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Incoherent Scattering — a Literature Study

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Incoherent Scattering — a Literature Study

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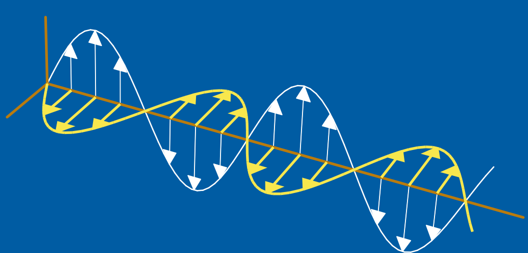

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1 Introduction

Many naturally occurring materials are characterized by a collection of randomly distributed discrete particles, for example clouds, either from dust, water vapour, or rain. Other materials seem continuous, such as milk or porous plastic, but consist of many small particles or holes. This paper deals with scattering by this type of materials. The research area is known as multiple scattering and describes many different phenomena that have a joint theoretical basis. For example many important works consider acoustic or particle scattering as well as electromagnetic scattering. The object of this paper is to serve as an introduction and road map for those who want to delve deeper into multiple scattering. Therefore, the most prominent literature on the subject is collected and briefly described here.

Measuring the scattering by discrete random distributions must by necessity be done over a finite time interval. Field data is collected over a time interval and an average value of the field is captured. However, if the particles in a distribution change position and orientation during that time, the field fluctuates, and it is natural to divide the field in two parts — the mean value of the field, the coherent field, and the remaining part with zero mean but non-zero variance, the incoherent field.[1]. This same effect can be found by averaging over volumes sufficiently larger than the length scale in the distribution. If a distribution of particles is sufficiently random, we can calculate and predict its scattering behaviour without having detailed information about the position of the individual particles.

To illustrate this phenomena, the electric field can be divided into two parts,

$$\boldsymbol{E} = \langle \boldsymbol{E} \rangle + \boldsymbol{e}, \tag{1.1}$$

the part of the field which has a non-zero average, the coherent field, $\langle \boldsymbol{E} \rangle$, and the part with zero average, $\langle \boldsymbol{e} \rangle = 0$, the incoherent field, $\boldsymbol{e}$. It is important to distinguish what quantity is being averaged over in. Often we wish to consider time averaged quantities as measurements are collected over time (note that measurements can be averaged over space if the measured volume is large enough [108]). However, modeling all the physical processes and mechanics that govern a systems development in time is not trivial. Therefore, the concept of *ergodicity* is employed in order to transform the system to more manageable theoretical considerations. The ergodic hypothesis states that, for a sufficiently random system, time averaging can be replaced by ensemble averaging [68]. To take an ensemble average means to average over different realizations of the systems, for example different orientations and positions of the particles in a distribution. Essentially, the ergodicity hypothesis states that the system varies in time and space in the same manner. This concept is used in incoherent scattering to calculate averaged quantities based on the systems realizations.

[1] Historically, another definition of the coherent and incoherent fields exists. This definition states that when a field is scattered by a random material it can be split into two parts, one part with an uniformly distributed phase function, and one part where the phase is correlated to the scattered wave. These are defined as the incoherent and coherent scattering, respectively.

Since the incoherent field has zero mean value, it cannot be measured by taking a time average measurement of the electric field, which is the method we normally use to measure at radio frequencies. However, the incoherent field can be measured through the time averaged intensity (the natural experimental quantity at optical frequencies),

$$\langle |\boldsymbol{E}|^2 \rangle = \langle (\langle \boldsymbol{E} \rangle + \boldsymbol{e}) \cdot (\langle \boldsymbol{E} \rangle + \boldsymbol{e})^* \rangle = |\langle \boldsymbol{E} \rangle|^2 + \langle |\boldsymbol{e}|^2 \rangle \,. \tag{1.2}$$

As a consequence, the theoretical evaluation of the incoherent field in measurable quantities is more involved than that of the coherent field. The coherent field is usually calculated through the ensemble average of an integral equation, and the incoherent field intensity is calculated through coupled integral equations. The computation of the incoherent contribution is very difficult due to these extra integral equations. No numerically effective method to calculate the incoherent contribution exists. Most papers on this topic deal with the evaluation of the total or the coherent fields, and one must glean information about the incoherent field from these works.

Some of the first papers on multiple scattering are Foldy's paper in 1945 [28] and Lax's paper in 1951 [53]. Foldy's paper is instrumental for the understanding of multiple scattering and for the general development of the theory of the subject. The theory is presented for scalar fields, but the concepts hold for vector fields after some modification. Only isotropic scattering is treated in [28], but the theory is generalized in Lax's paper. Additionally, there exists a wealth of textbook literature concerning multiple scattering, most prominently: Ishimaru in 1978, divided in two volumes [41, 42], Tsang's three-volume set from 2000-2001 [91, 92, 95], Martin in 2006 [60], and Mishchenko's two books from 2006 [68] and 2014 [66], respectively. These cover the basics of electromagnetic scattering as well as multiple scattering. The books covering incoherent phenomena are Ishimaru volume 2 [42], Tsang's third book [91], and Mishchenko's two books [66, 68]. There exist a few modern overview articles broadly covering multiple scattering and its historical development [19, 20, 74, 86, 90].

2 Applications

Multiple scattering of light occurs in many scenarios. Random distributions of particles often occur in nature and causes radiation to scatter with high incoherent components. In these cases, it is imperative to be able to predict the magnitude of such phenomena, or to glean information from the incoherent scattering. However, in other application areas, such as astronomy, incoherent electromagnetic scattering by random media is the only available data. It is then of interest to use the incoherent light to understand the distribution from which it was scattered. Here follows some examples of application areas explored in the literature.

2.1 Astronomy

Change to: Dust and particles in space are usually randomly distributed, and scattering of light by these particles reaches earth as incoherent radiation [66]. Analyzing this type of radiation is therefore of great interest [65, 84].

However, not all astronomical data is passively received. The use of incoherent radar in the ionosphere, and beyond, is well documented [4]. A powerful radar signal is sent into the atmosphere to detect free electrons in ionized materials. The scattered signal can be used to detect the properties of the ionized materials. Due to the distributions being random, the received signal is entirely incoherent [13, 22, 23, 25, 45, 79, 105, 113]. This technique has also been used to study objects outside the earth's atmosphere within the solar system [36].

Most of the radiation received from space is incoherent. However, interesting phenomena has been observed from the backscattering of radar on satellites. Instruments receive a distinct coherent component in the backscattering direction from, for example, Saturn's rings [65]. The explanation of this contribution is in-phase enhancement of the signal in the backscattering direction [65, 84].

2.2 Remote Sensing

One of the main applications for satellites and other spacecraft is to remotely analyze earth or other celestial bodies by either actively or passively scanning them. Due to random materials, such as clouds, and the roughness of the surfaces that are scanned, the received signal is often incoherent. This can be described using the theory of multiple scattering [93, 101]. From the incoherent component of the scattering it is possible to analyze the make up of the media being scanned, *e.g.*, terrain in a certain area [56, 102].

2.3 Meteorology

Many weather phenomenon include random distributions of particles, for example, rain, clouds, snow, or fog [8]. Scattering by these distributions often appears as incoherent if the intensity of the phenomenon is high enough. Multiple scattering has therefore been used to analyze, for example, how rain affects electromagnetic radiation at certain frequencies [1], or how weather radar is affected by different types of weather [5]. This type of analysis of weather effects is also beneficial to communication where attenuation due to scattering needs to be predicted [76].

2.4 Study of Aerosols and airborne particles

Aerosols are small particles suspended in gas or liquid. These particles are distributed randomly. Aerosols appear naturally as dust or clouds but are also created artificially and can appear as pollutants in, *e.g.*, cities or the atmosphere, as well as inside indoor environments. Studying them is important for understanding environmental effects. Investigations of aerosols are often done by illuminating the particles with light and studying the scattered components. Due to the randomness of the particle distributions, a part of the scattered light will be incoherent [7, 24, 81]. This field of research has many applications including studying sprays of particles [48].

2.5 Acoustics

Multiple scattering is often experienced by acoustic measurements [87], such as in ultrasound [10]. The theory that is used to describe the phenomena for acoustic waves is closely related to electromagnetic scattering, and many of the results and principles derived for acoustics can be used for both. Acoustic fields are scalar, which simplify some theoretical and computational aspects of multiple scattering. This has resulted in a wealth of literature [2, 32, 55, 61, 89, 110], with results within bulk parameter calculation and effective wave number estimation [12, 38]

3 Fundamental Equations

To understand the basic concepts of evaluating multiple scattering, we must first introduce a few equations. These equations form the basis of most methods to treat multiple scattering. They are included here as a reference, but also to help explain the approximations introduced later on. When scattering occurs from many particles, all particles in the configuration contribute to the field at the observations point. Mathematically this is handled by dividing up the phenomena into smaller parts relating to each particle. In this section, this division is illustrated for the Green's function, describing the mathematical connection between points, as well as the electric field. We will also briefly cover the radiative transfer equation that calculates scattering by the use of heuristic energy balance.

3.1 Dyson and Bethe-Salpeter equations

The Dyson equation is an exact expansion of the mean dyadic Green's function in a medium with random permittivity fluctuations [91]. However, it is perfectly valid for a discrete random medium where the permittivity fluctuations represent the particles. The dyadic Green's function is the mathematical function connecting two points electromagnetically, it occurs in almost all studies of electromagnetic theory.

The dyadic Green's function can be expanded in a series of terms that represent the mathematical field interacting at a different number of locations. This expansion is called a Neumann expansion. In reality, the field does not interact sequentially in this manner, but its mathematical representation can be interpreted this way. For example, the zeroth term in the expansion corresponds to no interaction, the first term corresponds to the field interacting with one particle, and so on. If the ensemble average of this expansion is taken, the Dyson equation can be derived,

$$\langle \mathbf{G}(\boldsymbol{r},\boldsymbol{r}_o)\rangle = \mathbf{G}^{(0)}(\boldsymbol{r},\boldsymbol{r}_o) + \int \mathrm{d}\boldsymbol{r}_1\,\mathrm{d}\boldsymbol{r}_2 \mathbf{G}^{(0)}(\boldsymbol{r},\boldsymbol{r}_1)\cdot\mathbf{Q}(\boldsymbol{r}_1,\boldsymbol{r}_2)\cdot\langle \mathbf{G}(\boldsymbol{r}_2,\boldsymbol{r}_o)\rangle\,, \tag{3.1}$$

where $\mathbf{G}$ is the dyadic Green's function, $\mathbf{G}^{(0)}$ is the homogeneous dyadic Green's function, *i.e.*, the Green's function for vacuum, and $\mathbf{Q}$ is the mass term quantifying the deviation of the electrical and magnetic properties of the medium to the background medium. The Dyson equation expands a single dyadic Green's function, dealing with the first moment of the field. Therefore, it has direct applications when calculating the average of the field (coherent field), as in (1.1).

Tthe Bethe-Salpeter equation deals with the second moment of the field, the intensity, as in (1.2). It is an exact equation of the mean correlation or covariance between two dyadic Green's functions. The Bethe-Salpeter and Dyson equations are derived in [91] for a medium with fluctuating permittivity, but are valid also in the case of discrete particles. The Dyson and Bethe-Salpeter equations are often simplified for application by ladder approximations that are based on describing the equations in Feynman diagrams and discarding some of the diagrammatic terms [30, 91, 104]

3.2 Foldy-Lax Equations

The Foldy-Lax equations create what is known as the hierarchy of equations. This hierarchy is what will be approximated in different ways for all analytical evaluation of multiple scattering.

The Foldy-Lax equations are a set of equations that form the basis for evaluating multiple scattering from discrete random media. These equations are derived from Maxwell's equations. Methods derived from Maxwell's equations are collectively called analytical theory of multiple scattering. The Foldy-Lax equations were first presented by Foldy in [28] and later expanded by Lax in [53]. They can also be found in many textbooks [66, 91]. The equations are best understood from the basic division of the total electric field as,

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}^{\text{inc}}(\boldsymbol{r}) + \boldsymbol{E}^{\text{sca}}(\boldsymbol{r}), \tag{3.2}$$

where $\boldsymbol{E}$ is the total electric field, $\boldsymbol{E}^{\text{inc}}$ is the incident electric field, the field with no particles present, and $\boldsymbol{E}^{\text{sca}}$ is the electric field scattered from the particles. The Foldy-Lax equations expand the scattered field into a sum of terms describing how each particle interacts with the field scattered from each other particle,

$$\boldsymbol{E}^{\text{sca}}(\boldsymbol{r}) = \sum_{i=1}^{N} \int_{V_i} \mathrm{dV}' \mathbf{G}(\boldsymbol{r}, \boldsymbol{r}') \cdot \int_{V_i} \mathrm{dV}'' \mathbf{T}_i(\boldsymbol{r}', \boldsymbol{r}'') \cdot \boldsymbol{E}_i^{\text{exc}}(\boldsymbol{r}''), \tag{3.3}$$

where N is the number of particles in the medium, $\mathbf{T}_i$ is the ith particle's transition operator describing how that particle interacts with the field, and $\boldsymbol{E}_i^{\text{exc}}$ is the electric field exciting the ith particle. The field exciting each particle is in turn expanded as,

$$\boldsymbol{E}_i^{\text{exc}}(\boldsymbol{r}) = \boldsymbol{E}^{\text{inc}}(\boldsymbol{r}) + \sum_{j(\neq i)=1}^{N} \boldsymbol{E}_{ij}^{\text{exc}}(\boldsymbol{r}), \tag{3.4}$$

where $\boldsymbol{E}_{ij}^{\text{exc}}$ are partially exciting fields impinging on particle i from all other particles except for particle i given as,

$$\boldsymbol{E}_{ij}^{\text{exc}}(\boldsymbol{r}) = \int_{V_j} \mathrm{dV}' \mathbf{G}(\boldsymbol{r}, \boldsymbol{r}') \cdot \int_{V_j} \mathrm{dV}'' \mathbf{T}_j(\boldsymbol{r}', \boldsymbol{r}'') \cdot \boldsymbol{E}_j^{\text{exc}}(\boldsymbol{r}''). \tag{3.5}$$

These equations essentially order the scattered field into sums of different levels of interactions between the field and the particles in the distribution. This division into levels of equations is known as the hierarchy of equations. In the first level of the hierarchy, (3.3) sums over each particle and the field impinging on it. However, to calculate the field impinging on one particle, we must sum over how two particles interact in (3.5). This can iteratively be expanded to include as many levels as there are particles in the distribution. To include all of these levels is obviously too cumbersome and cannot be calculated in most cases. At some point in the hierarchy further levels must be discarded, such a procedure is called truncating the hierarchy. Truncating the hierarchy at a specific level of interaction forms the back bone of many approximations used for multiple scattering [54, 91]. It is important to note that the field does not propagate between each particle in a causal time ordered sense, but it can be mathematically expanded as such [66]. Many papers deal with solving the Foldy-Lax equations for different cases and types of particles [15, 27, 89, 108].

3.3 T-matrix

In the previous section, the Foldy-Lax equations were introduced. These equations formally solves the multiple scattering problem. However, the form and structure of the integral equations are cumbersome, and the equations do not lend themselves to a straightforward numerical implementation. During the last decades, an alternative, more direct formulation has been popular. This is the Null-field approach, the Waterman method, or the T-matrix method. This approach focuses on the expansion coefficients of the incident or exciting fields and the scattered field in terms of appropriate spherical vector waves. The method was originally formulated by Peter Waterman in a series of well-known papers [106, 107, 109, 110], and it utilizes the integral representation of the solution, especially its extinction part, systematically. An explicit way of constructing the mapping (transition matrix) between the expansion coefficients of the incident and the scattered field, respectively, is presented. Later, this method has been extended to deal with multiple scattering [78]. A brief introduction to the original method and to the treatment of the multiple scattering case are given in [50, Ch. 9]. The method has now matured, and comprehensive databases are available [69, 70, 71, 72, 73, 75, 114]. Alternative approaches to find the transition matrix based on the far field are presented in [31, 33, 34]. Utilizing the T-matrix method in calculating multiple scattering is a computationally powerful tool and is widely used [9, 16, 39, 47, 49, 51, 85, 103, 104].

3.4 Radiative Transfer

Multiple scattering is analyzed in two different ways, analytical theory, the main topic of this report, and transport theory. Analytical theory is based on deriving a solution to the scattering problem from Maxwell's equations under certain assumptions. Transport theory [11, 40, 68, 82] is not based on the same mathematical foundation and is instead a heuristic theory based on the transfer of energy. The

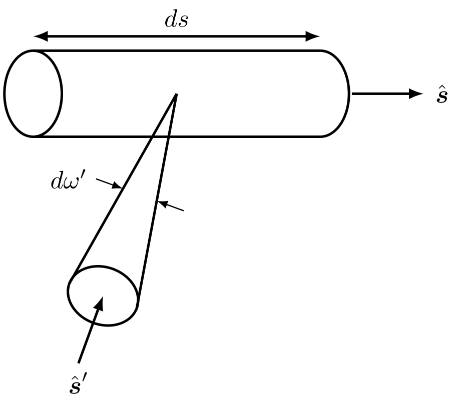

Figure 1: Illustration of the cylinder encapsulating the beam of energy in the material. The variable $\hat{\boldsymbol{s}}$ denotes the considered direction of radiation, ds is the cylinder length which the intensity variation is considered for, $\mathrm{d}\omega'$ is the angle element integrated over when considering scattering, and $\hat{\boldsymbol{s}}'$ is the direction of incoming incoherent scattering integrated over.

main equation, the radiative transfer equation, is an energy balance relation for a beam of radiation, where terms for each source or loss of energy are balanced. This beam of radiation can be particles scattered through a medium as well as waves. The theory was first developed to deal with particle scattering in astronomy. However, we consider here the radiative transfer equation for electromagnetic radiation. It has been shown that, even though analytical theory and transport theory have different origins, there exists a relationship between them [41]. It is in fact even possible to derive radiative transfer equations from Maxwell's equations [17, 18, 21, 41].

The transfer equation is derived by considering a beam travelling in a medium where each particle lies in the farzone of the other particles. Such a beam will loose energy due to absorption, gain energy by emission from the medium, and distribute its energy due to scattering. From these basic observations, we can consider a cylinder in the material that the beam is travelling through, as in Fig. 1, and write its energy balance as [41],

$$\frac{dI(\boldsymbol{r},\hat{\boldsymbol{s}})}{ds} = -\rho\sigma_{\mathrm{t}} I(\boldsymbol{r},\hat{\boldsymbol{s}}) + \frac{\rho\sigma_{\mathrm{t}}}{4\pi}\int p(\hat{\boldsymbol{s}},\hat{\boldsymbol{s}}')I(\boldsymbol{r},\hat{\boldsymbol{s}})\,\mathrm{d}\omega' + \epsilon(\boldsymbol{r},\hat{\boldsymbol{s}}) \tag{3.6}$$

where I is the specific intensity, ds is the length of the cylinder, $\boldsymbol{r}$ is the spacial coordinate, $\hat{\boldsymbol{s}}$ is the direction of the cylinder, $\hat{\boldsymbol{s}}'$ is the scattering direction integrated over, $\mathrm{d}\omega'$ is the increment of that integration, see Fig. 1, ρ is the density of the particles in the medium, σ_{t} is the extinction cross section, *i.e.*, the sum of the absorption and scattering cross sections, $p(\hat{\boldsymbol{s}},\hat{\boldsymbol{s}}')$ is the phase function, and $\epsilon(\boldsymbol{r},\hat{\boldsymbol{s}})$ is the power radiated by the medium. The first term on the right-hand side of (3.6) is the power lost due to absorption and scattering from particles in the beams path, it relates directly to the density of particles in the medium. The second part of the right hand side is an integral that sums all the contributions to the intensity of the beam due to scattering incident on the cylinder, due to scattering by all other particles in the configuration. Lastly, $\epsilon(\boldsymbol{r}, \hat{\boldsymbol{s}})$ describes the contribution to the intensity due to power radiated by the particles inside the cylinder. This equation is formulated for a scalar intensity, but the radiative transfer equation can also be written in vector form for the stokes parameters [67].

The radiative transfer equation is a systematic way of considering multiple scattering by random media. It can be very useful when exact modelling of all physical processes, such as in analytical theory, is too complex. This has lead to a multitude of papers and results utilizing this method [3, 44, 83], including many experimental studies [29, 48, 80]. However, the equation has never been solved exactly due to the inherent complexity of the integro-differential equation, and some assumptions must always be made.

4 Approximations

To evaluate multiple scattering analytically some assumptions or approximations have to be made. This is due to the immense amount of information that needs to be known to exactly calculate the scattering from a random distribution of particles. For example, all the particles in a cloud have a position, and a certain shape. It is clear that some of this information must be discarded in order to calculate the scattered field from such a distribution. In this section we will cover some of the most important approximations used within multiple scattering.

4.1 Truncating the Hierarchy of Equations

The hierarchy of equations is a structure that arises from the Foldy-Lax equations introduced in Sec. 3.2. This hierarchy essentially orders how field information is calculated based on how many particles are fixed in the distribution. The total scattered field is calculated in (3.3), by fixing one particle, the field with one particle fixed is calculated in (3.4) with two particles fixed, and so on. This can go on, fixing the positions of more and more particles, until all the particle positions are fixed. Resolving the calculation in this way is equivalent to evaluating the scattered field using all the available information. In practice, what is done is to truncate the hierarchy by discarding any extra information after a certain point in the hierarchy, *e.g.*, calculate the total scattered field by fixing one particle. This works because the particles are indistinguishable from each other when only one of them are fixed. Thus allowing us to calculate the operators in (3.4) once and reusing the results for all other particles.

There are two main ways of truncating the hierarchy of equations that are widely used. The first one was introduced by Foldy [28] and is called the effective field approximation or Foldy's approximation. This is the simplest way to truncate the hierarchy of equations. The approximation assumes that full field information can be gained by fixing one particle. This approximation works well for tenuous matter and estimates some quantities correctly, but has been shown to be insufficient for many problems [91, 108].

The second method of truncating the hierarchy of equations is known as the quasi-crystalline approximation and was introduced to multiple scattering by Lax [54]. This approximation truncates the hierarchy one step down, when two particles are fixed. This is equivalent to truncating the hierarchy of equations at (3.5). When two particles are fixed instead of one, their relation in regards to each other must be taken into account and are therefore evaluated in pairs. This gives special importance to how particles are located in a distribution in relation to each other, which is discussed in Sec. 5. Many papers using the quasi-crystalline approximation have been published and its validity has been proven theoretically [62, 91], and experimentally [112]. This approximation is exact when particles lie on a crystal lattice, a fact that may contribute to its name [37, 54].

4.2 Twersky approximation

The Twersky approximation is an approximation that simplifies how the field interacts with particles inside a distribution. When a field scatters by a random distribution, it interacts with the distribution as a whole to create the scattered field. We can imagine this complex interaction as a casual series of events, *i.e.*, the field is scattered by one particle, creating a scattered field that then interacts with a second particle and so on. In such a scenario, the field could scatter back against a particle that it had already interacted with. The Twersky approximation simplifies the field interaction by excluding all such contributions that interact multiple times with the same particle. However, this is not a physical way of regarding the scattered field [66], but it is possible to decompose it mathematically to terms that can be viewed in this manner. If all particles in a distribution are assumed to lie in the far zones of each other, and we are observing the scattered field in the far zone of all the particles, a Neumann expansion of the far-field can be applied to the field [66]. This expansion divides the field into different terms that can be seen as the field interacting with different numbers of particles in the distribution. The Twersky approximation discards the terms in this expansion that correspond to multiple interactions with the same particle [96, 98, 99, 100]. The approximation is increasingly accurate for large particle groups where it includes most interaction terms [91].

4.3 Low Frequency Approximation and Bulk Parameters

When observing complex materials or particle distributions, it is often desirable to simplify their description. For random distributions this can be done by replacing the complex distribution of particles that have different permittivities by an effective permittivity or material parameter [52]. These method is known as calculating the bulk parameters of a medium [97], and includes effective wave number approximations. If the frequency is low, the effective permittivity adequately describes the electromagnetic properties of the medium [91]. Therefore, many of these homogenization methods are known as low frequency approximations [57, 63, 88].

However, due to the widely varying differences in material parameters in discrete random distributions, this method often provides inadequate results. A different way to simplify the evaluation of scattering by random distributions is to calculate an effective wave number for the waves propagating through them [12, 38, 55, 58, 99, 108]. This can be done by assuming the existence of such a solution [19, 20]. However, it has been proven that more than one such solution exists for random media [37].

5 Particle Distribution

When simulating random distributions of particles, their distributions have to be modeled. This essentially means that they must be confined in a certain volume. If they are assumed to be distributed evenly, the probability density of a particle to exist at any point in that medium is

$$p(\boldsymbol{r}) = \frac{1}{V}, \tag{5.1}$$

where V is the volume of the confinement. However, once the first particle has been located the probability of the second particle to exist can be described in relation to the first particle as,

$$p(\boldsymbol{r}, \boldsymbol{r}') = \frac{g(\boldsymbol{r}, \boldsymbol{r}')}{V^2} \frac{N}{N-1}, \tag{5.2}$$

where $g(\boldsymbol{r}, \boldsymbol{r}')$ is the pair distribution function, and N is the number of particles in the medium. Different models of $g(\boldsymbol{r}, \boldsymbol{r}')$ form another cornerstone for the approximations used in multiple scattering [95].

5.1 Pair distribution estimation

Pair distribution is a measure of how the positions of different particles in a distribution relate to each other. This topic is of great interest in statistical mechanics [64]. Accurately describing matter in its different states, gas, liquid, and amorphous and crystalline solids, is done by using appropriate pair distributions functions. Gas, for example, is matter in extreme disorder where a function with a great amount of freedom is appropriate. Whereas, particles in crystalline solids are completely fixed in relation to each other, and therefore require a different type of pair distribution function [95].

One of the simplest pair distribution methods is known as the *hole correction method*. The hole correction approximation prevents overlap between two circumscribed spheres of any two particles in the medium, but otherwise does not restrict particle position [26, 55, 95]. Such a pair distribution function is appropriate when dealing with gases. This method gives good results for effective permittivity estimations, but does not adequately describe all random media and produces errors when calculating electromagnetic attenuation rates [89]. Many more complex pair distribution functions exist, notably the Perkus-Yevick equation [15, 77, 111] based on forces in statistical mechanics [64]. Monte Carlo type simulations are also used for random particle distributions and their multiple scattering [94].

5.2 Tenuous and Dense Media

Tenuous and dense media are the two classes of random media that are treated in the literature. Tenuous media are media with a small number of particles. On the other hand, dense media contain many particles and therefore cause a high degree of scattering. Each of these cases cause different difficulties and can be handled by appropriate approximations.

Tenuous media approximations are typically modeling media encountered in weather phenomena, such as mist, clouds or rain. In such cases, a wave travelling through the medium interacts with very few or none of the particles present. Therefore, it can be appropriate to use the single particle approximation [41] that assumes that waves only scatter at most once when travelling through a distribution. This greatly simplifies the calculations required to estimate the scattered field and has been used with radiative transfer, see Sec. 3.4, in both weather radar [5, 8] and ocean acoustics [14, 87]. In tenuous media, the effective field approximation, see Sec. 4.1, is also accurate. However, it has been observed that these approximations are not adequate once the particle density in the media increases [35, 43, 46, 59, 112]. For increased particle density, the particles in the media are correlated to each other and a greater degree of multiple scattering occurs. These effects can be compensated for by introducing a correction term based on the Percus-Yevick approximation [77, 111]. This produces accurate results at optical frequencies for tenuous media [29, 80].

Dense media is media where the volume fraction of the particles is above 5% [112]. To accurately calculate the scattering from such media, low scattering approximations cannot be used. To include all of the physics of the field interacting with the particles approximations such as the effective field approximation or the Twersky approximation cannot be employed. Instead the quasi-cyrstalline approximation is more appropriate [6, 15, 16, 35, 43, 46, 59].

6 Open Access Codes

Many codes have been created to evaluate electromagnetic scattering. A repository of these, together with articles and conference presentations can be found at Scattport.org. A comprehensive archive of Null-field methods for evaluating the T-matrix of different scatterers and distributions can be found in [69, 70, 71, 72, 73, 75, 114].

7 Conclusions

In this literature study the basics of multiple scattering is covered. We have briefly introduced the two main methods of evaluating multiple scattering, radiative transfer theory and analytical theory. Even though these are connected they represent a major difference in how multiple scattering is treated. Radiative transfer theory is based on heuristics and has been thoroughly investigated. Analytical theory is derived directly from the Maxwell equations and caries with it a more exact formulation of the multiple scattering problem. However, due to this greater theoretical and numerical complexity there is still a lot of work to be done to definitively model multiple scattering.

Research in multiple scattering has its roots in the mid 20th century, and many papers have been published since then. Currently there is great interest in the field and many new papers are being published. This literature study serves as an introduction to the existing papers and should provide a guide for further inquiry.

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