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Kristensson, Gerhard; Stratis, Ioannis; Wellander, Niklas; Yannacopoulos, Athanasios

2017

CODEN:LUTEDX/(TEAT-7259)/1-43/(2017)

Revision No. 1: January 2019

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Gerhard Kristensson, Ioannis G. Stratis, Niklas Wellander, and Athanasios N. Yannacopoulos

Dedicated to Professor Paul A. Martin on his 65th birthday

Electromagnetic Theory
Department of Electrical and Information Technology
Lund University
Sweden

(TEAT-7259)/1-43/(2017): G. Kristensson *et al.*, The exterior Calderón operator for non-spherical objects

Gerhard Kristensson
gerhard.kristensson@eit.lth.se

Department of Electrical and Information Technology
Electromagnetic Theory
Lund University
P.O. Box 118
SE-221 00 Lund
Sweden

Ioannis G. Stratis
istratis@math.uoa.gr

Department of Mathematics
National and Kapodistrian University of Athens
Panepistimioupolis
GR-157 84 Zographou
Athens
Greece

Niklas Wellander
niklas.wellander@foi.se

Swedish Defense Research Agency, FOI
P.O. Box 1165
SE-581 11 Linköping
Sweden

Athanasios N. Yannacopoulos
ayannaco@aueb.gr

Department of Statistics
Athens University of Economics and Business
Patision 76
GR-104 34 Athens
Greece

Editor: Mats Gustafsson

Abstract

This paper deals with the exterior Calderón operator for not necessarily spherical domains. We present a new approach of finding the norm of the exterior Calderón operator for a wide class of surfaces. The basic tool in the treatment is the set of eigenfunctions and eigenvalues to the Laplace-Beltrami operator for the surface. The norm is obtained in view of an eigenvalue problem of a quadratic form containing the exterior Calderón operator. The connection of the exterior Calderón operator to the transition matrix for a perfectly conducting surface is analyzed.

1 Introduction

The exterior Calderón operator maps the tangential scattered electric surface field to the corresponding magnetic surface field. This operator is also called the Poincaré-Steklov operator, and its discretization is often called the Schur complement. It has been studied intensively during many years, see *e.g.*, [9,18,20].

It is related to the Dirichlet-to-Neumann map for the scalar Helmholtz equation. The exterior Calderón map is instrumental in the analysis of the solution to the exterior solution of the scattering problem. In fact, it is strongly related to the solution of the scattering problem by a perfectly conducting (PEC) obstacle, which is a subject we analyze in Section 5.

The norm of the exterior Calderón operator determines the largest amplification factor of the surface fields. This norm specifies the largest impedance (the quotient between scattered tangential magnetic and electric fields) that can exist for a given scattering geometry. In several numerical implementations of the scattering problem, such as the Methods of Moments (MoM), the impedance matrix represents the exterior Calderón operator and this matrix is instrumental for the numerical solution of the problem. This observation gives a physical interpretation of the value of the norm of the exterior Calderón operator.

A new way of finding this norm is presented in this paper. The key ingredient in this analysis is the set of eigenfunctions to the Laplace-Beltrami operator of the surface. These eigenfunctions and the corresponding eigenvalues are intrinsic to the surface and constitute an excellent tool for further analysis; the literature on this subject of finding these eigenfunctions and eigenvalues is extensive, see, e.g., [4,11,19,28]. Explicit values of the norm of the exterior Calderón operator have only been obtained for the sphere case [18,20] and the planar case [3,9], and we refer to these bibliographical items for the explicit techniques of computing the norm. In this paper, we present a new way to explicitly find the norm for non-spherical obstacles. The final expression of the norm for a non-spherical obstacle is related to an eigenvalue problem of a quadratic form containing the exterior Calderón matrix.

An outline of the organization of the contents in this paper is now presented. In Section 2, the statement of the problem is introduced, the exterior Calderón operator is defined, and the useful integral representation of the scattered field is presented. The intrinsic generalized harmonics (both scalar and vector valued) are introduced in Section 3, and these functions are used in Section 4. The generalized harmonics developed in Section 3 constitute a great asset, and they serve as a natural orthonormal basis for the expansion of the surface fields in many scattering problems. A matrix representation of the exterior Calderón problem in terms of the generalized harmonics is presented in Section 4, and this matrix has many valuable properties that are useful in the solution of the exterior scattering problem. Section 4 also contains a constructive method to compute the norm of the exterior Calderón operator for non-spherical obstacles. The connection between the exterior Calderón operator and the transition matrix of the corresponding perfectly conducting obstacle is clarified in Section 5. The spherical geometry is explicitly treated in Section 6. The paper is concluded with some final remarks in Section 7 and several appendices where some relevant details of the analysis are presented.

2 Formulation of the scattering problem

In this section, we present the geometry of the problem and the solution of the scattered field in the exterior region.

2.1 Statement of problem (E)

Let Ω be an open, bounded, piecewise smooth[1] domain in $\mathbb{R}^3$ with simply connected[2] boundary Γ. The outward pointing unit normal is denoted by $\hat{\boldsymbol{\nu}}$.[3] We denote the exterior of the domain Ω by $\Omega_\mathrm{e} = \mathbb{R}^3 \setminus \overline{\Omega}$, which is assumed to be simply connected. See Figure 1 for a typical geometry.

The Maxwell equations in the exterior region are given by[4] (we adopt the time convention $\mathrm{e}^{-\mathrm{i}\omega t}$)

$$\begin{cases} \nabla \times \boldsymbol{E}(\boldsymbol{x}) = \mathrm{i}k\boldsymbol{H}(\boldsymbol{x}) \\ \nabla \times \boldsymbol{H}(\boldsymbol{x}) = -\mathrm{i}k\boldsymbol{E}(\boldsymbol{x}) \end{cases} \quad \boldsymbol{x} \in \Omega_\mathrm{e}. \tag{2.1}$$

The wave number $k = \omega/c$ is assumed to be a positive constant, where ω is the angular frequency of the fields, and c is the speed of light in the exterior medium.

In the region Ω_e, the (scattered) fields satisfy the time-harmonic Maxwell equations (2.1) and the Silver-Müller radiation condition at infinity, and we are looking for solutions $\boldsymbol{E}_\mathrm{sc}$ and $\boldsymbol{H}_\mathrm{sc}$ in the space $H_\mathrm{loc}(\mathrm{curl}, \overline{\Omega}_\mathrm{e})$.

[1] *i.e.*, the image of a polyhedron under a $C^{1,1}$ mapping.

[2] For non-simply connected boundary, see Remark 4.5.

[3] Throughout this paper vector-valued quantities are typed in italic boldface (*e.g.*, $\boldsymbol{E}$ and $\boldsymbol{x}$), and dyadics (matrices) in roman boldface (*e.g.*, $\mathbf{I}$ and $\mathbf{G}_\mathrm{e}$). Scalar-valued quantities are typed in italics (*e.g.*, k). Vectors with unit length have a "hat" or caret ($\hat{}$) over the symbol.

[4] We use scaled electric and magnetic fields, *i.e.*, the SI-unit fields $\boldsymbol{E}_\mathrm{SI}$ and $\boldsymbol{H}_\mathrm{SI}$ are related to the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ used in this paper by

$$\boldsymbol{E}_\mathrm{SI}(\boldsymbol{x}) = \frac{\boldsymbol{E}(\boldsymbol{x})}{\sqrt{\epsilon_0 \epsilon}}, \qquad \boldsymbol{H}_\mathrm{SI}(\boldsymbol{x}) = \frac{\boldsymbol{H}(\boldsymbol{x})}{\sqrt{\mu_0 \mu}},$$

where the permittivity and permeability of vacuum are denoted ϵ_0 and μ_0, respectively, and the relative permittivity and permeability of the exterior material are denoted ϵ and μ, respectively.

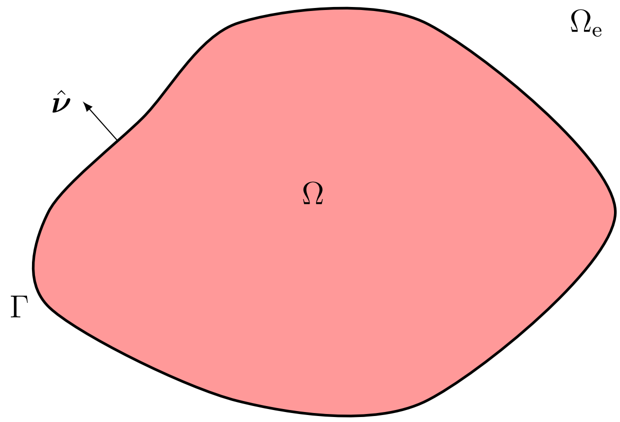

Figure 1: Typical geometry of the scattering problem in this paper. The domain Ω, its boundary Γ and the exterior Ω_{e}.

The trace operators $\boldsymbol{\pi}$ and $\boldsymbol{\gamma}$ on $C(\overline{\Omega_{\mathrm{e}}})$ are given by $\boldsymbol{\pi}(\boldsymbol{u}) = \hat{\boldsymbol{\nu}} \times (\boldsymbol{u}|_{\partial\Omega} \times \hat{\boldsymbol{\nu}})$ and $\boldsymbol{\gamma}(\boldsymbol{u}) = \hat{\boldsymbol{\nu}} \times \boldsymbol{u}|_{\partial\Omega}$, respectively,[5] and in the case that $\boldsymbol{u}$ belongs to $H_{\mathrm{loc}}(\mathrm{curl}, \overline{\Omega}_{\mathrm{e}})$, the fields have traces on $\partial\Omega$ belonging to $H^{-1/2}(\mathrm{div}, \Gamma)$; more precisely we have $(\boldsymbol{\gamma}(\boldsymbol{E}_{\mathrm{sc}}), \boldsymbol{\gamma}(\boldsymbol{H}_{\mathrm{sc}})) \in H^{-1/2}(\mathrm{div}, \Gamma) \times H^{-1/2}(\mathrm{div}, \Gamma)$, see [21] for the definition and the properties of the trace operators in $H_{\mathrm{loc}}(\mathrm{curl}, \overline{\Omega}_{\mathrm{e}})$. For non-smooth domains, see [7, 8].

The exterior Calderón operator or admittance operator, $\mathbf{C}^{\mathrm{e}}$, is defined as the mapping of the tangential component of the scattered electric field to the tangential component of the scattered magnetic field on the boundary of Ω [9]. We use the solution of a specific exterior problem to make the definition precise.

Consider the following exterior problem where the trace of the scattered electric field on the boundary is given by a fixed vector $\boldsymbol{m} \in H^{-1/2}(\mathrm{div}, \Gamma)$,[6]

$$
\begin{aligned}
&1)\ (\boldsymbol{E}_{\mathrm{sc}}, \boldsymbol{H}_{\mathrm{sc}}) \in H_{\mathrm{loc}}(\mathrm{curl}, \overline{\Omega}_{\mathrm{e}}) \times H_{\mathrm{loc}}(\mathrm{curl}, \overline{\Omega}_{\mathrm{e}}) \\
&2)\ \begin{cases} \nabla \times \boldsymbol{E}_{\mathrm{sc}}(\boldsymbol{x}) = \mathrm{i}k\boldsymbol{H}_{\mathrm{sc}}(\boldsymbol{x}) \\ \nabla \times \boldsymbol{H}_{\mathrm{sc}}(\boldsymbol{x}) = -\mathrm{i}k\boldsymbol{E}_{\mathrm{sc}}(\boldsymbol{x}) \end{cases} \qquad \boldsymbol{x} \in \Omega_{\mathrm{e}} \\
&3)\ \begin{cases} \hat{\boldsymbol{x}} \times \boldsymbol{E}_{\mathrm{sc}}(\boldsymbol{x}) - \boldsymbol{H}_{\mathrm{sc}}(\boldsymbol{x}) = o(1/x) \\ \text{or} \\ \hat{\boldsymbol{x}} \times \boldsymbol{H}_{\mathrm{sc}}(\boldsymbol{x}) + \boldsymbol{E}_{\mathrm{sc}}(\boldsymbol{x}) = o(1/x) \end{cases} \qquad \text{as } x \to \infty \qquad \text{(Problem (E))}, \\
&\qquad \text{uniformly w.r.t. } \hat{\boldsymbol{x}} \\
&4)\ \boldsymbol{\gamma}(\boldsymbol{E}_{\mathrm{sc}}) = \boldsymbol{m} \in H^{-1/2}(\mathrm{div}, \Gamma)
\end{aligned}
\tag{2.2}
$$

where $x = |\boldsymbol{x}|$. This problem has a unique solution [3, 9, 14], and a brief sketch of the proof is found in Appendix C.

The following theorem represents the solution to Problem (E):

[5] Some authors [14] use γ_t for $\boldsymbol{\gamma}$ and also use $\gamma_T = -\hat{\boldsymbol{\nu}} \times \boldsymbol{\gamma}$.

[6] The source $\boldsymbol{m}$ can be interpreted as a magnetic current density.

Theorem 2.1. *Let* $\boldsymbol{E}_{\rm sc}$ *and* $\boldsymbol{H}_{\rm sc}$ *be the solution of Problem (E). Then the fields satisfy the integral representations*

$$-\frac{1}{\mathrm{i}k}\nabla\times\left\{\nabla\times\int_\Gamma g(k,|\boldsymbol{x}-\boldsymbol{x}'|)\boldsymbol{\gamma}(\boldsymbol{H}_{\rm sc})(\boldsymbol{x}')\,\mathrm{d}S'\right\}$$
$$+\nabla\times\int_\Gamma g(k,|\boldsymbol{x}-\boldsymbol{x}'|)\boldsymbol{\gamma}(\boldsymbol{E}_{\rm sc})(\boldsymbol{x}')\,\mathrm{d}S'=\begin{cases}\boldsymbol{E}_{\rm sc}(\boldsymbol{x}), & \boldsymbol{x}\in\Omega_{\rm e}\\ \boldsymbol{0}, & \boldsymbol{x}\in\Omega,\end{cases}$$

and

$$\frac{1}{\mathrm{i}k}\nabla\times\left\{\nabla\times\int_\Gamma g(k,|\boldsymbol{x}-\boldsymbol{x}'|)\boldsymbol{\gamma}(\boldsymbol{E}_{\rm sc})(\boldsymbol{x}')\,\mathrm{d}S'\right\}$$
$$+\nabla\times\int_\Gamma g(k,|\boldsymbol{x}-\boldsymbol{x}'|)\boldsymbol{\gamma}(\boldsymbol{H}_{\rm sc})(\boldsymbol{x}')\,\mathrm{d}S'=\begin{cases}\boldsymbol{H}_{\rm sc}(\boldsymbol{x}), & \boldsymbol{x}\in\Omega_{\rm e}\\ \boldsymbol{0}, & \boldsymbol{x}\in\Omega,\end{cases}$$

where the scalar Green function is

$$g(k,|\boldsymbol{x}-\boldsymbol{x}'|)=\frac{\mathrm{e}^{\mathrm{i}k|\boldsymbol{x}-\boldsymbol{x}'|}}{4\pi\,|\boldsymbol{x}-\boldsymbol{x}'|}.$$

The proof of this theorem is found in *e.g.*, [14]. The second (lower) term of the integral representation, *i.e.*, when $\boldsymbol{x}\in\Omega$, is usually called the extinction part of the integral representation.

2.2 Definition of the exterior Calderón operator

We now define the exterior Calderón operator $\mathbf{C}^{\rm e}$. As usual, $TL^2(\Gamma)$ and $TH^s(\Gamma)$, ($s\in\mathbb{R}$), denote the trace spaces of elements $\boldsymbol{v}$ in $(L^2(\Gamma))^3$ and $(H^s(\Gamma))^3$, respectively, such that $\hat{\boldsymbol{\nu}}\cdot\boldsymbol{v}=0$ on Γ (see also Appendix A). Further, let $\mathrm{div}_\Gamma\boldsymbol{v}$ denote the surface divergence, defined *e.g.*, in [4, 9, 21, 25]. Then $H^{-1/2}(\mathrm{div},\Gamma):=\{\boldsymbol{v}\in TH^{-1/2}(\Gamma):\mathrm{div}_\Gamma\boldsymbol{v}\in H^{-1/2}(\Gamma)\}$. This is the natural trace space, which occurs in electromagnetic theory.

Definition 2.1. *The exterior Calderón operator* $\mathbf{C}^{\rm e}$ *is defined as*

$$\mathbf{C}^{\rm e}:\boldsymbol{m}\mapsto\boldsymbol{\gamma}(\boldsymbol{H}_{\rm sc}),\qquad H^{-1/2}(\mathrm{div},\Gamma)\to H^{-1/2}(\mathrm{div},\Gamma),$$

where $\boldsymbol{m}=\boldsymbol{\gamma}(\boldsymbol{E}_{\rm sc})$ *and the fields* $\boldsymbol{E}_{\rm sc}$ *and* $\boldsymbol{H}_{\rm sc}$ *satisfy Problem (E) in* (2.2).

We notice that the exterior Calderón operator $\mathbf{C}^{\rm e}$ is uniquely defined for all $\boldsymbol{m}\in H^{-1/2}(\mathrm{div},\Gamma)$, since Problem (E) has a unique solution in $H_{\rm loc}(\mathrm{curl},\overline{\Omega}_{\rm e})\times H_{\rm loc}(\mathrm{curl},\overline{\Omega}_{\rm e})$ for any $\boldsymbol{m}\in H^{-1/2}(\mathrm{div},\Gamma)$. Details on the space $H^{-1/2}(\mathrm{div},\Gamma)$ and its dual space $H^{-1/2}(\mathrm{curl},\Gamma)$ are given in [9] and [20].

Theorem 2.2. *The exterior Calderón operator defined in Definition* 2.1 *has the following properties [9]:*

1. *Positivity:*

$$\mathrm{Re}\int_\Gamma \mathbf{C}^{\mathrm{e}}(\boldsymbol{m})\cdot(\hat{\boldsymbol{\nu}}\times\boldsymbol{m}^*)\,\mathrm{d}S > 0 \quad \textit{for all}\ \boldsymbol{m}\in H^{-1/2}(\mathrm{div},\Gamma),\ \boldsymbol{m}\neq\mathbf{0}, \tag{2.3}$$

where $\mathrm{d}S$ *denotes the surface measure of* Γ*, and the star denotes the complex conjugation.*

2. $$(\mathbf{C}^{\mathrm{e}})^2 = -\mathbf{I}\ \textit{on}\ H^{-1/2}(\mathrm{div},\Gamma), \tag{2.4}$$

3. *The exterior Calderón operator is a boundedly invertible linear map in the space* $H^{-1/2}(\mathrm{div},\Gamma)$*, and consequently there exist constants* $0<\theta_{\mathrm{C}}\le\Theta_{\mathrm{C}}$*, such that*

$$\theta_{\mathrm{C}}\|\boldsymbol{m}\|_{H^{-1/2}(\mathrm{div},\Gamma)} \le \|\mathbf{C}^{\mathrm{e}}(\boldsymbol{m})\|_{H^{-1/2}(\mathrm{div},\Gamma)} \le \Theta_{\mathrm{C}}\|\boldsymbol{m}\|_{H^{-1/2}(\mathrm{div},\Gamma)}.$$

4. *The exterior Calderón operator is independent of the material properties inside the domain* Ω*.*

From Item 2 we conclude that the norm of the exterior Calderón operator satisfies $\|\mathbf{C}^{\mathrm{e}}\|_{H^{-1/2}(\mathrm{div},\Gamma)}\ge 1$, and also that the constants in Item 3 can be chosen as $\theta_{\mathrm{C}} = 1/\|\mathbf{C}^{\mathrm{e}}\|_{H^{-1/2}(\mathrm{div},\Gamma)}$ and $\Theta_{\mathrm{C}} = \|\mathbf{C}^{\mathrm{e}}\|_{H^{-1/2}(\mathrm{div},\Gamma)}$. Notice, that if we define the exterior Calderón operator with an extra imaginary unit (i), the exterior Calderón operator becomes its own inverse, *i.e.*, $\mathbf{C}^{\mathrm{e}} : \boldsymbol{m}\mapsto\boldsymbol{\gamma}(\mathrm{i}\boldsymbol{H}_{\mathrm{sc}})$. This is a correction for the $\pi/2$ phase shift between the fields.

2.3 Integral equation approach

The results in Theorem 2.1 can be used to put the exterior Calderón operator in a surface integral equation setting.

The following theorem is important for the analysis in this paper and proved in [14, Th. 5.52] (important results are also found in [10, 12, 27]):

Theorem 2.3. *Let* Q *be a bounded domain such that* $\Gamma\subseteq Q$*.*

1. *Define the operators* $\widetilde{\mathbf{L}},\widetilde{\mathbf{M}} : H^{-1/2}(\mathrm{div},\Gamma)\to H(\mathrm{curl},Q)$*, by*

$$\begin{cases} \left(\widetilde{\mathbf{L}}\boldsymbol{f}\right)(\boldsymbol{x}) = \nabla\times\left\{\nabla\times\int_\Gamma g(k,|\boldsymbol{x}-\boldsymbol{x}'|)\boldsymbol{f}(\boldsymbol{x}')\,\mathrm{d}S'\right\} \\ \left(\widetilde{\mathbf{M}}\boldsymbol{f}\right)(\boldsymbol{x}) = \nabla\times\int_\Gamma g(k,|\boldsymbol{x}-\boldsymbol{x}'|)\boldsymbol{f}(\boldsymbol{x}')\,\mathrm{d}S' \end{cases} \qquad \boldsymbol{x}\in Q.$$

These operators are well defined and bounded from the space $H^{-1/2}(\mathrm{div},\Gamma)$ *into the space* $H(\mathrm{curl},Q)$*.*

2. *For* $\boldsymbol{f} \in H^{-1/2}(\mathrm{div}, \Gamma)$*, the fields* $\boldsymbol{F} = \widetilde{\mathbf{M}}\boldsymbol{f}$ *and* $\nabla \times \boldsymbol{F} = \widetilde{\mathbf{L}}\boldsymbol{f}$ *satisfy*

$$\boldsymbol{\gamma}(\boldsymbol{F})|_+ - \boldsymbol{\gamma}(\boldsymbol{F})|_- = \boldsymbol{f}, \qquad \boldsymbol{\gamma}(\nabla \times \boldsymbol{F})|_+ - \boldsymbol{\gamma}(\nabla \times \boldsymbol{F})|_- = \mathbf{0}.$$

The notation $|_\pm$ *refers to the trace of the field taken from the outside* $(+)$ *or the inside* $(-)$ *of* Γ*, respectively. In particular,* $\boldsymbol{F} \in C^\infty(\mathbb{R}^3 \setminus \Gamma)$*, and* $\boldsymbol{F}$ *satisfies* $\nabla \times (\nabla \times \boldsymbol{F}) - k^2\boldsymbol{F} = \mathbf{0}$ *in* $\mathbb{R}^3 \setminus \Gamma$*. Furthermore, the functions* $\boldsymbol{F}$ *and* $\nabla \times \boldsymbol{F}$ *satisfy one of the two Silver-Müller radiation conditions*

$$\begin{cases} \mathrm{i}k\hat{\boldsymbol{x}} \times \boldsymbol{F} - \nabla \times \boldsymbol{F} = o(1/x) \\ \text{or} \\ \hat{\boldsymbol{x}} \times (\nabla \times \boldsymbol{F}) + \mathrm{i}k\boldsymbol{F} = o(1/x) \end{cases} \qquad \text{as } x \to \infty,$$

uniformly w.r.t. $\hat{\boldsymbol{x}}$*.*

3. *The traces* $\mathbf{L}$ *and* $\mathbf{M}$ *defined by*

$$\begin{cases} \mathbf{L}\boldsymbol{f} = \boldsymbol{\gamma}(\widetilde{\mathbf{L}}\boldsymbol{f}) \\ \mathbf{M}\boldsymbol{f} = \dfrac{1}{2}\left(\boldsymbol{\gamma}(\widetilde{\mathbf{M}}\boldsymbol{f})\Big|_+ + \boldsymbol{\gamma}(\widetilde{\mathbf{M}}\boldsymbol{f})\Big|_-\right) \end{cases} \qquad \boldsymbol{f} \in H^{-1/2}(\mathrm{div}, \Gamma),$$

are bounded from $H^{-1/2}(\mathrm{div}, \Gamma)$ *into itself.*

4. *For* $\boldsymbol{f} \in H^{-1/2}(\mathrm{div}, \Gamma)$*, the fields* $\boldsymbol{F} = \widetilde{\mathbf{M}}\boldsymbol{f}$ *and* $\nabla \times \boldsymbol{F} = \widetilde{\mathbf{L}}\boldsymbol{f}$ *have traces*

$$\begin{cases} \boldsymbol{\gamma}(\boldsymbol{F})|_\pm = \pm\dfrac{1}{2}\boldsymbol{f} + \mathbf{M}\boldsymbol{f} \\ \boldsymbol{\gamma}(\nabla \times \boldsymbol{F})|_\pm = \mathbf{L}\boldsymbol{f}. \end{cases}$$

5. *The operator* $\mathbf{L}$ *is the sum* $\mathbf{L} = \widehat{\mathbf{I}} + \mathbf{K}$ *of an isomorphism* $\widehat{\mathbf{I}}$ *from* $H^{-1/2}(\mathrm{div}, \Gamma)$ *onto itself and a compact operator* $\mathbf{K}$*.*

6. *The operator* $\widetilde{\mathbf{L}}$ *can be written as*

$$\widetilde{\mathbf{L}}\boldsymbol{f} = \nabla\left(\mathcal{S}\mathrm{div}_\Gamma\boldsymbol{f}\right) + k^2\mathbf{S}\boldsymbol{f}, \quad \boldsymbol{f} \in H^{-1/2}(\mathrm{div}, \Gamma),$$

where the scalar single layer potential operator $\mathcal{S}$ *is defined as*

$$\left(\mathcal{S}f\right)(\boldsymbol{x}) = \int_\Gamma g(k, |\boldsymbol{x} - \boldsymbol{x}'|)f(\boldsymbol{x}')\,\mathrm{d}S', \quad \boldsymbol{x} \in \Gamma,$$

where the surface integral is interpreted as a generalized integral (punctured surface by a circle). The corresponding vector-valued operator $\mathbf{S}$ *is denoted by*

$$\left(\mathbf{S}\boldsymbol{f}\right)(\boldsymbol{x}) = \int_\Gamma g(k, |\boldsymbol{x} - \boldsymbol{x}'|)\boldsymbol{f}(\boldsymbol{x}')\,\mathrm{d}S', \quad \boldsymbol{x} \in \Gamma,$$

which is interpreted as the operator $\mathcal{S}$ *applied to each Cartesian component of the tangential vector field* $\boldsymbol{f}$*.*

Theorem 2.4. *The exterior Calderón operator satisfies*

$$\frac{1}{2}\mathbf{C}^{\mathrm{e}}(\boldsymbol{m}) - \mathbf{M}\mathbf{C}^{\mathrm{e}}(\boldsymbol{m}) = \frac{1}{\mathrm{i}k}\mathbf{L}\boldsymbol{m},$$

for each $\boldsymbol{m} = \boldsymbol{\gamma}(\boldsymbol{E}_{\mathrm{sc}}) \in H^{-1/2}(\mathrm{div}, \Gamma)$*, where*

$$\mathbf{L}\boldsymbol{m} = \boldsymbol{\gamma}\left(\nabla\left(\mathcal{S}\mathrm{div}_\Gamma \boldsymbol{m}\right)\right) + k^2\boldsymbol{\gamma}\left(\mathbf{S}\boldsymbol{m}\right).$$

Proof. From the second representation in Theorem 2.1, we get by letting $\boldsymbol{m} = \boldsymbol{\gamma}(\boldsymbol{E}_{\mathrm{sc}})$ and $\mathbf{C}^{\mathrm{e}}(\boldsymbol{m}) = \boldsymbol{\gamma}(\boldsymbol{H}_{\mathrm{sc}})$,

$$\nabla \times \int_\Gamma g(k, |\boldsymbol{x} - \boldsymbol{x}'|)\mathbf{C}^{\mathrm{e}}(\boldsymbol{m})(\boldsymbol{x}')\ \mathrm{d}S' - \begin{cases} \boldsymbol{H}_{\mathrm{sc}}(\boldsymbol{x}), & \boldsymbol{x} \in \Omega_{\mathrm{e}} \\ \mathbf{0}, & \boldsymbol{x} \in \Omega \end{cases}$$
$$= -\frac{1}{\mathrm{i}k}\nabla \times \left\{\nabla \times \int_\Gamma g(k, |\boldsymbol{x} - \boldsymbol{x}'|)\boldsymbol{m}(\boldsymbol{x}')\ \mathrm{d}S'\right\}.$$

We intend to take the trace $\boldsymbol{\gamma}$ of this equation. In this limit process, the left-hand side becomes $-\frac{1}{2}\mathbf{C}^{\mathrm{e}}(\boldsymbol{m}) + \mathbf{M}\mathbf{C}^{\mathrm{e}}(\boldsymbol{m})$, by the the result of Theorem 2.3. This result holds, irrespectively from which side the limit is taken. The right-hand side has the limit

$$-\frac{1}{\mathrm{i}k}\mathbf{L}\boldsymbol{m} = -\frac{1}{\mathrm{i}k}\left\{\boldsymbol{\gamma}\left(\nabla\left(\mathcal{S}\mathrm{div}_\Gamma \boldsymbol{m}\right)\right) + k^2\boldsymbol{\gamma}\left(\mathbf{S}\boldsymbol{m}\right)\right\}, \quad \boldsymbol{m} \in H^{-1/2}(\mathrm{div}, \Gamma),$$

and the result of the theorem follows. □

3 Generalized harmonics

The vector spherical harmonics constitute a well-established and important tool for the expansion of tangential vector fields on a spherical surface [17]. The main motivation behind this section is to generalize this tool to include also non-spherical surfaces.

We start this section by a review of two introduced differential operators that act on scalars and vectors, respectively. For simplicity, we assume that the surface Γ is simply connected. The eigenfunctions of these operators provide bases for $L^2(\Gamma)$ and $TL^2(\Gamma)$, respectively. They are well suited for expansion of the traces of solutions to the Maxwell equations. The spherical surface case yields the well known vector spherical harmonics, see Appendix B.

The scalar Laplace-Beltrami operator Δ_Γ on Γ acting on a scalar field f is defined as [21]

$$\Delta_\Gamma f \stackrel{\mathrm{def}}{=} \mathrm{div}_\Gamma\, \mathbf{grad}_\Gamma f = -\mathrm{curl}_\Gamma\, \mathbf{curl}_\Gamma f, \tag{3.1}$$

The four intrinsic surface differential operators, $\mathrm{div}_\Gamma, \mathrm{curl}_\Gamma, \mathbf{grad}_\Gamma, \mathbf{curl}_\Gamma$ are defined in Appendix A.1, see also [4,9,21,25]. The vector Laplace-Beltrami operator $\boldsymbol{\Delta}_\Gamma$ on Γ acting on a tangential vector field $\boldsymbol{f}$ is defined as

$$\boldsymbol{\Delta}_\Gamma \boldsymbol{f} \stackrel{\mathrm{def}}{=} \mathbf{grad}_\Gamma\, \mathrm{div}_\Gamma \boldsymbol{f} - \mathbf{curl}_\Gamma \mathrm{curl}_\Gamma \boldsymbol{f}.$$

The scalar Laplace-Beltrami operator has a countable set of eigenfunctions in $L^2(\Gamma)$, which we denote $\{Y_n\}_{n=1}^\infty$, and they satisfy, see Appendix D and [21]

$$-\Delta_\Gamma \, Y_n = k^2 \lambda_n Y_n. \tag{3.2}$$

The eigenvalues are all real, positive, and the only possible accumulation point of the eigenvalues is at infinity [16,21]. We order the eigenvalues as $\lambda_1 \leq \lambda_2 \leq \ldots$, and normalizing the eigenfunctions $\{Y_n\}_{n=1}^\infty$ in $L^2(\Gamma)$, *i.e.*,

$$\int\limits_\Gamma Y_n Y_{n'}^* \, \mathrm{d}S = \delta_{nn'}, \tag{3.3}$$

we obtain an orthonormal basis in $L^2(\Gamma)$, where, as above, a star $*$ denotes complex conjugation. Notice that the eigenvalues are scaled with the wave number k^2 in order to have a dimensionless quantity, and moreover the functions Y_n have dimension inverse length, *i.e.*, $[\mathrm{m}^{-1}]$.

The following lemma is easily verified with the definitions of the scalar and vector Laplace-Beltrami operators.

Lemma 3.1. *If f satisfies*

$$-\Delta_\Gamma \, f = \Lambda f,$$

for some $\Lambda \in \mathbb{R}$, then

$$-\boldsymbol{\Delta}_\Gamma \, \mathbf{curl}_\Gamma f = \Lambda \, \mathbf{curl}_\Gamma f, \quad -\boldsymbol{\Delta}_\Gamma \, \mathbf{grad}_\Gamma f = \Lambda \, \mathbf{grad}_\Gamma f.$$

Proof. Start with

$$\begin{aligned} -\boldsymbol{\Delta}_\Gamma \, \mathbf{curl}_\Gamma f &= -\mathbf{grad}_\Gamma \, \mathrm{div}_\Gamma \, \mathbf{curl}_\Gamma f + \mathbf{curl}_\Gamma \mathrm{curl}_\Gamma \, \mathbf{curl}_\Gamma f \\ &= \mathbf{curl}_\Gamma \mathrm{curl}_\Gamma \, \mathbf{curl}_\Gamma f = -\mathbf{curl}_\Gamma \Delta_\Gamma f = \Lambda \, \mathbf{curl}_\Gamma f. \end{aligned}$$

since $\mathrm{div}_\Gamma \mathbf{curl}_\Gamma f \equiv 0$. We also have

$$\begin{aligned} -\boldsymbol{\Delta}_\Gamma \, \mathbf{grad}_\Gamma f &= -\mathbf{grad}_\Gamma \, \mathrm{div}_\Gamma \, \mathbf{grad}_\Gamma f + \mathbf{curl}_\Gamma \mathrm{curl}_\Gamma \, \mathbf{grad}_\Gamma f \\ &= -\mathbf{grad}_\Gamma \, \mathrm{div}_\Gamma \, \mathbf{grad}_\Gamma f = -\mathbf{grad}_\Gamma \Delta_\Gamma f = \Lambda \, \mathbf{grad}_\Gamma f. \end{aligned}$$

since $\mathrm{curl}_\Gamma \mathbf{grad}_\Gamma f \equiv 0$, and the lemma is proved. □

By the use of this lemma, we can construct a set of eigenfunctions to the vector Laplace-Beltrami operator. In the sequel, unless otherwise stated, we will consider that $\tau = 1, 2$ and $n, n' \in \mathbb{N} = \{1, 2, 3, \ldots\}$.

Definition 3.1. *The vector generalized harmonics are defined as*

$$\boldsymbol{Y}_{1n} = \frac{1}{k\sqrt{\lambda_n}} \mathbf{curl}_\Gamma Y_n, \quad \boldsymbol{Y}_{2n} = \frac{1}{k\sqrt{\lambda_n}} \mathbf{grad}_\Gamma Y_n.$$

These functions have dimension inverse length, i.e., $[\mathrm{m}^{-1}]$.

Remark 3.1. *Note that* $\boldsymbol{Y}_{1n}$ *and* $\boldsymbol{Y}_{2n}$ *are eigenfunctions to the* $\mathbf{curl}_\Gamma \mathrm{curl}_\Gamma$ *and* $-\mathbf{grad}_\Gamma\, \mathrm{div}_\Gamma$ *operators, respectively. We also observe that* $\boldsymbol{Y}_{1n}$ *belongs to the kernel of the* $-\mathbf{grad}_\Gamma\, \mathrm{div}_\Gamma$ *operator, and that* $\boldsymbol{Y}_{2n}$ *belongs to the kernel of the* $\mathbf{curl}_\Gamma \mathrm{curl}_\Gamma$ *operator. Note also that for a simply-connected surface* Γ*, there is no eigenvalue* $\lambda = 0$*, see the end of proof of Lemma 3.2.*

The following lemma proves that the set $\{\boldsymbol{Y}_{\tau n}, \tau = 1,2, n = 1,2,\ldots\}$ is an orthonormal system on $TL^2(\Gamma)$:

Lemma 3.2. *The vector functions* $\boldsymbol{Y}_{1n}$ *and* $\boldsymbol{Y}_{2n}$ *defined in Definition 3.1 constitute an orthonormal basis on* $TL^2(\Gamma)$*, i.e.,*

$$\int\limits_\Gamma \boldsymbol{Y}_{\tau n} \cdot \boldsymbol{Y}^*_{\tau' n'}\ \mathrm{d}S = \delta_{\tau\tau'}\delta_{nn'}.$$

The vector functions satisfy

$$\hat{\boldsymbol{\nu}} \times \boldsymbol{Y}_{\tau n} = (-1)^{\tau+1}\boldsymbol{Y}_{\overline{\tau} n}, \tag{3.4}$$

where the dual index $\overline{\tau}$ *is* $\overline{1} = 2$ *and* $\overline{2} = 1$.

Moreover,

$$\mathrm{curl}_\Gamma \boldsymbol{Y}_{\tau n} = k\delta_{\tau,1}\sqrt{\lambda_n} Y_n, \quad \mathrm{div}_\Gamma \boldsymbol{Y}_{\tau n} = -k\delta_{\tau,2}\sqrt{\lambda_n} Y_n,$$

and

$$-\boldsymbol{\Delta}_\Gamma\, \boldsymbol{Y}_{\tau n} = k^2 \lambda_n \boldsymbol{Y}_{\tau n}.$$

Proof. We start by noticing that $\boldsymbol{Y}_{1n}$ and $\boldsymbol{Y}_{2n'}$ both are tangential to Γ, by the definition of the operators $\mathbf{curl}_\Gamma$ and $\mathbf{grad}_\Gamma$. Equations (3.1), (3.2), (3.3), the relations $\langle \mathbf{curl}_\Gamma u, \boldsymbol{v}\rangle_{TL^2(\Gamma)} = \langle u, \mathrm{curl}_\Gamma \boldsymbol{v}\rangle_{L^2(\Gamma)}$, $\langle \mathrm{div}_\Gamma \boldsymbol{u}, \phi\rangle_{L^2(\Gamma)} = -\langle \boldsymbol{u}, \mathbf{grad}_\Gamma \phi\rangle_{TL^2(\Gamma)}$, together with $\mathrm{curl}_\Gamma \mathbf{grad}_\Gamma \phi = 0$, and $\mathbf{curl}_\Gamma u = \mathbf{grad}_\Gamma u \times \hat{\boldsymbol{\nu}}$ imply

$$\begin{aligned}
\int\limits_\Gamma \boldsymbol{Y}_{1n} \cdot \boldsymbol{Y}^*_{1n'}\ \mathrm{d}S &= \frac{1}{k^2\sqrt{\lambda_n \lambda_{n'}}} \int\limits_\Gamma \mathbf{curl}_\Gamma Y_n \cdot \mathbf{curl}_\Gamma Y^*_{n'}\ \mathrm{d}S \\
&= \frac{1}{k^2\sqrt{\lambda_n \lambda_{n'}}} \int\limits_\Gamma Y_n \mathrm{curl}_\Gamma \mathbf{curl}_\Gamma Y^*_{n'}\ \mathrm{d}S = -\frac{1}{k^2\sqrt{\lambda_n \lambda_{n'}}} \int\limits_\Gamma Y_n \Delta_\Gamma Y^*_{n'}\ \mathrm{d}S \\
&= \frac{\lambda_{n'}}{\sqrt{\lambda_n \lambda_{n'}}} \int\limits_\Gamma Y_n Y^*_{n'}\ \mathrm{d}S = \delta_{nn'},
\end{aligned}$$

and

$$\begin{aligned}
\int\limits_\Gamma \boldsymbol{Y}_{2n} \cdot \boldsymbol{Y}^*_{2n'}\ \mathrm{d}S &= \frac{1}{k^2\sqrt{\lambda_n \lambda_{n'}}} \int\limits_\Gamma \mathbf{grad}_\Gamma Y_n \cdot \mathbf{grad}_\Gamma Y^*_{n'}\ \mathrm{d}S \\
&= -\frac{1}{k^2\sqrt{\lambda_n \lambda_{n'}}} \int\limits_\Gamma Y_n \mathrm{div}_\Gamma \mathbf{grad}_\Gamma Y^*_{n'}\ \mathrm{d}S = -\frac{1}{k^2\sqrt{\lambda_n \lambda_{n'}}} \int\limits_\Gamma Y_n \Delta_\Gamma Y^*_{n'}\ \mathrm{d}S \\
&= \frac{\lambda_{n'}}{\sqrt{\lambda_n \lambda_{n'}}} \int\limits_\Gamma Y_n Y^*_{n'}\ \mathrm{d}S = \delta_{nn'},
\end{aligned}$$

and

$$\int_\Gamma \boldsymbol{Y}_{1n} \cdot \boldsymbol{Y}^*_{2n'} \, \mathrm{d}S = \frac{1}{k^2\sqrt{\lambda_n \lambda_{n'}}} \int_\Gamma \mathbf{curl}_\Gamma Y_n \cdot \mathbf{grad}_\Gamma Y^*_{n'} \, \mathrm{d}S$$
$$= \frac{1}{k^2\sqrt{\lambda_n \lambda_{n'}}} \int_\Gamma Y_n \mathrm{curl}_\Gamma \mathbf{grad}_\Gamma Y^*_{n'} \, \mathrm{d}S = 0.$$

Moreover,

$$\hat{\boldsymbol{\nu}} \times \boldsymbol{Y}_{1n} = \frac{1}{k\sqrt{\lambda_n}} \hat{\boldsymbol{\nu}} \times \mathbf{curl}_\Gamma Y_n = \frac{1}{k\sqrt{\lambda_n}} \hat{\boldsymbol{\nu}} \times (\mathbf{grad}_\Gamma Y_n \times \hat{\boldsymbol{\nu}})$$
$$= \frac{1}{k\sqrt{\lambda_n}} \mathbf{grad}_\Gamma Y_n = \boldsymbol{Y}_{2n},$$

and

$$\hat{\boldsymbol{\nu}} \times \boldsymbol{Y}_{2n} = \hat{\boldsymbol{\nu}} \times (\hat{\boldsymbol{\nu}} \times \boldsymbol{Y}_{1n}) = -\boldsymbol{Y}_{1n}.$$

The final statements are easily proven by

$$\mathrm{curl}_\Gamma \boldsymbol{Y}_{\tau n} = -\frac{1}{k\sqrt{\lambda_n}} \delta_{\tau,1} \Delta_\Gamma Y_n = k \delta_{\tau,1} \sqrt{\lambda_n} Y_n,$$

and

$$\mathrm{div}_\Gamma \boldsymbol{Y}_{\tau n} = \frac{1}{k\sqrt{\lambda_n}} \delta_{\tau,2} \Delta_\Gamma Y_n = -k \delta_{\tau,2} \sqrt{\lambda_n} Y_n,$$

and

$$-\boldsymbol{\Delta}_\Gamma \, \boldsymbol{Y}_{\tau n} = -\mathbf{grad}_\Gamma \, \mathrm{div}_\Gamma \boldsymbol{Y}_{\tau n} + \mathbf{curl}_\Gamma \left(\mathrm{curl}_\Gamma \boldsymbol{Y}_{\tau n} \right)$$
$$= k\delta_{\tau,2}\sqrt{\lambda_n} \mathbf{grad}_\Gamma \, Y_n + k\delta_{\tau,1}\sqrt{\lambda_n} \mathbf{curl}_\Gamma Y_n = k^2 \lambda_n \boldsymbol{Y}_{\tau n},$$

The completeness of the set of vector generalized harmonics $\{\boldsymbol{Y}_{1n}, \boldsymbol{Y}_{2n}\}_{n=1}^\infty$ can be proved by investigating which $\boldsymbol{f}$ satisfies

$$\langle \boldsymbol{f}, \boldsymbol{Y}_{\tau n} \rangle = 0, \quad \tau = 1, 2, \ \forall n \in \mathbb{N}.$$

If this statement implies $\boldsymbol{f} = \mathbf{0}$, the set of vector generalized harmonics will be dense in $TL^2(\Gamma)$. We start with $\tau = 1$, and get

$$0 = \langle \boldsymbol{f}, \boldsymbol{Y}_{1n} \rangle = \frac{1}{k\sqrt{\lambda_n}} \langle \boldsymbol{f}, \mathbf{curl}_\Gamma Y_n \rangle = \frac{1}{k\sqrt{\lambda_n}} \langle \mathrm{curl}_\Gamma \boldsymbol{f}, Y_n \rangle, \quad \forall n \in \mathbb{N}.$$

From the completeness of the generalized harmonics Y_n (see, *e.g.*, [21]), *i.e.*, from the fact that $\langle g, Y_n \rangle = 0$, $\forall n \in \mathbb{N}$ renders $g = 0$, we obtain that $\mathrm{curl}_\Gamma \boldsymbol{f} = 0$. In the above, as well in the following relation, the brackets ($\langle \cdot, \cdot \rangle$) denote the suitable inner product or the appropriate duality pairing between the involved function spaces. We continue with $\tau = 2$.

$$0 = \langle \boldsymbol{f}, \boldsymbol{Y}_{2n} \rangle = \frac{1}{k\sqrt{\lambda_n}} \langle \boldsymbol{f}, \mathbf{grad}_\Gamma Y_n \rangle$$
$$= -\frac{1}{k\sqrt{\lambda_n}} \langle \mathrm{div}_\Gamma \boldsymbol{f}, Y_n \rangle, \quad \forall n \in \mathbb{N}.$$

Again, the completeness of the generalized harmonics Y_n implies $\operatorname{div}_\Gamma \boldsymbol{f} = 0$. However, a function $\boldsymbol{f}$, which satisfies $\operatorname{curl}_\Gamma \boldsymbol{f} = \operatorname{div}_\Gamma \boldsymbol{f} = 0$ on a simply connected surface Γ, is zero [21, p. 206], and the lemma is proved. $\square$

4 Trace spaces and the exterior Calderón matrix

4.1 Spectral characterization of trace spaces

We redefine (in the spirit of [21]) the pertinent function spaces used frequently in this paper in terms of the orthogonal bases Y_n and $\boldsymbol{Y}_{\tau n}$. The generalized Fourier series of a function f is

$$f = \sum_n a_n Y_n, \qquad a_n = \langle f, Y_n \rangle_{L^2(\Gamma)},$$

where convergence is in the $L^2(\Gamma)$ norm (defined below). The space $L^2(\Gamma)$ is characterized as

$$L^2(\Gamma) = \left\{ f \in \mathcal{D}'(\Gamma) : \sum_n |a_n|^2 < \infty \right\},$$

equipped with the norm

$$\|f\|^2_{L^2(\Gamma)} = \sum_n |a_n|^2 ,$$

and the space $H^s(\Gamma)$ is characterized as

$$H^s(\Gamma) = \left\{ f \in \mathcal{D}'(\Gamma) : \sum_n (1+\lambda_n)^s |a_n|^2 < \infty \right\},$$

equipped with the norm [21, p. 206]

$$\|f\|^2_{H^s(\Gamma)} = \sum_n (1+\lambda_n)^s |a_n|^2 .$$

Similarly, the generalized Fourier series of a tangential vector function $\boldsymbol{f}$ is

$$\boldsymbol{f} = \sum_{\tau n} a_{\tau n} \boldsymbol{Y}_{\tau n}, \qquad a_{\tau n} = \langle \boldsymbol{f}, \boldsymbol{Y}_{\tau n} \rangle_{TL^2(\Gamma)},$$

where convergence is in the $TL^2(\Gamma)$ norm. The space $TL^2(\Gamma)$ is characterized as

$$TL^2(\Gamma) = \left\{ \boldsymbol{f} \in \mathcal{D}'(\Gamma) : \sum_{\tau n} |a_{\tau n}|^2 < \infty \right\},$$

equipped with the norm

$$\|\boldsymbol{f}\|^2_{TL^2(\Gamma)} = \sum_{\tau n} |a_{\tau n}|^2 ,$$

and the space $TH^s(\Gamma)$ is characterized as

$$TH^s(\Gamma) = \left\{ \boldsymbol{f} \in \mathcal{D}'(\Gamma) : \sum_{\tau n} (1+\lambda_n)^s |a_{\tau n}|^2 < \infty \right\},$$

equipped with the norm

$$\|\boldsymbol{f}\|^2_{TH^s(\Gamma)} = \sum_{\tau n} (1+\lambda_n)^s |a_{\tau n}|^2 . \tag{4.1}$$

Remark 4.1. *In [21] is this norm defined as*

$$\|\boldsymbol{f}\|^2_{TH^s(\Gamma)} = \sum_{\tau n} (\lambda_n)^s |a_{\tau n}|^2 .$$

which is equivalent with (4.1) *as long as the smallest eigenvalue is strictly positive.*

The operations of curl_Γ and div_Γ imply, using Lemma 3.2,

$$\mathrm{curl}_\Gamma \boldsymbol{f} = \sum_{\tau n} a_{\tau n} \mathrm{curl}_\Gamma \boldsymbol{Y}_{\tau n} = k \sum_n \sqrt{\lambda_n} a_{1n} Y_n,$$

and

$$\mathrm{div}_\Gamma \boldsymbol{f} = \sum_{\tau n} a_{\tau n} \mathrm{div}_\Gamma \boldsymbol{Y}_{\tau n} = -k \sum_n \sqrt{\lambda_n} a_{2n} Y_n.$$

Note that only one of $\boldsymbol{Y}_{1n}$ and $\boldsymbol{Y}_{2n}$ survives the respective differentiation. This motivates the following redefinition of the involved spaces in terms of the corresponding suitable norms.

Definition 4.1. *We define* $H^{-1/2}(\mathrm{div},\Gamma)$ *and* $H^{-1/2}(\mathrm{curl},\Gamma)$ *as*

$$H^{-1/2}(\mathrm{div},\Gamma) = \left\{ \boldsymbol{f} \in TH^{-1/2}(\Gamma), \mathrm{div}_\Gamma \boldsymbol{f} \in H^{-1/2}(\Gamma) \right\},$$

equipped with the norm

$$\|\boldsymbol{f}\|^2_{H^{-1/2}(\mathrm{div},\Gamma)} = \sum_{\tau n} (1+\lambda_n)^{\tau-3/2} |a_{\tau n}|^2 ,$$

and

$$H^{-1/2}(\mathrm{curl},\Gamma) = \left\{ \boldsymbol{f} \in TH^{-1/2}(\Gamma), \mathrm{curl}_\Gamma \boldsymbol{f} \in H^{-1/2}(\Gamma) \right\},$$

equipped with the norm

$$\|\boldsymbol{f}\|^2_{H^{-1/2}(\mathrm{curl},\Gamma)} = \sum_{\tau n} (1+\lambda_n)^{\overline{\tau}-3/2} |a_{\tau n}|^2 .$$

We also employ the weighted space $\boldsymbol{\ell}^{-1/2}(\mathrm{div})$ defined by

$$\boldsymbol{\ell}^{-1/2}(\mathrm{div}) = \left\{ a_{\tau n} \in \mathbb{C} : \sum_{\tau n} (1+\lambda_n)^{\tau-3/2} |a_{\tau n}|^2 < \infty \right\}.$$

We notice that the spaces $\boldsymbol{\ell}^{-1/2}(\mathrm{div})$ and $H^{-1/2}(\mathrm{div},\Gamma)$ are equivalent in the sense that $\boldsymbol{f} \in H^{-1/2}(\mathrm{div},\Gamma)$ if and only if its Fourier coefficients $a_{\tau n} \in \boldsymbol{\ell}^{-1/2}(\mathrm{div})$. We have the following Parseval type of identity

Lemma 4.1. *Let $\boldsymbol{u}, \boldsymbol{v} \in L^2(\Gamma)$ with expansions*

$$\begin{cases} \boldsymbol{u} = \sum_{\tau n} e_{\tau n} \boldsymbol{Y}_{\tau n} \\ \boldsymbol{v} = \sum_{\tau n} h_{\tau n} \boldsymbol{Y}_{\tau n}, \end{cases}$$

then

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{L^2(\Gamma)} = \sum_{\tau n} e_{\tau n} h^*_{\tau n}.$$

Proof. The proof follows the proof of the orthogonality of the vector generalized harmonics in Lemma 3.2. □

Remark 4.2. *Let $\boldsymbol{u} \in H^{-1/2}(\mathrm{div}, \Gamma)$ and $\boldsymbol{v} \in H^{-1/2}(\mathrm{curl}, \Gamma)$. The two norms are explicitly given as*

$$\|\boldsymbol{u}\|^2_{H^{-1/2}(\mathrm{div},\Gamma)} = \sum_n \frac{1}{\sqrt{1+\lambda_n}} |e_{1n}|^2 + \sum_n \sqrt{1+\lambda_n}\, |e_{2n}|^2 ,$$

and

$$\|\boldsymbol{v}\|^2_{H^{-1/2}(\mathrm{curl},\Gamma)} = \sum_n \sqrt{1+\lambda_n} |h_{1n}|^2 + \sum_n \frac{1}{\sqrt{1+\lambda_n}} \, |h_{2n}|^2 ,$$

respectively. A duality pairing between the spaces $H^{-1/2}(\mathrm{div}, \Gamma)$ and $H^{-1/2}(\mathrm{curl}, \Gamma)$ yields

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{H^{-1/2}(\mathrm{div},\Gamma), H^{-1/2}(\mathrm{curl},\Gamma)} = \langle \boldsymbol{u}, \boldsymbol{v} \rangle_{L^2(\Gamma)}$$

Lemma 4.2. *Let $\boldsymbol{u} \in H^{-1/2}(\mathrm{div}, \Gamma)$ and $\boldsymbol{v} \in H^{-1/2}(\mathrm{curl}, \Gamma)$ with expansions*

$$\begin{cases} \boldsymbol{u} = \sum_{\tau n} e_{\tau n} \boldsymbol{Y}_{\tau n} \\ \boldsymbol{v} = \sum_{\tau n} h_{\tau n} \boldsymbol{Y}_{\tau n}, \end{cases}$$

then

$$\|\boldsymbol{u}\|^2_{H^{-1/2}(\mathrm{div},\Gamma)} = \|\hat{\boldsymbol{\nu}} \times \boldsymbol{u}\|^2_{H^{-1/2}(\mathrm{curl},\Gamma)} = \sum_{\tau n} (1+\lambda_n)^{\tau - 3/2} \, |e_{\tau n}|^2 ,$$

and

$$\|\boldsymbol{v}\|^2_{H^{-1/2}(\mathrm{curl},\Gamma)} = \|\hat{\boldsymbol{\nu}} \times \boldsymbol{v}\|^2_{H^{-1/2}(\mathrm{div},\Gamma)} = \sum_{\tau n} (1+\lambda_n)^{\bar{\tau} - 3/2} \, |h_{\tau n}|^2 .$$

Proof. The proof follows from the construction of the vector generalized harmonics in Definition 3.1 and Lemma 3.2. □

4.2 The exterior Calderón matrix

For simplicity, we assume that the surface Γ is simply connected.[7]

Any $\boldsymbol{m} \in H^{-1/2}(\mathrm{div}, \Gamma) \cap TL^2(\Gamma)$ has a convergent Fourier expansion in terms of $\boldsymbol{Y}_{\tau n}$, *i.e.*,

$$\boldsymbol{m} = \sum_{\tau n} e_{\tau n} \boldsymbol{Y}_{\tau n}, \quad e_{\tau n} = \langle \boldsymbol{m}, \boldsymbol{Y}_{\tau n} \rangle_{TL^2(\Gamma)} = \int_\Gamma \boldsymbol{m} \cdot \boldsymbol{Y}_{\tau n}^* \, \mathrm{d}S, \tag{4.2}$$

Using Riesz representation, any $\boldsymbol{m} \in H^{-1/2}(\mathrm{div}, \Gamma)$ has a generalized Fourier expansion in terms of the same basis as (4.2), where $e_{\tau n} \in \boldsymbol{\ell}^{-1/2}(\mathrm{div})$.

With the solution of Problem (E), the image of the exterior Calderón map $\mathbf{C}^{\mathrm{e}}(\boldsymbol{m}) \in H^{-1/2}(\mathrm{div}, \Gamma)$ has an expansion

$$\mathbf{C}^{\mathrm{e}}(\boldsymbol{m}) = \boldsymbol{\gamma}(\boldsymbol{H}_{\mathrm{sc}}) = \mathrm{i} \sum_{\tau n} h_{\tau n} \boldsymbol{Y}_{\overline{\tau} n}, \quad h_{\tau n} = -\mathrm{i} \langle \boldsymbol{\gamma}(\boldsymbol{H}_{\mathrm{sc}}), \boldsymbol{Y}_{\overline{\tau} n} \rangle_{TL^2(\Gamma)}, \tag{4.3}$$

and $h_{\tau n} \in \boldsymbol{\ell}^{-1/2}(\mathrm{div})$. Note the bar over the index τ, which denotes the dual index in τ ($\overline{1} = 2$ and $\overline{2} = 1$), and an extra factor of i. The reason for this choice is that the expansion coefficients of the magnetic surface field then has a simple relation to the corresponding coefficients of the electric surface field.

Remark 4.3. *We note that the expansion in* (4.2) *is a Helmholtz-Hodge decomposition of the elements* $\boldsymbol{m}$ *in* $H^{-1/2}(\mathrm{div}, \Gamma)$ *(and similarly of* $H^{-1/2}(\mathrm{curl}, \Gamma)$*) and that the* L^2*-projection can be interpreted as a duality pairing between* $H^{-1/2}(\mathrm{div}, \Gamma)$ *and* $H^{-1/2}(\mathrm{curl}, \Gamma)$*, see Remark 4.2.*

The mapping $\boldsymbol{\ell}^{-1/2}(\mathrm{div}) \ni e_{\tau n} \mapsto h_{\tau n} \in \boldsymbol{\ell}^{-1/2}(\mathrm{div})$ is a realization of the exterior Calderón operator. To every set of coefficients $e_{\tau n}$ there exists a unique set of coefficients $h_{\tau n}$, and this association defines a linear relation between $e_{\tau n} \mapsto h_{\tau n}$ manifested by a matrix C (the exterior Calderón matrix) and

$$h_{\tau n} = \sum_{\tau' n'} C_{\tau n, \tau' n'} e_{\tau' n'}. \tag{4.4}$$

The explicit form of the matrix is

$$C_{\tau n, \tau' n'} = -\mathrm{i} \langle \mathbf{C}^{\mathrm{e}}(\boldsymbol{Y}_{\tau' n'}), \boldsymbol{Y}_{\overline{\tau} n} \rangle_{TL^2(\Gamma)}. \tag{4.5}$$

By the use of Lemma E.1 in Appendix E, we conclude that the exterior Calderón matrix C is invertible in $\boldsymbol{\ell}^{-1/2}(\mathrm{div})$.

Lemma 4.3. *The exterior Calderón matrix* $C_{\tau n, \tau' n'} = -\mathrm{i} \langle \mathbf{C}^{\mathrm{e}}(\boldsymbol{Y}_{\tau' n'}), \boldsymbol{Y}_{\overline{\tau} n} \rangle_{TL^2(\Gamma)}$ *defined by* (4.4) *and* (4.5) *satisfies*

$$\sum_{\tau'' n''} C_{\overline{\tau} n, \overline{\tau''} n''} C_{\tau'' n'', \tau' n'} = \delta_{\tau \tau'} \delta_{n n'},$$

and its inverse is

$$C^{-1}_{\tau n, \tau' n'} = C_{\overline{\tau} n, \overline{\tau'} n'}.$$

[7] For the generalization of the analysis to not simply connected surfaces, see Remark 4.5.

Proof. The lemma is a consequence of $(\mathbf{C}^{\mathrm{e}})^2 = -\mathbf{I}$ on $H^{-1/2}(\mathrm{div}, \Gamma)$, the expansions in (4.2), (4.3), and the map (4.4). We have

$$\boldsymbol{m} = -\mathbf{C}^{\mathrm{e}}\left(\mathbf{C}^{\mathrm{e}}(\boldsymbol{m})\right), \quad \forall \boldsymbol{m} \in H^{-1/2}(\mathrm{div}, \Gamma),$$

or due to continuity of the exterior Calderón operator

$$\begin{aligned}\sum_{\tau n} e_{\tau n}\boldsymbol{Y}_{\tau n} &= -\mathrm{i}\sum_{\tau n} h_{\overline{\tau} n}\mathbf{C}^{\mathrm{e}}\left(\boldsymbol{Y}_{\tau n}\right) = -\mathrm{i}\sum_{\tau n}\sum_{\tau' n'} C_{\overline{\tau} n,\tau' n'} e_{\tau' n'}\mathbf{C}^{\mathrm{e}}\left(\boldsymbol{Y}_{\tau n}\right)\\ &= \sum_{\tau n}\sum_{\tau' n'}\sum_{\tau'' n''} C_{\overline{\tau} n,\tau' n'} e_{\tau' n'} C_{\tau'' n'',\tau n}\boldsymbol{Y}_{\overline{\tau''} n''} = \sum_{\tau n}\sum_{\tau' n'}\sum_{\tau'' n''} C_{\overline{\overline{\tau''}} n'',\tau' n'} e_{\tau' n'} C_{\overline{\tau} n,\tau'' n''}\boldsymbol{Y}_{\tau n},\end{aligned}$$

since by (4.3) and (4.4)

$$\mathbf{C}^{\mathrm{e}}\left(\boldsymbol{Y}_{\tau n}\right) = \mathrm{i}\sum_{\tau'' n''} C_{\tau'' n'',\tau n}\boldsymbol{Y}_{\overline{\tau''} n''}. \tag{4.6}$$

Orthogonality then implies

$$e_{\tau n} = \sum_{\tau' n'}\sum_{\tau'' n''} C_{\overline{\overline{\tau''}} n'',\tau' n'} C_{\overline{\tau} n,\tau'' n''} e_{\tau' n'},$$

or, since $e_{\tau n}$ is arbitrary

$$\sum_{\tau'' n''} C_{\overline{\tau} n,\overline{\overline{\tau''}} n''} C_{\tau'' n'',\tau' n'} = \sum_{\tau'' n''} C_{\overline{\overline{\tau''}} n'',\tau' n'} C_{\overline{\tau} n,\tau'' n''} = \delta_{\tau,\tau'}\delta_{n,n'}.$$

which ends the proof. □

Moreover, we have

Lemma 4.4. *The matrix*

$$\frac{1}{2\mathrm{i}}\left\{(-1)^{\tau} C_{\tau n,\tau' n'} - (-1)^{\tau'} C^*_{\tau' n',\tau n}\right\},$$

is positive definite.

Proof. The exterior Calderón operator satisfies (2.3)

$$\mathrm{Re}\int_{\Gamma} \mathbf{C}^{\mathrm{e}}(\boldsymbol{m})\cdot(\hat{\boldsymbol{\nu}}\times\boldsymbol{m}^*)\,\mathrm{d}S > 0 \qquad \text{for all } \boldsymbol{m}\in H^{-1/2}(\mathrm{div},\Gamma)\,\boldsymbol{m}\neq\boldsymbol{0}.$$

Insert the expansions of $\boldsymbol{m}$ and $\mathbf{C}^{\mathrm{e}}(\boldsymbol{m})$, see (4.2) and (4.3). We obtain

$$\mathrm{Re}\,\mathrm{i}\sum_{\tau n}\sum_{\tau' n'} h_{\tau n} e^*_{\tau' n'}\underbrace{\int_{\Gamma}\boldsymbol{Y}_{\overline{\tau} n}\cdot(\hat{\boldsymbol{\nu}}\times\boldsymbol{Y}^*_{\tau' n'})\,\mathrm{d}S}_{=\delta_{\tau\tau'}\delta_{nn'}(-1)^{\tau'+1}} = \mathrm{Re}\,\mathrm{i}\sum_{\tau n}(-1)^{\tau+1} h_{\tau n} e^*_{\tau n} > 0,$$

where we used $\hat{\boldsymbol{\nu}} \times \boldsymbol{Y}_{\tau'n'} = (-1)^{\tau'+1}\boldsymbol{Y}_{\overline{\tau'}n'}$, see (3.4) in Lemma 3.2. This implies

$$\operatorname{Im}\sum_{\tau n}\sum_{\tau'n'} e_{\tau n}^{*}(-1)^{\tau} C_{\tau n,\tau'n'} e_{\tau'n'} > 0, \quad \forall e_{\tau n} \in \boldsymbol{\ell}^{-1/2}(\mathrm{div}) \text{ not all } e_{\tau n} = 0.$$

Rewrite the imaginary part explicitly and change summation indices. We get

$$\frac{1}{2\mathrm{i}}\sum_{\tau n}\sum_{\tau'n'} e_{\tau n}^{*}\left\{(-1)^{\tau} C_{\tau n,\tau'n'} - (-1)^{\tau'} C_{\tau'n',\tau n}^{*}\right\} e_{\tau'n'} > 0$$
$$\forall e_{\tau n} \in \boldsymbol{\ell}^{-1/2}(\mathrm{div}) \text{ not all } e_{\tau n} = 0,$$

which proves the lemma. □

Theorem 4.1. *The norm of the exterior Calderón operator in* $H^{-1/2}(\mathrm{div},\Gamma)$ *is determined by the square root of the largest eigenvalue of the Hermitian matrix* $P = D^{-1/2}C^{\dagger}D^{-1}CD^{-1/2}$, *i.e., the matrix*

$$P_{\tau n,\tau'n'} = \sum_{\tau''n''} (1+\lambda_n)^{-\tau/2+3/4}\, C_{\tau''n'',\tau n}^{*}\, (1+\lambda_{n''})^{-\tau''+3/2}\, C_{\tau''n'',\tau'n'}\, (1+\lambda_{n'})^{-\tau'/2+3/4},$$

where the diagonal matrix D *is*

$$D_{\tau n,\tau'n'} = \delta_{nn'}\delta_{\tau\tau'}\,(1+\lambda_n)^{\tau-3/2}.$$

Proof. The norms of the trace of the scattered electric and magnetic field are

$$\|\boldsymbol{\gamma}(\boldsymbol{E}_{\mathrm{sc}})\|_{H^{-1/2}(\mathrm{div},\Gamma)}^{2} = \sum_{\tau n} (1+\lambda_n)^{\tau-3/2}\,|e_{\tau n}|^{2},$$

and

$$\|\boldsymbol{\gamma}(\boldsymbol{H}_{\mathrm{sc}})\|_{H^{-1/2}(\mathrm{div},\Gamma)}^{2} = \sum_{\tau n} (1+\lambda_n)^{\overline{\tau}-3/2}\,|h_{\tau n}|^{2} = \sum_{\tau n} (1+\lambda_n)^{-\tau+3/2}\,|h_{\tau n}|^{2},$$

or in short-hand matrix notation

$$\|\boldsymbol{\gamma}(\boldsymbol{E}_{\mathrm{sc}})\|_{H^{-1/2}(\mathrm{div},\Gamma)}^{2} = e^{\dagger}De, \quad \|\boldsymbol{\gamma}(\boldsymbol{H}_{\mathrm{sc}})\|_{H^{-1/2}(\mathrm{div},\Gamma)}^{2} = h^{\dagger}D^{-1}h,$$

where e and h are the column vectors of the coefficients $e_{\tau n}$ and $h_{\tau n}$, respectively, and the matrix D is defined above. The Hermitian conjugate of these column vectors are denoted $e^{\dagger}$ and $h^{\dagger}$. The norm of the exterior Calderón operator in $H^{-1/2}(\mathrm{div},\Gamma)$ can then be formed, *viz.*,

$$\|\mathbf{C}^{\mathrm{e}}\|_{H^{-1/2}(\mathrm{div},\Gamma)}^{2} = \sup_{e}\frac{(Ce)^{\dagger}D^{-1}(Ce)}{e^{\dagger}De} = \sup_{e}\frac{e^{\dagger}D^{1/2}\left(D^{-1/2}C^{\dagger}D^{-1}CD^{-1/2}\right)D^{1/2}e}{e^{\dagger}D^{1/2}D^{1/2}e}.$$

This is a quadratic form and the largest eigenvalue of $D^{-1/2}C^{\dagger}D^{-1}CD^{-1/2}$ determines the norm. □

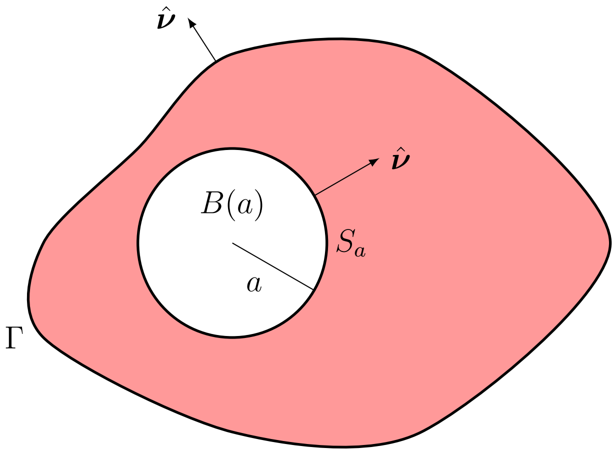

Figure 2: The spherical surface S_a and the domain Ω.

4.3 Calculation of the exterior Calderón matrix

The goal now is to find an explicit representation of the exterior Calderón matrix $C_{\tau n,\tau' n'}$ in terms of the geometry of the surface Γ. A number of lemmata and propositions guide us.

Denote by S_r the sphere of radius r centered at the origin, see Figure 2. The restriction of $\boldsymbol{\gamma}(\widetilde{\mathbf{M}\boldsymbol{f}})$ to S_r defines an operator $\mathbf{A}_r : H^{-1/2}(\mathrm{div},\Gamma) \to H^{-1/2}(\mathrm{div},S_r)$. The explicit expression of the operator is, for $\boldsymbol{f} \in H^{-1/2}(\mathrm{div},\Gamma)$

$$\boldsymbol{f} \mapsto (\mathbf{A}_r\boldsymbol{f})(\boldsymbol{x}) = \hat{\boldsymbol{x}} \times \left(\nabla \times \int_\Gamma g(k,|\boldsymbol{x}-\boldsymbol{x}'|)\boldsymbol{f}(\boldsymbol{x}')\,\mathrm{d}S' \right), \qquad \boldsymbol{x} \in S_r, \tag{4.7}$$

where the radius $0 < r < R$, $R = \min_{\boldsymbol{x}'\in\Gamma} |\boldsymbol{x}'|$.

Define the radius $a \in (0,R)$ such that the functions $\psi_l(ka) \neq 0$ and $\psi_l'(ka) \neq 0$ for all $l = 1,2,\ldots$, where $\psi_l(z)$ are the Riccati-Bessel functions [17, 22]. This is always possible for small enough $ka > 0$.

Lemma 4.5. *The operator* $\mathbf{A}_a : H^{-1/2}(\mathrm{div},\Gamma) \to H^{-1/2}(\mathrm{div},S_a)$*, defined by* (4.7)*, is compact and injective with dense range.*

Proof. The kernel of the operator $\mathbf{A}_a$ is continuous (analytic in the variable $\boldsymbol{x}$) and hence $\mathbf{A}_a$ is compact. The operator is injective if we can prove that

$$(\mathbf{A}_a\boldsymbol{f})(\boldsymbol{x}) = \mathbf{0}, \quad \forall \boldsymbol{x} \in S_a \quad \Rightarrow \quad \boldsymbol{f} = \mathbf{0}.$$

To accomplish this, define

$$\boldsymbol{F}(\boldsymbol{x}) = \nabla \times \int_\Gamma g(k,|\boldsymbol{x}-\boldsymbol{x}'|)\boldsymbol{f}(\boldsymbol{x}')\,\mathrm{d}S', \quad \boldsymbol{x} \in \mathbb{R}^3 \setminus \Gamma.$$

By assumption, $\boldsymbol{\gamma}(\boldsymbol{F}) = \mathbf{0}$ on S_a (the same limit from both sides). We proceed by proving that the only $\boldsymbol{f}$ that satisfies this condition is $\boldsymbol{f} = \mathbf{0}$.

Let $B(a)$ denote the ball, centered at the origin, of radius a, see Figure 2. The function $\boldsymbol{F}(\boldsymbol{x})$ satisfies, see Theorem 2.3

$$\nabla \times (\nabla \times \boldsymbol{F}(\boldsymbol{x})) - k^2 \boldsymbol{F}(\boldsymbol{x}) = \mathbf{0}, \quad \boldsymbol{x} \in \mathbb{R}^3 \setminus \Gamma,$$

therefore also in the ball $B(a)$. Inside the ball $B(a)$, the field $\boldsymbol{F}(\boldsymbol{x})$ has an expansion in regular spherical vector waves $\boldsymbol{v}_n(k\boldsymbol{x})$, defined by

$$\begin{cases} \boldsymbol{v}_{1n}(k\boldsymbol{x}) = x j_l(kx) \boldsymbol{Y}_{1n}(\boldsymbol{x}) \\ \boldsymbol{v}_{2n}(k\boldsymbol{x}) = \dfrac{1}{k} \nabla \times \left(x j_l(kx) \boldsymbol{Y}_{1n}(\boldsymbol{x}) \right), \end{cases} \tag{4.8}$$

where $j_l(kx)$ is the spherical Bessel function of the first kind [23], and $\boldsymbol{Y}_{\tau n}(\boldsymbol{x})$ are vector harmonics for the sphere (vector spherical harmonics), see Appendix B. Due to orthogonality of the vector spherical harmonics, and the choice of a such that $\psi_l(ka) \neq 0$ and $\psi_l'(ka) \neq 0$ for all $l = 1, 2, \ldots$, the expansion coefficients of this expansion are all zero. Therefore, the interior boundary value problem has a unique solution $\boldsymbol{F}(\boldsymbol{x}) = \mathbf{0}$, $\boldsymbol{x} \in B(a)$. By analyticity, $\boldsymbol{F}(\boldsymbol{x}) = \mathbf{0}$ for all $\boldsymbol{x} \in \Omega$ [24]. As a consequence, the traces $\boldsymbol{\gamma}(\boldsymbol{F})|_- = \mathbf{0}$ and $\boldsymbol{\gamma}(\nabla \times \boldsymbol{F})|_- = \mathbf{0}$. By Theorem 2.3, we also conclude that $\boldsymbol{\gamma}(\nabla \times \boldsymbol{F})|_+ = \mathbf{0}$.

As a function of $\boldsymbol{x} \in \Omega_{\mathrm{e}}$, $\nabla \times \boldsymbol{F}(\boldsymbol{x})$ satisfies the correct radiation conditions at infinity and $\boldsymbol{\gamma}(\nabla \times \boldsymbol{F})|_+ = \mathbf{0}$ on Γ. Due to unique solvability of the exterior problem (Problem (E)), $\nabla \times \boldsymbol{F}(\boldsymbol{x}) = \mathbf{0}$ in Ω_{e}. Since $\boldsymbol{F} = k^{-2} \nabla \times (\nabla \times \boldsymbol{F})$, $\boldsymbol{F}(\boldsymbol{x}) = \mathbf{0}$ in Ω_{e}, and, consequently, $\boldsymbol{\gamma}(\boldsymbol{F})|_+ = \mathbf{0}$. Finally, the jump condition on the trace of $\boldsymbol{F}$ shows, see Theorem 2.3

$$\mathbf{0} = \boldsymbol{\gamma}(\boldsymbol{F})|_+ - \boldsymbol{\gamma}(\boldsymbol{F})|_- = \boldsymbol{f}.$$

This proves the injectivity of the operator $\mathbf{A}_a$.

To prove that the range is dense, and for this purpose, we define the adjoint operator $\mathbf{A}_a^\dagger : H^{-1/2}(\mathrm{curl}, S_a) \to H^{-1/2}(\mathrm{curl}, \Gamma)$ of $\mathbf{A}_a$ w.r.t. to the dual spaces $(H^{-1/2}(\mathrm{div}, \Gamma), H^{-1/2}(\mathrm{div}, S_a))$. The explicit form of the adjoint operator is

$$\begin{aligned} \left(\mathbf{A}_a^\dagger \boldsymbol{g} \right)(\boldsymbol{x}) &= -\hat{\boldsymbol{\nu}}(\boldsymbol{x}) \times \left(\hat{\boldsymbol{\nu}}(\boldsymbol{x}) \times \int_{S_a} \nabla' g(k, |\boldsymbol{x} - \boldsymbol{x}'|) \times \left[\hat{\boldsymbol{x}}' \times \boldsymbol{g}(\boldsymbol{x}') \right] \, \mathrm{d}S' \right) \\ &= \hat{\boldsymbol{\nu}}(\boldsymbol{x}) \times \left(\mathbf{B}(\hat{\boldsymbol{x}} \times \boldsymbol{g}) \right)(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Gamma, \end{aligned}$$

where (use $\nabla g(k, |\boldsymbol{x} - \boldsymbol{x}'|) = -\nabla' g(k, |\boldsymbol{x} - \boldsymbol{x}'|)$)

$$\begin{aligned} \left(\mathbf{B} \boldsymbol{g} \right)(\boldsymbol{x}) &= -\hat{\boldsymbol{\nu}}(\boldsymbol{x}) \times \int_{S_a} \nabla' g(k, |\boldsymbol{x} - \boldsymbol{x}'|) \times \boldsymbol{g}(\boldsymbol{x}') \, \mathrm{d}S' \\ &= \hat{\boldsymbol{\nu}}(\boldsymbol{x}) \times \left(\nabla \times \int_{S_a} g(k, |\boldsymbol{x} - \boldsymbol{x}'|) \boldsymbol{g}(\boldsymbol{x}') \, \mathrm{d}S' \right), \qquad \boldsymbol{x} \in \Gamma. \end{aligned}$$

We now prove that $\mathbf{A}_a^\dagger$ is injective, *i.e.*, $\mathbf{B}$ is injective, namely

$$(\mathbf{B}\boldsymbol{g})(\boldsymbol{x}) = \mathbf{0}, \quad \boldsymbol{x} \in \Gamma \quad \Rightarrow \quad \boldsymbol{g} = \mathbf{0}.$$

To this end assume that $(\mathbf{B}\boldsymbol{g})(\boldsymbol{x}) = \mathbf{0}$, $\boldsymbol{x} \in \Gamma$, and similarly as above, define the function

$$\widetilde{\boldsymbol{F}}(\boldsymbol{x}) = \nabla \times \int_{S_a} g(k, |\boldsymbol{x} - \boldsymbol{x}'|)\boldsymbol{g}(\boldsymbol{x}')\ \mathrm{d}S', \quad \boldsymbol{x} \in \mathbb{R}^3 \setminus S_a,$$

so that by assumption, $\boldsymbol{\gamma}(\widetilde{\boldsymbol{F}})|_\pm = (\mathbf{B}\boldsymbol{g})(\boldsymbol{x}) = \mathbf{0}$ on Γ (same limit from both sides).

The function $\widetilde{\boldsymbol{F}}(\boldsymbol{x})$ satisfies

$$\nabla \times (\nabla \times \widetilde{\boldsymbol{F}}(\boldsymbol{x})) - k^2 \widetilde{\boldsymbol{F}}(\boldsymbol{x}) = \mathbf{0}, \quad \boldsymbol{x} \in \mathbb{R}^3 \setminus S_a.$$

Moreover, the function satisfies the appropriate radiation condition at infinity and $\boldsymbol{\gamma}(\widetilde{\boldsymbol{F}})|_+ = \mathbf{0}$ on Γ. The uniqueness of the exterior scattering problem (Problem (E)), $\widetilde{\boldsymbol{F}}(\boldsymbol{x}) = \mathbf{0}$, $\boldsymbol{x} \in \Omega_{\mathrm{e}}$, and by analyticity, $\widetilde{\boldsymbol{F}} = \mathbf{0}$ also outside S_a.

As above, by Theorem 2.3, the curl of $\widetilde{\boldsymbol{F}}$ has a continuous tangential component at S_a. The interior problem is uniquely solvable, since $\psi_l(ka) \neq 0$ and $\psi_l'(ka) \neq 0$ for all $l = 1, 2, \ldots$, which implies that $\widetilde{\boldsymbol{F}}(\boldsymbol{x}) = \mathbf{0}$, $\boldsymbol{x} \in B(a)$. The tangential components of $\widetilde{\boldsymbol{F}}(\boldsymbol{x})$ have a jump discontinuity on S_a, Theorem 2.3.

$$\mathbf{0} = \hat{\boldsymbol{x}} \times \widetilde{\boldsymbol{F}}(\boldsymbol{x})\Big|_+ - \hat{\boldsymbol{x}} \times \widetilde{\boldsymbol{F}}(\boldsymbol{x})\Big|_- = \boldsymbol{g}(\boldsymbol{x}), \quad \boldsymbol{x} \in S_a.$$

This proves the injectivity of the operator $\mathbf{B}$, and, consequently, that the operator $\mathbf{A}_a$ has a dense range, since $N(\mathbf{A}_a^\dagger) = R(\mathbf{A}_a)^\perp$ [6, p. 241]. □

Lemma 4.6. *The expansion coefficients $e_{\tau n}$ and $h_{\tau n}$, see* (4.2), (4.3), *and* (4.4), *are related by*

$$\sum_{\tau' n'} A_{\overline{\tau} n, \overline{\tau}' n'} h_{\tau' n'} = \sum_{\tau' n'} A_{\tau n, \tau' n'} e_{\tau' n'}, \tag{4.9}$$

where the dimensionless matrix $A_{\tau n, \tau' n'}$ is defined as

$$A_{\tau n, \tau' n'} = k \int_\Gamma \boldsymbol{u}_{\tau n} \cdot \boldsymbol{Y}_{\tau' n'}\ \mathrm{d}S. \tag{4.10}$$

The bar over the index τ denotes the dual index in τ ($\overline{1} = 2$ and $\overline{2} = 1$).

Here $\boldsymbol{u}_{\tau n}(k\boldsymbol{x})$ are the radiating spherical vector waves, defined by

$$\begin{cases} \boldsymbol{u}_{1n}(k\boldsymbol{x}) = x h_l^{(1)}(kx) \boldsymbol{Y}_{1n}(\boldsymbol{x}) \\ \boldsymbol{u}_{2n}(k\boldsymbol{x}) = \dfrac{1}{k} \nabla \times \left(x h_l^{(1)}(kx) \boldsymbol{Y}_{1n}(\boldsymbol{x}) \right), \end{cases} \tag{4.11}$$

where $h_l^{(1)}(kx)$ is the spherical Hankel function of the first kind [23], see also Appendix B. The matrix $A_{\tau n, \tau' n'}$ plays a central role in the procedure of calculating the norm of the exterior Calderón operator and it deserves a thorough study. This is done in Proposition 4.1 and Theorem 4.2 below.

Proof. The extinction part of Theorem 2.1 reads

$$\begin{aligned}
&\nabla\times\int_\Gamma g(k,|\boldsymbol{x}-\boldsymbol{x}'|)\boldsymbol{\gamma}(\boldsymbol{H}_{\mathrm{sc}})(\boldsymbol{x}')\,\mathrm{d}S'\\
&=-\frac{1}{\mathrm{i}k}\nabla\times\left\{\nabla\times\int_\Gamma g(k,|\boldsymbol{x}-\boldsymbol{x}'|)\boldsymbol{\gamma}(\boldsymbol{E}_{\mathrm{sc}})(\boldsymbol{x}')\,\mathrm{d}S'\right\},\quad \boldsymbol{x}\in\Omega.
\end{aligned}$$

Introduce the Green dyadic for the electric field in free space [17]

$$\mathbf{G}_{\mathrm{e}}(k,\boldsymbol{x}-\boldsymbol{x}')=\left(\mathbf{I}_3+\frac{1}{k^2}\nabla\nabla\right)g(k,|\boldsymbol{x}-\boldsymbol{x}'|)=\left(\mathbf{I}_3+\frac{1}{k^2}\nabla'\nabla'\right)g(k,|\boldsymbol{x}-\boldsymbol{x}'|),$$

where $\mathbf{I}_3$ is the unit dyadic in $\mathbb{R}^3$. Consequently, the extinction part is

$$\begin{aligned}
&\nabla\times\int_\Gamma \mathbf{G}_{\mathrm{e}}(k,\boldsymbol{x}-\boldsymbol{x}')\cdot\boldsymbol{\gamma}(\boldsymbol{H}_{\mathrm{sc}})(\boldsymbol{x}')\,\mathrm{d}S'\\
&=-\frac{1}{\mathrm{i}k}\nabla\times\left\{\nabla\times\int_\Gamma \mathbf{G}_{\mathrm{e}}(k,\boldsymbol{x}-\boldsymbol{x}')\cdot\boldsymbol{\gamma}(\boldsymbol{E}_{\mathrm{sc}})(\boldsymbol{x}')\,\mathrm{d}S'\right\},\quad \boldsymbol{x}\in\Omega.
\end{aligned}\tag{4.12}$$

In fact, the curl on $\mathbf{G}_{\mathrm{e}}(k,\boldsymbol{x}-\boldsymbol{x}')$ gives $\nabla\times\mathbf{G}_{\mathrm{e}}(k,\boldsymbol{x}-\boldsymbol{x}')=\nabla\times(\mathbf{I}_3 g(k,|\boldsymbol{x}-\boldsymbol{x}'|))$, which verifies (4.12).

The Green dyadic for the electric field is [17, (7.24) on p. 370]

$$\begin{aligned}
\mathbf{G}_{\mathrm{e}}(k,\boldsymbol{x}-\boldsymbol{x}')&=\mathrm{i}k\sum_{\tau n}\boldsymbol{v}^*_{\tau n}(k\boldsymbol{x}_<)\boldsymbol{u}_{\tau n}(k\boldsymbol{x}_>)\\
&=\mathrm{i}k\sum_{\tau n}\boldsymbol{u}_{\tau n}(k\boldsymbol{x}_>)\boldsymbol{v}^*_{\tau n}(k\boldsymbol{x}_<),\quad x\neq x',
\end{aligned}\tag{4.13}$$

where $\boldsymbol{x}_<$ ($\boldsymbol{x}_>$) is the position vector with the smallest (largest) distance to the origin, *i.e.*, if $x<x'$ then $\boldsymbol{x}_<=\boldsymbol{x}$ and $\boldsymbol{x}_>=\boldsymbol{x}'$. The definition of the spherical vector waves is given in Appendix B, and a star $*$ denotes complex conjugate. This expansion is uniformly convergent in compact (bounded and closed) domains, provided $x\neq x'$ in the domain [15, 20].

Apply (4.13) to (4.12) for an $\boldsymbol{x}$ inside the inscribed sphere of Γ and use the dual property of the spherical vector waves, *i.e.*,

$$\nabla\times\boldsymbol{v}_{\tau n}(k\boldsymbol{x})=k\boldsymbol{v}_{\bar{\tau}n}(k\boldsymbol{x}),\qquad \nabla\times\boldsymbol{u}_{\tau n}(k\boldsymbol{x})=k\boldsymbol{u}_{\bar{\tau}n}(k\boldsymbol{x}).$$

We get

$$\begin{aligned}
&\mathrm{i}k^2\sum_{\tau n}\boldsymbol{v}^*_{\tau n}(k\boldsymbol{x})\int_\Gamma \boldsymbol{u}_{\bar{\tau}n}(k\boldsymbol{x}')\cdot\boldsymbol{\gamma}(\boldsymbol{H}_{\mathrm{sc}})(\boldsymbol{x}')\,\mathrm{d}S'\\
&=-k^2\sum_{\tau n}\boldsymbol{v}^*_{\tau n}(k\boldsymbol{x})\int_\Gamma \boldsymbol{u}_{\tau n}(k\boldsymbol{x}')\cdot\boldsymbol{\gamma}(\boldsymbol{E}_{\mathrm{sc}})(\boldsymbol{x}')\,\mathrm{d}S',\quad \boldsymbol{x}\in\Omega.
\end{aligned}$$

Orthogonality of the vector spherical harmonics on the inscribed sphere implies

$$\int\limits_\Gamma \boldsymbol{u}_{\overline{\tau}n} \cdot \boldsymbol{\gamma}(\boldsymbol{H}_{\rm sc})\ {\rm d}S = {\rm i} \int\limits_\Gamma \boldsymbol{u}_{\tau n} \cdot \boldsymbol{\gamma}(\boldsymbol{E}_{\rm sc})\ {\rm d}S, \quad \forall n,\ \tau = 1,2. \tag{4.14}$$

Insert the expansion of the field in their Fourier series, (4.2) and (4.3), and we obtain

$$\sum_{\tau' n'} h_{\tau' n'} \int\limits_\Gamma \boldsymbol{u}_{\overline{\tau}n} \cdot \boldsymbol{Y}_{\overline{\tau}' n'}\ {\rm d}S = \sum_{\tau' n'} e_{\tau' n'} \int\limits_\Gamma \boldsymbol{u}_{\tau n} \cdot \boldsymbol{Y}_{\tau' n'}\ {\rm d}S, \quad \forall n,\ \tau = 1,2,$$

which is identical to the statement in the lemma. □

Remark 4.4. *Equation* (4.14) *in Lemma 4.6 allows a simple proof of Item 2 of Theorem 2.2.*

Integration by parts gives an alternative form of the matrix $A_{\tau n,\tau' n'}$, see (4.10) and use Definition 3.1.

$$A_{\tau n,1n'} = \frac{1}{\sqrt{\lambda_{n'}}} \int\limits_\Gamma \left(\mathrm{curl}_\Gamma \boldsymbol{\pi}(\boldsymbol{u}_{\tau n})\right) Y_{n'}\ {\rm d}S, \quad \forall n,\ \tau = 1,2,$$

and

$$A_{\tau n,2n'} = -\frac{1}{\sqrt{\lambda_{n'}}} \int\limits_\Gamma \left(\mathrm{div}_\Gamma \boldsymbol{\pi}(\boldsymbol{u}_{\tau n})\right) Y_{n'}\ {\rm d}S, \quad \forall n,\ \tau = 1,2.$$

Proposition 4.1. *The mapping*

$$a_{\tau n} \mapsto \sum_{\tau' n'} A_{\tau n,\tau' n'} a_{\tau' n'},$$

is injective, where the matrix $A_{\tau n,\tau' n'}$ is defined in (4.10)*.*

Proof. We prove the proposition by showing

$$\sum_{\tau' n'} A_{\tau n,\tau' n'} a_{\tau' n'} = 0, \quad \forall n,\ \tau = 1,2,$$

implies that $a_{\tau n} = 0$ for $\tau = 1,2$ and all n.

Multiply this relation with $\boldsymbol{v}^*_{\tau n}(k\boldsymbol{x})$, where $\boldsymbol{x}$ lies inside the inscribed sphere of the scatterer, and sum over τ and n. We obtain, see (4.13)

$$\frac{1}{{\rm i}k} \int\limits_\Gamma \mathbf{G}_{\rm e}(k, \boldsymbol{x} - \boldsymbol{x}') \cdot \boldsymbol{a}(\boldsymbol{x}')\ {\rm d}S' = \boldsymbol{0}, \quad \forall \boldsymbol{x} \text{ inside the inscribed sphere},$$

where

$$\boldsymbol{a} = \sum_{\tau n} a_{\tau n} \boldsymbol{Y}_{\tau n}.$$

Now consider the vector-valued function

$$\boldsymbol{A}(\boldsymbol{x}) = \int_\Gamma \mathbf{G}_\mathrm{e}(k, \boldsymbol{x} - \boldsymbol{x}') \cdot \boldsymbol{a}(\boldsymbol{x}') \, \mathrm{d}S', \quad \boldsymbol{x} \in \mathbb{R}^3 \setminus \Gamma,$$

which is defined everywhere in $\mathbb{R}^3 \setminus \Gamma$. This function is, by definition, zero inside the inscribed sphere of the scatterer. By analyticity, the function $\boldsymbol{A}(\boldsymbol{x}) = \mathbf{0}$ for all $\boldsymbol{x} \in \Omega$ [24]. As a consequence, the traces $\boldsymbol{\gamma}(\boldsymbol{A})|_- = \mathbf{0}$ and $\boldsymbol{\gamma}(\nabla \times \boldsymbol{A})|_- = \mathbf{0}$.

The vector field $\boldsymbol{A}(\boldsymbol{x})$ satisfies

$$\nabla \times (\nabla \times \boldsymbol{A}(\boldsymbol{x})) - k^2 \boldsymbol{A}(\boldsymbol{x}) = \mathbf{0}, \quad \boldsymbol{x} \in \mathbb{R}^3 \setminus \Gamma.$$

Moreover, $\boldsymbol{A}(\boldsymbol{x})$ satisfies the correct radiation conditions at infinity. Due to unique solvability of the exterior problem, $\boldsymbol{A}(\boldsymbol{x}) = \mathbf{0}$ in the entire exterior region Ω_e. As a consequence, the traces $\boldsymbol{\gamma}(\boldsymbol{A})|_+ = \mathbf{0}$ and $\boldsymbol{\gamma}(\nabla \times \boldsymbol{A})|_+ = \mathbf{0}$.

The curl of $\boldsymbol{A}(\boldsymbol{x})$ is

$$\boldsymbol{F}(\boldsymbol{x}) = \nabla \times \boldsymbol{A}(\boldsymbol{x}) = -\int_\Gamma \nabla' g(k, |\boldsymbol{x} - \boldsymbol{x}'|) \times \boldsymbol{a}(\boldsymbol{x}') \, \mathrm{d}S', \quad \boldsymbol{x} \in \mathbb{R}^3 \setminus \Gamma,$$

The trace of $\boldsymbol{F}(\boldsymbol{x})$ has a jump discontinuity on Γ, see Theorem 2.3.

$$\mathbf{0} = \boldsymbol{\gamma}(\nabla \times \boldsymbol{A})|_+ - \boldsymbol{\gamma}(\nabla \times \boldsymbol{A})|_- = \boldsymbol{\gamma}(\boldsymbol{F})|_+ - \boldsymbol{\gamma}(\boldsymbol{F})|_- = \boldsymbol{a}, \quad \boldsymbol{x} \in \Gamma.$$

and consequently, by orthogonality of the vector generalized harmonics, $a_{\tau n} = 0$, which implies the injectivity of the mapping above. □

To simplify the analysis in the theorem below, we introduce a special notation for the matrix with dual τ indices. To this end, define the matrix

$$\overline{A}_{\tau n, \tau' n'} = A_{\overline{\tau} n, \overline{\tau'} n'},$$

Theorem 4.2. *The exterior Calderón matrix C can be approximated by*

$$C^\alpha_{\tau n, \tau' n'} = \sum_{\tau'' n''} \sum_{\tau''' n'''} (\alpha I + \overline{A}^\dagger \overline{A})^{-1}_{\tau n, \tau'' n''} \overline{A}^*_{\tau''' n''', \tau'' n''} A_{\tau''' n''', \tau' n'},$$

for adequately small $\alpha > 0$, where $\dagger$ denotes the Hermitian conjugated matrix. In shorthand matrix notation $C^\alpha = (\alpha I + \overline{A}^\dagger \overline{A})^{-1} \overline{A}^\dagger A$.

Proof. The expansion coefficients $e_{\tau n}$ and $h_{\tau n}$ are related by, see (4.9)

$$\sum_{\tau' n'} \overline{A}_{\tau n, \tau' n'} h_{\tau' n'} = \sum_{\tau' n'} A_{\tau n, \tau' n'} e_{\tau' n'}, \tag{4.15}$$

This equation consists of a countable set of linear equations, the solution of which may be used to express $h_{\tau n}$ in terms of $e_{\tau n}$, thus providing a matrix form representation of the exterior Calderón operator in terms of the chosen basis of generalized harmonics. Assuming the invertibility of the matrix $\overline{A}_{\tau n,\tau' n'}$, we write the equation as, see Appendix F

$$h_{\tau n} = \sum_{\tau' n} \sum_{\tau'' n''} \overline{A}^{-1}_{\tau n,\tau'' n''} A_{\tau'' n'',\tau' n'} e_{\tau' n'},$$

so that $\mathbf{C}^{\mathrm{e}}$ admits the matrix representation

$$C_{\tau n,\tau' n'} = \sum_{\tau' n} \sum_{\tau'' n''} \overline{A}^{-1}_{\tau n,\tau'' n''} A_{\tau'' n'',\tau' n'},$$

In shorthand matrix notation $C = \overline{A}^{-1} A$, where C is the exterior Calderón matrix.

However, by the definition of the matrix operator $\overline{A}$ and the connection of the spherical vector waves $\boldsymbol{u}_{\tau,n}$ with the Green dyadic for the electric field, see left-hand side of (4.12) and (4.7), we see that A, and therefore also $\overline{A}$, is related to a compact operator; hence $\overline{A}$ is not expected, in general, to be invertible and, even if it were, it would lead to an ill-posed problem which could not provide a well defined numerical scheme.

We may, however, resort to a Tikhonov regularization approach of the solution of (4.15), which leads to a, well-suited for numerical approaches, approximation of the exterior Calderón operator. According to the theory of the Tikhonov regularization, see [16, Ch. 16], the regularized approximate solution of (4.15) is

$$h^{\alpha}_{\tau n} = \sum_{\tau'' n''} \sum_{\tau''' n'''} (\alpha I + \overline{A}^{\dagger} \overline{A})^{-1}_{\tau n,\tau'' n''} \overline{A}^{*}_{\tau''' n''',\tau'' n''} A_{\tau''' n''',\tau' n'} e_{\tau' n'}, \quad \alpha > 0,$$

or in shorthand matrix notation $h^{\alpha} = (\alpha I + \overline{A}^{\dagger}\overline{A})^{-1}\overline{A}^{\dagger} A e$, which leads to an approximation of C by C^{α}, where

$$C^{\alpha} := \sum_{\tau'' n''} \sum_{\tau''' n'''} (\alpha I + \overline{A}^{\dagger} \overline{A})^{-1}_{\tau n,\tau'' n''} \overline{A}^{*}_{\tau''' n''',\tau'' n''} A_{\tau''' n''',\tau' n'}, \quad \alpha > 0,$$

or in shorthand matrix notation $C^{\alpha} = (\alpha I + \overline{A}^{\dagger}\overline{A})^{-1}\overline{A}^{\dagger} A$. The invertibility of the matrix $\alpha I + \overline{A}^{\dagger}\overline{A}$ is easily obtained by the Lax-Milgram Lemma, since the regularization term αI introduces coercivity into the problem and the numerical inversion can be performed in terms of a variational approach related to the minimization problem

$$\min_{z \in \boldsymbol{\ell}^{-1/2}(\mathrm{div})} \alpha \|z\|^2_{\boldsymbol{\ell}^{-1/2}(\mathrm{div})} + \langle \overline{A}^{\dagger} \overline{A}\, z, z \rangle_{\boldsymbol{\ell}^{-1/2}(\mathrm{div})}.$$

□

The behavior as $\alpha \to 0$ follows the general case setting of [16, Chap. 16].

4.4 The finite dimensional problem

This section contains a generalization of the result presented in [13] for a spherical surface to a general surface Γ. Denote

$$S_N = \left\{ \boldsymbol{f}_N : \boldsymbol{f}_N = \sum_{\tau n}^{N} a_{\tau n} \boldsymbol{Y}_{\tau n}, a_{\tau n} = \langle \boldsymbol{f}, \boldsymbol{Y}_{\tau n} \rangle_{TL^2(\Gamma)} \right\}.$$

We define the orthogonal projection $\mathbf{P}_N : H^{-1/2}(\mathrm{div},\Gamma) \to H^{-1/2}(\mathrm{div},\Gamma)$ where $\boldsymbol{f} \mapsto \boldsymbol{f}_N = \mathbf{P}_N\boldsymbol{f}$ in the $H^{-1/2}(\mathrm{div},\Gamma)$ inner product.

The following proposition holds:

Proposition 4.2.

$$\mathbf{P}_N\boldsymbol{f} \to \boldsymbol{f} \text{ in } H^{-1/2}(\mathrm{div},\Gamma) \text{ as } N \to \infty,$$

and

$$\|(I-\mathbf{P}_N)\boldsymbol{f}\|_{H^{-1/2}(\mathrm{div},\Gamma)} \le \lambda_N^{-(s+1/2)/2}\|\boldsymbol{f}\|_{H^s(\mathrm{div},\Gamma)},$$

holds for any $s \ge -1/2$, *where*

$$\|\boldsymbol{f}\|^2_{H^s(\mathrm{div},\Gamma)} = \sum_{\tau n}(1+\lambda_n)^{s+\tau-1}\left|a_{\tau n}\right|^2.$$

Proof. The convergence

$$\mathbf{P}_N\boldsymbol{f} \to \boldsymbol{f} \text{ in } H^{-1/2}(\mathrm{div},\Gamma) \text{ as } N \to \infty,$$

is a consequence of the generalized Fourier transform properties.

We estimate for every $s \ge -1/2$

$$\begin{aligned}\|(I-\mathbf{P}_N)\boldsymbol{f}\|^2_{H^{-1/2}(\mathrm{div},\Gamma)} &= \sum_{\substack{n>N\\ \tau=1,2}}(1+\lambda_n)^{\tau-3/2}\left|a_{\tau n}\right|^2 \\ &= \sum_{\substack{n>N\\ \tau=1,2}}(1+\lambda_n)^{-s-1/2}(1+\lambda_n)^{s+\tau-1}\left|a_{\tau n}\right|^2 \\ &\le (1+\lambda_N)^{-s-1/2}\sum_{\substack{n>N\\ \tau=1,2}}(1+\lambda_n)^{s+\tau-1}\left|a_{\tau n}\right|^2 \le \lambda_N^{-s-1/2}\|\boldsymbol{f}\|^2_{H^s(\mathrm{div},\Gamma)}.\end{aligned}$$

□

Remark 4.5. *The analysis can be extended for the case of non-simply-connected surfaces* Γ, *by extending the proposed orthonormal basis with the finite-dimensional basis of the kernel of the Laplace-Beltrami operator on* Γ, *see [21, p. 206].*

5 Connection to the transition matrix for a PEC obstacle

Scattering by a perfectly conducting obstacle (PEC) with bounding surface Γ is related to the exterior Calderón operator $\mathbf{C}^{\mathrm{e}}$. This section develops and clarifies this connection.

The transition matrix (T-matrix), $T_{\tau n,\tau' n'}$, connects the expansion coefficients of the incident field $\boldsymbol{E}_{\mathrm{inc}}$, with sources in Ω_{e} and the scattering $\boldsymbol{E}_{\mathrm{sc}}$ in terms of the regular spherical vector waves, $\boldsymbol{v}_{\tau n}(k\boldsymbol{x})$, and the radiating spherical vector waves,

$\boldsymbol{u}_{\tau n}(k\boldsymbol{x})$, respectively. The definition of the spherical vector waves is given in Appendix B. Specifically,

$$\boldsymbol{E}_{\mathrm{inc}}(\boldsymbol{x}) = \sum_{\tau n} a_{\tau n} \boldsymbol{v}_{\tau n}(k\boldsymbol{x}), \qquad \boldsymbol{E}_{\mathrm{sc}}(\boldsymbol{x}) = \sum_{\tau n} f_{\tau n} \boldsymbol{u}_{\tau n}(k\boldsymbol{x}),$$

where the regular and radiating spherical vector waves, $\boldsymbol{v}_{\tau n}$ and $\boldsymbol{u}_{\tau n}$, are defined in (4.8) and (4.11), respectively, see also Appendix B, and where the expansion coefficients $f_{\tau n}$ and $a_{\tau n}$ are related as

$$f_{\tau n} = \sum_{\tau' n'} T_{\tau n, \tau' n'} a_{\tau' n'}.$$

The expansion of the incident field is absolutely convergent, at least, inside the inscribed sphere of the PEC obstacle,[8] and the expansion of the scattered field converges, at least, outside the circumscribed sphere of the PEC obstacle. The transition matrix completely characterizes the scattering process.

The following theorem shows that when the exterior Calderón operator is known, the transition matrix for a PEC obstacle is obtained by some simple operations:

Theorem 5.1. *The transition matrix for a PEC obstacle, $T_{\tau n,\tau' n'}$, with bounding surface Γ and the corresponding exterior Calderón matrix, $C_{\tau n,\tau' n'}$, is:*

$$T_{\tau n,\tau' n'} = \mathrm{i} \sum_{\tau'' n''} \left\{ W_{\tau n,\tau'' n''} V_{\overline{\tau'} n', \overline{\tau''} n''} + V_{\tau' n',\tau'' n''} \sum_{\tau''' n'''} C_{\overline{\tau'''} n''', \overline{\tau''} n''} W_{\tau n, \tau''' n'''} \right\},$$

where the dimensionless matrices $W_{\tau n,\tau' n'}$ and $V_{\tau n,\tau' n'}$ are

$$W_{\tau n,\tau' n'} = k \int_\Gamma \boldsymbol{v}^*_{\tau n} \cdot \boldsymbol{Y}_{\tau' n'} \, \mathrm{d}S, \qquad V_{\tau n,\tau' n'} = k \int_\Gamma \boldsymbol{\gamma}(\boldsymbol{v}_{\tau n}) \cdot \boldsymbol{Y}^*_{\overline{\tau'} n'} \, \mathrm{d}S.$$

Notice that $W_{\tau n,\tau' n'}$ and $V_{\tau n,\tau' n'}$ are related, *i.e.*, $V_{\tau n,\tau' n'} = (-1)^{\tau+1} W^*_{\tau n,\tau' n'}$.

Proof. For a given incident field $\boldsymbol{E}_{\mathrm{inc}}$, the boundary condition on the surface Γ is $\boldsymbol{\gamma}(\boldsymbol{E}_{\mathrm{inc}} + \boldsymbol{E}_{\mathrm{sc}}) = \boldsymbol{0}$, which implies

$$\boldsymbol{\gamma}(\boldsymbol{E}_{\mathrm{sc}}) = -\boldsymbol{\gamma}(\boldsymbol{E}_{\mathrm{inc}}).$$

The trace of the scattered magnetic field on Γ is

$$\boldsymbol{\gamma}(\boldsymbol{H}) = \boldsymbol{\gamma}(\boldsymbol{H}_{\mathrm{inc}} + \boldsymbol{H}_{\mathrm{sc}}) = \boldsymbol{\gamma}(\boldsymbol{H}_{\mathrm{inc}}) - \mathbf{C}^{\mathrm{e}}(\boldsymbol{\gamma}(\boldsymbol{E}_{\mathrm{inc}})).$$

The expansion coefficients of the scattered electric field for a PEC surface, $f_{\tau n}$, are [17, (9.3) on p. 481]

$$f_{\tau n} = -k^2 \int_\Gamma \boldsymbol{v}^*_{\tau n} \cdot \boldsymbol{\gamma}(\boldsymbol{H}) \, \mathrm{d}S = -k^2 \int_\Gamma \boldsymbol{v}^*_{\tau n} \cdot \{\boldsymbol{\gamma}(\boldsymbol{H}_{\mathrm{inc}}) - \mathbf{C}^{\mathrm{e}}(\boldsymbol{\gamma}(\boldsymbol{E}_{\mathrm{inc}}))\} \, \mathrm{d}S.$$

[8] More precisely, the convergence is guaranteed inside the largest inscribable ball not including the sources of the incident field.

Inserting the expansions of the incident fields, we obtain an explicit form of the transition matrix, *viz.*,

$$T_{\tau n,\tau' n'} = k^2 \int_\Gamma \boldsymbol{v}^*_{\tau n} \cdot \{\mathrm{i}\boldsymbol{\gamma}(\boldsymbol{v}_{\overline{\tau'} n'}) + \mathbf{C}^{\mathrm{e}}(\boldsymbol{\gamma}(\boldsymbol{v}_{\tau' n'}))\}\ \mathrm{d}S,$$

where we also used the explicit form of the trace of the incident magnetic and electric fields

$$\boldsymbol{H}_{\mathrm{inc}}(\boldsymbol{x}) = -\mathrm{i}\sum_{\tau n} a_{\tau n}\boldsymbol{v}_{\overline{\tau} n}(k\boldsymbol{x}), \quad \boldsymbol{E}_{\mathrm{inc}}(\boldsymbol{x}) = \sum_{\tau n} a_{\tau n}\boldsymbol{v}_{\tau n}(k\boldsymbol{x}).$$

The regular spherical vector wave $\boldsymbol{\gamma}(\boldsymbol{v}_{\tau n})$ has a Fourier series expansion in $\boldsymbol{Y}_{\overline{\tau} n}$.

$$k\boldsymbol{\gamma}(\boldsymbol{v}_{\tau n}) = \sum_{\tau' n'} V_{\tau n,\tau' n'}\boldsymbol{Y}_{\overline{\tau'} n'}, \qquad V_{\tau n,\tau' n'} = k\int_\Gamma \boldsymbol{\gamma}(\boldsymbol{v}_{\tau n}) \cdot \boldsymbol{Y}^*_{\overline{\tau'} n'}\ \mathrm{d}S,$$

and (4.6) yields

$$\mathbf{C}^{\mathrm{e}}\left(\boldsymbol{Y}_{\tau n}\right) = \mathrm{i}\sum_{\tau'' n''} C_{\tau'' n'',\tau n}\boldsymbol{Y}_{\overline{\tau''} n''}.$$

Combine these expansions

$$k\mathbf{C}^{\mathrm{e}}\left(\boldsymbol{\gamma}(\boldsymbol{v}_{\tau n})\right) = \mathrm{i}\sum_{\tau' n',\tau'' n''} V_{\tau n,\tau' n'}C_{\tau'' n'',\overline{\tau'} n'}\boldsymbol{Y}_{\overline{\tau''} n''} = \mathrm{i}\sum_{\tau' n',\tau'' n''} V_{\tau n,\tau' n'}C_{\overline{\tau''} n'',\overline{\tau'} n'}\boldsymbol{Y}_{\tau'' n''}.$$

These expressions lead to

$$T_{\tau n,\tau' n'} = \mathrm{i}k\sum_{\tau'' n''}\int_\Gamma \boldsymbol{v}^*_{\tau n} \cdot \left\{ V_{\overline{\tau'} n',\overline{\tau''} n''}\boldsymbol{Y}_{\tau'' n''} + V_{\tau' n',\tau'' n''}\sum_{\tau''' n'''} C_{\overline{\tau'''} n''',\overline{\tau''} n''}\boldsymbol{Y}_{\tau''' n'''} \right\}\ \mathrm{d}S.$$

If we denote

$$W_{\tau n,\tau' n'} = k\int_\Gamma \boldsymbol{v}^*_{\tau n} \cdot \boldsymbol{Y}_{\tau' n'}\ \mathrm{d}S,$$

we get in matrix notation

$$T_{\tau n,\tau' n'} = \mathrm{i}\sum_{\tau'' n''}\left\{ W_{\tau n,\tau'' n''}V_{\overline{\tau'} n',\overline{\tau''} n''} + V_{\tau' n',\tau'' n''}\sum_{\tau''' n'''} C_{\overline{\tau'''} n''',\overline{\tau''} n''}W_{\tau n,\tau''' n'''} \right\},$$

which proves the theorem. □

6 The spherical geometry — an explicit example

The spherical geometry is well-known and, so far, the only known geometry, where we can test the theory analytically. In this section, we apply the results above to a sphere of radius r. The eigenvalues for the sphere are[9] $\lambda_n = l(l+1)/(kr)^2$, and the vector spherical harmonics $\boldsymbol{Y}_{\tau n}(\boldsymbol{x})$, see [17] and Appendix B.

For the sphere, the matrix A is diagonal. Specifically,

$$A_{\tau n,\tau' n'} = \delta_{nn'}\delta_{\tau\tau'} \begin{cases} \xi_l(kr), & \tau = 1 \\ \xi_l'(kr), & \tau = 2, \end{cases}$$

and

$$C_{\tau n,\tau' n'} = \delta_{nn'}\delta_{\tau\tau'} \begin{cases} \dfrac{\xi_l(kr)}{\xi_l'(kr)}, & \tau = 1 \\ \dfrac{\xi_l'(kr)}{\xi_l(kr)}, & \tau = 2, \end{cases}$$

where $\xi_l(z) = zh_l^{(1)}(z)$ is the Riccati-Hankel function [17, 22]. Notice the result of Lemma 4.3, *i.e.*,

$$C^{-1}_{\tau n,\tau' n'} = C_{\overline{\tau} n,\overline{\tau'} n'}.$$

Moreover,

$$P_{\tau n,\tau' n'} = \delta_{nn'}\delta_{\tau\tau'} \begin{cases} (1+\lambda_n)\left|\dfrac{\xi_l(kr)}{\xi_l'(kr)}\right|^2, & \tau = 1 \\ (1+\lambda_n)^{-1}\left|\dfrac{\xi_l'(kr)}{\xi_l(kr)}\right|^2, & \tau = 2, \end{cases}$$

which is, apart from a different normalization, in agreement with [18], see Figure 3.

The static limit of the exterior Calderón operator for a spherical geometry is of interest. We have

$$\lim_{kr\to 0} P_{\tau n,\tau' n'} = \delta_{nn'}\delta_{\tau\tau'} \begin{cases} \dfrac{l+1}{l}, & \tau = 1 \\ \dfrac{l}{l+1}, & \tau = 2, \end{cases}$$

and consequently $\lim_{kr\to 0}\|\mathbf{C}^{\mathrm{e}}\|_{H^{-1/2}(\mathrm{div},\partial B_r)} = \sqrt{2}$.

We can also check the validity of Lemma 4.4.

$$(-1)^\tau C_{\tau n,\tau' n'} = \delta_{nn'}\delta_{\tau\tau'} \begin{cases} -\dfrac{\xi_l(kr)}{\xi_l'(kr)}, & \tau = 1 \\ \dfrac{\xi_l'(kr)}{\xi_l(kr)}, & \tau = 2. \end{cases}$$

[9] We here adopt the standard indexing of the eigenvalues λ_n of the spherical harmonics, where $n = \{l, m\}$, $l = 1, 2, \ldots$, $m = -l, -l+1, \ldots, l-1, l$.

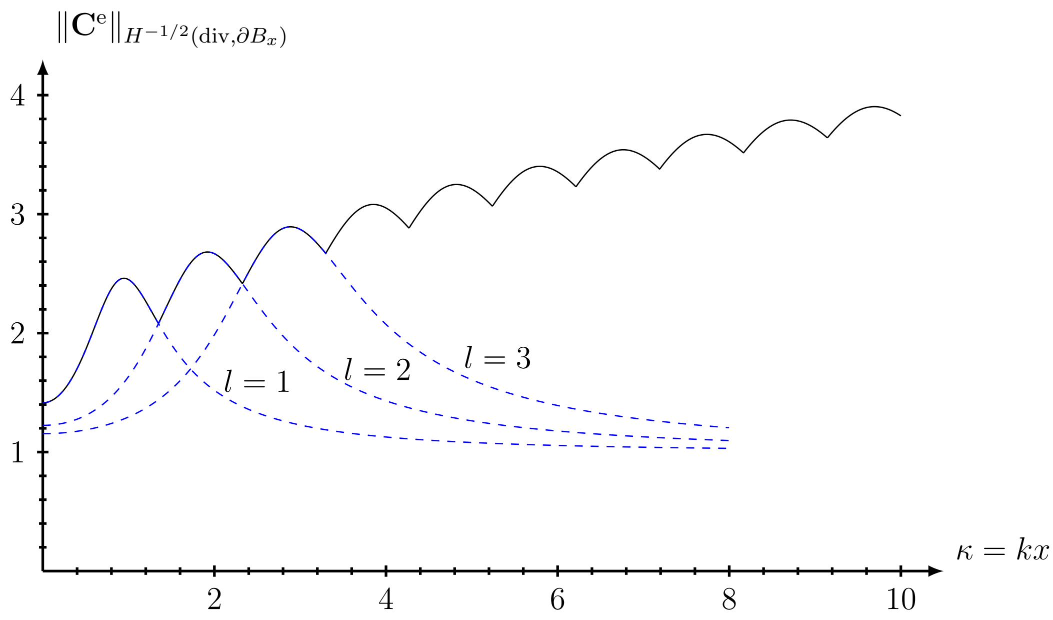

Figure 3: The norm of the exterior Calderón operator $\|\mathbf{C}^{\mathrm{e}}\|_{H^{-1/2}(\mathrm{div},\partial B_x)}$ for a sphere of radius x is depicted. The dashed blue lines depict the function $P_{1l,1l}$ for $l = 1, 2, 3$.

Therefore,

$$\frac{1}{2\mathrm{i}}\left\{(-1)^{\tau} C_{\tau n,\tau' n'} - (-1)^{\tau'} C^{*}_{\tau' n',\tau n}\right\} = \delta_{nn'}\delta_{\tau\tau'}\begin{cases} -\operatorname{Im}\dfrac{\xi_l(kr)}{\xi_l'(kr)} \\ \operatorname{Im}\dfrac{\xi_l'(kr)}{\xi_l(kr)} \end{cases}$$

$$= -\mathrm{i}\delta_{nn'}\delta_{\tau\tau'}\begin{cases} -\dfrac{\xi_l(kr)\psi_l'(kr) - \xi_l'(kr)\psi_l(kr)}{|\xi_l'(kr)|^2} \\ \dfrac{\xi_l'(kr)\psi_l(kr) - \xi_l(kr)\psi_l'(kr)}{|\xi_l(kr)|^2} \end{cases} = \delta_{nn'}\delta_{\tau\tau'}\begin{cases} \dfrac{1}{|\xi_l'(kr)|^2}, & \tau = 1 \\ \dfrac{1}{|\xi_l(kr)|^2}, & \tau = 2, \end{cases}$$

by the use of $\xi_l^*(kr) = 2\psi_l(kr) - \xi_l(kr)$ and the Wronskian for the Riccati-Bessel functions $\psi_l(z)\xi_l'(z) - \psi_l'(z)\xi_l(z) = \mathrm{i}$ [17]. Obviously, this matrix is positive definite.

We also illustrate the result in Theorem 5.1 with a sphere of radius r. From above, we have

$$C_{\tau n,\tau' n'} = \delta_{nn'}\delta_{\tau\tau'}\begin{cases} \dfrac{\xi_l(kr)}{\xi_l'(kr)}, & \tau = 1 \\ \dfrac{\xi_l'(kr)}{\xi_l(kr)}, & \tau = 2. \end{cases}$$

Moreover, we have

$$V_{\tau n,\tau' n'} = \delta_{nn'}\delta_{\tau\tau'}\begin{cases} \psi_l(kr), & \tau = 1 \\ -\psi_l'(kr), & \tau = 2, \end{cases}$$

and

$$W_{\tau n,\tau' n'} = \delta_{nn'}\delta_{\tau\tau'} \begin{cases} \psi_l(kr), & \tau = 1 \\ \psi_l'(kr), & \tau = 2. \end{cases}$$

where $\psi_l(z) = zj_l(z)$ is the Riccati-Bessel function [17, 22]. The transition matrix becomes

$$T_{\tau n,\tau' n'} = \mathrm{i}\delta_{nn'}\delta_{\tau\tau'} \begin{cases} -\psi_l(kr)\psi_l'(kr) + \psi_l(kr)\psi_l(kr)\dfrac{\xi_l'(kr)}{\xi_l(kr)}, & \tau = 1 \\ \psi_l(kr)\psi_l'(kr) - \psi_l'(kr)\psi_l'(kr)\dfrac{\xi_l(kr)}{\xi_l'(kr)}, & \tau = 2, \end{cases}$$

which by the use of the Wronskian for the Riccati-Bessel functions

$$\psi_l(z)\xi_l'(z) - \psi_l'(z)\xi_l(z) = \mathrm{i},$$

simplifies to

$$T_{\tau n,\tau' n'} = -\delta_{nn'}\delta_{\tau\tau'} \begin{cases} \dfrac{\psi_l(kr)}{\xi_l(kr)}, & \tau = 1 \\ \dfrac{\psi_l'(kr)}{\xi_l'(kr)}, & \tau = 2, \end{cases}$$

in agreement with the result of Mie scattering [17].

7 Conclusions

This paper deals with a novel approach to compute the exterior Calderón operator, and, in particular, the computation of its norm in the space $H^{-1/2}(\mathrm{div}, \Gamma)$. This operator is instrumental in the understanding of the scattering problem. The approach is constructive, and employs the eigenfunctions of the Beltrami-Laplace operator of the surface. These functions are intrinsic to the surface, and constitute the natural orthonormal set for a matrix representation of the operator. The norm of the operator is explicitly given as the largest eigenvalue of a quadratic form that contains this representation of the exterior Calderón operator. The paper is closed by an investigation of the connection between the exterior Calderón operator and the transition matrix of the same perfectly conducting surface. In a future paper, the numerical behavior of the suggested algorithm is intended to be conducted. The results of the present work can be used for treating different challenging problems, including a new natural coupling formulation between integral equations and finite elements, in the spirit of the results introduced by Ammari and Nédélec in [1]; see also [2].

Acknowledgements

To pursue this work, the Royal Physiographic Society of Lund, Sweden, has financially supported the stay for one of the authors (GK) in Athens. This support is gratefully acknowledged.

Appendix A Function spaces

In this appendix, we list the various function spaces used in this paper. Let Ω be an open, bounded domain in $\mathbb{R}^3$ with a piecewise smooth boundary $\partial\Omega$, see [4].

The space $C(\Omega)$ is the space of continuous functions in Ω. We also use $C_0(\overline{\Omega})$ which consists of all uniformly continuous functions, which are zero at the boundary. The space $C^\infty(\Omega)$ is the space of infinitely continuously differentiable functions in Ω, and $C_0^\infty(\Omega)$ are the functions in this space with compact support in Ω, which we also denote $D(\Omega)$.

Several function spaces with square integrable functions are used in this paper. The basic space is given by functions u ($\boldsymbol{x} \mapsto u(\boldsymbol{x})$) defined on $\Omega \subset \mathbb{R}^3 \to \mathbb{C}$

$$L^2(\Omega) \stackrel{\text{def}}{=} \left\{ u \text{ Lebesgue integrable in } \Omega, \int_\Omega |u|^2 \, \mathrm{d}v < \infty \right\},$$

with scalar product and norm

$$\langle u, v \rangle_{L^2(\Omega)} = \int_\Omega u\overline{v} \, \mathrm{d}v, \qquad \|u\|_{L^2(\Omega)} = \left\{ \int_\Omega |u|^2 \, \mathrm{d}v \right\}^{1/2},$$

where bar denotes the complex conjugate. Similarly for vector-valued spaces we have the scalar product

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{L^2(\Omega)} = \int_\Omega \boldsymbol{u} \cdot \overline{\boldsymbol{v}} \, \mathrm{d}v,$$

and the norm

$$\|\boldsymbol{u}\|_{L^2(\Omega)} = \left\{ \int_\Omega |\boldsymbol{u}|^2 \, \mathrm{d}v \right\}^{1/2},$$

where $\cdot$ and $|\cdot|$ denotes the Euclidean scalar product and norm in $\mathbb{C}^3$, respectively. We also define the function spaces

$$\begin{cases} H(\mathrm{div}, \Omega) \stackrel{\text{def}}{=} \left\{ \boldsymbol{u} \in L^2(\Omega) : \nabla \cdot \boldsymbol{u} \in L^2(\Omega) \right\} \\ H(\mathrm{curl}, \Omega) \stackrel{\text{def}}{=} \left\{ \boldsymbol{u} \in L^2(\Omega) : \nabla \times \boldsymbol{u} \in L^2(\Omega) \right\}, \end{cases}$$

which are Hilbert spaces with norms

$$\begin{cases} \|\boldsymbol{u}\|_{H(\mathrm{div},\Omega)} = \left(\|\boldsymbol{u}\|^2_{L^2(\Omega)} + \|\nabla \cdot \boldsymbol{u}\|^2_{L^2(\Omega)} \right)^{1/2} \\ \|\boldsymbol{u}\|_{H(\mathrm{curl},\Omega)} = \left(\|\boldsymbol{u}\|^2_{L^2(\Omega)} + \|\nabla \times \boldsymbol{u}\|^2_{L^2(\Omega)} \right)^{1/2}. \end{cases}$$

The curl and the divergence are defined in the weak sense as

$$\begin{cases} \langle \nabla \times \boldsymbol{u}, \boldsymbol{\phi} \rangle_{L^2(\Omega)} = \langle \boldsymbol{u}, \nabla \times \boldsymbol{\phi} \rangle_{L^2(\Omega)}, & \forall \boldsymbol{\phi} \in D(\Omega) \\ \langle \nabla \cdot \boldsymbol{u}, \phi \rangle_{L^2(\Omega)} = -\langle \boldsymbol{u}, \nabla \phi \rangle_{L^2(\Omega)}, & \forall \phi \in D(\Omega). \end{cases}$$

In the exterior region, we define spaces of locally integrable functions as

$$\begin{cases} H_{\text{loc}}(\text{div}, \overline{\Omega}_{\text{e}}) \stackrel{\text{def}}{=} \left\{ \boldsymbol{u} \in D'(\Omega_{\text{e}}) : \xi \boldsymbol{u} \in H(\text{div}, \Omega_{\text{e}}), \forall\, \xi \in D(\mathbb{R}^3) \right\} \\ H_{\text{loc}}(\text{curl}, \overline{\Omega}_{\text{e}}) \stackrel{\text{def}}{=} \left\{ \boldsymbol{u} \in D'(\Omega_{\text{e}}) : \xi \boldsymbol{u} \in H(\text{curl}, \Omega_{\text{e}}), \forall\, \xi \in D(\mathbb{R}^3) \right\}, \end{cases}$$

where $\Omega_{\text{e}} = \mathbb{R}^3 \setminus \overline{\Omega}$ and $D'(\Omega_{\text{e}})$ is the space of distributions with finite support in Ω_{e}.

A.1 Surface spaces, trace and lifting operators

On the boundary, we have the L^2 spaces

$$L^2(\Gamma) = \left\{ u : \int_\Gamma |u|^2 \, \mathrm{d}S < \infty \right\},$$

where $\mathrm{d}S$ denotes the surface measure of Γ. For the vector-valued functions, we have

$$\begin{cases} L^2(\Gamma) = \left\{ \boldsymbol{u} : \displaystyle\int_\Gamma |\boldsymbol{u}|^2 \, \mathrm{d}S < \infty \right\} \\ TL^2(\Gamma) = \left\{ \boldsymbol{u} : \boldsymbol{u} \cdot \hat{\boldsymbol{\nu}} = 0 \text{ and } \displaystyle\int_\Gamma |\boldsymbol{u}|^2 \, \mathrm{d}S < \infty \right\}, \end{cases}$$

where $\hat{\boldsymbol{\nu}}$ is the outward pointing unit normal to Γ. The scalar products and norms are

$$\langle u, v \rangle_{L^2(\Gamma)} = \int_\Gamma u v^* \, \mathrm{d}S, \qquad \|u\|_{L^2(\Gamma)} = \left\{ \int_\Gamma |u|^2 \, \mathrm{d}S \right\}^{1/2},$$

and

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{L^2(\Gamma)} = \int_\Gamma \boldsymbol{u} \cdot \boldsymbol{v}^* \, \mathrm{d}S, \qquad \|\boldsymbol{u}\|_{L^2(\Gamma)} = \left\{ \int_\Gamma |\boldsymbol{u}|^2 \, \mathrm{d}S \right\}^{1/2},$$

With our assumptions on Ω and Γ, there exists a unique linear continuous map $\gamma_0 : H^1(\Omega) \to L^2(\Gamma)$, such that for any $u \in H^1(\Omega) \cap C(\overline{\Omega})$ one has $\gamma_0(u) = u|_\Gamma$. The function $\gamma_0(u)$ is called the *trace* of u on Γ. Note that γ_0 is not onto $L^2(\Gamma)$.

Now, define

$$H^{1/2}(\Gamma) := \gamma_0(H^1(\Omega)).$$

This is a Banach space for the norm defined by

$$\|u\|_{H^{1/2}(\Gamma)} = \int_\Gamma |u(x)|^2 \, \mathrm{d}S_x + \int_\Gamma \int_\Gamma \frac{|u(x) - u(y)|^2}{|x - y|^4} \, \mathrm{d}S_x \, \mathrm{d}S_y \, .$$

The second term (double integral) is the so-called Gagliardo semi-norm in the particular case we are considering (*i.e.*, $n = 3, s = 1/2, p = 2$).

In view of the above, one has the following definitions

$$H_0^1(\Omega) = \{u : u \in H^1(\Omega) \text{ and } \gamma_0(u) = 0\},$$
$$H^{-1}(\Omega) = (H_0^1(\Omega))',$$

and

$$H^{-1/2}(\Gamma) = (H^{1/2}(\Gamma))',$$

where by prime is denoted the dual space.

For important properties related to the above spaces, we refer to [4, 9]. We just note here that

(i) $H^{1/2}(\Gamma) \subset L^2(\Gamma) \subset H^{-1/2}(\Gamma)$, where the injections are compact.

(ii) For $\boldsymbol{u} \in H(\text{div}, \Omega)$ it holds that $\hat{\boldsymbol{\nu}} \cdot \boldsymbol{u} \in H^{-1/2}(\Gamma)$ and the map $\boldsymbol{u} \mapsto \hat{\boldsymbol{\nu}} \cdot \boldsymbol{u}$ is linear and continuous.

Using (ii) (which is an important result, due to Jacques-Louis Lions and Enrico Magenes, of the late 1960's), we note that even though, as mentioned above, elements $\boldsymbol{u}$ of $L^2(\Omega)$ do not necessarily have a trace on the boundary, nevertheless $\hat{\boldsymbol{\nu}} \cdot \boldsymbol{u}$ makes sense, if, additionally, $\nabla \cdot \boldsymbol{u}$ is also in $L^2(\Omega)$.

The appropriate trace spaces which we use in this paper are $H^{-1/2}(\text{div}, \Gamma)$ and $H^{-1/2}(\text{curl}, \Gamma)$ defined by

$$\begin{cases} H^{-1/2}(\text{div}, \Gamma) \stackrel{\text{def}}{=} \left\{\boldsymbol{u} \in H^{-1/2}(\Gamma),\ \hat{\boldsymbol{\nu}} \cdot \boldsymbol{u} = 0,\ \text{div}_\Gamma \boldsymbol{u} \in H^{-1/2}(\Gamma)\right\} \\ H^{-1/2}(\text{curl}, \Gamma) \stackrel{\text{def}}{=} \left\{\boldsymbol{u} \in H^{-1/2}(\Gamma),\ \hat{\boldsymbol{\nu}} \cdot \boldsymbol{u} = 0,\ \text{curl}_\Gamma \boldsymbol{u} \in H^{-1/2}(\Gamma)\right\}, \end{cases}$$

where the surface divergence, div_Γ, and the surface curl, curl_Γ, are defined by duality and restriction, see [9, 21, 25]

$$\begin{cases} \langle \text{div}_\Gamma \boldsymbol{u}, \phi \rangle_{L^2(\Gamma)} \stackrel{\text{def}}{=} -\langle \boldsymbol{u}, \mathbf{grad}_\Gamma \phi \rangle_{TL^2(\Gamma)}, \qquad \forall \phi \in D(\Gamma) \\ \text{curl}_\Gamma \boldsymbol{u} \stackrel{\text{def}}{=} \hat{\boldsymbol{\nu}} \cdot (\nabla \times \boldsymbol{u})|_\Gamma, \end{cases} \tag{A.1}$$

and the surface gradient, $\mathbf{grad}_\Gamma$, is defined by the orthogonal projection of ∇ on the surface Γ, *i.e.*, $\mathbf{grad}_\Gamma \phi = \boldsymbol{\pi}(\nabla \phi)$, where $\boldsymbol{\pi}$ is defined in Theorem A.1 below. Notice that the surface curl operator, curl_Γ, provides a scalar quantity. We can define a vector valued curl operator acting on scalars,

$$\begin{cases} \mathbf{curl}_\Gamma u \stackrel{\text{def}}{=} \mathbf{grad}_\Gamma u \times \hat{\boldsymbol{\nu}}, \\ \text{alternatively by duality} \\ \langle \mathbf{curl}_\Gamma u, \boldsymbol{v} \rangle_{TL^2(\Gamma)} \stackrel{\text{def}}{=} \langle u, \text{curl}_\Gamma \boldsymbol{v} \rangle_{L^2(\Gamma)}, \qquad \forall \boldsymbol{v} \in D^2(\Gamma). \end{cases} \tag{A.2}$$

The space $H^{-1/2}(\text{div}, \Gamma)$ is defined as the completion of the tangential fields in $H^1(\Gamma)$ w.r.t. the norm

$$\|\boldsymbol{m}\|^2 = \|\boldsymbol{m}\|^2_{H^{-1/2}(\Gamma)} + \|\nabla_\Gamma \cdot \boldsymbol{m}\|^2_{H^{-1/2}(\Gamma)},$$

With the assumptions made on the boundary Γ, the space $H^{-1/2}(\text{curl}, \Gamma)$ is the dual of $H^{-1/2}(\text{div}, \Gamma)$, *i.e.*, $\left(H^{-1/2}(\text{div}, \Gamma)\right)' = H^{-1/2}(\text{curl}, \Gamma)$. In [9, Lemma 4, p. 34], we have the following result

Lemma A.1. *For any* $\boldsymbol{u} \in H(\text{curl}, \Omega)$ *holds*

$$\text{curl}_\Gamma \boldsymbol{\pi}(\boldsymbol{u}) = -\text{div}_\Gamma\left(\hat{\boldsymbol{\nu}} \times \boldsymbol{u}|_\Gamma\right) = -\text{div}_\Gamma\left(\boldsymbol{\gamma}(\boldsymbol{u})\right).$$

It follows that, see [9, Corollary 2, p. 38], for $\boldsymbol{v} \in H^{-1/2}(\text{curl}, \Gamma)$ we have

$$\text{curl}_\Gamma \boldsymbol{v} = -\text{div}_\Gamma\left(\hat{\boldsymbol{\nu}} \times \boldsymbol{v}\right),$$

which implies that if $\text{curl}_\Gamma \boldsymbol{\pi}(\boldsymbol{u}) \in H^{-1/2}(\Gamma)$ then $\text{div}_\Gamma\left(\boldsymbol{\gamma}(\boldsymbol{u})\right) \in H^{-1/2}(\Gamma)$ as well, or in other words,

$$\|\boldsymbol{\pi}(\boldsymbol{u})\|_{H^{-1/2}(\text{curl},\Gamma)} = \|\boldsymbol{\gamma}(\boldsymbol{u})\|_{H^{-1/2}(\text{div},\Gamma)},$$

and if $\text{div}_\Gamma \boldsymbol{\pi}(\boldsymbol{u}) \in H^{-1/2}(\Gamma)$ then $\text{curl}_\Gamma\left(\boldsymbol{\gamma}(\boldsymbol{u})\right) \in H^{-1/2}(\Gamma)$ and

$$\|\boldsymbol{\pi}(\boldsymbol{u})\|_{H^{-1/2}(\text{div},\Gamma)} = \|\boldsymbol{\gamma}(\boldsymbol{u})\|_{H^{-1/2}(\text{curl},\Gamma)}.$$

The following theorem is proved in [21]:

Theorem A.1. *1. The trace mapping* $\boldsymbol{\pi} : H(\text{curl}, \Omega) \to H^{-1/2}(\text{curl}, \Gamma)$*, that assigns to any* $\boldsymbol{u} \in H(\text{curl}, \Omega)$ *its tangential component* $\hat{\boldsymbol{\nu}} \times (\boldsymbol{u} \times \hat{\boldsymbol{\nu}})$*, is continuous and surjective from* $H(\text{curl}, \Omega)$ *onto* $H^{-1/2}(\text{curl}, \Gamma)$*. That is*

$$\|\boldsymbol{\pi}(\boldsymbol{u})\|_{H^{-1/2}(\text{curl},\Gamma)} \leq C_\pi \|\boldsymbol{u}\|_{H(\text{curl},\Omega)}, \quad \forall \boldsymbol{u} \in H(\text{curl}, \Omega).$$

2. *The trace mapping* $\boldsymbol{\gamma} : H(\text{curl}, \Omega) \to H^{-1/2}(\text{div}, \Gamma)$*, that takes* $\boldsymbol{u} \in H(\text{curl}, \Omega)$ *to its (rotated) tangential component* $\hat{\boldsymbol{\nu}} \times \boldsymbol{u}$*, is continuous and surjective from* $H(\text{curl}, \Omega)$ *onto* $H^{-1/2}(\text{div}, \Gamma)$*. That is*

$$\|\boldsymbol{\gamma}(\boldsymbol{u})\|_{H^{-1/2}(\text{div},\Gamma)} \leq C_\gamma \|\boldsymbol{u}\|_{H(\text{curl},\Omega)}, \quad \forall \boldsymbol{u} \in H(\text{curl}, \Omega).$$

3. *In both cases, a continuous lifting with zero divergence for these trace operators in* $H(\text{curl}, \Omega)$ *exists. More precisely, there exists an operator* $\mathcal{R} : H^{-1/2}(\text{div}, \Gamma) \to H(\text{curl}, \Omega)$ *such that for every* $\boldsymbol{m} \in H^{-1/2}(\text{div}, \Gamma)$ *there exists a* $\boldsymbol{u} \in H(\text{curl}, \Omega)$ *satisfying* $\boldsymbol{\gamma}(\boldsymbol{u}) = \boldsymbol{m}$*, and*

$$\|\mathcal{R}(\boldsymbol{m})\|_{H(\text{curl},\Omega)} \leq C \|\boldsymbol{m}\|_{H^{-1/2}(\text{div},\Gamma)}, \quad \forall \boldsymbol{m} \in H^{-1/2}(\text{div}, \Gamma),$$

and similarly from $H^{-1/2}(\text{curl}, \Gamma)$ *to* $H(\text{curl}, \Omega)$*, corresponding to the* $\boldsymbol{\pi}$*-trace.*

4. *For any* $\boldsymbol{u}, \boldsymbol{v} \in H(\text{curl}, \Omega)$*, the following Stokes' formula holds:*

$$\langle \nabla \times \boldsymbol{u}, \boldsymbol{v} \rangle_{L^2(\Omega)} - \langle \boldsymbol{u}, \nabla \times \boldsymbol{v} \rangle_{L^2(\Omega)} = \langle \boldsymbol{\gamma}(\boldsymbol{u}), \boldsymbol{\pi}(\boldsymbol{v}) \rangle_{L^2(\Gamma)}.$$

Appendix B Spherical vector waves

The spherical harmonics $Y_n(\boldsymbol{x})$ are defined as

$$Y_n(\boldsymbol{x}) = \frac{1}{x}\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)\mathrm{e}^{\mathrm{i}m\phi},$$

in terms of the spherical angles θ (polar angle) and ϕ (azimuthal angle) of the unit vector $\hat{\boldsymbol{x}}$. The associated Legendre function is denoted $P_l^m(\cos\theta)$. The index n is a multi-index for the integer indices $l = 0, 1, 2, 3, \dots$, $m = -l, -l+1, \dots, -1, 0, 1, \dots, l$. Note, the extra factor $1/x$ in the definition of the spherical harmonics, which makes the spherical harmonics orthonormal on the sphere of radius x.

The vector spherical harmonics are defined by, *cf.* [5, 17]

$$\begin{cases} \boldsymbol{Y}_{1n}(\boldsymbol{x}) = \dfrac{\nabla_{S^2} Y_n(\boldsymbol{x}) \times \hat{\boldsymbol{x}}}{\sqrt{l(l+1)}} \\ \boldsymbol{Y}_{2n}(\boldsymbol{x}) = \dfrac{\nabla_{S^2} Y_n(\boldsymbol{x})}{\sqrt{l(l+1)}}, \end{cases}$$

where ∇_{S^2} is the nabla-operator on the unit sphere.

The radiating solutions to the Maxwell equations in vacuum are defined as (outgoing spherical vector waves)

$$\begin{cases} \boldsymbol{u}_{1n}(k\boldsymbol{x}) = \dfrac{\xi_l(kx)}{k}\boldsymbol{Y}_{1n}(\hat{\boldsymbol{x}}) \\ \boldsymbol{u}_{2n}(k\boldsymbol{x}) = \dfrac{1}{k}\nabla \times \left(\dfrac{\xi_l(kx)}{k}\boldsymbol{Y}_{1n}(\hat{\boldsymbol{x}})\right). \end{cases}$$

Here, we use the Riccati-Bessel functions $\xi_l(kx) = kxh_l^{(1)}(kx)$, where $h_l^{(1)}(kx)$ is the spherical Hankel function of the first kind [23]. These vector waves satisfy

$$\nabla \times (\nabla \times \boldsymbol{u}_{\tau n}(k\boldsymbol{x})) - k^2\boldsymbol{u}_{\tau n}(k\boldsymbol{x}) = \boldsymbol{0}, \qquad \tau = 1, 2,$$

and they also satisfy the Silver-Müller radiation condition [10, 17]. Another representation of the definition of the vector waves is

$$\begin{cases} \boldsymbol{u}_{1n}(k\boldsymbol{x}) = \dfrac{\xi_l(kx)}{k}\boldsymbol{Y}_{1n}(\boldsymbol{x}) \\ \boldsymbol{u}_{2n}(k\boldsymbol{x}) = \dfrac{\xi_l'(kx)}{k}\boldsymbol{Y}_{2n}(\boldsymbol{x}) + \sqrt{l(l+1)}\dfrac{\xi_l(kx)}{k^2x}Y_n(\boldsymbol{x}). \end{cases}$$

A simple consequence of these definitions is

$$\begin{cases} \boldsymbol{u}_{1n}(k\boldsymbol{x}) = \dfrac{1}{k}\nabla \times \boldsymbol{u}_{2n}(k\boldsymbol{x}) \\ \boldsymbol{u}_{2n}(k\boldsymbol{x}) = \dfrac{1}{k}\nabla \times \boldsymbol{u}_{1n}(k\boldsymbol{x}). \end{cases}$$

In a similar way, the regular spherical vector waves $\boldsymbol{v}_{\tau n}(k\boldsymbol{x})$ are defined [5, 17].

$$\begin{cases} \boldsymbol{v}_{1n}(k\boldsymbol{x}) = xj_l(kx)\boldsymbol{Y}_{1n}(\boldsymbol{x}) \\ \boldsymbol{v}_{2n}(k\boldsymbol{x}) = \dfrac{1}{k}\nabla \times (xj_l(kx)\boldsymbol{Y}_{1n}(\boldsymbol{x})) \,, \end{cases}$$

where $j_l(kx)$ is the spherical Bessel function of the first kind [23].

Appendix C Variational solvability of Problem (E)

In order to solve Problem (E), we first obtain a variational formulation. By multiplying $\nabla \times \boldsymbol{H}_{\mathrm{sc}}(\boldsymbol{x}) = -\mathrm{i}k\boldsymbol{E}_{\mathrm{sc}}(\boldsymbol{x})$ with a test function $\boldsymbol{\psi} \in H_0(\mathrm{curl}; \overline{\Omega}_{\mathrm{e}})$, which consists of the functions $\boldsymbol{u}$ in $H_{\mathrm{loc}}(\mathrm{curl}, \overline{\Omega}_{\mathrm{e}})$ such that $\boldsymbol{\gamma}(\boldsymbol{u}) = 0$, with compact support, integrating over Ω_{e}, using Green identities and then substituting $\boldsymbol{H}_{\mathrm{sc}}$ using $\nabla \times \boldsymbol{E}_{\mathrm{sc}}(\boldsymbol{x}) = \mathrm{i}k\boldsymbol{H}_{\mathrm{sc}}(\boldsymbol{x})$, we obtain the variational formulation of problem (E): Find $\boldsymbol{E}_{\mathrm{sc}} \in H_{\mathrm{loc}}(\mathrm{curl}, \overline{\Omega}_{\mathrm{e}})$ satisfying $\boldsymbol{\gamma}(\boldsymbol{E}_{\mathrm{sc}}) = \boldsymbol{m} \in H^{-1/2}(\mathrm{div}, \Gamma)$ and the Silver-Müller boundary conditions at infinity such that

$$\int_{\Omega_e} [\nabla \times \boldsymbol{E}_{\mathrm{sc}}(\boldsymbol{x}) \cdot \nabla \times \boldsymbol{\psi}(\boldsymbol{x}) - k^2 \boldsymbol{E}_{\mathrm{sc}}(\boldsymbol{x}) \cdot \boldsymbol{\psi}(\boldsymbol{x})] d\boldsymbol{x} = 0, \quad \forall \boldsymbol{\psi} \in H_0(\mathrm{curl}; \overline{\Omega}_{\mathrm{e}}). \tag{C.1}$$

Since $\boldsymbol{\gamma} : H_{\mathrm{loc}}(\mathrm{curl}, \overline{\Omega}_{\mathrm{e}}) \to H^{-1/2}(\mathrm{div}, \Gamma)$ is onto, there exists a $\mathbf{U} \in H(\mathrm{curl}, \Omega_{\mathrm{e}})$ such that $\boldsymbol{\gamma}(\mathbf{U}) = \boldsymbol{m}$. We now express the solution of (C.1) as $\boldsymbol{E}_{\mathrm{sc}} = \boldsymbol{E}_0 + \mathbf{U}$, where $\boldsymbol{E}_0 \in H_0(\mathrm{curl}; \overline{\Omega}_{\mathrm{e}})$. Substituting this Ansatz in (C.1), we obtain the following variational problem for $\boldsymbol{E}_0$: Find $\boldsymbol{E}_0 \in H_0(\mathrm{curl}; \overline{\Omega}_{\mathrm{e}})$ such that

$$\begin{aligned} &\int_{\Omega_e} [\nabla \times \boldsymbol{E}_0(\boldsymbol{x}) \cdot \nabla \times \boldsymbol{\psi}(\boldsymbol{x}) - k^2 \boldsymbol{E}_0(\boldsymbol{x}) \cdot \boldsymbol{\psi}(\boldsymbol{x})] d\boldsymbol{x} \\ &\quad = - \int_{\Omega_e} [\nabla \times \boldsymbol{U} \cdot \nabla \times \boldsymbol{\psi}(\boldsymbol{x}) - k^2 \boldsymbol{U}(\boldsymbol{x}) \cdot \boldsymbol{\psi}(\boldsymbol{x})] d\boldsymbol{x}, \quad \forall \boldsymbol{\psi} \in H_0(\mathrm{curl}; \overline{\Omega}_{\mathrm{e}}). \end{aligned} \tag{C.2}$$

Furthermore (see Corollary p. 37 in [9]), $\boldsymbol{U}$ can be chosen so that $\nabla \cdot \boldsymbol{U} = 0$, so that $\nabla \cdot \boldsymbol{E}_0$ also vanishes, by which we conclude that (C.2) is equivalent to the vector Helmholtz equation, with homogeneous tangential boundary condition. The unique solvability of this problem is obtained in terms of the sequilinear form $\langle \nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{v} \rangle - k^2 \langle \boldsymbol{u}, \boldsymbol{v} \rangle$ for $u, v \in H_0(\mathrm{curl}; \overline{\Omega}_{\mathrm{e}})$, for k such that k^2 is not an eigenvalue of $-\Delta$, with the considered boundary conditions (for details see Theorem 5.60 in [14] or Theorem 6, p. 107 in [9]).

Appendix D The Laplace-Beltrami operator and its eigenvalue problem

Let (M, g) be a compact smooth manifold without boundary, g being the Riemannian metric. On M we may define the Lebesgue space $L^2(M)$, as

$$L^2(M) = \{u : M \to \mathbb{R} \ : \ \int_M |u|^2 \, \mathrm{d}\mu_g < \infty\},$$

where μ_g is the Riemann canonical measure $\mu_g = \sqrt{g}\,\mathrm{d}\mu_{\mathcal{L}^d}$ where $\mu_{\mathcal{L}^d}$ is the Lebesgue measure on $\mathbb{R}^d$. The function space $L^2(M)$ is a Hilbert space for the scalar produce

$$\langle u, v\rangle_{L^2(M)} = \int_M uv^* \,\mathrm{d}\mu_g.$$

We may further define Sobolev spaces on M. In particular, we may define the Sobolev space $H^1(M)$ by $H^1(M) := \overline{C^\infty(M)}$, with respect to the norm $\|\cdot\|_{H^1}$ defined by $\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2$, where ∇u denotes the gradient on (M, g). This is a Hilbert space as well, for the scalar product

$$\langle u, v\rangle_{H^1} = \langle u, v\rangle_{L^2} + \langle \nabla u, \nabla v\rangle_{L^2}.$$

This Sobolev space satisfies a version of the Rellich-Kondrachev embedding theorem,

Theorem D.1. *If (M, g) is compact then the embeddings $H^1(M) \hookrightarrow L^2(M)$ are compact.*

We now consider the Laplace-Beltrami operator on the manifold (M, g), defined as $\Delta_g f := \operatorname{div}(\nabla f)$ or in terms of a local chart $\phi : U \subset M \to \mathbb{R}^d$ of M as

$$\Delta_g f = \frac{1}{\sqrt{g}} \sum_{j,k} \frac{\partial}{\partial x_j}\left(\sqrt{g} g^{jk} \frac{\partial}{\partial x_k}(f \circ \phi^{-1})\right),$$

where $g = \det g_{jk}$ and $g^{jk} = g_{jk}^{-1}$.

We now consider the so called closed spectral problem (a compact manifold without boundary is called closed), which consists of finding $\lambda \in \mathbb{R}$ such that there exist $u \in C^\infty(M)$, $u \neq 0$ for which

$$\Delta_g u = \lambda u.$$

Theorem D.2. *The following assertions hold true [19]*

1. *The spectrum and the point spectrum of $L := -\Delta_g$ coincide and consist of a real infinite sequence*

$$0 \leq \lambda_1 \leq \lambda_2 \leq \ldots,$$

such that $\lambda_k \to \infty$ as $k \to \infty$, and the eigenfunctions $u_k \in H^1(M)$ and are also analytic.

2. *Each eigenvalue has finite multiplicity and the eigenspaces corresponding to each eigenvalue are L^2-orthogonal. If we denote by $E(\lambda_k)$ the eigenspace corresponding to the eigenvalue λ_k, then*

$$\overline{\bigoplus_{k\geq 1} E(\lambda_k)} = L^2(M),$$

where the closure is taken for the $L^2(M)$ norm.

The second assertion of the above theorem implies that the eigenfunctions of the operator $L := -\Delta_g$ form a complete orthonormal basis for the space $L^2(M)$.

We sketch a variational proof of the above theorem following [19].

Proof. Given $f \in H^1(M)'$, consider the problem of finding $u \in L^2(M)$ such that

$$-\Delta_g u + u = f.$$

Using Green identity and the density of $C^\infty(M)$ in $H^1(M)$, this can be expressed in weak form as the problem of finding $u \in H^1(M)$ such that

$$a(u,v) = \langle f, v\rangle_{H^1(M)',H^1(M)}, \quad \forall\, v \in H^1(M),$$

where $a : H^1(M) \times H^1(M) \to \mathbb{R}$ is the bilinear form

$$a(u,v) = \langle u, v\rangle_{H^1(M)}.$$

By its definition in terms of an inner product this bilinear form is continuous and coercive, so by a standard application of the Lax-Milgram lemma there exists a unique solution of the above equation $u_f \in H^1(M)$. Defining the mapping $T : L^2(M) \to H^1(M)$ by $f \mapsto u_f =: T(f)$ we see that $T = (-\Delta_g + I)^{-1}$, and furthermore by the Rellich-Kontrachev embedding we note that T is a compact operator. By the Fredholm theorem there exists a sequence of real numbers $\mu_k > 0$, such that

$$(-\Delta_g + I)^{-1} u_k = T u_k = \mu_k u_k, \tag{D.1}$$

with $u_k \in L^2(M)$, $\mu_k \to 0$ as $k \to \infty$, and furthermore $\{u_k\}_{k\in\mathbb{N}}$ forms an orthonormal basis of $L^2(M)$. However, (D.1) implies that

$$-\Delta_g u_k = \lambda_k u_k,$$

with $\lambda_k = 1 - \frac{1}{\mu_k}$. □

Let us mention that in the case of the n-sphere

$$\mathrm{S}^n := \{(x_1, x_2, \cdots x_{n+1}) : \sum_{j=1}^{n+1} x_j^2 = 1\}$$

in $\mathbb{R}^{n+1}$, there is a metric, induced by the standard metric on $\mathbb{R}^{n+1}$, and a Laplace operator Δ_{S}, arising from Δ on $\mathbb{R}^{n+1}$, as

$$\Delta_{\mathrm{S}} f = \Delta \widetilde{f}\,|_{\mathrm{S}^n}\,, \quad \widetilde{f}(x) := f\left(\frac{x}{|x|}\right).$$

Regarding the eigenvalues of $-\Delta_{\mathrm{S}}$ it is known that these are [26, Theorem 22.1 and Corollary 22.1(a)]

$$\lambda_k = k\,(k + n - 1)\,, \; k = 0, 1, 2, \cdots,$$

with the multiplicity of λ_k being

$$m(\lambda_k) = \binom{n+k}{n} - \binom{n+k-2}{n},$$

where, as usual, $\binom{n+k}{n} := \frac{(n+k)!}{n!\,k!}$.

Appendix E Proof of the invertibility of the matrix representation

Lemma E.1. *Let $T : H \to H$ be a bounded linear and invertible operator and define the infinite dimensional matrix $A = (A_{nm})$, defined by $A_{nm} = \langle Te_n, e_m \rangle$, where $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of the Hilbert space H. Then the matrix A^{t} is invertible, where A^{t} denotes the transpose of the matrix A.*

Proof. Since T is invertible for every $v \in H$ there exists a unique $u \in H$ such that $Tu = v$. Let $v = \sum_n v_n e_n$, for $\{v_n\} \in \ell^2$. Since the solution of $Tu = v$ exists in H, it also has an expansion of the form $u = \sum_n u_n e_n$, for some $\{u_n\} \in \ell^2$. Using this expansion, we rewrite $Tu = v$ as

$$T\left(\sum_n u_n e_n\right) = \sum_n u_n (Te_n) = \sum_n v_n e_n. \tag{E.1}$$

Let us call $Te_n = a_n \in H$, so that a_n admits an expansion as

$$a_n = \sum_m \langle Te_n, e_m \rangle e_m = \sum_m A_{nm} e_m. \tag{E.2}$$

Introducing (E.2) into (E.1) we conclude that

$$\sum_n u_n \left(\sum_m A_{nm} e_m\right) = \sum_n v_n e_n,$$

which upon rearranging can be expressed as

$$\sum_m \sum_n A_{nm} u_n e_m = \sum_n v_n e_n,$$

and by interchanging n with m leads to

$$\sum_n \sum_m A_{mn} u_m e_n = \sum_n v_n e_n.$$

Projecting on the basis, we then conclude that

$$\sum_m A_{mn} u_m = v_n \ \ \forall\, n \in \mathbb{N}. \tag{E.3}$$

Since $\{v_n\} \in \ell^2$ is arbitrary, the above considerations lead us to the conclusion that the infinite system of linear equations (E.3) admits a unique solution $\{u_m\} \in \ell^2$ for every $\{v_n\} \in \ell^2$, hence the matrix A^{t} is invertible. □

An example of the application of the lemma in this appendix is the invertibility of the exterior Calderón matrix $C_{\tau n,\tau' n'}$ in (4.5). To be explicit, due to the linearity and invertibility of T

$$u = \sum_n u_n e_n = T^{-1}\left(\sum_n v_n e_n\right) = \sum_n v_n (T^{-1} e_n). \tag{E.4}$$

Let us define $b_n = T^{-1}e_n \in H$, so that b_n admits an expansion as

$$b_n = \sum_m \langle T^{-1}e_n, e_m\rangle e_m = \sum_m B_{nm}e_m. \tag{E.5}$$

Introducing (E.5) into (E.4) we conclude that

$$\sum_n u_n e_n = \sum_n v_n \left(\sum_m B_{nm}e_m\right),$$

which upon rearranging can be expressed as

$$\sum_m \sum_n B_{nm} v_n e_m = \sum_n u_n e_n,$$

and by interchanging n with m leads to

$$\sum_n \sum_m B_{mn} v_m e_n = \sum_n u_n e_n.$$

Projecting on the basis, we then conclude that

$$\sum_m B_{mn} v_m = u_n \;\; \forall n \in \mathbb{N}. \tag{E.6}$$

Comparing (E.3) with (E.6) we conclude

$$(A^{\mathrm{t}})^{-1} = B^{\mathrm{t}} = \langle T^{-1}e_m, e_n\rangle. \tag{E.7}$$

Appendix F The infinite matrices A and $\overline{A}$

The compactness of the integral operator defined with the Green function as kernel shows the matrix $\overline{A}$ is not expected to be invertible. This appendix develops the details on this subject.

Initially, we consider the operator equations

$$\sum_{\tau' n'} A_{\tau n,\tau' n'} a_{\tau' n'} = b_{\tau n}, \quad \tau = 1, 2,\; n \in \mathbb{N} = 1, 2, \ldots. \tag{F.1}$$

or in shorthand notation $Aa = b$, where the matrix A is defined in Lemma 4.6, *viz.*,

$$A_{\tau n,\tau' n'} = k \int_\Gamma \boldsymbol{u}_{\tau n} \cdot \boldsymbol{Y}_{\tau' n'} \,\mathrm{d}S.$$

The vectors $a_{\tau n}$ and $b_{\tau n}$ are assumed to belong to the space $\boldsymbol{\ell}^{-1/2}(\mathrm{div})$.

We want to identify these operator equations as discretizations, in a proper orthogonal base, of appropriate integral equations.

Consider any point $\boldsymbol{x}$ inside the inscribed sphere S of the scatterer. Multiply the equation (F.1) by $\mathrm{i}\boldsymbol{v}^*_{\tau n}(k\boldsymbol{x})$ and sum over $\tau = 1, 2$ and over all n. Recalling the definition of the components $A_{\tau n, \tau' n'}$, we obtain

$$\sum_{\tau n}\sum_{\tau' n'} \mathrm{i}k \int_\Gamma \boldsymbol{v}^*_{\tau n}(k\boldsymbol{x})\boldsymbol{u}_{\tau n}(k\boldsymbol{x}') \cdot (a_{\tau' n'}\boldsymbol{Y}_{\tau' n'}(\boldsymbol{x}'))\ \mathrm{d}S' = \mathrm{i}\sum_{\tau n} b_{\tau n}\boldsymbol{v}^*_{\tau n}(k\boldsymbol{x}).$$

Recall the representation of Green dyadic for the electric field, see (4.13), and we obtain

$$\int_\Gamma \mathbf{G}_{\mathrm{e}}(k, \boldsymbol{x} - \boldsymbol{x}') \cdot \boldsymbol{f}(\boldsymbol{x}')\ \mathrm{d}S' = \boldsymbol{g}(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in S, \tag{F.2}$$

where S is the inscribed sphere and

$$\begin{cases} \boldsymbol{f}(\boldsymbol{x}') = \sum\limits_{\tau' n'} a_{\tau' n'}\boldsymbol{Y}_{\tau' n'}(\boldsymbol{x}') \\ \boldsymbol{g}(\boldsymbol{x}) = \mathrm{i}\sum\limits_{\tau n} b_{\tau n}\boldsymbol{v}^*_{\tau n}(k\boldsymbol{x}). \end{cases}$$

So in order to solve $Aa = b$, construct the function $\boldsymbol{g}(\boldsymbol{x}) = \mathrm{i}\sum_{\tau n} b_{\tau n}\boldsymbol{v}^*_{\tau n}(k\boldsymbol{x})$, solve the integral operator equation (F.2) to obtain $\boldsymbol{f}$ and then expand the solution in generalized harmonics to obtain $a_{\tau n}$. The compactness of the integral operator defined with the Green dyadic for the electric field as kernel shows the matrix A is not expected to be invertible.

We now consider $\overline{A}a = b$, which in coordinate form is

$$\sum_{\tau' n'} \overline{A}_{\tau n, \tau' n'} a_{\tau' n'} = b_{\tau n}, \quad \tau = 1, 2,\ n \in \mathbb{N} = 1, 2, \ldots.$$

where

$$\overline{A}_{\tau n, \tau' n'} = A_{\overline{\tau} n, \overline{\tau'} n'} = k\int_\Gamma \boldsymbol{u}_{\overline{\tau} n} \cdot \boldsymbol{Y}_{\overline{\tau'} n'}\ \mathrm{d}S.$$

Consider any point $\boldsymbol{x}$ inside the inscribed sphere S of the scatterer. Multiply the equation (F.1) by $\mathrm{i}\boldsymbol{v}^*_{\overline{\tau} n}(k\boldsymbol{x})$ and sum over $\tau = 1, 2$ and over all n. Recalling the definition of the components $\overline{A}_{\tau n, \tau' n'}$, we obtain

$$\sum_{\tau n}\sum_{\tau' n'} \mathrm{i}k \int_\Gamma \boldsymbol{v}^*_{\overline{\tau} n}(k\boldsymbol{x})\boldsymbol{u}_{\overline{\tau} n}(k\boldsymbol{x}') \cdot (a_{\tau' n'}\boldsymbol{Y}_{\overline{\tau'} n'}(\boldsymbol{x}'))\ \mathrm{d}S' = \mathrm{i}\sum_{\tau n} b_{\tau n}\boldsymbol{v}^*_{\overline{\tau} n}(k\boldsymbol{x}).$$

Recall the representation of Green dyadic for the electric field, see (4.13), and we obtain

$$\int_\Gamma \mathbf{G}_{\mathrm{e}}(k, \boldsymbol{x} - \boldsymbol{x}') \cdot \overline{\boldsymbol{f}}(\boldsymbol{x}')\ \mathrm{d}S' = \overline{\boldsymbol{g}}(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in S, \tag{F.3}$$

where S is the inscribed sphere and

$$\begin{cases} \overline{\boldsymbol{f}}(\boldsymbol{x}') = \sum\limits_{\tau' n'} a_{\tau' n'}\boldsymbol{Y}_{\overline{\tau'} n'}(\boldsymbol{x}') \\ \overline{\boldsymbol{g}}(\boldsymbol{x}) = \mathrm{i}\sum\limits_{\tau n} b_{\tau n}\boldsymbol{v}^*_{\overline{\tau} n}(k\boldsymbol{x}). \end{cases}$$

So to solve $\overline{A}a = b$, one can construct the function $\overline{\boldsymbol{g}}(\boldsymbol{x}) = \mathrm{i}\sum_n (b_{\tau n}\boldsymbol{v}^*_{\overline{\tau} n}(k\boldsymbol{x})$, solve the integral operator equation (F.3) to obtain $\overline{f}$ and then expand the solution in generalized harmonics to obtain $a_{\tau n}$.

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