators with a (+g) suffix (lower panel) make use of the $\overline{g}_t$ variable to project out the factors.

(ii) Dotted red lines indicate the population parameter value ($\rho = 0.8$). The boxplot 'whiskers' extend to the most extreme data point which is no more than 1.5 times the interquartile range from the box.

Figure F-4: Monte Carlo results for ρ: Boxplots for CCEP and CCEPbc estimators over N for two strong factors ($m = 2$, $RI = 3$) with $T = 10$.



Notes:

(i) Reported are simulation results for estimating $\rho = 0.8$ with $m = 2$ and $RI = 3$ for $N = 25, 50, 100, ..., 50.000$. The CCEP estimators with a (+g) suffix (lower panel) make use of the $\overline{g}_t$ variable to project out the factors.

(ii) Dotted red lines indicate the population parameter value ($\rho = 0.8$). The boxplot 'whiskers' extend to the most extreme data point which is no more than 1.5 times the interquartile range from the box.