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## Splitting of dissipative evolution equations

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(joint work with Tony Stillfjord)

The aim of this note is to give an overview of some recent progress when analyzing splitting schemes applied to nonlinear evolution equations. Such equations are frequently encountered in biology, chemistry and physics, as they describe reaction-diffusion systems, as well as the damped wave equation. More precisely, we will consider splitting based discretizations for evolution equations of the form

$$(1) \quad \dot{u} = (F + G)u, \quad u(0) = u_0,$$

where  $F$  is typically the vector field of a nonlinear diffusion process and  $G$  is a nonlinear source term. Due to the nonlinearity of  $F + G$ , we can not expect the solution to exhibit any higher-order temporal regularity. For example, if

$$(F + G)u = \Delta(|u|^m u) + 0,$$

then the solution of (1) is given by the classical Barenblatt solution which is not continuously differentiable in time nor space. Because of the lack of time-regularity it is not, in general, possible to prove that a time discretization of the problem converges with an order greater than  $p = 1$ . Furthermore, due to the presence of diffusion, a spatial discretization of the equation will result in a stiff ODE system and therefore implicit schemes are required. Of the few remaining numerical methods the implicit Euler scheme is then the natural choice, but it is often computationally costly. An alternative is given by splitting methods, where the flows related to  $F$  and  $G$  are approximated separately. This can dramatically reduce the computational cost. We consider several (formally) first-order splitting schemes given by the time stepping operators

$$S_h = (I - hF)^{-1}P_{hG},$$

where  $S_h^n u_0$  is an approximation of the solution  $u$  at time  $t = nh$  and the operator  $P_{hG}$  is chosen depending on the structure of  $G$ . Three standard choices are  $P_{hG} = I + hG$ ,  $(I - hG)^{-1}$  or  $e^{hG}$ .

The foundation of our numerical analysis is to assume that (1) is given on a real Banach space  $X$  and the nonlinear operator  $F + G$  is  $m$ -dissipative, i.e., the resolvent

$$R_h = (I - h(F + G))^{-1}$$

is defined on  $X$  and it has a Lipschitz constant of the form  $L[R_h] \leq 1 + Ch$  for sufficiently small values of  $h$ . The  $m$ -dissipativity of the operator  $F + G$  guarantees the existence of a unique mild solution to (1), which is one of the core results of the nonlinear semigroup theory [1, 4]. The main idea of the existence proof is to establish that  $\{R_{t/n}^n u_0\}_{n \geq 1}$  is a Cauchy sequence in  $X$ , and then define the mild solution as

$$u(t) = \lim_{n \rightarrow \infty} R_{t/n}^n u_0.$$

In other words, the solution is given by the limit of the implicit Euler discretization. Furthermore, as part of the original proof [4, Theorem I] a convergence order of  $p = 1/2$  was derived for the implicit Euler approximation  $R_h^n u_0$ , i.e.,

$$\|R_h^n u_0 - u(nh)\| \leq C_T h^{1/2}, \quad 0 \leq nh \leq T,$$

when  $u_0 \in \mathcal{D}(F + G)$ . Note that a convergence order of  $p = 1$  is typically observed when  $X$  is finite dimensional, but the obtained order of one half is in fact optimal for general  $m$ -dissipative vector fields, as exemplified by [8].

The convergence of splitting schemes can be proven in this  $m$ -dissipative framework, see e.g. [3] where the case  $P_{hG} = (I - hG)^{-1}$  is treated. Also [2] and [9] prove convergence of several similar splitting schemes in different contexts. However, the aim of this work is to prove *orders of convergence*. Our basic idea, which unifies the theory for different  $P_{hG}$ , is to prove that the splitting approximation  $S_h^n u_0$  is within an  $\mathcal{O}(h^q)$ -vicinity of the implicit Euler approximation  $R_h^n u_0$  when the operator  $G$  has some further structure:

**Theorem 1.** *Let  $F$ ,  $G$  and  $F + G$  all be  $m$ -dissipative operators on the real Banach space  $X$ ,  $u_0 \in \mathcal{D}(F + G)$  be given and  $h \leq h_0$ . If  $P_{hG}$  is stable, i.e.,  $L[P_{hG}] \leq 1 + Ch$ , and satisfies the consistency bound*

$$(2) \quad \|(hGR_h + I - P_{hG})R_h^j u_0\| \leq Ch^{1+q},$$

for all  $j = 0, \dots, n$ , then

$$\|S_h^n u_0 - u(nh)\| \leq C_T (h^p + h^q), \quad 0 \leq nh \leq T,$$

where  $u$  is the mild solution of (1) and  $p \in [1/2, 1]$  is the convergence order of the implicit Euler scheme.

The proof follows by the telescopic sum

$$\begin{aligned} \|R_h^n u_0 - S_h^n u_0\| &\leq \sum_{j=1}^n \|S_h^{n-j} R_h^j u_0 - S_h^{n-j+1} R_h^{j-1} u_0\| \\ &\leq \sum_{j=1}^n L[S_h]^{n-j} L[(I - hF)^{-1}] \|((I - hF)R_h - P_{hG})R_h^{j-1} u_0\| \\ &\leq C \sum_{j=1}^n \|(hGR_h + I - P_{hG})R_h^{j-1} u_0\|. \end{aligned}$$

While the stability assumption  $L[P_{hG}] \leq 1 + Ch$  is natural for a well-defined scheme, the question of when the consistency (2) is true remains. We give three different examples of this, which illustrate the wide applicability of the theory.

**Example 1.** Let  $G$  be Lipschitz continuous and choose  $P_{hG} = I + hG$ . This gives the implicit-explicit Euler method where the diffusive term  $F$  is approximated by the implicit scheme and the non-stiff perturbation by the explicit scheme. In this case it is seen that Equation (2) holds with  $q = 1$ . For example, the evolution of  $s$  competing species can be modeled by the system

$$\dot{u}_\ell = \Delta(u_\ell)^{m+1} + G_\ell(u_1, u_2, \dots, u_s), \quad \ell = 1, \dots, s,$$

which can be cast into the setting of an  $m$ -dissipative evolution equation on  $X = [L^1(\Omega)]^s$ . Splitting the system means that the diffusion terms decouple, and hence their approximations can be parallelized. See [5] for details.

**Example 2.** A nonlinear parabolic equation with delay such as

$$\dot{u}(t) = \nabla \cdot (|\nabla u(t)|^m \nabla u(t)) + G_1 u(t-1) + G_2 \int_{-1}^0 u(t+s) ds$$

can be formulated on an appropriate Banach space so that the operators are again  $m$ -dissipative. With  $P_{hG} = (I - hG)^{-1}$ , we get that the consistency assumption (2) holds with  $q > 1/2$ . Such equations e.g. model electrical circuits and more realistic population dynamics which takes gestation periods into account. We refer to [6].

**Example 3.** The abstract Riccati equation

$$\dot{u} = A^* \circ u + u \circ A + v - u \circ u$$

can be treated in the setting of Hilbert-Schmidt operators on  $L^2(\Omega)$  when  $-A$  is a linear elliptic differential operator. If  $Fu = A^* \circ u + u \circ A + v$  and  $Gu = -u \circ u$ , respectively, then  $P_{hG} = e^{hG}$  has a closed-form expression and it can be proven that Equation (2) holds with  $q = 1$ . Riccati equations arise in e.g. linear quadratic regulator problems, where their solutions provide the link between the system states and the optimal feedback. For more details, see [7].

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