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Scattering from inhomogeneities below a non-planar interface

by

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Abstract

A previously developed formalism for calculation of the scattering from a buried inhomogeneity is taken as a starting point for an iterative scheme for treating the influence of a deviation from a flat interface. This deviation can be a hill, or a depression, of finite extent on an otherwise flat surface. Each iteration takes into account all the multiple interactions between the hill and the inhomogeneity. Extensions in various directions are briefly discussed.

I. Introduction

In previous papers a general formalism for studying scattering from buried inhomogeneities has been developed [1], [2]. This formalism is in principle applicable to very general geometrical shapes of the air-ground interface. However, in order to solve a general case one has to invert a two-dimensional integral transform and this usually constitutes a formidable analytic and numerical problem. It is therefore of great interest to find special approximate methods which apply to specific classes of non-planar surfaces. We recall that for a plane ground surface, the integral transformation degenerates and an explicit algebraic inversion can be made [1]. We also note that this inversion is independent of the inhomogeneity.

In the present article we give a preliminary report on approximate methods of solution for the case when the deviation from the plane is of finite extent. In the approximation introduced here an algebraic solution similar to the one for the planar case can still be obtained. As a consequence, the interference between the influence of the inhomogeneity and on the other hand of the deviation from a plane surface can be studied. Since the approximation is expected to become more accurate as the wavelength increases, our method is expected to be useful e.g. in VLF prospecting situation, where the wavelength is of the order of 10-30 km.

II. Statement of the problem

For simplicity we consider the case of a scalar wave (as is clear from [2], the electromagnetic case can be treated in an analogous fashion). For the configuration depicted in Fig. 1, the solution of the stationary scattering problem, with the source lying above the ground surface S, is given in terms of the following quantities (cf. Ref. [1] for more details).

i) $a(\vec{k}_o)$, $f(\vec{k}_o)$: the plane wave expansion coefficients of the incoming and scattered field respectively (harmonic and evanescent plane waves)

ii) $\alpha(k)$, $\beta(k)$: the plane wave expansion coefficients of the field ψ_1^- on the lower side of S.

iii) the operator

$$Q(\vec{k}_0,\vec{k}_1) \equiv \frac{k_0}{8\pi^2} \int_S \left[e^{-i\vec{k}_0\cdot\vec{r}'} \nabla'\left(e^{i\vec{k}_1\cdot\vec{r}'}\right) - C_{01}\left(\nabla' e^{-i\vec{k}_0\cdot\vec{r}'}\right) e^{i\vec{k}_1\cdot\vec{r}'} \right]\cdot d\vec{S}' \tag{1}$$

which can be thought of as describing the passage of a plane wave through S from below. Similarly its inverse Q^{-1} can be thought of describing a passage thorugh S from above. Furthermore, the reflection coefficient of S for plane waves coming from above is in this notation formally given by

$$\tilde{R}(\vec{k}_0,\vec{k}_0') \equiv -\int_0^{2\pi} d\beta_1 \int_{C_-} Q(\vec{k}_0,\vec{k}_1)\, Q^{-1}(\vec{k}_0',\vec{k}_1) \sin\alpha_1\, d\alpha_1$$

$$\hat{k}_0 \in C_+ ,\ \hat{k}_0' \in C_- \tag{2}$$

Similarly, the reflection coefficient of S for waves coming from below is

$$\mathcal{R}(\vec{k}_1,\vec{k}_1') \equiv -\int_0^{2\pi} d\beta_0 \int_{C_-} Q(\vec{k}_0,\vec{k}_1') Q^{-1}(\vec{k}_0,\vec{k}_1) \sin\alpha_0\, d\alpha_0$$

$$\hat{k}_1 \in C_-\,, \hat{k}_1' \in C_+ \tag{3}$$

iv) the transition matrix $T_{nn'}(1)$ of the inhomogeneity, referring to spherical waves.

The scattered field above $Z=Z_>$ is then given by

$$\psi^{sc}(\vec{r}) =$$

$$\int_0^{2\pi} d\beta_0 \int_{C_+} \Big\{ \int_0^{2\pi} d\beta_0' \int_{C_-} \tilde{\mathcal{R}}(\vec{k}_0,\vec{k}_0') a(\vec{k}_0') \cdot \sin\alpha_0'\, d\alpha_0' +$$

$$+ 2\sum_{nn'} i^{n'-n-1} T_{nn'}(1) C_{n'} \cdot \int_0^{2\pi} d\beta_1 \int_{C_+} Y_n(\hat{k}_1) \Big[Q(\vec{k}_0,\vec{k}_1) +$$

$$+ \int_0^{2\pi} d\beta_1' \int_{C_-} \mathcal{R}(\vec{k}_1',\vec{k}_1) Q(\vec{k}_0,\vec{k}_1') \sin\alpha_1'\, d\alpha_1' \Big] \sin\alpha_1\, d\alpha_1 \Big\} e^{i\vec{k}_0\cdot\vec{r}} \sin\alpha_0\, d\alpha_0$$

$$\tag{4}$$

Here C_n is the spherical wave projection of $\alpha(\vec{k}_1)$. The $\beta(\vec{k}_1)$:s have been expressed in terms of $\alpha(\vec{k}_1)$ i.e. C_n by means of the scattering properties of the inhomogeneity. Furthermore, C_n is determined in terms of the incoming field by taking into account the scattering properties of the inhomogeneity and of the ground surface. One obtains a equation of the form

$$C_n = d_n - \sum_{n'} A_{nn'} C_{n'} \tag{5}$$

where d_n represents the spherical wave components of the incoming field after it has passed through S. $A_{nn'}$ is independent of the fields and depends only on the geometry and scattering properties of S and S_1 [1].

From (1) it can be seen that in the special case of a plane surface S, $Q(\vec{k}_o,\vec{k}_1)$ degenerates and becomes proportional to $\delta^{(2)}(\hat{n}\times(\vec{k}_o-\vec{k}_1))$ (which is just an expression of Snell's law of refraction at a plane interface). Several of the previous expressions then simplify substantially and one obtains for instance

$$R(\vec{k}_1,\vec{k}_1') = -R(\lambda_1)\exp\{i2z_o(k_1^2-\lambda_1^2)^{1/2}\}\cdot\delta(\beta_1-\beta_1')\cdot$$
$$\cdot\,\delta(\alpha_1+\alpha_1'-\pi)/\sin\alpha_1 \qquad , \hat{k}_1\in C_-, \hat{k}_1'\in C_+ . \tag{6}$$

Here $Z=Z_o$ defines the plane and $R(\lambda_1)$, $\lambda_1=k_1\sin\alpha$, is the well-known reflection coefficient of the plane [1]. Refs. [1] and [2] contain further details of the resulting expression for ψ^{sc}, as well as numerical results for particular inhomogeneities.

From (2), (3) and (4) it is seen that in order to obtain an explicit solution for ψ^{sc} for a general surface S, it is necessary to invert, in an explicit manner, integral transformations with the kernel $Q(\vec{k}_o,\vec{k}_1)$. Since, furthermore, the functions on which this kernel acts are of a complicated nature, we wish to find approximate procedures useful for treating particular classes of nonplanar surfaces S. In these procedures we want to make maximum use of the solution obtained previously for the planar case.

One generalization of the planar case which immediately suggests itself is when the deviation from a plane is confined to a finite region. It is then natural to try to use the planar case as a starting point in an iterative scheme and we shall later concentrate on that possibility.

However, we first make a few remarks on some general features of the problem. Consider the case when there is no inhomogeneity (cf. Fig. 2). Several approaches are possible here. First we note that one might treat the situation in Fig. 2 as a limiting case of that in Fig. 3. This latter case can be treated by means of the formulation in Ref. [1] by making use of the alternative which applies when the source and the inhomogeneity lie in the same half-space and by then taking the limit of the lower part of the hill approaching the plane. However for several reasons this approach is expected to be of limited use. The main reason for this is that one would then violate the geometrical constraints which are basic to the T matrix approach. Suitable modifications of the geometry etc could be considered but this would then jeopardize the simplicity and usefulness of the results. One may note, however, that in applying the method of Ref. [1], the case when the hill consists of a material different from that of the lower half-space could also be accommodated. In general, comparison with the method of Ref. [1] mainly serves to indicate a way of looking at the problem and to define a possible set of quantities which can be used in constructing the solution, even if this is not made exactly as in Ref. [1]. For instance, for a low hill one would, in a direct application of the T matrix method, want to compute the T matrix of a very oblate object which would increase the demand on the matrix size. However one could here still use expansions for the fields on the top and the botton of the hill in terms of a discrete set of functions and then eliminate these by some procedure different from that of forming the T matrix for the hill. A search for suitable discretization methods for the hill problem is presently pursued in related work.

We thus conclude that it is desirable to develop an approximate approach which differs from the more or less direct application of the transition matrix scheme of Ref. [1].

III. Approximate treatment of the influence of the hill

One obvious way of making use of the plane surface solution is to write the integrals over S in the expression (1) for $Q(\vec{k}_o,\vec{k}_1)$ as one integral over the plane, cutting through the hill, plus the difference between the integrals over the hill and over the bottom of the hill (cf. Fig. 2). Let S_p denote the flat part of S and let ΔS be the hill so that $S=S_p+\Delta S$. Let furthermore S_o denote the whole plane and ΔS_o that part of S_o while lies underneath ΔS, so that $S_o=S_p+\Delta S$. As is easily seen, the treatment given below applies to a depression as well as to a hill. However, for brevity we shall always refer to the finite deviation from the plane as the "hill".

It is instructive to first consider the case when the two half-spaces separated by S are homogeneous. We construct an approximation scheme for this case and then study how the full solution of the "plane surface plus buried inhomogeneity" - problem can be introduced into this scheme so that interactions between the hill and the inhomogeneity can be taken into account. In the absence of an inhomogeneity we have [1]

$$a(\vec{k}_o) = +i \int_0^{2\pi} d\beta_1 \int_{C_-} \alpha(\vec{k}_1) Q(\vec{k}_o,\vec{k}_1) \sin\alpha_1 \, d\alpha_1 \quad , \hat{k}_o \in C_- \tag{7}$$

According to the division

$$\int_S [\ldots]\cdot d\vec{S} = \int_{S_p} [\ldots]\cdot d\vec{S} + \int_{\Delta S_o} [\ldots]\cdot d\vec{S} + \int_{\Delta S} [\ldots]\cdot d\vec{S} - \int_{\Delta S_o} [\ldots] d\vec{S} =$$

$$= \int_{S_o} [\ldots]\cdot d\vec{S} + \int_{\Delta S-\Delta S_o} [\ldots]\cdot d\vec{S}$$

of an integral over S, we have a corresponding division of $Q(\vec{k}_o, \vec{k}_1)$:

$$Q(\vec{k}_0, \vec{k}_1) = Q_0(\vec{k}_0, \vec{k}_1) + \Delta Q(\vec{k}_0, \vec{k}_1)$$
$$\hat{k}_0 \in C_-, \hat{k}_1 \in C_+ \qquad (8)$$

where Q_o is the Q-function obtained in the plane surface case (as was noted above, $Q_o \sim \delta^{(2)}(\hat{n}_o \times (\vec{k}_o - \vec{k}_1)))$. If the decomposition (8) is introduced into (7) the first term on the right hand side is

$$i\int_0^{2\pi}\int_{C_-} \alpha(\vec{k}_1)\, Q_0(\vec{k}_0, \vec{k}_1) \sin\alpha_1 \, d\alpha_1 \equiv D(\lambda)\, \alpha(\tilde{\vec{k}}_1) \qquad (9)$$

where $\lambda = k_o \sin\alpha_o = k_1 \sin\alpha_1, \lambda \in (0,\infty)$,

$$D(\lambda) = \frac{k_0\left[C_{01}(k_0^2-\lambda^2)^{1/2} + (k_1^2-\lambda^2)^{1/2}\right]}{2k_1(k_1^2-\lambda^2)^{1/2}}\, e^{i z_0\left[(k_0^2-\lambda^2)^{1/2}-(k_1^2-\lambda^2)^{1/2}\right]} \qquad (10)$$

and where $\tilde{\vec{k}}_1$ denotes the "Snell-transformed" $\vec{k}_1$ vector i.e.

$$\tilde{\vec{k}}_1 = (k_0 \sin\alpha_0 \cos\beta_0, k_0 \sin\alpha_0 \sin\beta_0, -(k_1^2-\lambda^2)^{1/2}) \qquad (11)$$

Thus the relation between $a(\vec{k}_o)$ and $\alpha(\vec{k}_1)$ can be written

$$a(\vec{k}_0) = D(\lambda)\,\alpha(\tilde{\vec{k}}_1) + i\int_0^{2\pi} d\beta_1 \int_{C_-} \alpha(\vec{k}_1)\, \Delta Q(\vec{k}_0, \vec{k}_1) \sin\alpha_1 \, d\alpha_1$$
$$\vec{k}_0 \in C_- \qquad (12)$$

So far, no approximation has been made. Eq. (12) can be regarded as an integral equation of the second kind for α. The driving term $a(\vec{k}_o)\cdot N^{-1}(\lambda\ )$ corresponds to the solution for the planar case. If one is primarily interested in solutions for the long wavelength case (as in VLF prospecting), the solution for the planar case is a suitable starting point for an interative approach to (12). We note that, according to (1) and (8), Q contains terms proportional to k_o^2 and $k_o \cdot k_1$ and therefore "long wavelength" in this context will mean that λ^o, λ^1, where $\lambda^i = 2\pi k_i^{-1}$, $i=0,1$, are greater than the main dimensions of ΔS. The once-iterated solution of (12) is thus obtained by introducing the approximation $\alpha = a\cdot N^{-1}$ in the integrand. The integration involves only the direction angles of the $\vec{k}$ vectors and with notation $a(\vec{k}_o) \equiv a(k_o, \alpha_o, \beta_o)$ etc, the first correction term can be written

$$\Delta^1 \equiv \int_0^{2\pi} d\beta' \int_{C_-} \sin\alpha' d\alpha' \, \frac{a(k_o, \alpha', \beta')}{D(\lambda'; k_o, k_1)} \Delta Q(k_o, \alpha_o, \beta_o; k_1, \alpha', \beta') \tag{13}$$

$(\lambda' \equiv k_o \sin\alpha')$ and the first order approximation for α is $\alpha^1 = (a - \Delta^1)\cdot N^{-1}$. This process can now be continued, but in view of the computer time requirements which are typical for iteration schemes, one would in practice have to limit one-self to cases where one or two iterations would suffice (the drawback is of course that the solution obtained in each step must be computed at sufficiently many $\vec{k}$-values for the integration in the next step to be accurate enough).

IV. The combined influence of the hill and the inhomogeneity

In the previous section the known solution for a plane interface was taken as the starting point for the iteration. Since the solution for a plane interface plus an inhomogeneity is also known, it is natural to investigate the possibility of using this solution as a starting point in a similar iteration process which would then include both the influences of the inhomogeneity and of the hill, as well as interactions between these two structures.

When V_1 contains an inhomogeneity, the surface field just below S is described in terms of two sets of plane wave coefficients, $\alpha(\vec{k}_1)$ and $\beta(\vec{k}_1)$ and instead of (7) one has now [1]

$$a(\vec{k}_0) = i \int_0^{2\pi} d\beta_1 \left[\int_{C_-} \alpha(\vec{k}_1) + \int_{C_+} \beta(\vec{k}_1) \right] Q(\vec{k}_0, \vec{k}_1) \sin\alpha_1 \, d\alpha_1 \qquad \hat{k}_0 \in C_- \tag{14}$$

Here $\beta(\vec{k}_1)$ represents the total upgoing component of the surface field. This component is built up from reflections from the inhomogeneity. The technical expression of this fact is the following relation between β and the spherical projections C_n of α.

$$\beta(\vec{k}_1) = 2 \sum_{nn'} i^{n'-n} Y_n(\hat{k}_1) T_{nn'}(1) C_{n'} \tag{15}$$

Thus, (14) can be written

$$a(\vec{k}_0) = i\int_0^{2\pi} d\beta_1 \Big[\int_{C_-} \alpha(\vec{k}_1) +$$

$$+ \int_{C_+} \ell \sum_{nn'} i^{n'-n} Y_n(\hat{k}_1) T_{nn'}(1) C_{n'} \Big] Q(\vec{k}_0, \vec{k}_1) \sin\alpha_1 d\alpha_1$$

$$\hat{k}_0 \in C_- \tag{16}$$

We now wish to make use of the separation $Q=Q_o+\Delta Q$ in an approximate treatment of (16). We note again that the crucial problem is that of inverting Q in the term $\int_{C-} \alpha \cdot Q$ in (16). In the other terms where α has been projected into the C_n:s, it suffices just to calculate Q itself. However, the division $Q=Q_o+\Delta Q$ is still useful here since it separates out one part that will correspond to the $A_{nn'}$-matrix [1] of the plane surface case so that previous calculations concerning that matrix can be used again in the present problem. With $Q=Q_o+\Delta Q$ we thus have

$$a(\vec{k}_0) = \int_0^{2\pi} d\beta_1 \int_{C_-} \alpha(\vec{k}_1) Q_o(\vec{k}_0, \vec{k}_1) \sin\alpha_1 d\alpha_1 +$$

$$+ \int_0^{2\pi} d\beta_1 \int_{C_-} \alpha(\vec{k}_1) \Delta Q(\vec{k}_0, \vec{k}_1) \sin\alpha_1 d\alpha_1 +$$

$$+ \int_0^{2\pi} d\beta_1 \int_{C_+} \sum_{nn'} \ell i^{n'-n} Y_n(\hat{k}_1) T_{nn'}(1) C_{n'} Q_o(\vec{k}_0, \vec{k}_1) \sin\alpha_1 d\alpha_1$$

$$+ \int_0^{2\pi} d\beta_1 \int_{C_+} \sum_{nn'} \ell \cdot i^{n'-n} Y_n(\hat{k}_1) T_{nn'}(1) C_{n'} \Delta Q(\vec{k}_0, \vec{k}_1) \sin\alpha_1 d\alpha_1 \tag{17}$$

The first term on the right hand side of (17) reduces as before to an algebraic expression so that α can be extracted. If we then introduce the flat surface plus inhomogeneity solution into the second term, we obtain an equation which can be solved by a procedure similar to the one used in Ref. [1]. Here we note that if we would also introduce the flat surface plus inhomogeneity solution as an approximation for the C_n:s in the fourth term on the right hand side of (17), we would, after projecting on spherical waves, obtain an equation of the form

$$d_n = C_n + \sum_{n'} A_{nn'} C_{n'} + \delta_n^{(o)} \tag{18}$$

Here $\delta_n^{(o)}$ is what we get by introducing the flat surface plus inhomogeneity solution $\alpha=\alpha_o$ into the two terms in (17) which contain ΔQ. Furthermore $A_{nn'}$ is the matrix given by Eq. (60) in Ref. [1]. This means that in comparison with the flat surface case, the equation for C_n would be modified only a trivial way, namely only a change in the value of d_n. This means, in particular, that the multiple reflections would have exactly the same structure as in the flat surface case (this structure is given by $(1+A)^{-1}$). Such a scheme is expected to give a much poorer result than a scheme in which the influence of the hill is taken into account in each multiple scattering. This is achieved by not making any approximations for C_n in the fourth term on the right hand side of (17), but rather include it in the multiple scattering equation. We then obtain

$$d_n = C_n + \sum_{n'} \tilde{A}_{nn'} C_{n'} + \delta_n \tag{19}$$

i.e.

$$C_n = \left[(1+\tilde{A})^{-1}\cdot \tilde{d}\right]_n \tag{20}$$

where

$$\tilde{A}_{nn'} = A_{nn'} + (\delta A)_{nn'} \tag{21}$$

$$\tilde{d}_n = d_n + \delta_n \tag{22}$$

Again, $A_{nn'}$ and d_n are those given in Ref. [1] for the flat surface plus inhomogeneity case. For δ_n and $(\delta A)_{nn'}$ we get

$$(\delta A)_{nn'} = \int_0^{2\pi} d\beta_1 \int_{C_-} \frac{z_0 k_1}{k_0}(1-R(\lambda))\, e^{-iz_0(k_0^2-\lambda^2)^{1/2}+iz_0(k_1^2-\lambda^2)^{1/2}} \cdot$$

$$\cdot Y_n(\hat{k}_1)\sum_{n''} i^{n'-n''}\cdot T_{n''n'}(1)\int_0^{2\pi} d\beta_1' \int_{C_+} Y_{n''}(\hat{k}_1')\,\Delta Q(\tilde{\vec{k}}_0,\vec{k}_1')\sin\alpha_1'\, d\alpha_1' \tag{23}$$

$$\delta_n = \int_0^{2\pi} d\beta_1 \int_{C_-} \frac{k_1}{k_0}(1-R(\lambda))\, e^{-iz_0(k_0^2-\lambda^2)^{1/2}+iz_0(k_1^2-\lambda^2)^{1/2}} \cdot$$

$$\cdot Y_n(\hat{k}_1)\int_0^{2\pi} d\beta_1' \int_{C_-} \alpha(\vec{k}_1')\,\Delta Q(\tilde{\vec{k}}_0,\vec{k}_1')\sin\alpha_1'\, d\alpha_1' \tag{24}$$

$$\left(\tilde{\vec{k}}_0 \equiv \left(k_1\sin\alpha_1\cos\beta_1,\ k_1\sin\alpha_1\sin\beta_1,\ -(k_0^2-\lambda^2)^{-1/2}\right)\right)$$

By introducing $\alpha=\alpha_0$ in (24), δ_n becomes a known quantity and the solution for C_n is obtained from (20). The corresponding solution for ψ^{sc} is obtained by means of Eq. (54) in Ref. [1] where the separation $Q=Q_0+\Delta Q$ can be used again.

V. Concluding remarks

The approximation scheme suggested above should next be tried numerically in some representative cases. Furthermore it should be extended to the electromagnetic case and tried for parameter choices which are relevant to the VLF prospecting situation.

In the problem treated above, the hill was assumed to consist of the same material as the rest of V_1. However, in geophysical wave propagation problems and in VLF prospecting problems in particular, the situations depicted in Figs. 4 and 5 are of interest. The situation in Fig. 5 can for instance be used as a model for the influence of a lake on the response from an ore body, a problem which is encountered very often in practice. We suggest that similar iteration techniques be developed for these cases. We must warn, however, that although the geometry is in many respects similar to that studied in the present note, this similarity may prove to be deceptive: we know from previously studied multiple scattering problems that the full solution for the configuration in Fig. 4 is considerably more complicated.

In this context we also note that it is natural to try to develope analogous approximation schemes for the case depicted in Fig. 6. Of course, problems of this nature have been treated by a large number of different methods in the literature. However, one is then most often interested in the propagation characteristics along the waveguide with non-constant cross-section, formed by the two surfaces and this is strongly reflected in the methods used. We are mainly interested in the reflection and transmission characteristics across the layer and a formalism like ours where the propagation along the z-axis plays a special role may then have some advantages.

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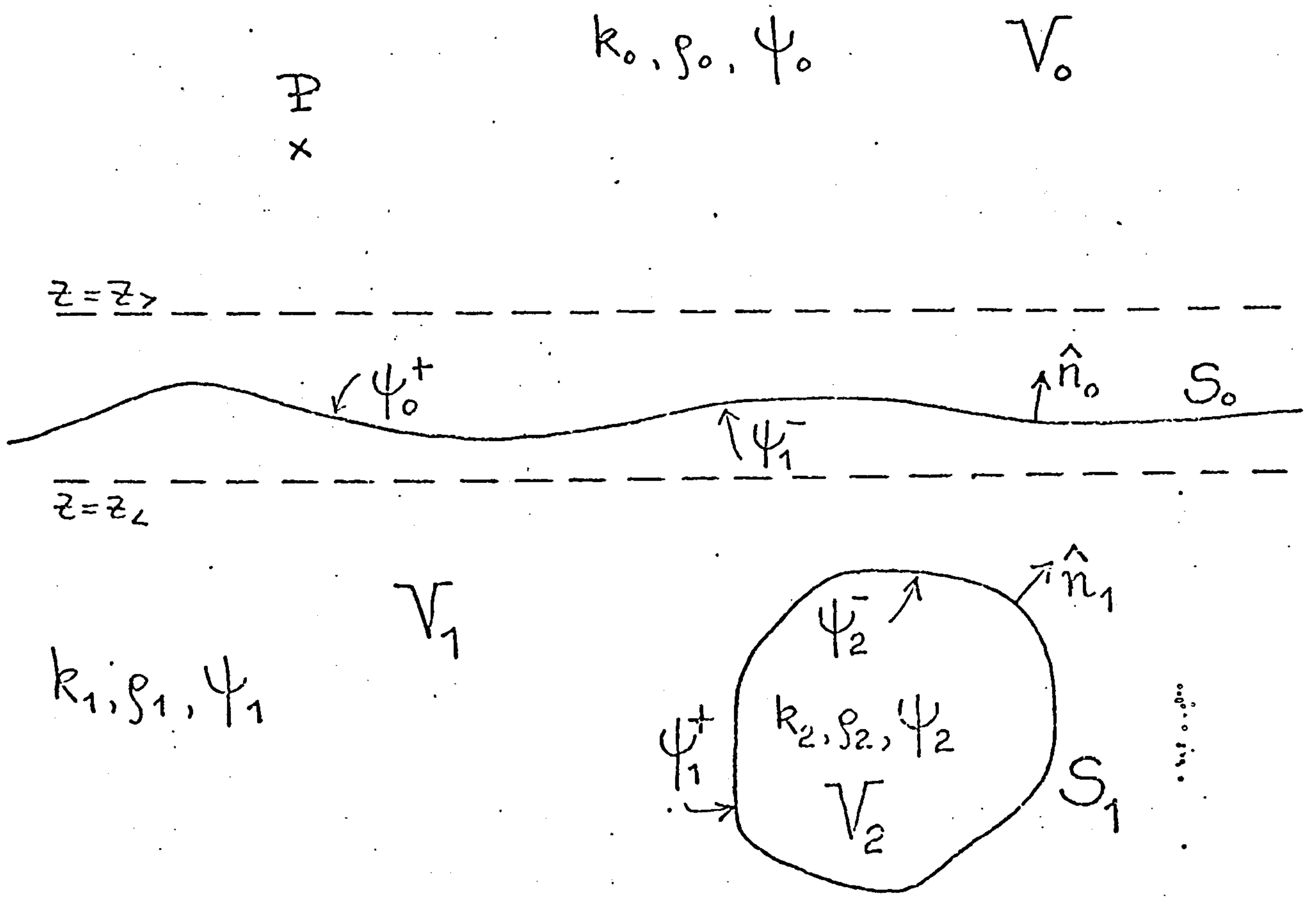
FIG. 1



k_0, ρ_0, ψ_0
V_0
P
$z = z_>$
ψ_0^+
$\hat{n}_0$
S_0
ψ_1^-
$z = z_<$
V_1
$\hat{n}_1$
ψ_2^-
k_1, ρ_1, ψ_1
ψ_1^+
k_2, ρ_2, ψ_2
S_1
V_2

FIG. 2

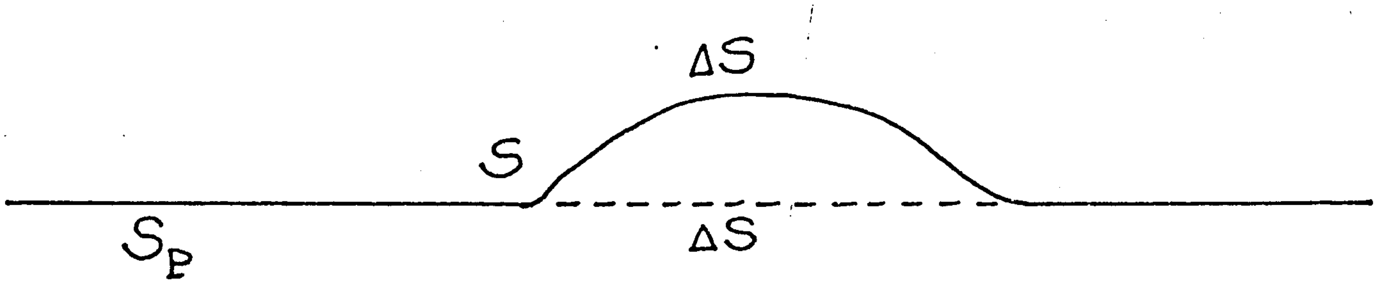


FIG 3

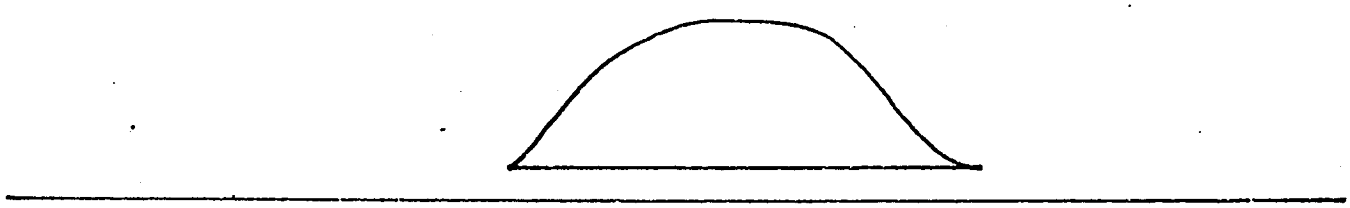


FIG 4

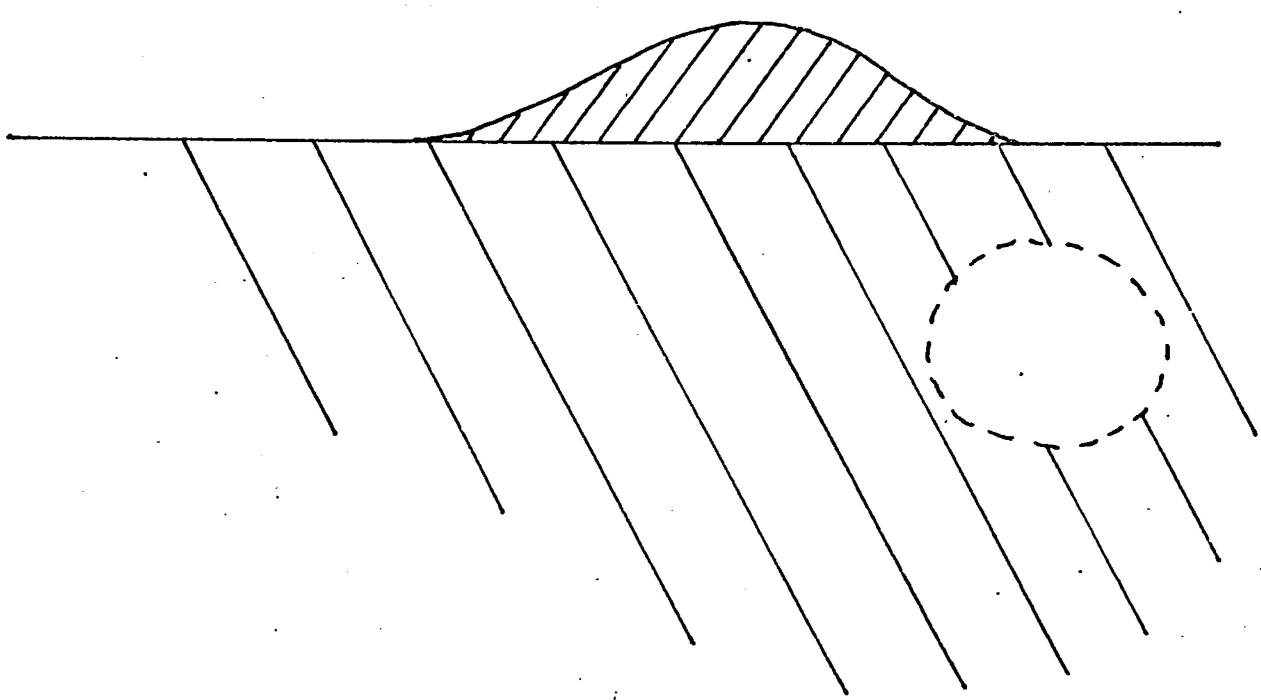


FIG 5

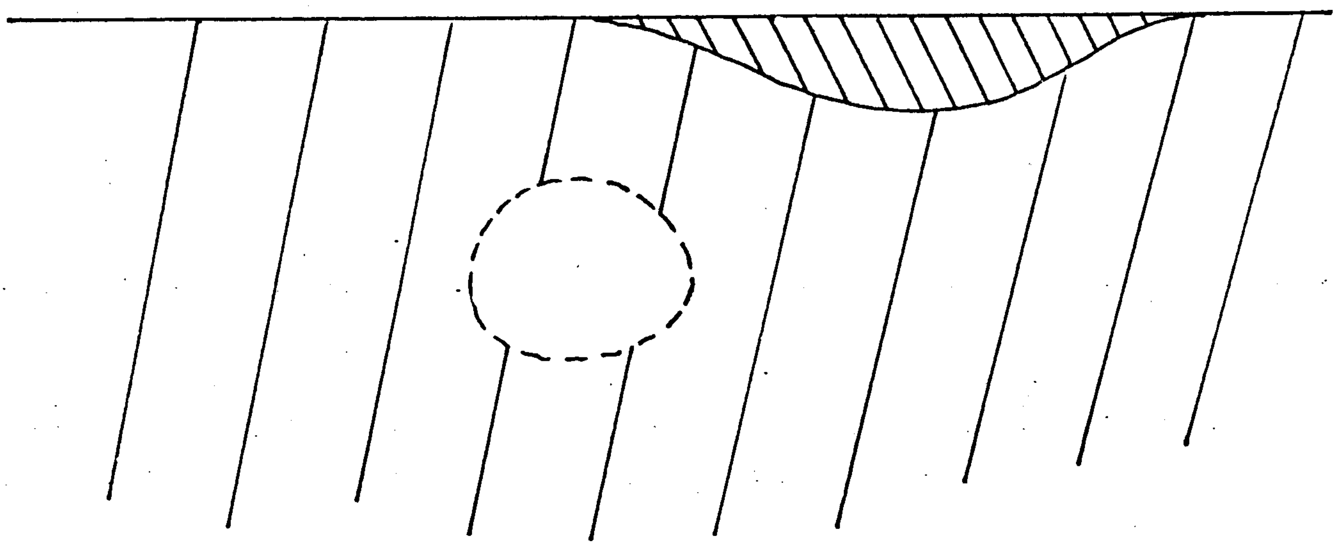


FIG 6