Improving Numerical Accuracy of Gröbner Basis Polynomial Equation Solvers

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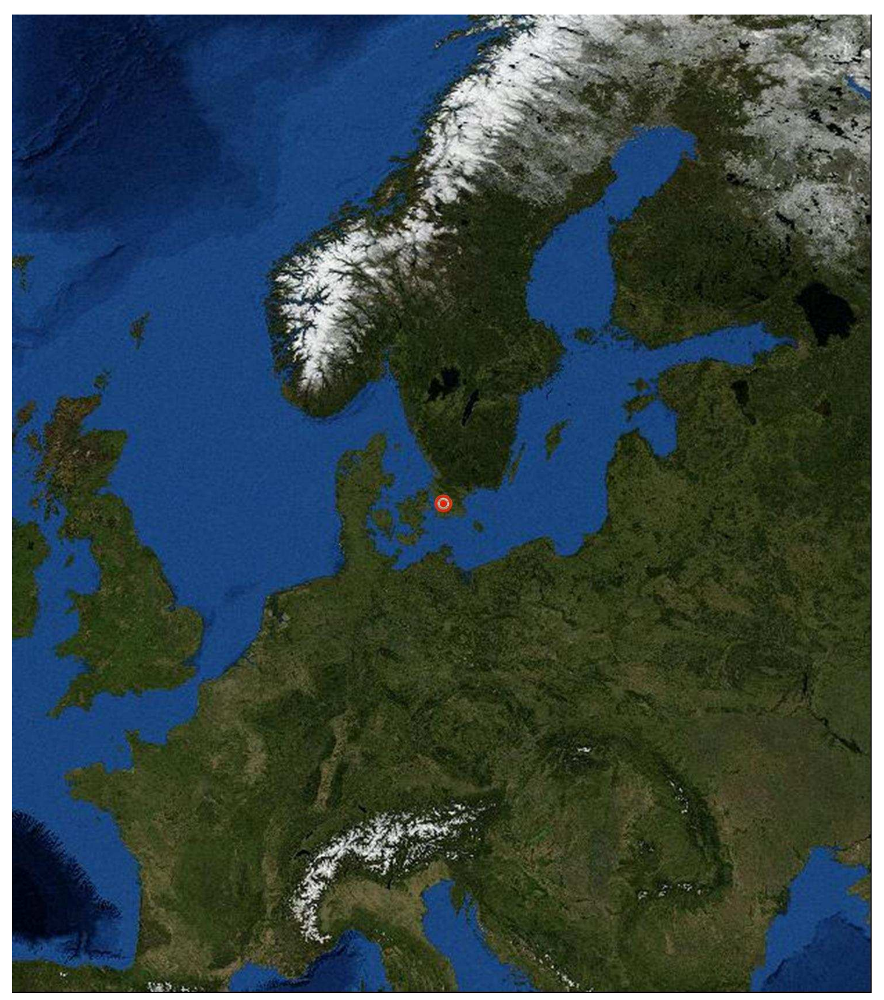


Overview

This work is about solving systmes of polynomial equations arising in many geometric vision problems.

Problem: State-of-the-art methods use Gröbner basis techniques, but are still numerically unstable in many cases.

Solution: We propose to make a change of basis in $\mathbb{R}[\mathbf{x}]/I$ to improve the conditioning of a crucial elimination step of the Gröbner basis computation and thereby gain roughly a factor 10^5 in precision.

Contribution:

- Theory for how to change basis.
- A strategy for how to choose the new basis

Problem Statement

Find the complete set of solutions to a system of equations on the following form

$$\begin{aligned} c_{11}\mathbf{x}^{\alpha_1} + c_{12}\mathbf{x}^{\alpha_2} + \ldots + c_{1n}\mathbf{x}^{\alpha_n} &= 0, \\ &\vdots \\ c_{m1}\mathbf{x}^{\alpha_1} + c_{m2}\mathbf{x}^{\alpha_2} + \ldots + c_{mn}\mathbf{x}^{\alpha_n} &= 0, \end{aligned} \qquad (1)$$

where $\mathbf{x}^{\alpha_1}, \ldots, \mathbf{x}^{\alpha_n}$ are a given set of monomials with $\mathbf{x}^{\alpha_k} = x_1^{\alpha_{k1}} \cdot \cdots \cdot x_p^{\alpha_{kn}}$. Ensure high numerical accuracy in the process.

Motivation

- Polynomial equations arise in *e.g.* minimal cases of structure from motion and in global optimisation.
- Numerical stability of existing solvers in many cases poor.
- The Gröbner basis technique for equation solving not yet fully understood.

Gröbner Basis Equation Solving

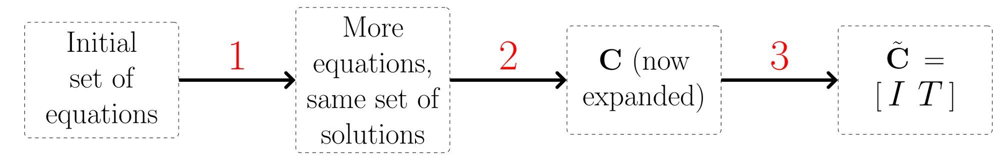
1. All is based on polynomial divison in several variables.
2. The remainders under division by the set of equations form a linear space with dimension equal to the number of solutions.
3. Multiplication by a variable x_k in this space is a linear operation which by choosing a basis can be represented as the matrix $\mathbf{m}_{x_k}$.
4. The eigenvalues of $\mathbf{m}_{x_k}$ yield the solutions to our system of equations.
5. We need a Gröbner basis to compute $\mathbf{m}_{x_k}$. This is the numerically difficult part!

Buchberger's Algorithm

Buchberger's algorithm computes a Gröbner basis, but for our purposes we need to reformulate it using matrices. Note that Equation 1 can be written using matrix notation as

$$\mathbf{C}\begin{pmatrix}\mathbf{x}^{\alpha_1}\\ \vdots \\ \mathbf{x}^{\alpha_n}\end{pmatrix} = 0. \qquad (2)$$

With this notation, we can use a variation of the Buchberger algorithm like so:



1. Multiply with a set of monomials, 2. Stack coefficients in a matrix, 3. Put $\mathbf{C}$ *on reduced row echelon form (elimination step).*

If (1) was ok, then we now have a Gröbner basis given by $\tilde{\mathbf{C}}$, but!

PROBLEM: Step (3) might be very ill conditioned.

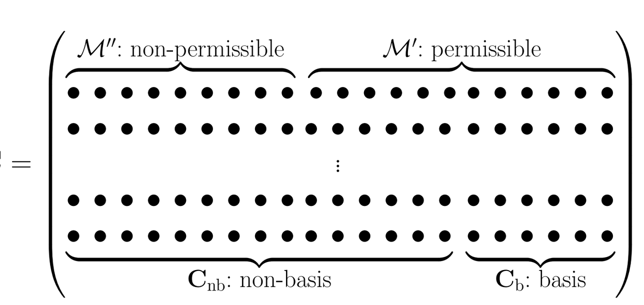
REMEDY: Change basis in the column space of C. This induces a change of basis in $\mathbb{R}[\mathbf{x}]/I$.

Changing Basis

- Question: What monomials can we use in our basis for $\mathbb{R}[\mathbf{x}]/I$ and still be able to perform remainder arithmetic?
- Answer: If $\mathcal{M}$ is the set of monomials present in our equations (1), then we can use any monomials that stay in $\mathcal{M}$ under multiplication by x_k.
- We call these special monomials *permissible* and define the set as

$$\mathcal{M}' = \{\mathbf{x}^\alpha \in \mathcal{M} : x_k \cdot \mathbf{x}^\alpha \in \mathcal{M}\}.$$

- The basis for $\mathbb{R}[\mathbf{x}]/I$ can consist of arbitrary (linearly independent) linear combinations of monomials in $\mathcal{M}'$.

$\mathbf{C} =$ 

- The goal is to decrease the condition number of $\mathbf{C}_{\text{nb}}$.
- We do this by a change of basis $\hat{\mathbf{C}}_{\text{nb}} = \mathbf{C}_{\text{nb}}\mathbf{Q}$ with an orthogonal matrix Q, subject to the constraint that $\mathbf{Q}$ does not mix permissible and non-permissible columns.
- This influences the computation of the action matrix, which requires some theory (see the paper for details).

How to Choose a Basis

For any linear system $Ax = b$, the error is controlled by the condition number of the matrix. Hence, we would like to minimize the condition number $\kappa(\mathbf{C}_{\text{nb}})$. We therefore employ the following goal:

Make the columns of $\mathbf{C}_{\text{nb}}$ "as linearly independent as possible".

We propose a heuristic strategy based on singular value decomposition (SVD) which tries to reach this goal. Consider the coefficient matrix $\mathbf{C}$ and denote the permissible columns by $\mathbf{C}'$ and the non-permissible columns by $\mathbf{C}''$. Then do the following.

1. Write $\mathbf{C}'$ as $\mathbf{C}' = \mathbf{C}'^{//} + \mathbf{C}'^{\perp}$, where $\mathbf{C}'^{\perp}$ is the projection of the column vectors of $\mathbf{C}'$ onto the orthogonal complement of the subspace spanned by the columns of $\mathbf{C}''$.
2. Decompose $\mathbf{C}'^{\perp}$ as $\mathbf{C}'^{\perp} = \mathbf{U}\Sigma\mathbf{V}^{\mathbf{t}}$.
3. Discard $\mathbf{C}'^{\perp}$ but use $\mathbf{V}$ from this decomposition to form $\mathbf{V}_e = \begin{pmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{V}\end{pmatrix}$ and $\hat{\mathbf{C}} = \mathbf{C}\mathbf{V}_e$.

Eigenvalues

Extracting the solutions from the eigenvalues is numerically more stable than extracting them from the eigenvectors. Using the fact that the eigenvectors of $\mathbf{m}_{x_k}$ are equal for all $\mathbf{m}_{x_k}$ all eigenvalues can be calculated fast by the following method.

1. Compute $\mathbf{m}_{x_k}$ for all x_k.
2. Make the eigenvalue decomposition for one k: $\mathbf{m}_{x_k}\mathbf{V} = \mathbf{D}\mathbf{V}$.
3. For all $\mathbf{m}_{x_k}$ calculate $\mathbf{m}_{x_k}\mathbf{V}$ followed by element-wise division to get eigenvalues for all matrices $\mathbf{m}_{x_k}$.

The experiments show that this improves the numerical accuracy.

Generalized Cameras

- Minimal case: 6 points in two views [SNOÅ05].
- 3 unknowns, 64 solutions.
- 15 equations with total degree 6.
- After expansion: 101 equations in 165 monomials with total degree 8.

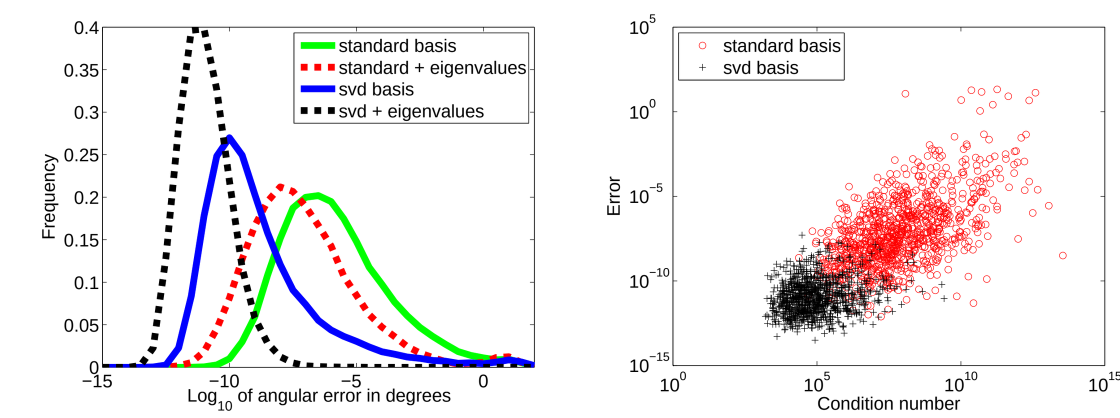


FIGURE 1: *Left*: Histogram over the angular error in degrees of the estimated rotation matrix in the solver for relative pose for generalized cameras. *Right*: The angular error plotted versus the condition number of $\mathbf{C}_{nb}$.

Improvement: 10^5

Unknown Focal Length

- Minimal case: 6 points in 2 views [SKNS05].
- 3 unknowns, 15 solutions.
- 10 equations with total degree 5.
- After expansion: 34 equations in 50 monomials with total degree 7.

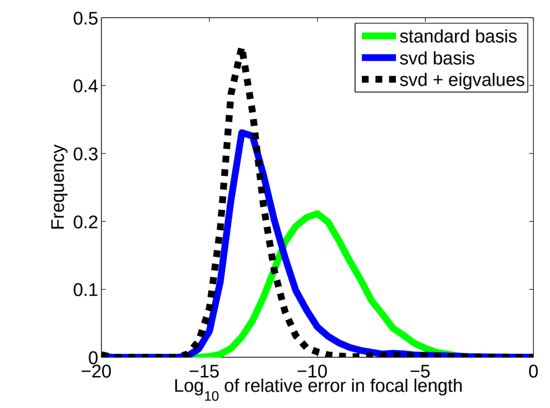


FIGURE 2: Histogram over the error in relative focal length estimated in the solver for relative pose for standard cameras with unknown focal length. Note how all significant errors are eliminated when the improved method is used.

Improvement: 10^5

Three View Triangulation

- Optimal L_2-triangulation by calculation of all stationary points [SSN05].
- 3 unknowns, 50 solutions.
- 3 equations with total degree 6.
- After expansion: 225 equations in 209 monomials with total degree 9.

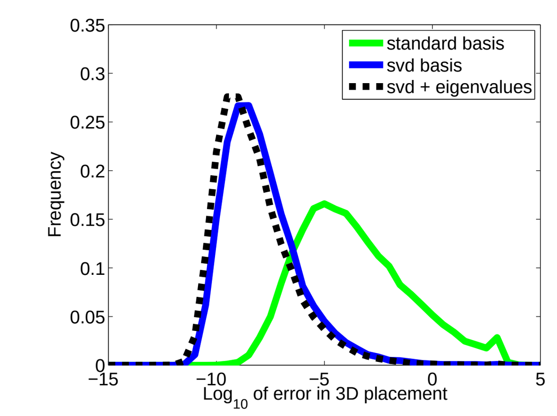


FIGURE 3: Histogram over the error in 3D placement of the unknown point obtained using optimal three view triangulation. With the improvement of the new method this problem is now solvable in standard double arithmetic.

Improvement: 10^6

References

[CLO07] D. Cox, J. Little, and D. O'Shea. *Ideals, Varieties, and Algorithms.* Springer, 2007.

[SKNS05] H. Stewénius, F. Kahl, D. Nistér, and F. Schaffalitzky. A minimal solution for relative pose with unknown focal length. In *Proc. Conf. Computer Vision and Pattern Recognition, San Diego, USA*, 2005.

[SNOÅ05] H. Stewénius, D. Nistér, M. Oskarsson, and K. Åström. Solutions to minimal generalized relative pose problems. In *Workshop on Omnidirectional Vision*, Beijing China, October 2005.

[SSN05] H. Stewénius, F. Schaffalitzky, and D. Nistér. How hard is three-view triangulation really? In *Proc. Int. Conf. on Computer Vision*, pages 686–693, Beijing, China, 2005.

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