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Ordell, Astrid

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LUND UNIVERSITY

PO Box 117
221 00 Lund
+46 46-222 00 00

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ASTRID ORDELL
FACULTY OF SCIENCE | LUND UNIVERSITY

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Faculty of Science
Department of Astonomy
and Theoretical Physics

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by Astrid Ordell

LUND UNIVERSITY

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Faculty opponent: Prof. David J. Miller

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Abstract

This thesis is composed of four papers, which all treat various extensions to the Standard Model (SM). The first two papers concern a particular supersymmetric, grand-unified theory (GUT), while in the latter two, we classify anomaly-free implementations of two-Higgs doublet models (2HDM) with a gauged abelian symmetry.

Paper I. In this paper, we present an alternative solution for avoiding GUT-scale lepton masses in so-called trinification based GUT-models (T-GUTs). Rather than introducing several copies of the tri-triplets, which is the conventional approach, we solve the issue by embedding the symmetries into an E_8 group.

Paper II. This paper involves a more detailed study of the model proposed in Paper I. Some of its features is an absence of the μ-problem and a universal Yukawa coupling for all chiral fermions.

Paper III. Here, we classify all anomaly-free implementations of a 2HDM with a gauged Abelian symmetry and the SM fermion content. The resulting 11 models are then compared under a range of experimental bounds, from which we identify the most promising ones.

Paper IV. In this paper, we again classify and compare all anomaly-free implementations of a 2HDM with a gauged Abelian symmetry, but this time around also including two generations of right-handed neutrinos. The right-handed neutrinos are incorporated to provide a mechanism for the neutrinos to obtain a mass, here via a type-I seesaw mechanism. In total, there are 16 valid models, which are all compared, both for normal- and inverted ordering. Two models are identified as the most promising candidates for future studies.

Key words
Two-Higgs-doublet models, Z' models, Flavour physics, Anomaly constraints, SUSY

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A doctoral thesis at a university in Sweden takes either the form of a single, cohesive research study (monograph) or a summary of research papers (compilation thesis), which the doctoral student has written alone or together with one or several other author(s).

In the latter case the thesis consists of two parts. An introductory text puts the research work into context and summarizes the main points of the papers. Then, the research publications themselves are reproduced, together with a description of the individual contributions of the authors. The research papers may either have been already published or are manuscripts at various stages (in press, submitted, or in draft).

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Populärvetenskaplig sammanfattning

Inom kvantfältsteori, precis som inom fysik i övrigt, är modeller bara giltiga i ett begränsat energiintervall. Med andra ord, så finns det för närvarande ingen fulländad teori, utan våra matematiska ramverk är begränsade till att beskriva ett beteende över, eller under, en viss energinivå. Det som däremot finns, är mer eller mindre omfattande modeller.

Genom historien har det begränsade intervallet gång på gång expanderats. Allmän relativitetsteori expanderade intervallet för Newtons gravitationslag, icke-relativistisk kvantmekanik intervallet för klassisk mekanik, och relativistisk kvantmekanik intervallet för dess icke-relativistiska motsvarighet.

Det är däremot inte sant att en mer omfattande modell nödvändigtvis är bättre. Till exempel är det mycket smidigare att räkna ut energinivåerna hos väteatomen i kvantmekanik än i kvantfältsteori, och planetbanor fås enklare från Newtons gravitationslag än från allmän relativitetsteori. Det vill säga, på vissa längdskalor är den mer omfattande modellen onödigt komplicerad och det optimala valet för att utföra beräkningar är istället att arbeta i det enklaste möjliga ramverk som fångar all relevant fysik.

Även om man väljer att använda den förenklade modellen, är det dock fortfarande användbart av att känna till den mer omfattande teorin. Till exempel, om vi gör vår kvantmekaniska beräkning genom att ta den icke-relativistiska gränsen av kvantfältsteori, så besitter vi möjligheten att korrigera vårt svar till önskad precision, då vår mindre omfattande modell blir en funktion av parameterarna hos den mer omfattande modellen. Vi behåller på så sätt bekvämligheten av en enkel modell, men utan att förlora i noggrannhet.

Inom kvantfältsteori kallas förenklade modeller för *effektiva fältteorier*, och spelar en central roll i denna avhandling – både i dess mer, och i dess mindre, omfattande format. I dess mer omfattande form används effektiva fältteorier i ett försök att utöka det nuvarande paradigmet, den så kallade Standard Modellen, vilket länge har varit ett mål inom partikelfysiken, och i dess mindre omfattande form används fältteorier för att jämföra ens modeller med experiment. Då själva poängen med fysik är att beskriva verkligheten, är det sistnämnda användningsområdet av yttersta relevans.

Contents

Phenomenology in multi-scalar extensions of the Standard Model

The Standard Model (SM) has been, more or less, in its modern format since the early 1970s. One of the last pieces to fall into place, on the theoretical side, was the prediction of a third generation of quarks by Kobayashi and Maskawa in 1973 [1]. Experimentally, the quark model was confirmed via deep inelastic scattering experiments in 1968 [2,3], followed by the discovery of the charm quark in 1974 [4,5], the bottom quark in 1977 [6] and the W and Z bosons in 1983 [7–10], by which the majority of the physics community were already convinced that the SM was a valid, low-energy effective description of the fundamental interactions.

In the wake of the W and Z discoveries, the experimental community instead turned the majority of their attention to New Physics (NP). They were hoping to find signatures of something even more comprehensive than the SM and – from the enormous success of the previous decade – people were optimistic. In the fall of 1984, Carlo Rubbia even claimed that the UA1 experiment would provide substantial proof of supersymmetry (SUSY) before the end of the year [11]. But the task proved more difficult than expected.

Today, three decades later, we are still in the dark as to what that high-energy theory might be. There has been plenty of experimental advancement – from the discovery of the top quark in 1994 [12,13] and the Higgs boson in 2012 [14,15] to a continually increasing precision on all existing observables – but nothing conclusive in terms of NP. Yet, as the SM lacks mechanisms for features such as neutrino mass generation and dark matter, it is obviously not complete.

This thesis treats two rather different approaches to beyond the SM (BSM) physics, even if they of course share the same low-energy realization. Paper I and II concern a grand unified theory (GUT), which greatly extends the SM, while Paper III and IV treat minimal extensions. While the approaches differ, the validity of both is still decided by their agreement with experiments at low energies. To this end, this thesis is, to a large part, dedicated to low-energy phenomenology.

1 Effective Field Theories

"*Effective field theory is more than a convenience.*" – Howard Georgi

As we do not yet have (and might never have) a theory about everything, all models used in particle physics have a limited range of validity. As such, they are all effective field theories (EFTs) – including the SM. This section is focused mainly on an EFT referred to as the Weak Hamiltonian, which plays a central role in Paper III and IV.

Having a limited range of validity is also what leads to one of the most beautiful concepts in quantum field theory (QFT), namely the renormalization group (RG). This is a key feature in all four papers and will be the subject of Sec. 2.

1.1 The Weak Hamiltonian

To introduce the Weak Hamiltonian, let us consider the amplitude of $c \to su\bar{d}$ at tree-level, momentarily ignoring any effects from quantum chromodynamics (QCD). For this particular process, the effective description corresponds to having integrated out the W-boson from the electroweak (EW) theory, resulting in the so-called 4-Fermi theory. In 4-Fermi theory there is only one relevant operator for this process, namely

$$\mathcal{H}_{\mathit{eff}} = C\,(\bar{s}\gamma^{\mu}P_L c)\left(\bar{u}\gamma_{\mu}P_L d\right), \tag{1.1}$$

where C is the so-called Wilson coefficient, $P_L \equiv (1-\gamma_5)/2$ is a projection operator, and where, throughout the text, Hamiltonian *densities* are denoted by the letter $\mathcal{H}$ while ordinary Hamiltonians are denoted by H. The effective action is then given by

$$\begin{aligned} S_{\mathit{eff}} &= -iC\int \mathrm{d}^4x \left\{ \langle su\bar{d}|(\bar{s}\gamma^{\mu}P_L c)(\bar{u}\gamma_{\mu}P_L d)|c\rangle \right\} \\ &= -iC(2\pi)^4\delta^{(4)}(p_f - p_i)\,(\bar{u}_s\gamma^{\mu}P_L u_c)\left(\bar{u}_u\gamma_{\mu}P_L v_d\right) \\ &\equiv i\mathcal{M}_{\mathit{eff}}(2\pi)^4\delta^{(4)}(p_f-p_i), \end{aligned} \tag{1.2}$$

where the lines denote Wick contractions, and where the effective scattering amplitude hence ends up being

$$i\mathcal{M}_{eff} = -iC\,(\bar{u}_s\gamma^{\mu}P_L u_c)\left(\bar{u}_u\gamma_{\mu}P_L v_d\right). \qquad (1.3)$$

Note that the subscripts on the basis spinors are flavor indices rather than spin labels, which is the conventional notation in the flavor physics community.

If we obtain the amplitude for the same tree-level process in the UV-completion, which now involves a W propagator, we have the non-local result

$$i\mathcal{M}^{NL}_{full} = \left(\frac{ig_2}{\sqrt{2}}\right)^2 V^*_{cs}V_{ud}\left(\frac{-ig_{\mu\nu}}{p^2 - m^2_W + i\epsilon}\right)(\bar{u}_s\gamma^{\mu}P_L u_c)\,(\bar{u}_u\gamma^{\nu}P_L v_d)\,, \qquad (1.4)$$

where V_{ij} are Cabibbo-Kobayashi-Maskawa (CKM) matrix elements, $g_{\mu\nu}$ the metric tensor, m_W the mass of the W boson, g_2 the $SU(2)_W$ gauge coupling, and where the W propagator is given in the Feynman-'t Hooft gauge. In the limit of $p^2 \ll m_W^2$, the amplitude can approximately be replaced by the local expression

$$i\mathcal{M}^{L} \simeq -\frac{ig_2^2}{2}V^*_{cs}V_{ud}\frac{1}{m^2_W}(\bar{u}_s\gamma^{\mu}P_L u_c)\left(\bar{u}_u\gamma_{\mu}P_L v_d\right). \qquad (1.5)$$

The Wilson coefficient is then determined by the tree-level matching between the effective amplitude in eq. (1.3) and the local amplitude in eq. (1.5), resulting in

$$C = \frac{g_2^2}{2}V^*_{cs}V_{ud}\frac{1}{m^2_W}. \qquad (1.6)$$

Or, equivalently, if we use the convention of having $C \equiv (4G_F V^*_{cs}V_{ud})/\sqrt{2}$, the Fermi constant is given by $G_F = g_2^2/(4\sqrt{2}m_W^2)$. Either way, the non-relevant degrees of freedom are incorporated into the effective theory via the matching procedure.

1.2 Operator Product Expansion

To formalize the procedure in Sec. 1.1, let us now introduce the concept of operator product expansion (OPE). For the type of effective field theories treated in this thesis, the move from one EFT to the next involves discarding some heavy degree of freedom (dof), such as the W

boson in the previous subsection. Using the notation in Ref. [16], we begin by integrating out the heavy fields χ, leaving us with a non-local action S_{NL}

$$\int [\mathrm{d}\chi]\, e^{\,iS(\chi,\xi)} = e^{\,iS_{NL}(\xi)} \,, \tag{1.7}$$

such that the heavy degrees of freedom exist only as intermediate, virtual particles, and not as external states. The external states are instead solely constituted by the light fields ξ. In other words – while "integrating out" is commonly referred to as the process of obtaining the local effective theory *below* the matching scale (i.e. as the process of discarding the heavy field completely, even as an internal state) – what a Gaussian integration actually amounts to, is producing the non-local theory *above* the matching scale, corresponding to eq. (1.4).

To perform the Gaussian integral in eq. (1.7), let us begin with the case of χ being a massive spin-0 field, $\chi \equiv \phi$. Here, the generating functional is given by [17]

$$Z[J] = \int \mathcal{D}\phi \exp\left[i \int \mathrm{d}^4x \left(\mathcal{L}_0^{\phi} + J(x)\phi(x)\right)\right], \quad \mathcal{L}_0^{\phi} = \frac{1}{2}\partial^{\mu}\phi\partial_{\mu}\phi - \frac{m}{2}\phi^2 \,, \tag{1.8}$$

where $\mathcal{L}_0^{\phi}$ is the free Klein-Gordon Lagrangian, and where $J(x)$ is typically an external classical source, but can equally well be composed of other fields in the theory. In fact, in the context of integrating out $\phi(x)$, we have no intention of taking functional derivatives of eq. (1.8), as this would generate n-point functions with ϕ as an external state. As such, we interpret $J(x)$ as being composed of some fields ξ that are not being integrated out. To indicate that $\mathcal{L}_0$ contain only the kinetic terms for the scalar field, and not the kinetic terms for this other field (that constitutes the current), we have used a superscript ϕ.

Integrating by parts, we then have

$$Z[J] = \int \mathcal{D}\phi \exp\left[i \int \mathrm{d}^4x \left(\frac{1}{2}\phi\left(-\partial^2 - m^2 + i\epsilon\right)\phi + J(x)\phi(x)\right)\right], \tag{1.9}$$

where the $+i\epsilon$ has been added as a necessary convergence factor for Gaussian integrals. Then, shifting the field as

$$\phi(x) \to \phi(x) - i \int \mathrm{d}^4y\, D_F(x-y)J(y) \,, \tag{1.10}$$

and again using integration by parts for any terms not yet on the form of having the Klein-Gordon operator, (∂^2+m^2), on the left-hand side of the Feynman propagator. Then, using that $(\partial^2+m^2)D_F(x-y) = -i\delta^4(x-y)$, we get

$$Z[J] = Z[0]\exp\left[-\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y\ J(x)D_F(x-y)J(y)\right], \tag{1.11}$$

or equivalently, that the non-local action after having integrated out the scalar field ϕ, is given by

$$S_{NL} = \int \mathrm{d}^4x\ \mathcal{L}_0^\xi + \frac{i}{2}\int \mathrm{d}^4x\mathrm{d}^4y\ J(x)D_F(x-y)J(y)\ , \tag{1.12}$$

where $\mathcal{L}_0^\xi$ denotes the kinetic terms for whichever field that constitutes the current, i.e. the field that has not been integrated out but remains as both external and internal states.

Now, if we wish to carry out the equivalent procedure for a massive vector boson, to compare it with the result in Sec. 1.1, we instead have the Lagrangian

$$\begin{aligned}\mathcal{L}_W = &-\frac{1}{2}(\partial^\mu W^{+\nu} - \partial^\nu W^{+\mu})(\partial_\mu W^-_\nu - \partial_\nu W^-_\mu) + m_W^2 W^+_\mu W^{-\mu} \\ &+ \frac{g_2}{\sqrt{2}}\left(J^+_\mu W^{-\mu} + J^-_\mu W^{+\mu}\right)\ ,\end{aligned} \tag{1.13}$$

where we have included only the terms relevant for either tree-level or 1-loop diagrams with internal W-bosons and external- or internal quarks. The kinetic terms for the fields that constitute the currents have once again been left out, with the quark currents defined as

$$J^+_\mu = V_{ij}\bar{u}_i\gamma_\mu P_L d_j, \qquad J^-_\mu = \left(J^+_\mu\right)^\dagger, \tag{1.14}$$

where $i,j=1,2,3$ are flavor indices, $V_{ij} = (U^\dagger_{uL}U_{dL})_{ij}$ is the CKM matrix, and $U_{fL(R)}$ the unitary field transformation between the flavor eigenbasis and the mass eigenbasis for the quark type $f_{L(R)}, f = u, d$.

With the kinetic operator being invertible, we can simply take the kinetic part of the Lagrangian in eq. (1.13), $\mathcal{L}_0^W$, and after integration by parts obtain

$$\int \mathrm{d}^4x\ \mathcal{L}_0^W = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\tilde{W}^+_\mu(k)\left((-k^2+m_W^2)g^{\mu\nu} + k^\mu k^\nu\right)\tilde{W}^-_\nu(-k), \tag{1.15}$$

where the tilde denotes Fourier-transformed quantities. With the W propagator being the Green's function of this linear operator, i.e. with $\left((-k^2 + m_W^2)g_{\mu\nu} + k_\mu k_\nu\right)\tilde{\Delta}^{\nu\rho}(k) = i\delta_\mu^{\ \rho}$, its form in momentum space ends up being[1]

$$\tilde{\Delta}_{\mu\nu}(k) = \frac{-i}{k^2 - m_W^2 + i\epsilon}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{m_W^2}\right) . \tag{1.16}$$

The non-local action, i.e. the equivalent of eq. (1.12) is then given by[2]

$$S_{NL} = \int \mathrm{d}^4x\, \mathcal{L}_0^\psi + \frac{ig_2^2}{2}\int \mathrm{d}^4x \mathrm{d}^4y\, J_\mu^-(x)\Delta^{\mu\nu}(x-y)J_\nu^+(y) , \tag{1.17}$$

where $\mathcal{L}_0^\psi$ are the kinetic terms for the quarks in eq. (1.14). To lowest order, the propagator in position space is given by

$$\Delta^{\mu\nu}(x-y) \simeq \frac{g^{\mu\nu}}{m_W^2}\,\delta^{(4)}(x-y) , \tag{1.18}$$

such that eq. (1.17) simplifies to

$$S_L = \int \mathrm{d}^4x\, \mathcal{L}_0^\psi - \frac{g_2^2}{2m_W^2}\int \mathrm{d}^4x\, J_\mu^-(x)J^{+\mu}(x) , \tag{1.19}$$

which is local and, after a Legendre transform, given by

$$\mathcal{H}_{\mathit{eff}}(x) = \frac{G_F}{\sqrt{2}} J_\mu^-(x)J^{+\mu}(x) = V_{ji}^* V_{k\ell}\left(\bar{d}_k\gamma^\mu P_L u_\ell\right)\left(\bar{u}_i\gamma_\mu P_L d_j\right) , \tag{1.20}$$

in agreement with eq. (1.1) for $\{i, j, k, \ell\} = \{2, 2, 1, 1\}$.

[1] To see this, write the propagator on the general form $\tilde{\Delta}_F^{\nu\rho} = a(k^2)g^{\nu\rho} + b(k^2)k^\mu k^\nu$, and then apply the operator and solve for a, b. Note that this is only possible in the massive case.

[2] Note that the prefactor of the second term differs from that in Ref. [18] simply because a factor of $1/2$ has been included into the definition of the projection operator in eq. (1.14).

2 The Renormalization Group

"*Ingenting bör hända, och det gör det ju inte heller. Då är allt som det ska!*"
– Carl-Erik Magnusson

The mathematical procedure of avoiding divergent integrals, until they eventually cancel, was for a long time viewed upon with suspicion. It was not until the early 1970s that physicists came to rest with what quantum field theories actually are – not an omniscient description of the universe, but an effective approach valid only in a certain energy range. With this, renormalization lost the reputation of sweeping infinities under the carpet and was instead understood as a natural consequence of our model's ignorance.

Before introducing the concept of renormalization and the renormalization group, which is central to all four papers, we will first cover the main features of the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula. It is one of the central results in QFT and will be referred back to throughout this thesis.

2.1 The LSZ reduction formula

The LSZ formula gives the relation between S-matrix elements and n-point correlation functions. In broad strokes, the proof consists of a series of reductions, where each reduction corresponds to the conversion of a pole in the n-point Green's function into a particle in an asymptotic state. The key steps of the derivation are presented below.

To obtain the first step in the reduction, we Fourier transform the n-point correlation function with respect to one argument only, i.e. considering the amplitude

$$\int \mathrm{d}^4x\, e^{ip\cdot x} \langle \Omega | T\{\phi(x)\phi(z_1)\phi(z_2)\ldots\} |\Omega\rangle \ . \tag{2.1}$$

In general, our field operator $\phi(x)$ can create both one-particle states (simple poles) and multi-particle states (branch cuts). However, as we wish to arrive at the S-matrix for m-to-n scattering, we are interested solely in one-particle states with momentum $\vec{p}_i$, which corresponds to isolated poles at $p_i^0 = E_{\vec{p}_i}$. With the analyticity determined by the time component, it is convenient to split the time-integral into three parts,

$$\int \mathrm{d}x^0 = \int_{T_+}^{\infty} \mathrm{d}x^0 + \int_{T_-}^{T_+} \mathrm{d}x^0 + \int_{-\infty}^{T_-} \mathrm{d}x^0 \ , \tag{2.2}$$

where poles occur only in the first- and third interval, as the middle interval is analytic in p^0. Starting with the first interval, and inserting a completeness relation, we then have

$$\int_{T_+}^{\infty} \mathrm{d}x^0 \int \mathrm{d}^3x \, e^{ip\cdot x} \sum_{\lambda} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1}{2E_{\vec{q}}(\lambda)} \langle \Omega | \phi(x) | \lambda_{\vec{q}} \rangle \langle \lambda_{\vec{q}} | T\{\phi(z_1)\phi(z_2)...\} \, |\Omega\rangle \,, \quad (2.3)$$

with

$$\langle \Omega | \phi(x) | \lambda_{\vec{q}} \rangle = \langle \Omega | e^{iP\cdot x} \phi(0) e^{-iP\cdot x} | \lambda_{\vec{q}} \rangle = \langle \Omega | \phi(0) | \lambda_{\vec{0}} \rangle \cdot e^{-iq\cdot x} \,, \quad (2.4)$$

as $|\Omega\rangle$ and $|\lambda_{\vec{q}}\rangle$ are both eigenstates of momentum operator P, with eigenvalues 0 and q, respectively, and with spin-0 particles transforming trivially under the Lorentz group. Furthermore, we define

$$\langle \Omega | \phi(0) | \lambda_{\vec{0}} \rangle \equiv \sqrt{Z} \,, \quad (2.5)$$

such that the field strength corresponds to how much our field operator is creating a one-particle state. After a few additional manipulations, we arrive at the endpoint for the first step of the reduction, namely

$$\int \mathrm{d}^4x \, e^{ip\cdot x} \langle \Omega | T\{\phi(x)\phi(z_1)\phi(z_2)...\} | \Omega \rangle \sim \frac{\sqrt{Z}}{p^2 - m^2 + i\varepsilon} \langle \vec{p} | T\{\phi(z_1)\phi(z_2)..\} | \Omega \rangle, \quad (2.6)$$

where the tilde corresponds to both sides having the same pole structure, by taking the on-shell limit $p^0 \to E_{\vec{p}}$. Repeating the procedure for the third time-interval in eq. (2.2), we again convert a pole of the Green's function into an asymptotic state, but this time into an asymptotic *initial* state, rather than an asymptotic *final* state.

Finally, once the procedure has been repeated for all field operators, to produce n final states and m initial states, we end up with the LSZ reduction formula

$$\begin{aligned} &\left(\prod_{i=1}^{n} \int \mathrm{d}^4 x_i \, e^{ip_i \cdot x_i} \right) \left(\prod_{j=1}^{m} \int \mathrm{d}^4 y_j \, e^{-ik_j \cdot y_j} \right) \langle \Omega | T\{\phi(x_1)...\phi(x_n)\phi(y_1)...\phi(y_m)\} \, |\Omega\rangle \\ &\sim \left(\prod_{i=1}^{n} \frac{\sqrt{Z}}{p_i^2 - m^2 + i\varepsilon} \right) \left(\prod_{j=1}^{m} \frac{\sqrt{Z}}{k_j^2 - m^2 + i\varepsilon} \right) \langle \vec{p}_1 ... \vec{p}_n | \, S \, | \vec{k}_1 ... \vec{k}_m \rangle \,, \end{aligned} \quad (2.7)$$

in the on-shell limit $p_i^0 \to E_{\vec{p}_i}$ and $k_i^0 \to E_{\vec{k}_i}$. In the case of having particles with non-zero spin, there are also basis spinors and spin sums.

2.2 Renormalization

To illustrate the concept of renormalization, let us consider quantum electrodynamics (QED) to 1-loop order. In QED, before renormalizing, there are a total of three UV-divergent n-point functions – the amputated electron two-point function, the amputated photon two-point function, and the trilinear vertex correction, all logarithmically divergent.[3] Starting with the Lagrangian expressed in terms of bare parameters,

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}\left(i\partial\!\!\!/ - m_0\right)\psi - e_0\bar{\psi}\gamma_\mu\psi A^\mu, \tag{2.8}$$

we rescale the fields as $\psi \equiv Z_2^{1/2}\psi_r$ and $A^\mu \equiv Z_3^{1/2}A_r^\mu$, such that the field strength renormalizations are absorbed into the Lagrangian and hence no longer present in eq. (2.7). After this, use the replacements

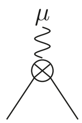
$$e_0 Z_2 Z_3^{1/2} \equiv \mu^{(d-4)/2} e Z_1, \quad Z_2 m_0 \equiv m_r + \delta_m, \quad Z_i \equiv 1 + \delta_i, \tag{2.9}$$

for i=1,2,3, by which the counterterms δ_i have absorbed all infinite shifts between the bare- and renormalized parameters. The factor of $\mu^{(d-4)/2}$ is included in order for the mass dimension of the coupling constant to remain zero in the fractal dimension introduced in dimensional regularization. The Lagrangian is then given by

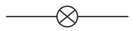
$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_r^{\mu\nu}F^r_{\mu\nu} + \bar{\psi}_r\left(i\partial\!\!\!/ - m_r\right)\psi_r - \mu^{(d-4)/2}e\bar{\psi}_r\gamma_\mu\psi_r A_r^\mu - \frac{1}{4}\delta_3 F_r^{\mu\nu}F^r_{\mu\nu} \\ &+ \bar{\psi}_r\left(i\delta_2\partial\!\!\!/ - \delta_m\right)\psi_r - \mu^{(d-4)/2}e\delta_1\bar{\psi}_r\gamma_\mu\psi_r A_r^\mu,\end{aligned} \tag{2.10}$$

where the first three terms are in the same format as in the bare Lagrangian. The Feynman rules for the photon propagator, electron propagator, and the trilinear fermion-gauge vertex hence have the same form as in bare perturbation theory, but with bare parameters exchanged for their renormalized counterparts, denoted by an index r. For the counterterms, on the other hand, we have

$= -i\left(g^{\mu\nu}q^2 - q^\mu q^\nu\right)\delta_3,$



$= -i\mu^{(d-4)/2}e\gamma^\mu\delta_1,$ (2.11)



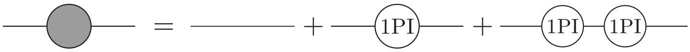
$= i\left(p\!\!\!/\delta_2 - \delta_m\right).$

[3] From the superficial degree of divergence, $D = 4 - N_\gamma - 3N_e/2$, where N_γ are the number of external photon legs and N_e the number of external electron legs, we would expect for the electron- and photon two-point functions to be linearly and quadratically divergent, respectively. However, the degree of divergence is reduced by the custodial chiral symmetry and by the Ward identity.

We now arrive at the key step in renormalization, which is to assign finite values for the divergent amplitudes at some scale. For a renormalizable theory, independently on what value we choose, or what scale we pick, this will ensure that all amplitudes are rendered finite – provided that we introduce as many conditions as there are counterterms. In QED, with four infinite constants, there are hence four so-called renormalization conditions, given by, for example

$$\Sigma(\not{p} = m_p) = 0, \qquad \left.\frac{\Sigma(\not{p})}{d\not{p}}\right|_{\not{p}=m_p} = 0, \qquad \Pi(q^2 = 0) = 0, \qquad -ie\Gamma^\mu(0) = -ie\gamma^\mu, \tag{2.12}$$

where the first condition sets the renormalized mass m_r equal to the pole mass m_p, while the second condition fixes the residue of the electron propagator to 1, as apparent from considering the two-point function for the electron in renormalized perturbation theory

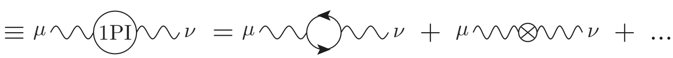


$$= \frac{i}{\not{p} - m_r - \Sigma(\not{p})} \sim \frac{i}{\not{p} - m_p} + \{\text{finite}\}, \tag{2.13}$$

where – in contrast to bare perturbation theory – there is no factor of Z in the numerator, we have the renormalized mass rather than the bare one, and where $-i\Sigma(\not{p})$ now also incorporates counterterm diagrams. Similarly, the third condition in eq. (2.12) fixes the residue of the photon propagator to 1. These are so-called *on-shell* renormalization conditions – we will return to other possible options in Sec. 2.3.

The four counterterms are then determined through the three divergent n-point functions. Starting with $\Pi^{\mu\nu}$, we have

$$i\Pi^{\mu\nu}_{\gamma\gamma}(q^2) \equiv i\left(g^{\mu\nu}q^2 - q^\mu q^\nu\right)\Pi(q^2)$$



$$= i\left(g^{\mu\nu}q^2 - q^\mu q^\nu\right)\left[\Pi_2(q^2) - \delta_3\right] + \dots, \tag{2.14}$$

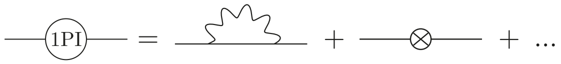
where $\Pi_2(q^2)$ is defined as the 1-loop photon self-energy correction. The third renormalization condition in eq. (2.12) hence results in $\delta_3 = \Pi_2(0)$, such that

$$i\Pi^{\mu\nu}_{\gamma\gamma}(q^2) \simeq i\left(g^{\mu\nu}q^2 - q^\mu q^\nu\right)\left[\Pi_2(q^2) - \Pi_2(0)\right], \tag{2.15}$$

which is *finite*, since $\Pi_2(q^2)$ and $\Pi_2(0)$ carry the same divergent behavior, but come with a relative sign in eq. (2.15).

The remaining three counterterms are determined from the string of 1PI-diagrams

$$-i\Sigma(\not{p}) \equiv$$

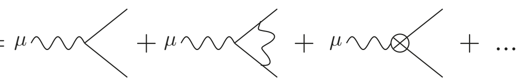


$$= -i\Sigma_2(\not{p}) + i\left(\not{p}\delta_2 - \delta_m\right) + \ldots \tag{2.16}$$

and the vertex correction

$$-ie\Gamma^\mu(p,p') = -ie\left(\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m_r}F_2(q^2)\right)$$

$$=$$



$$= -ie\gamma^\mu - ie\delta\Gamma^\mu(p,p') - ie\gamma^\mu\delta_1 + \ldots \tag{2.17}$$

with $\delta\Gamma(p,p') \equiv \gamma^\mu \delta F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m_r}\delta F_2(q^2)$. Combined with the renormalization conditions in eq. (2.12) this results in

$$m_p\delta_2 - \delta_m = \Sigma_2(m_p), \quad \delta_2 = \left.\frac{\Sigma_2(\not{p})}{d\not{p}}\right|_{\not{p}=m_p}, \quad \delta_1 = -\delta F_1(0), \tag{2.18}$$

by which all amplitudes in QED at 1-loop are rendered finite.

2.3 Physical and Unphysical Renormalization Schemes

> "*Close your eyes and regularize!*" – Alexey Vladimirov

To explicitly see the cancellation of divergences in eq. (2.15), let us evaluate $\Pi_2(q^2)$. As always, the process involves introducing Feynman parameters, completing the square to make the integrand spherically symmetric, simplifying the numerator using a number of identities, and then Wick rotating from Minkowski space to Euclidean space. If the resulting momentum integral(s) are divergent, we then need to introduce a regulator to render the Feynman integrals finite, so that they can be dealt with until the divergences cancel in a later stage. Using dimensional regularization, we end up with [17]

$$
\begin{aligned}
\Pi_2(q^2) &= -\frac{e^2\mu^{4-d}}{(4\pi)^{d/2}}\int_0^1 \mathrm{d}x\, \frac{\Gamma\left(2-d/2\right)}{(m_r^2-x(1-x)q^2)^{2-d/2}}\, 8x(1-x) \\
&\simeq -\frac{e^2}{2\pi^2}\left[\frac{1}{6}\left(\frac{2}{\epsilon}-\gamma_E+\ln(4\pi)\right)+\int_0^1 \mathrm{d}x\,\ln\left(\frac{\mu^2}{m_r^2-x(1-x)q^2}\right)x(1-x)\right],
\end{aligned}
\tag{2.19}
$$

where the diagram is regularized by extending it to $d = 4 - \varepsilon$ dimensions, and not taking the limit of $\varepsilon \to 0$ until all $1/\varepsilon$ dependence has cancelled. Using that

$$
\varepsilon\Gamma(\varepsilon) = \Gamma(1+\varepsilon) \simeq \Gamma(1) + \varepsilon\Gamma'(1) + \mathcal{O}(\varepsilon^2), \quad \Gamma'(1) \equiv -\gamma_E, \tag{2.20}
$$

where γ_E is the Euler-Mascheroni constant, eq. (2.19) evaluates to[4]

$$
\Pi_2(q^2=0) \simeq -\frac{e^2}{2\pi^2}\left[\frac{1}{6}\left(\frac{2}{\epsilon}-\gamma_E+\ln(4\pi)\right)+\int_0^1 \mathrm{d}x\ \ln\left(\frac{\mu^2}{m_p^2}\right)x(1-x)\right], \tag{2.21}
$$

for $q^2 = 0$, such that eq. (2.15) results in

$$
\begin{aligned}
i\Pi^{\mu\nu}_{\gamma\gamma}(q^2) &\simeq i\left(g^{\mu\nu}q^2 - q^\mu q^\nu\right)\left[\Pi_2(q^2)-\Pi_2(0)\right] \\
&\simeq i\left(g^{\mu\nu}q^2 - q^\mu q^\nu\right)\left[-\frac{e^2}{2\pi^2}\int_0^1 \mathrm{d}x\ \ln\left(\frac{m_p^2}{m_r^2-x(1-x)q^2}\right)x(1-x)\right],
\end{aligned}
\tag{2.22}
$$

which is finite in the limit of $\varepsilon \to 0$.

Let us now address a few things regarding eq. (2.22). The μ-scale, introduced to deal with the fractal dimension, has cancelled – and while that scale is arbitrary, it is commonly set equal to the renormalization scale, which in our case is the pole mass of the electron, m_p. For this choice, the integral in eq. (2.21) is zero, and we obtain eq. (2.22) in an even simpler manner. For the remainder of this work, we will always assume that the dimensional regularization scale μ is set equal to the renormalization scale, using the two concepts interchangeably. Also, note that, m_p and m_r has been kept separate throughout the calculation,

[4]Note that the integral can be evaluated, but is still kept unevaluated for the convenience of the subtraction in the following step.

even though $m_r = m_p$ in an on-shell scheme, simply to clarify what the end result would look like in an off-shell scheme.

As the finite piece of the counterterm anyway always cancels for any physical observables, the sole requirement is for it to contain the pole, such that the divergence is taken care of. One possible off-shell renormalization scheme is hence the minimal subtraction scheme, *MS*, where only the $(1/\epsilon)$-term, and its prefactor, is included in the counterterm. Another popular choice is the modified minimal subtraction scheme, $\overline{MS}$, where the counterterm is instead proportional to $(1/\epsilon - \gamma_E + \ln 4\pi)$.

2.4 The Continuum Renormalization Group

In Sec. 2.2, the amplitudes were renormalized by the use of renormalization conditions at some scale μ. As this scale is arbitrary, we demand that the bare parameters cannot depend on it. This, in turn, tells us exactly how the renormalized couplings evolve with energy.[5]

Take for example the fermion-gauge interaction in a non-abelian gauge theory, $\bar{\psi}_i \gamma^\mu T^a_{ij} \psi_j A^a_\mu$, where the bare- and renormalized gauge couplings g_0 and g are related in the same way as e_0 and e in eq. (2.9). From demanding the bare coupling to be independent of μ, we hence have that

$$0 = \mu \frac{d}{d\mu} g_0 = \mu \frac{d}{d\mu} \left(g \, \frac{Z_1}{Z_2 \sqrt{Z_3}} \mu^{\frac{\varepsilon}{2}} \right), \tag{2.23}$$

where, after using the product rule, $Z_i = 1 + \delta_i$ and expanding perturbatively (with $\ln(1 + \delta_i) \simeq \delta_i$ and keeping only terms linear in δ_i), the beta function is given by

$$\beta(g) \equiv \mu \frac{d}{d\mu} g \simeq g \left[-\frac{\varepsilon}{2} - \mu \frac{d}{d\mu} \left(\delta_1 - \delta_2 - \frac{1}{2} \delta_3 \right) \right], \tag{2.24}$$

such that the beta function is known once all counterterms have been established [20]. In other words, from demanding the bare coupling to be independent of μ, we have found a differential equation telling us exactly how the renormalized coupling must evolve.

Using this procedure, or an equivalent method, it can be shown that the beta function for an $SU(N)$ gauge coupling at 1-loop, once all contributions are accounted for, is given by [21],

[5] Note that this differs slightly from Wilson's approach, with a floating finite UV cut-off, where instead the bare parameters are cut-off dependent in such a way that physical parameters are cut-off independent [19].

$$\beta(g_i) \equiv \mu \frac{dg_i}{d\mu} = -\frac{g_i^3}{(4\pi)^2}\left(\frac{11}{3}C_2(G) - \frac{4}{3}\kappa S_2(F) - \frac{1}{3}S_2(S)\right) \equiv \frac{b_i}{16\pi^2}g_i^3, \tag{2.25}$$

which is a separable differential equation, with the solution (using $g_i^2 \equiv 4\pi\alpha_i$)

$$\alpha_i^{-1}(\mu) = \alpha_i^{-1}(\mu_0) - \frac{b_i}{2\pi}\log\left(\frac{\mu}{\mu_0}\right), \tag{2.26}$$

or, equivalently,

$$\alpha_i(\mu) = \frac{\alpha_i(\mu_0)}{1 - \frac{b_i\alpha_i(\mu_0)}{2\pi}\log\left(\frac{\mu}{\mu_0}\right)}, \tag{2.27}$$

where $C_2(G)$ is the Casimir operator for the adjoint representation (rep), κ is a constant equal to $1/2$ for two-component fermions and 1 for four-component fermions, $S_2(F)$ is the Dynkin index for a fermion and $S_2(S)$ is the Dynkin index for a scalar, with

$$S_2(S)\delta^{ab} = \mathrm{tr}[\theta^a\theta^b], \quad S_2(F)\delta^{ab} = \mathrm{tr}[t^a t^b], \quad C_2(G)\delta^{ab} = S_2(G)\delta^{ab} = f^{acd}f^{bcd}, \tag{2.28}$$

with an implicit summation over repeated indices, and where θ^a and t^a are generators defined as

$$D_\mu\phi_i = \partial_\mu\phi_i - ig\theta^a_{ij}V^a_\mu\phi_j, \quad D_\mu\psi_i = \partial_\mu\psi_i - igt^a_{ij}V^a_\mu\psi_j, \tag{2.29}$$

with i, j being rep indices and a an adjoint index. In other words, the Casimir operator and the Dynkin index are both group theoretical invariants, constructed from the bilinear $T^a_{ij}T^b_{jk}$, with the Dynkin index corresponding to contracting i with k, such that $T^a_{ij}T^b_{ji} = \mathrm{tr}[T^aT^b]$, but with no contraction in the adjoint indices, while Casimir operators contracts a with b. In the adjoint representation, the Casimir operator and the Dynkin index are equal to each other, as the rep indices are also adjoint indices. Hence, we can obtain the final relation in eq. (2.28) by again contracting i with k, and summing over repeated indices. Renaming i, j as c, d to emphasize that they are adjoint indices, we hence have that $(T^a)^{cd}(T^b)^{dc} = (-if^{acd})(-if^{bdc}) = f^{acd}f^{bcd}$.

From eq. (2.25) we see that the running is fully determined by the number of fermions and scalars in the theory, and by what representation they are in. For example, the running of the $SU(3)_C$ gauge coupling, with 18 Weyl fermions and 18 scalars in the fundamental rep of $SU(3)_C$ and one Weyl fermion in the adjoint rep of $SU(3)_C$, is given by

$$\begin{aligned} b_{g_C} &= -\left(\frac{11}{3}\cdot 3 - \frac{4}{3}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot 18 - \frac{4}{3}\cdot\frac{1}{2}\cdot 3\cdot 1 - \frac{1}{3}\cdot\frac{1}{2}\cdot 18 - \frac{1}{3}\cdot 3\cdot 0\right) \\ &= 0\ , \end{aligned} \tag{2.30}$$

using that $C_2(G) = N$, the Dynkin index is equal to $1/2$ for a field in the fundamental rep of $SU(N)$, and equal to N for a field in the adjoint rep of $SU(N)$, and that $\kappa = 1/2$ for Weyl fermions. As such, in Paper II, the $SU(3)_C$ gauge coupling in "Region I" runs flat, while e.g. an additional fermion in the adjoint rep would have changed this value to $b_{g_C} = 3$.

It is important to mention that eq. (2.25) was derived in an off-shell renormalization scheme, which means that Appelquist and Carazzone's decoupling theorem [22] does not apply. Hence, to get a sensible outcome of an RG evolution using these equations, one has to manually discard any particles that are heavier than the current mass scale of the evolution and each time perform a matching procedure. In other words, in an unphysical scheme the beta function must be adjusted by hand through switching from one EFT to the next [23, 24]. If we instead would have been using an on-shell scheme, the beta function is mass dependent, which results in the heavy states automatically having a negligible effect at lower scales. In a physical scheme, it hence does not matter whether heavy states are kept or not.

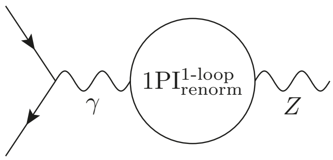
3 Electroweak Observables

When extending the SM, we are not always so fortunate as to have all NP effects for the EW sector entirely parameterized by the Peskin-Takeuchi (PT) parameters. For example, in cases where the NP scale is not decoupled from the EW scale, or in cases where the Z-boson mass is altered by the presence of neutral NP gauge bosons, the PT parameters are not valid, and we instead need to go to the trouble of evaluating the theoretical prediction for various Z-pole observables, as in Paper III and IV. To demonstrate this procedure, we consider below one of the most accurately measured observables in the EW sector, namely the polarization asymmetry in Z-boson production.

Sections 3.2 and 3.3 then treat various features of dealing with unstable particles, such as the narrow-width approximation and its connection to the optical theorem.

3.1 Polarization Asymmetry in Z-boson Production

The procedure of obtaining the theoretical prediction for an observable begins with the perturbative calculation of the corresponding amplitude in terms of the (unphysical) parameters of the Lagrangian. For the polarization asymmetry in Z-boson production, the only relevant contribution at 1-loop comes from the diagram



, (3.1)

where "1PI$^{\text{1-loop}}_{\text{renorm}}$" denotes all amputated 1PI diagrams at 1-loop, in addition to all the corresponding counterterms. From this, we obtain the asymmetry

$$A^e_{LR} \equiv \frac{\sigma(e^-_L e^+_L \to Z) - \sigma(e^-_R e^+_R \to Z)}{\sigma(e^-_L e^+_L \to Z) + \sigma(e^-_R e^+_R \to Z)} = \frac{\left(\frac{1}{2} - s^2_{\text{eff}}\right)^2 - s^4_{\text{eff}}}{\left(\frac{1}{2} - s^2_{\text{eff}}\right)^2 + s^4_{\text{eff}}} , \tag{3.2}$$

with s_{eff} defined as

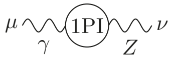
$$s^2_{\text{eff}} \equiv s^2_W - s_W c_W \frac{\Pi^{\text{1-loop}}_{\gamma Z}(m^2_Z)}{m^2_Z} , \tag{3.3}$$

where s_W is the (unphysical) renormalized $\overline{MS}$ value for the (sine of the) weak mixing angle, i.e. the quantity appearing in the Lagrangian when expressed in terms of renormalized $\overline{MS}$ parameters

$$\begin{aligned} \mathcal{L} = &- \frac{e}{s_W c_W} \left[\left(\frac{1}{2} - s^2_W \right) \bar{e}_L \gamma^\mu e_L - s^2_W \bar{e}_R \gamma^\mu e_R \right] Z_\mu - e \left[\bar{e}_L \gamma^\mu e_L - \bar{e}_R \gamma^\mu e_R \right] A_\mu \\ &+ \left\{ \text{Counterterms} \right\} , \end{aligned} \tag{3.4}$$

and where $\Pi_{\gamma Z}$ is defined via the renormalized 1PI vacuum polarization diagrams, as

$$i\Pi^{\mu\nu}_{\gamma Z}(q^2) \equiv$$



$$\equiv i(g^{\mu\nu} q^2 - q^\mu q^\nu) \left(- \frac{\Pi_{\gamma Z}(q^2)}{q^2} \right) \tag{3.5}$$

i.e. defined with a factor $(-1/q^2)$ relative to the form used in eq. (2.14), which is the convention in EW precision physics.

The next step is to express the amplitude in eq. (3.2) in terms of physical parameters, here denoted by a circumflex, $\hat{s}_W$ and $\hat{c}_W$. In other words, we need to define an expression for the physical weak mixing angle, in addition to finding a relation between the physical and unphysical parameters. Starting with the definition of the physical weak mixing angle, one of the most common choices is

$$\hat{s}_W^2 \hat{c}_W^2 \equiv \frac{\pi \hat{\alpha}(m_Z)}{\sqrt{2}\hat{G}_F \hat{m}_Z^2} , \tag{3.6}$$

that is, to express $\hat{s}_W$ in terms of the most accurately measured EW observables – the pole mass of the Z-boson $\hat{m}_Z$, the fine structure constant $\hat{\alpha}$ at the m_Z-scale and the Fermi constant $\hat{G}_F$ [25]

$$\begin{aligned} \hat{m}_Z &= 91.1876(21) \text{ GeV}, \qquad \hat{\alpha}(m_Z)^{-1} = 127.916(15), \\ \hat{G}_F &= 1.1663787(6) \times 10^{-5} \text{ GeV}^{-2} , \end{aligned} \tag{3.7}$$

with the latter two extracted from measurements of the electric magnetic dipole moment and the lifetime of the muon, respectively.

Finally, we need the expression relating the physical and the unphysical parameters, i.e. the relation between s_W and $\hat{s}_W$. As we defined $\hat{s}_W$ in terms of the pole mass of the Z-boson, the Fermi constant and the fine structure constant, this process involves finding expressions for $\hat{m}_Z$, $\hat{G}_F$ and $\hat{\alpha}$ in terms of their unphysical counterparts m_Z, G_F and α, and then inverting the equations. This results in, using the notation in Ref. [20]

$$s_W^2 = \hat{s}_W^2 \left(1 + \frac{\hat{c}^2}{\hat{c}^2 - \hat{s}^2}\,\Pi_R\right), \quad c_W^2 = \hat{c}_W^2 \left(1 + \frac{\hat{s}^2}{\hat{c}^2 - \hat{s}^2}\,\Pi_R\right), \tag{3.8}$$

with

$$\Pi_R \equiv -\frac{\Pi_{\gamma\gamma}^{\text{1-loop}}(\hat{m}_Z^2)}{\hat{m}_Z^2} + \frac{\Pi_{ZZ}^{\text{1-loop}}(\hat{m}_Z^2)}{\hat{m}_Z^2} - \frac{\Pi_{WW}^{\text{1-loop}}(0)}{\hat{m}_W^2} , \tag{3.9}$$

allowing us to, finally, express the asymmetry in eq. (3.2) in terms of physical parameters, with

$$s^2_{\text{eff}} \equiv \hat{s}^2_W + \frac{\hat{s}^2_W \hat{c}^2_W}{\hat{c}^2_W - \hat{s}^2_W} \Pi_R - \hat{s}_W \hat{c}_W \frac{\Pi^{\text{1-loop}}_{\gamma Z}(\hat{m}^2_Z)}{\hat{m}^2_Z} \,, \tag{3.10}$$

by which we have our 1-loop theoretical prediction for the polarization asymmetry, $A^e_{LR}(\hat{s}_W)$.

3.2 The Optical Theorem

Before being able to discuss the Breit-Wigner distribution and the narrow-width approximation, we must first cover the optical theorem. The generalized optical theorem follows straightforwardly from the unitarity of the S-matrix, $S^\dagger S = 1$, with $S \equiv 1 + iT$, by inserting a complete set of states. It is given by

$$\begin{aligned} &\mathcal{M}(a \to b) - \mathcal{M}^*(b \to a) \\ &= i \sum_f \left(\prod_{j=1}^{n} \int \frac{\mathrm{d}^3 p_j}{(2\pi)^3} \frac{1}{2E_{\vec{p}_j}} \right) (2\pi)^4 \delta^4(p_i - p_n) \mathcal{M}(a \to f) \mathcal{M}^*(b \to f), \end{aligned} \tag{3.11}$$

where the naming of a, b and f is a bit awkward, as in an uncut diagram f is an intermediate state, and a and b the initial and final states, respectively, while in a cut diagram, f represents final state particles. We are then interested in the special case of having an identical initial and final state, $|a\rangle = |b\rangle = |A\rangle$. In the case of $|A\rangle$ being a two-particle state, we have what is commonly referred to as the optical theorem

$$\operatorname{Im}\left[\mathcal{M}(A \to A)\right] = 2E_{\text{cm}} p_{\text{cm}} \sum_f \sigma(A \to f), \tag{3.12}$$

relating the total cross section with the imaginary part of the scattering amplitude. Here, p_{cm} is the momentum of either of the two particles in the center-of-mass frame, the sum is over all possible final states f, and E_{cm} the total center-of-mass energy. For the case of $|A\rangle$ being a one-particle state, the derivations differ as to whether they assume the decay rate to be previously known from scattering theory, as in e.g. Ref. [20], or by it being the end-result of the derivation, as in e.g. Ref. [17]. We will follow the former approach, by which the generalized optical theorem simplifies to

$$\operatorname{Im}\left[\mathcal{M}(A \to A)\right] = m_A \sum_f \Gamma(A \to f) = m_A \Gamma_{tot} \,. \tag{3.13}$$

3.3 The Narrow-Width Approximation

To simplify the discussion of the narrow-width approximation, this section considers a fictitious, unstable spin-0 particle. The result, however, generalizes easily to the spin-1 case.

Combining eq. (2.7) and eq. (3.13), we have, in bare perturbation theory and in the vicinity of the pole,

$$\begin{aligned}
\Gamma_{tot} &= \frac{1}{m_p}\text{Im}\left[(-iZ)\text{—(Amp)—}\right] \\
&= \frac{1}{m_p}\text{Im}\left[(-iZ)\left(\text{———} + \text{—(1PI)—} + \text{—(1PI)—(1PI)—} + \ldots\right)\right] \\
&\simeq \frac{1}{m_p}\text{Im}\left[(-iZ)\left(\text{—(1PI)—}\right)\right] = \frac{1}{m_p}\text{Im}\left[-ZM^2(m_p^2)\right] ,
\end{aligned} \tag{3.14}$$

using that the tree-level propagator has no imaginary part, and neglecting higher-order terms. Switching to renormalized perturbation theory gives the same relationship, but with $Z = 1$ and with counterterm diagrams now being incorporated into the definition of $M^2(p^2)$, as discussed in Sec. 2.2. Hence, in renormalized perturbation theory,

$$\text{Im}\left[M^2(m_p^2)\right] \simeq -m_p\Gamma_{tot} . \tag{3.15}$$

With the amplitude having a non-zero imaginary part for unstable particles, let us then alter the definition of the pole mass to be

$$m_p^2 - m_r^2 + \text{Re}\left[M^2(m_p^2)\right] = 0, \tag{3.16}$$

such that, in the vicinity of the pole, with eq. (3.15),

$$\text{—}\bullet\text{—} = \frac{i}{p^2 - m_r^2 - M^2(p^2)} \simeq \frac{i}{p^2 - m_p^2 + im_p\Gamma_{tot}} , \tag{3.17}$$

where the cross-section for an s-channel process in the region of a resonance, is the so-called Breit-Wigner distribution

$$\sigma \propto \left|\frac{i}{p^2 - m_p^2 + im_p\Gamma_{tot}}\right|^2 , \tag{3.18}$$

with the full-width half-maximum corresponding to $2m_p\Gamma_{tot}$. In the limit of $\Gamma/m_p \to 0$, this then results in

$$\lim_{\Gamma/m_p\to 0}\left|\frac{i}{p^2 - m_p^2 + im_p\Gamma_{tot}}\right|^2 = \frac{\pi}{m_p\Gamma_{tot}}\delta(p^2 - m_p^2), \tag{3.19}$$

by using that

$$\lim_{\epsilon\to 0}\frac{\epsilon}{x^2+\epsilon^2}=\pi\delta(x)\ , \tag{3.20}$$

with $\epsilon \equiv \Gamma_{tot}/m_p$ and $x \equiv (p^2/m^2) - 1$.

The result in eq. (3.19) is the so-called narrow-width approximation, which says that, as long as $m_p \gg \Gamma_{tot}$, we can, to leading order near a resonance, treat the unstable particle as being on-shell. Under these conditions, the cross-section hence factorizes, such that there is no interference between production and decay. For example

$$\left[\sigma(e^+e^- \to Z' \to X)\right]_{\mathrm{NWA}} = \sigma(e^+e^- \to Z')\mathrm{Br}(Z' \to X)\ , \tag{3.21}$$

where the branching ratio is defined as the decay width of that particular decay, divided by the total decay width. The narrow-width approximation is used in Paper III and IV.

4 Extending the SM Higgs Sector

All four papers in this thesis are related to extensions of the SM Higgs sector, for which one of the strictest constraints comes from the mass ratio of the W and Z bosons. The SM tree-level prediction is given by $\rho_0 \equiv m_W^2/(c_W^2 m_Z^2) = 1$, which is very close to its measured value. In the SM this is explained by the parameter being protected from any sizable radiative corrections by a custodial $SU(2)$ symmetry.[6]

When extending the SM Higgs sector by n scalars ϕ_i – with weak hypercharge Y_i, weak isospin T_i and vacuum expectation value (VEV) v_i – the W and Z bosons acquire the masses [26]

$$m_Z^2 = \frac{g_1^2+g_2^2}{4}\sum_{i=1}^{n} Y_i^2 v_i^2, \quad m_W^2 = \frac{g_1^2}{2}\sum_{i=1}^{n}\left(T_i\left(T_i+1\right)-\frac{Y_i^2}{4}\right)v_i^2, \tag{4.1}$$

where g_1 and g_2 are gauge couplings of $SU(2)_\mathrm{L}$ and $U(1)_\mathrm{Y}$, respectively. Hence, to not disturb the SM tree-level prediction of $\rho_0 = 1$, with $c_W^2 = g_1^2/\left(g_1^2+g_2^2\right)$, any extension to the SM Higgs sector must obey the relation

[6]The SM scalar potential has an $SO(4)$ symmetry, that is spontaneously broken down to $SO(3)$ with EWSB, which is locally isomorphic to $SU(2)$. The symmetry is however not a symmetry of the full Lagrangian, but is explicitly broken by $g_2 \neq 0$ and $Y_u \neq Y_d$, where Y_u and Y_d are the Yukawa couplings for the up- and down sectors, respectively. The radiative corrections are then proportional to these symmetry breaking parameters, which protects them from becoming sizeable.

$$4\sum_{i=1}^{n} T_i(T_i+1) = 3\sum_{i=1}^{n} Y_i^2. \tag{4.2}$$

While there are infinite solutions to this relation [27], we will focus on the minimal cases $T_i = Y_i = 0$ and $T_i = Y_i = 1/2$, which corresponds to adding either scalar singlets with $Y = 0$, or Higgs doublets with $Y = 1/2$ (and the VEV positioned in its neutral component).

4.1 The *n*HDM

Let us begin with the scenario of extending the SM scalar sector with $n-1$ Higgs-doublets, in a so-called n-Higgs-doublet model (nHDM), which plays a central role in Paper III and IV, but also as the low-energy limit of Paper I and II. Without any imposed symmetries, the Yukawa interactions are given by

$$-\mathcal{L}_{\mathrm{Y}} = \bar{q}_{\mathrm{L}}\Gamma^k\phi^k d_{\mathrm{R}} + \bar{q}_{\mathrm{L}}\Delta^k\tilde{\phi}^k u_{\mathrm{R}} + \bar{\ell}_{\mathrm{L}}\Pi^k\phi^k e_{\mathrm{R}} + \mathrm{H.c.}, \tag{4.3}$$

where k runs from one to n, and where Γ, Δ and Π are Yukawa coupling matrices in flavor space with general complex entries. Furthermore, the scalar potential with no imposed symmetries is of the form

$$V = m_{ij}\,\phi_i^\dagger\phi_j + \lambda_{ijk\ell}\left(\phi_i^\dagger\phi_j\right)\left(\phi_k^\dagger\phi_\ell\right), \tag{4.4}$$

with indices once again running from one to n, $\tilde{\phi} \equiv i\sigma_2\phi^*$, and where $m_{ij} = m_{ji}^*$, and $\lambda_{ijkl} = \lambda_{klij} = \lambda_{jilk}^*$ follows from V being hermitian. Before accounting for possible field redefinitions, there are hence a total of $n^2 + n^2(n^2-1)/2$ independent, real parameters coming from the scalar potential and $54n$ from the Yukawa sector.[7] Furthermore, the parameters are required to fulfill stability constraints and perturbative unitary bounds [28].

After EWSB, the Higgs-doublets are commonly parameterized as

$$\phi_k = \frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{2}\varphi_k^+ \\ v_k e^{i\alpha_k} + \eta_k + i\chi_k\end{pmatrix}, \quad v^2 = \sum_{k=1}^{n} v_k^2 = (246\ \mathrm{GeV})^2 \tag{4.5}$$

[7]For example, with m_{ij} being an $n\times n$ hermitian matrix, there are $(n^2-n)/2$ complex entries (and hence n^2-n real parameters) above the main diagonal. Adding to this the n real diagonal entries (the diagonal elements of an Hermitian matrix is always real), we then end up with a total of n^2 independent, real parameters.

where v_k are the VEVs of the scalar fields, and where $U(1)_{\rm Y}$ allows for one of the phases α to be eliminated without loss of generality.[8] It is also common practice to further decrease the number of phases via field redefinitions of the Higgs-doublets, provided that these have not already been used up for eliminating phases in the couplings in the Yukawa- and scalar sectors.

A convenient way of establishing the maximum number of eliminated phases is to write all field redefinitions as one large system of equations. Take for example the condition coming from a parameter in the Yukawa sector, Γ^k_{ij},

$$-\mathcal{L}_{\rm Y} = \bar{q}_{{\rm L}i}\Gamma^k_{ij}\phi^k d_{{\rm R}_j} \quad \Rightarrow \quad -\theta_{q_i} + \gamma^k_{ij} + \theta_{d_j} + \theta_{\phi_k} = 0, \tag{4.6}$$

where θ_{q_i} is the phase of $i^{\rm th}$ generation of up-type quarks, θ_{d_j} the phase of $j^{\rm th}$ generation of down-type quarks, θ_{ϕ_k} the phase of $k^{\rm th}$ Higgs-doublet and γ^k_{ij} the phase of the corresponding Yukawa coupling Γ^k_{ij}. After carrying out the equivalent procedure for all other terms in the Lagrangian, the rank of the corresponding matrix corresponds to the maximum number of parameters that can be made simultaneously real.[9]

4.2 The Alignment Limit

Since the 2012 Higgs boson discovery [14, 15], there is yet to be any significant discrepancy between Higgs measurements and the SM prediction. As such, nHDMs are normally forced to be in the vicinity of the so-called alignment limit, where one of the scalars is taken to be SM-like.

For demonstrational purposes, let us consider the simplified scenario of having an nHDM with only real parameters. Here, the Higgs basis (denoted with a superscript H) is the basis in which the VEV resides solely in the first doublet, such that

$$\phi^H_1 \equiv \frac{1}{v}\sum_{k=1}^{N} v_k\phi_k, \qquad \langle\phi^H_k\rangle = \frac{v}{\sqrt{2}}\delta_{k1}, \tag{4.7}$$

with ϕ_k, v_k and v defined as in eq. (4.5). With this, the doublet becomes

[8] With all Higgs-doublets having degenerate charges under $U(1)_{\rm Y}$, only one phase can be eliminated by a global hypercharge rotation. This number can be increased by extending the SM gauge group with additional gauged abelian symmetries.

[9] An $a \times b$ matrix of rank m, with $a \leq b$, has $a - m$ zero-rows when expressed in row-echelon form.

$$\phi_1^H = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}G^+ \\ v + H_1 + iG^0 \end{pmatrix}, \qquad H_1 \equiv \frac{1}{v} \sum_{k=1}^{N} v_k \eta_k, \tag{4.8}$$

where G^+ is the charged Goldstone boson, G^0 the neutral Goldstone boson, and with η_k being the CP-even states in the original basis, defined in eq. (4.5). Note that H_1 has the same tree-level Yukawa couplings and gauge couplings as the SM Higgs, but that it is in general *not* a mass eigenstate. Here, the alignment limit is defined as the limit in which H_1 aligns itself with the lightest CP-even mass eigenstate h.

Take, for example, the familiar case of a 2HDM with an imposed Z_2-symmetry. By demanding the scalar potential to be invariant under $\phi_1 \to \phi_1$ and $\phi_2 \to -\phi_2$, we have that[10]

$$\begin{aligned} V = {} & m_{11}^2 \phi_1^\dagger \phi_1 + m_{22}^2 \phi_2^\dagger \phi_2 + \frac{\lambda_1}{2} (\phi_1^\dagger \phi_1)^2 + \frac{\lambda_2}{2} (\phi_2^\dagger \phi_2)^2 + \lambda_3 (\phi_1^\dagger \phi_1)(\phi_2^\dagger \phi_2) \\ & + \lambda_4 (\phi_1^\dagger \phi_2)(\phi_2^\dagger \phi_1) + \frac{\lambda_5}{2} (\phi_1^\dagger \phi_2)^2 + \frac{\lambda_5^*}{2} (\phi_2^\dagger \phi_1)^2, \end{aligned} \tag{4.9}$$

where the only complex parameters λ_5 and v_2 can be made real by trivial rephasings of the two doublets. With both VEVs real, the CP-even and CP-odd sectors separate, such that the mass eigenbasis relates to the original basis by

$$\begin{pmatrix} G^0 \\ A \end{pmatrix} = U_\beta \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \quad \begin{pmatrix} H \\ h \end{pmatrix} = U_\alpha \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \quad \begin{pmatrix} G^+ \\ H^+ \end{pmatrix} = U_\beta \begin{pmatrix} \varphi_1^+ \\ \varphi_2^+ \end{pmatrix}, \tag{4.10}$$

with $U_\theta \equiv \{\{\cos\theta, \sin\theta\}, \{-\sin\theta, \cos\theta\}\}$, $\tan\beta \equiv v_2/v_1$, and with the Higgs doublets parameterized as in eq. (4.5), with $k = 1, 2$ and $\alpha_1 = \alpha_2 = 0$. Here, A a pseudoscalar, h the lightest CP-even state and H^+ a charged Higgs. Note that the charged sector and the CP-odd sector share the same eigenvectors.

From eq. (4.8), we have that H_1, and its orthogonal combination H_2, relates to the original (CP-even) basis with U_β, such that

$$\begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = U_\beta \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = U_\beta U_\alpha^{\mathrm{T}} \begin{pmatrix} H \\ h \end{pmatrix} = U_{\beta-\alpha}^{\mathrm{T}} \begin{pmatrix} H \\ h \end{pmatrix}, \tag{4.11}$$

[10]Note that the prefactors have been recast with respect to eq. (4.4), as to match the popular convention.

where T denotes a transpose. Here we see that H_1 aligns with h for $\sin(\alpha - \beta) = 1$, which corresponds to the so-called alignment limit. For more details, see Ref. [28].

5 Supersymmetry

One of the motivations for working with nHDMs comes from supersymmetry, which also happens to be the framework used in Paper I and II. More specifically, as supersymmetry does not allow for the potential to depend both on a field and its complex conjugate, a minimum of two Higgs doublets are required for giving mass to both up-type and down-type quarks. Furthermore, having just a single Higgs doublet spoils anomaly cancellation for $U(1)_Y$ [29]. We will return to the subject of anomalies in Sec. 7.

This section covers how the supersymmetry algebra follows almost uniquely from Lorentz invariance and the Coleman-Mandula theorem. Using the algebra, we then develop the concepts of superfields and superspace. Supersymmetry plays a central role in Paper I and II.

5.1 Representations of the Lorentz Group

The inhomogeneous Lorentz group, also referred to as the Poincaré group, is of great interest when working with relativistic theories, as it is the group of coordinate transformations that preserves the Minkowski metric. Considering a coordinate transformation $x_\mu \to x'_\mu$, we have the length element

$$ds^2 = dx^\mu g_{\mu\nu}\, dx^\nu = dx'^\rho g_{\rho\sigma} dx'^\sigma = \frac{\partial x'^\rho}{\partial x^\mu} dx^\mu g_{\rho\sigma} \frac{\partial x'^\sigma}{\partial x^\nu} dx^\nu \equiv \Lambda^\rho{}_\mu dx^\mu g_{\rho\sigma} \Lambda^\sigma{}_\nu dx^\nu, \tag{5.1}$$

such that $g_{\mu\nu} = \Lambda^\rho{}_\mu g_{\rho\sigma} \Lambda^\sigma{}_\nu$, where in the last step we defined the Jacobi matrix $\partial x'^\mu / \partial x^\nu$ to be equal to $\Lambda^\mu_{\ \nu}$. Note that the Jacobi matrix does not fully determine our coordinate transformation; from integrating $dx'^\mu = (\partial x'^\mu / \partial x^\nu) dx^\nu$, we get $x'^\mu = \Lambda^\mu_{\ \nu} x^\nu + a^\mu$, where a^μ is a constant of integration. The homogeneous Lorentz group is then defined (through its four-vector representation) as the elements that fulfill $g_{\mu\nu} = \Lambda^\rho{}_\mu g_{\rho\sigma} \Lambda^\sigma{}_\nu$.[11] In other words, the Poincaré group consists of ten generators – the six generators of the homogeneous Lorentz group (three rotations and three boosts) and four spacetime translations.

If we constrain ourselves to the group elements connected to the identity, we instead have the proper orthochronous Lorentz group $SO^+(1,3)$, with $\det\Lambda = 1$ and $\Lambda^0_{\ 0} \geq 1$. Here,

[11] Of course, we must also verify that the group axioms are fulfilled before referring to it as a group.

the group elements can be generated by an exponentiation of the generators for rotations and boosts, $\Lambda = \exp(\Omega_{\rho\sigma} J^{\rho\sigma})$, or infinitesimally as

$$\Lambda^{\mu}_{\ \nu} = \delta^{\mu}_{\ \nu} + \frac{1}{2}\Omega_{\rho\sigma}(J^{\rho\sigma})^{\mu}_{\ \nu}, \tag{5.2}$$

where the indices ρ and σ indicate the two dimensions being mixed; the mixing of two spatial dimensions corresponds to rotations J_i, and the mixing between a time dimension and a spatial dimension corresponds to boosts K_i.

From using that $(J^{\rho\sigma})_{\mu\nu}$ is anti-symmetric in the μ, ν-indices, we then arrive at the Lie algebra $so^{+}(1,3)$

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \quad [K_i, K_j] = i\epsilon_{ijk} K_k, \quad [K_i, K_j] = i\epsilon_{ijk} K_k, \tag{5.3}$$

which, even though obtained in the four-vector representation, holds for any representation of the Lorentz group. Furthermore, we see that $so(1,3) = su(2) \oplus su(2)$ by forming the linear combinations $A_i = (J_i + iK_i)/2$ and $B_i = (J_i - iK_i)/2$, such that

$$[A_i, A_j] = i\epsilon_{ijk} A_k, \quad [B_i, B_j] = i\epsilon_{ijk} B_k, \quad [A_i, B_j] = 0, \tag{5.4}$$

where the final Lie bracket tells us that the subalgebras commute. Hence, representations of the homogeneous Lorentz group are labeled as (A, B) with dimension $(2A+1)(2B+1)$, where $s = A, B$ are the two spins and where a and b each run in unit steps over the spin projections for each vector space. That is, a runs from $-A$ to A and b from $-B$ to B.

For example, Weyl spinors belong to one of the irreducible representations (irreps) of $su(2) \oplus su(2)$, i.e. they are either in the $(\frac{1}{2}, 0)$ rep or the $(0, \frac{1}{2})$ rep, while Dirac spinors are in the $(\frac{1}{2}, 0) + (0, \frac{1}{2})$ rep, the generators $J^{\mu\nu}$ are in the $(1,0) + (0,1)$ rep, and the generators of translations P^{μ} are in the $(\frac{1}{2}, \frac{1}{2})$ rep.

Even though Weyl spinors are irreps, while Dirac spinors are a reducible representation, it is often more convenient to use Dirac spinors due to their transformation properties under parity. With parity affecting boosts, $PK_iP^{-1} = -K_i$, but not rotations, $PJ_iP^{-1} = J_i$, we have that

$$PA_iP^{-1} = B_i, \quad PB_iP^{-1} = A_i \quad \Rightarrow \quad P : (A, B) \to (B, A), \tag{5.5}$$

such that Dirac spinors are invariant under parity, while Weyl spinors are not (unless they are Majorana).

In supersymmetry, however, the convention is to use Weyl spinors. In Sec. 5.2 we will see the main reason for this – the symmetry generator that links fermionic and bosonic particles happens to be forced to be in the Weyl representation. It is of course still possible to formulate supersymmetry in terms of Dirac spinors, as done in e.g. Ref. [30], just a bit inconvenient.

5.2 The Graded Algebra

Let us begin by stating that we would like to have a symmetry generator that links bosonic and fermionic particles – i.e. an operator that acts on the infinite-dimensional Hilbert space of quantum states, whose specific action on one-particle states, $|p, s\rangle$, is to transform it to another one-particle state with the opposite statistics, where p is the momentum and s is the spin. As the symmetry generator is supposed to relate bosonic and fermionic states, it must be of half-integer spin (we have not yet specified it to be spin $1/2$ though!) and hence transforms non-trivially under the homogeneous Lorentz group. More specifically, the generator Q^{AB}_{ab} is in some representation (A, B), and its Hermitian adjoint in some representation (B, A), where either A is a half-integer and B an integer, or vice versa.

To find which representation (A, B) the generator is in, we can consider the anti-commutator of Q^{AB}_{ab} with its Hermitian adjoint, which must be proportional to a direct sum of bosonic irreps of the tensor product $(A, B) \times (B, A)$. Here, the Coleman-Mandula theorem tells that the only bosonic generators that can be in an extension of the Poincaré algebra is P^μ, $J^{\mu\nu}$ and so-called *internal* operators, i.e. generators that commute with the generators of the Poincaré group [30].

As an internal operator could not possibly give us a Lorentz covariant expression, we are left with only P^μ and $J^{\mu\nu}$, which are in the $(\frac{1}{2}, \frac{1}{2})$ rep and the $(1, 0) + (0, 1)$ rep, respectively. Combining this information with the fact that A is an half-integer and B an integer, or vice versa, and that they should be irreps of the tensor product $(A, B) \times (B, A)$, the only option is for Q^{AB}_{ab} to be in either of the Weyl representations, with $(0, \frac{1}{2}) \times (\frac{1}{2}, 0) = (\frac{1}{2}, \frac{1}{2})$.

In other words, we can denote the generator as Q_α, and its Hermitian adjoint as $\bar{Q}_{\dot{\alpha}}$, where the undotted and dotted spinor indices denote whether the generator belongs to the irrep $(\frac{1}{2}, 0)$ or the irrep $(0, \frac{1}{2})$, and where the barred notation has no other purpose than to clarify which subalgebra we are in whenever spinor indices are suppressed.

Q_α then acts on a so-called *supermultiplet* consisting of a one-particle state and its superpartner of the opposite statistics, i.e. the supermultiplet is in the irreducible representation. There are two types of supermultiplets – *chiral* supermultiplets consisting of a fermion and a sfermion, and *vector* supermultiplets consisting of a gauge boson and a gaugino. Since the one-particle states in a supermultiplet are related by Q_α, which commutes with the squared mass operator, the two components are guaranteed to have the same mass. In this text, we consider only one pair of generators, i.e. $N = 1$ supersymmetry.

In order for the anti-commutator to be proportional to the generator of translations P^μ, while simultaneously being Lorentz covariant, it must be of the form

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu \,, \tag{5.6}$$

where $(\sigma^\mu)_{\alpha\dot{\beta}}$ links the two subalgebras. Once we know that Q_α is in the Weyl representation, we can also determine the remainder of the SUSY algebra to be

$$\begin{aligned} \{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0 \,, \qquad & [P^\mu, Q_\alpha] = [P^\mu, \bar{Q}_{\dot{\alpha}}] = 0 \,, \\ [J^{\mu\nu}, Q_\alpha] = -i(\sigma^{\mu\nu})_\alpha^{\ \beta} Q_\beta \,, \qquad & [J^{\mu\nu}, \bar{Q}^{\dot{\alpha}}] = -i(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}} \bar{Q}^{\dot{\beta}} \,. \end{aligned} \tag{5.7}$$

Note that we have used the same notation for $J^{\mu\nu}$ as in Sec. 5.1, even though the generators here act on the Hilbert space, rather than on the four-vector representation. Note also that the algebra does not close beyond mass shell when acting on field operators. To solve this issue, we need to introduce auxiliary fields F and D, with only off-shell degrees of freedom, i.e. fields that lack a kinetic term. We will return to this subject in Sec. 5.3.

5.3 Superfields and Superspace

For the discussion of superfields and superspace, we want to switch to a different representation of the generators. Rather than having them acting on the Hilbert space, we now want to have them acting on fields on superspace. These generators will be denoted by a circumflex, but of course still fulfill the algebra in eq. (5.6) and eq. (5.7).

From comparing the graded algebra in Sec. 5.2 to the Poincaré algebra $i[P^\mu, J^{\nu\sigma}] = g^{\mu\nu} P^\sigma - g^{\mu\sigma} P^\nu$ and $[P^\mu, P^\nu] = 0$, we expect that $\hat{Q}_\alpha$ behaves in similarity to $\hat{P}^\mu$, which is the generator of ordinary spacetime translations, represented by $-i\partial_\mu$ on field operators. In fact, as worked out by Salam and Strathdee [31], a form for the representation of the fermionic generators that agrees with the graded algebra, is given by

$$\hat{Q}_\alpha = i\frac{\partial}{\partial\theta^\alpha} - (\sigma^\mu)_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial_\mu, \qquad \hat{\bar{Q}}_{\dot{\alpha}} = -i\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + \theta^\alpha (\sigma^\mu)_{\alpha\dot{\alpha}} \partial_\mu, \tag{5.8}$$

such that we can think of $\hat{Q}_\alpha$ and $\hat{\bar{Q}}_{\dot{\alpha}}$ as generators of translations in *superspace*, where superspace is defined as the extension of ordinary spacetime by four Grassman-valued coordinates, θ_1, θ_2, $\bar{\theta}_1$ and $\bar{\theta}_2$, contained in the two-component spinors θ^α and $\bar{\theta}_{\dot{\alpha}}$.

In other words, an infinitesimal supersymmetry transformation

$$\theta^\alpha \to \theta^\alpha + \epsilon^\alpha, \qquad \bar{\theta}_{\dot{\alpha}} \to \bar{\theta}_{\dot{\alpha}} + \bar{\epsilon}_{\dot{\alpha}}, \qquad x^\mu \to x^\mu + i\epsilon^\alpha (\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}, \tag{5.9}$$

induces a transformation on a so-called superfield, $\mathcal{S}(x^\mu, \theta, \bar{\theta})$, on the form

$$\mathcal{S} \to -i(\epsilon^\alpha \hat{Q}_\alpha + \bar{\epsilon}^{\dot{\alpha}} \hat{\bar{Q}}_{\dot{\alpha}})\mathcal{S} \tag{5.10}$$

in similarity to how an ordinary Lorentz transformation $x^\mu \to \Lambda^\mu_{\ \nu} x^\nu$ induces a transformation on e.g. a scalar field $\phi(x) \to \phi(\Lambda^{-1}x)$. Here, the general form of the superfield is given by

$$\mathcal{S}(x^\mu, \theta, \bar{\theta}) = a + \theta\xi + \bar{\theta}\bar{\chi} + \theta\theta b + \bar{\theta}\bar{\theta}c + \bar{\theta}\bar{\sigma}^\mu\theta v_\mu + \bar{\theta}\bar{\theta}\theta\eta + \theta\theta\bar{\theta}\bar{\zeta} + \theta\theta\bar{\theta}\bar{\theta}d, \tag{5.11}$$

where the subscripts for $\theta_{1,2}$ are suppressed, and where all lowercase Greek letters apart from σ are fermionic, while the lowercase Latin letters are bosonic.

For the purpose of eventually describing the chiral- and vector supermultiplets in Sec. 5.2, we apply constraints on the superfield $\mathcal{S}$. For chiral superfields, the constraints are on the form

$$\mathcal{D}_\alpha \mathcal{S}^* = 0, \quad \overline{\mathcal{D}}_{\dot{\alpha}} \mathcal{S} = 0, \tag{5.12}$$

where the super-derivative is defined as

$$\mathcal{D}_\alpha \equiv \frac{\partial}{\partial\theta^\alpha} + i\,(\sigma^\mu)_{\alpha\dot{\alpha}}\,\bar{\theta}^{\dot{\alpha}}\partial_\mu, \qquad \overline{\mathcal{D}}_{\dot{\beta}} \equiv -\frac{\partial}{\partial\bar{\theta}^{\dot{\beta}}} - i\theta^\beta\,(\sigma^\mu)_{\beta\dot{\beta}}\,\partial_\mu, \tag{5.13}$$

and where the corresponding superfield is commonly denoted as Φ. After applying this constraint to eq. (5.11), we have that the chiral superfield is given by

$$\Phi(y^\mu, \theta) \equiv \Phi(x^\mu + i\bar{\theta}\sigma^\mu\theta, \theta) = \phi(y^\mu) + \sqrt{2}\theta\psi(y^\mu) + \theta\theta F(y^\mu), \tag{5.14}$$

where the parameters a, ξ and b has been renamed as ϕ, $\sqrt{2}\psi$ and F, respectively, based on how they transform under supersymmetry transformations and with the factor of $\sqrt{2}$ being conventional. Note also that, since the super-derivative commutes with Q_α, it is consistent with our symmetry. The field F is the auxiliary field mentioned in Sec. 5.2.

In other words, the constraints in eq. (5.12) gave us the corresponding field operators for all one-particle states of the chiral multiplet, as desired. Similarly, superfields fulfilling the constraint $\mathcal{S} = \mathcal{S}^*$ are so-called vector superfields, and will result in the only remaining

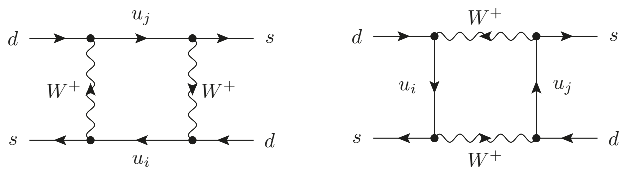


Figure 6.1: Box diagrams for $K^0 - \bar{K}^0$ mixing in the SM, using the convention that all momentum arrows on external legs are pointing to the right.

terms being those of field operators for the gauge boson, the gaugino and the auxiliary field D. The vector superfields are typically denoted as $\mathcal{V}$.

As any holomorphic function of Φ is again a chiral superfield, we can construct the super-Lagrangian, $W(y^\mu, \theta)$, as a holomorphic polynomial in Φ

$$S = \int d^4x d^2\theta \; W(y^\mu, \theta) = \int d^4x \; \mathcal{L}^{\text{SUSY}}, \tag{5.15}$$

where, in the last step, we obtained an ordinary, but supersymmetric Lagrangian, by integrating over θ.

6 Meson Observables

This section is focused on some of the key observables in the meson sector, namely mass splittings and CP violation. When allowing for non-zero NP flavor changing interactions, these observables are highly constraining for NP parameters such as NP scalar masses and Yukawa couplings between NP scalars and SM quarks, and play a central role in Paper III and IV.

6.1 Neutral Kaon Oscillation

With there being no difference in conserved quantum numbers for K^0 and $\bar{K}^0$, the two states can mix via weak interactions, as shown in figure 6.1. Starting with some linear combination of the two flavor eigenstates, the state vector after a time t is given by

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle \equiv A(t)|K^0\rangle + B(t)|\bar{K}^0\rangle, \tag{6.1}$$

where $|\psi(t)\rangle$ is the Schrödinger picture state vector, and where the time-dependent parameters A and B are determined by the insertion of a complete set of flavor eigenstates, such that

$$A(t) \equiv \langle K^0|e^{-iHt}|\psi(0)\rangle, \quad B(t) \equiv \langle \bar{K}^0|e^{-iHt}|\psi(0)\rangle. \tag{6.2}$$

Under charge conjugation and parity, we have that

$$C|K^0\rangle = |\bar{K}^0\rangle, \quad CP|K^0\rangle = -|\bar{K}^0\rangle, \tag{6.3}$$

which involves choosing a phase convention. Here, the convention for the charge conjugation differs between treatments, while almost everyone decides for the flavor eigenstates to transform with a minus sign under parity due to them being pseudoscalars. The *CP* eigenstates are then given by

$$|K_{\substack{1\\2}}\rangle = \frac{1}{\sqrt{2}}\left(|K^0\rangle \mp |\bar{K}^0\rangle\right), \quad CP|K_{\substack{1\\2}}\rangle = \pm|K_{\substack{1\\2}}\rangle\,. \tag{6.4}$$

With $|K^0\rangle$ being represented by $(1,0)$ and $|\bar{K}^0\rangle$ by $(0,1)$, the Hamiltonian in eq. (6.1) is given by

$$H_{ij} = M_{ij} - \frac{i}{2}\Gamma_{ij} \equiv \begin{pmatrix} M - \frac{i}{2}\Gamma & p^2 \\ q^2 & M - \frac{i}{2}\Gamma \end{pmatrix}, \tag{6.5}$$

with

$$p^2 \equiv M_{12} - \frac{i}{2}\Gamma_{12}, \quad q^2 \equiv M_{12}^* - \frac{i}{2}\Gamma_{12}^*, \tag{6.6}$$

where $\Gamma_{21} = \Gamma_{12}^*$ and $M_{21} = M_{12}^*$ follows from M_{ij} and Γ_{ij} being Hermitian matrices (such that $i\Gamma_{ij}$ is anti-Hermitian), and where $M_{11} = M_{22} \equiv M$ and $\Gamma_{11} = \Gamma_{22} \equiv \Gamma$ follows from *CPT* invariance. The eigenstates and eigenvalues of the Hamiltonian are then given by

$$|K_{\substack{L\\S}}\rangle = \frac{1}{\sqrt{|p|^2+|q|^2}}\left(p|K^0\rangle \pm q|\bar{K}^0\rangle\right), \quad \lambda_{\substack{L\\S}} = M - i\Gamma \pm pq, \tag{6.7}$$

where, as always in scattering theory, the decay width is given by the imaginary part of the eigenvalue, times a factor (-2), and the mass by the real part of the eigenvalue, such that

$$M_{\substack{L\\S}} = M \pm \mathrm{Re}(pq), \quad \Gamma_{\substack{L\\S}} = \Gamma \mp 2\mathrm{Im}(pq), \tag{6.8}$$

hence resulting in

$$\Delta M_K \equiv M_L - M_S = 2\mathrm{Re}(pq), \quad \Delta\Gamma_K \equiv \Gamma_S - \Gamma_L = 4\mathrm{Im}(pq), \tag{6.9}$$

where ΔM_K and $\Delta\Gamma_K$ are defined such that they are both positive quantities.

As apparent from eq. (6.7), the mass eigenstates are only *CP* eigenstates for $p = q$, for which K_L is identified with K_2, and K_S with K_1. As *CP* is only slightly violated in the SM, a convenient parameterization is

$$|K_{\substack{L\\S}}\rangle = \frac{1}{\sqrt{1+|\bar{\epsilon}|^2}}\left(|K_{\substack{2\\1}}\rangle + \bar{\epsilon}|K_{\substack{1\\2}}\rangle\right), \quad \bar{\epsilon} \equiv \frac{p-q}{p+q}. \tag{6.10}$$

Note, however, that $\bar{\epsilon}$ cannot be used as a measure of *CP* violation, as it depends on the phase convention in eq. (6.4). We will return to this subject in Sec. 6.3. Also, with $\bar{\epsilon} \ll 1$, we have that $M_{12} \simeq M_{12}^*$ and $\Gamma_{12} \simeq \Gamma_{12}^*$, or in other words that the real parts of M_{12} and Γ_{12} are much larger than the imaginary part, such that

$$\Delta M_K \equiv M_L - M_S \simeq 2\text{Re}M_{12}, \quad \Delta\Gamma_K \equiv \Gamma_S - \Gamma_L \simeq 2\text{Re}\Gamma_{12}. \tag{6.11}$$

Experimentally, ΔM_K and $\Delta\Gamma_K$ are measured by starting in a pure K^0 or $\bar{K}^0$ state at $t = 0$. By inverting eq. (6.7) and plugging it into eqs. (6.2) and (6.1), the state after some time t is given by

$$|K^0(t)\rangle = g_+(t)|K^0\rangle + \frac{q}{p}g_-(t)|\bar{K}^0\rangle, \quad |\bar{K}^0(t)\rangle = \frac{p}{q}g_-(t)|K^0\rangle + g_+(t)|\bar{K}^0\rangle, \tag{6.12}$$

with

$$g_\pm(t) \equiv \frac{1}{2}e^{-\Gamma_L t/2}e^{-iM_L t}\left[1 \pm e^{-\Delta\Gamma t/2}e^{i\Delta M t}\right], \tag{6.13}$$

resulting in $\Delta M_K = 3.484(6) \times 10^{-15}$ GeV [25].

To relate this system to an effective weak Hamiltonian, we can use second-order perturbation theory in the Fermi coupling G_F, as the weak interactions are feeble in comparison to the strong- and electromagnetic ones. The unperturbed system then corresponds to completely turning off the strangeness-violating weak interactions, for which K^0 and $\bar{K}^0$ are two degenerate mass eigenstates, and we have that [32, 33]

$$\left[M - \frac{i}{2}\Gamma\right]_{ij} = m_K^{(0)}\delta_{ij} + \frac{\langle K_i^0|\mathcal{H}_{eff}^{\Delta S=2}|\bar{K}_j^0\rangle}{2m_K} + \frac{1}{2m_K}\sum_n \frac{\langle K_i^0|\mathcal{H}_{eff}^{\Delta S=1}|n\rangle\langle n|\mathcal{H}_{eff}^{\Delta S=1}|\bar{K}_j^0\rangle}{m_K^{(0)} - E_n + i\epsilon}, \tag{6.14}$$

where the factor $2m_K$ comes from normalizing our states as $\langle\vec{p}\,', s|\vec{p}, r\rangle = 2E_{\vec{p}}\delta_{rs}(2\pi)^3\delta^{(3)}(\vec{p}\,' - \vec{p})$, and where the final term corresponds to the decay width only for physical intermediate states, as in Sec. 3.3. The term of interest from eq. (6.14) is

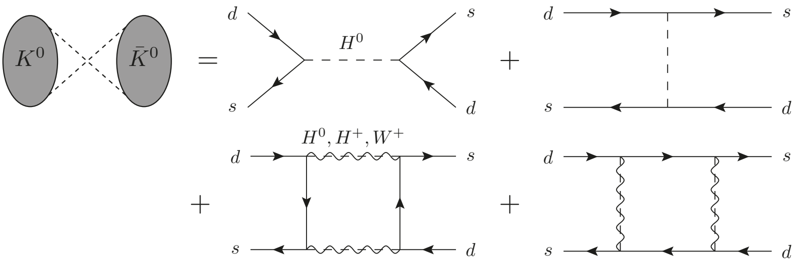
$$2m_K M_{12}^* \simeq \langle\bar{K}^0|\mathcal{H}_{eff}^{\Delta S=2}|K^0\rangle, \tag{6.15}$$

which, combined with eq. (6.11), allows us to relate the theoretical prediction with measurements. Note that this equation is commonly, but incorrectly, presented without the factor of $2m_K$.[12]

[12]From the relativistically invariant continuum normalization, with $\delta^3(\vec{p} - \vec{p}\,')$ having dimension -3 and $E_{\vec{p}}$ having dimension 1, the states must each be of dimension -1. As such, the dimensionality of the right-hand side of eq. (6.15) is 2, with the Hamiltonian density having dimension 4, which does not match the dimensionality of the left-hand unless the factor of m_K is present.

6.2 New Physics Contributions to Box Diagrams

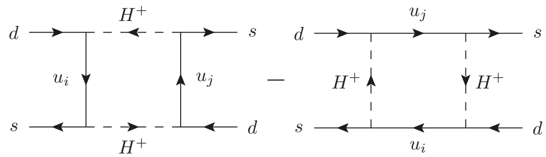
To check whether NP is compatible with the experimental values for ΔM_K, we need to evaluate all NP contributions to $\Delta S = 2$. Take for example a 2HDM, which in general allows for flavor-changing neutral currents at tree-level, and neutral- and charged currents at 1-loop,



(6.16)

where the both dashed and wiggled line denotes either a scalar or a gauge boson. Note that H^0 only comes in combination with itself, and with a sum over any up-type quark combination, while for the charged particles, we have every combination of H^+ and W together with every possible combination of down-type quarks. The WW-combination is, however, not an NP contribution.

To demonstrate the process of obtaining the Wilson coefficients, let us focus on only one of these contributions, e.g. the pure charged Higgs box diagram, given by



$$
= \int \frac{d^d k}{(2\pi)^d} \left(\frac{i}{k^2 - m_H^2 + i\varepsilon} \right)^2
$$

$$
\times \left[\bar{u}_s \left(-iP_L(\Pi_R^\dagger)_{2i} - iP_R(\Pi_L^\dagger)_{2i} \right) \frac{i(\not{k} - m_i)}{k^2 - m_i^2 + i\varepsilon} (-iP_R(\Pi_R)_{i1} - iP_L(\Pi_L)_{i1}) \, v_d \right]
$$

$$
\times \left[\bar{v}_s \left(-iP_L(\Pi_R^\dagger)_{2j} - iP_R(\Pi_L^\dagger)_{2j} \right) \frac{i(\not{k} - m_j)}{k^2 - m_j^2 + i\varepsilon} \left(-iP_R(\Pi_R)_{j1} - iP_L(\Pi_L)_{j1} \right) u_d \right] \qquad (6.17)
$$

$$
- \left\{ \text{Same but with } \left[\bar{u}_s(\ldots)u_d \right] \left[\bar{v}_s(\ldots)v_d \right] \text{ and } m_i \leftrightarrow m_j \right\},
$$

where m_H is the charged Higgs mass, $m_{i,j}$ an up-type quark mass with flavor index $i, j = 1, 2, 3$, and where Π_{ij} is the general Yukawa coupling for a charged scalar, defined as

$$-\mathcal{L}_Y = \bar{u}_i \left[P_L(\Pi_L)_{ij} + P_R(\Pi_R)_{ij}\right] d_j H^+ + \text{H.c.} \tag{6.18}$$

As in Sec. 1.1, the subscripts on the basis spinors denote the flavor of the corresponding quarks, rather than being spin indices. Note that the external momenta have been put to zero, even though all internal propagators are far from massive in the case of $i, j = 1, 2$. The reason why this is allowed, and a common practice for this process, is because it does not affect the Wilson coefficients. Note also that one does not need to perform external field renormalization, as both UV- and IR divergences cancel in the matching. For more details on this, see Ref. [34].

Then, using Fierz identities, the amplitude simplifies to

$$\begin{aligned}
i\mathcal{M} = \frac{1}{4}\sum_{ij}\Bigg(& a_1^{ij}\Big[(\bar{u}_s^\alpha\gamma^\mu P_R v_d^\alpha)(\bar{v}_s^\beta\gamma_\mu P_R u_d^\beta) - (\bar{u}_s^\alpha\gamma^\mu P_R u_d^\alpha)(\bar{v}_s^\beta\gamma_\mu P_R v_d^\beta)\Big] \\
& + a_4^{ij}\Big[L\leftrightarrow R\Big] + 2a_2^{ij}\Big[(\bar{u}_s^\alpha P_L u_d^\beta)(\bar{v}_s^\beta P_R v_d^\alpha) - (\bar{u}_s^\alpha P_L v_d^\beta)(\bar{v}_s^\beta P_R u_d^\alpha)\Big] \\
& + 2a_3^{ij}\Big[L\leftrightarrow R\Big]\Bigg) \times \frac{i}{16\pi^2} D_2(m_i^2, m_j^2, m_H^2, m_H^2) \\
+ \sum_{ij}\Bigg(& a_5^{ij}\Big[(\bar{u}_s^\alpha P_R v_d^\alpha)(\bar{v}_s^\beta P_R u_d^\beta) - (\bar{u}_s^\alpha P_R u_d^\alpha)(\bar{v}_s^\beta P_R v_d^\beta)\Big] + a_8^{ij}\Big[L\leftrightarrow R\Big] \\
& + a_6^{ij}\Big[(\bar{u}_s^\alpha P_R v_d^\alpha)(\bar{v}_s^\beta P_L u_d^\beta) - (\bar{u}_s^\alpha P_R u_d^\alpha)(\bar{v}_s^\beta P_L v_d^\beta)\Big] + a_7^{ij}\Big[L\leftrightarrow R\Big]\Bigg) \\
& \times \frac{i}{16\pi^2} D_0(m_i^2, m_j^2, m_H^2, m_H^2) \times m_i m_j \,,
\end{aligned} \tag{6.19}$$

where α, β are color indices, and where D_0, D_2 are Passarino-Veltman functions, defined as in Ref. [35]. Here, the couplings a_{1-8}^{ij} are all on the form

$$a_{XYZK}^{ij} \equiv (\Pi_X^\dagger)_{2i}(\Pi_Y)_{i1}(\Pi_Z^\dagger)_{2j}(\Pi_K)_{j1}, \tag{6.20}$$

with

$$\begin{aligned}
a_1^{ij} &\equiv a_{RRRR}^{ij}, \quad a_2^{ij} = a_3^{ij} \equiv a_{RRLL}^{ij}, \quad a_4^{ij} \equiv a_{LLLL}^{ij}, \\
a_5^{ij} &\equiv a_{LRLR}^{ij}, \quad a_6^{ij} = a_7^{ij} \equiv a_{LRRL}^{ij}, \quad a_8^{ij} \equiv a_{RLRL}^{ij}.
\end{aligned} \tag{6.21}$$

At the NP scale, i.e. the scale where the charged Higgses are integrated out, the full theory is then to be matched onto the effective theory. Here, the effective Hamiltonian density is given by

$$\mathcal{H}_{\text{eff}}^{\Delta S=2} = \sum_i^5 C_i O_i + \sum_i^3 C_i' O_i', \tag{6.22}$$

with the local operators

$$\begin{aligned}
O_1 &= (\bar{s}^\alpha \gamma^\mu P_L d^\alpha)(\bar{s}^\beta \gamma_\mu P_L d^\beta), \quad & O_2 &= (\bar{s}^\alpha P_L d^\alpha)(\bar{s}^\beta P_L d^\beta), \\
O_3 &= (\bar{s}^\alpha P_L d^\beta)(\bar{s}^\beta P_L d^\alpha), & O_4 &= (\bar{s}^\alpha P_L d^\alpha)(\bar{s}^\beta P_R d^\beta), \\
O_5 &= (\bar{s}^\alpha P_L d^\beta)(\bar{s}^\beta P_R d^\alpha), & &
\end{aligned} \tag{6.23}$$

and where the primed operators $O'_{1,2,3}$ has the same form as their non-primed counterparts, but with $L \leftrightarrow R$. Starting with the Wilson coefficient C_2, we have that

$$\begin{aligned}
S_{fi} &= -2iC_2 \int d^4x \left\{ \langle \bar{d}s|(\bar{s}P_L d)(\bar{s}P_L d)|\bar{s}d\rangle + \langle \bar{d}s|(\bar{s}P_L d)(\bar{s}P_L d)|\bar{s}d\rangle \right\} \\
&= -2iC_2 \Big[(\bar{u}_s P_L v_d)(\bar{v}_s P_L u_d) - (\bar{u}_s P_L u_d)(\bar{v}_s P_L v_d) \Big] (2\pi)^4 \delta^4(p_f - p_i) \\
&\equiv i\mathcal{M}_{fi}^{eff} (2\pi)^4 \delta^4(p_f - p_i),
\end{aligned} \tag{6.24}$$

where the color indices have been suppressed, as the color contractions are always within the same bracket. From comparing with eq. (6.19), the pure charged Higgs box contribution then results in

$$C_2 = -\frac{1}{32\pi^2} \sum_{ij} D_0(m_i^2, m_j^2, m_H^2, m_H^2)\; m_i m_j\; a_8^{ij}. \tag{6.25}$$

The remaining Wilson coefficients are determined in an equivalent fashion, with

$$\begin{aligned}
C_1 &= -\frac{1}{128\pi^2} \sum_{ij} D_2(m_i^2, m_j^2, m_H^2, m_H^2)\; a_4^{ij}\,, \\
C_4 &= -\frac{1}{32\pi^2} \sum_{ij} D_0(m_i^2, m_j^2, m_H^2, m_H^2)\; m_i m_j\; \left[a_6^{ij} + a_7^{ij} \right]\,, \\
C_5 &= \frac{1}{64\pi^2} \sum_{ij} D_2(m_i^2, m_j^2, m_H^2, m_H^2)\; \left[a_2^{ij} + a_3^{ij} \right]\,,
\end{aligned} \tag{6.26}$$

where $C'_{1,2,3}$ has the same form as $C_{1,2,3}$ but with $L \leftrightarrow R$, in agreement with Ref. [35]. Note that the Wilson coefficients above also receive box contributions from the charged Higgs in combination with the W-boson.

The Wilson coefficients are then RG-evolved from the NP-scale to the meson-scale, using that

$$\mu \frac{d}{d\mu} \left(C_i^{(\prime)} O_i^{(\prime)} \right) = 0, \tag{6.27}$$

with no implicit sum over i unless there is operator mixing. In other words, the scale dependence of the operators and the Wilson coefficients exactly cancel. However, rather than doing the RG-evolution for every single parameter point in a scan, a much more efficient (and common) procedure is to use already computed results for the evolution by matching onto a standardized set of operators. The two most common sets of operators for $\Delta S = 2$ is the one shown in eq. (6.23), used in e.g. Ref. [35], and one where O_5 is replaced with

$$O_1^{LR} = (\bar{s}^\alpha \gamma^\mu P_L d^\alpha)(\bar{s}^\beta \gamma_\mu P_R d^\beta) \quad \Rightarrow \quad C_1^{LR} = -2C_5, \tag{6.28}$$

used in e.g. Ref. [36]. Here, the relation between the Wilson coefficients comes from the Fierz identity

$$(P_{R,L})_{ij}(P_{L,R})_{kl} = \frac{1}{2}(\gamma^\mu P_{L,R})_{il}(\gamma_\mu P_{R,L})_{kj}, \tag{6.29}$$

such that

$$\begin{aligned}
&\left(\bar{u}_s^\alpha P_L v_d^\beta\right)\left(\bar{v}_s^\beta P_R u_d^\alpha\right) + \left(\bar{u}_s^\alpha P_R v_d^\beta\right)\left(\bar{v}_s^\beta P_L u_d^\alpha\right) - \left(\bar{u}_s^\alpha P_L u_d^\beta\right)\left(\bar{v}_s^\beta P_R v_d^\alpha\right) \\
&- \left(\bar{u}_s^\alpha P_R u_d^\beta\right)\left(\bar{v}_s^\beta P_L v_d^\alpha\right) \\
&= -\frac{1}{2}\Big[\left(\bar{u}_s^\alpha \gamma^\mu P_L v_d^\alpha\right)\left(\bar{v}_s^\beta \gamma_\mu P_R u_d^\beta\right) + \left(\bar{u}_s^\alpha \gamma^\mu P_R v_d^\alpha\right)\left(\bar{v}_s^\beta \gamma_\mu P_L u_d^\beta\right) \\
&\qquad - \left(\bar{u}_s^\alpha \gamma^\mu P_L u_d^\alpha\right)\left(\bar{v}_s^\beta \gamma_\mu P_R v_d^\beta\right) - \left(\bar{u}_s^\alpha \gamma^\mu P_R u_d^\alpha\right)\left(\bar{v}_s^\beta \gamma_\mu P_L v_d^\beta\right)\Big].
\end{aligned} \tag{6.30}$$

Finally, to compare with experiments, we have that

$$2m_K M_{12}^* \simeq \langle \bar{K}^0 | \mathcal{H}_{\mathit{eff}}^{\Delta S=2} | K^0 \rangle = \sum_i C_i^{(\prime)}(\mu) \, \langle \bar{K}^0 | O_i^{(\prime)}(\mu) | K^0 \rangle, \tag{6.31}$$

where the hadronic matrix element $\langle \bar{K}^0 | O_i^{(\prime)}(\mu) | K^0 \rangle$ is evaluated with some non-perturbative method, e.g. lattice.

6.3 Direct versus Indirect CP violation

There are two kinds of *CP* violation in the Kaon system, direct and indirect. The source of indirect *CP* violation comes from the fact that K_L and K_S, due to *CP* violation, are not *CP* eigenstates. In other words, even if K_L is almost equal to the *CP*-odd eigenstate K_2, it also has a small element of the *CP*-even eigenstate K_1, and can hence (with a small branching fraction) decay to the *CP*-even final state of two pions, without the decay itself being *CP*-violating. Similarly, K_S can, with a small branching fraction, decay to the *CP*-odd final state of three pions via K_2. Direct *CP*-violation, on the other hand, comes from the decay itself being *CP*-violating, i.e. by involving $K_1 \to 3\pi$, or $K_2 \to 2\pi$.

There are a great number of ways to study *CP* violation phenomenologically. In this section, we will focus solely on one type of process, namely the ratios transition amplitudes

$$\eta_{00} = \frac{A(K_L \to \pi^0\pi^0)}{A(K_S \to \pi^0\pi^0)}, \quad \eta_{+-} = \frac{A(K_L \to \pi^+\pi^-)}{A(K_S \to \pi^+\pi^-)}, \tag{6.32}$$

where, experimentally, the norm of η_{00} and η_{+-} are extracted from the corresponding decay widths, resulting in [25]

$$|\eta_{00}| = 2.220(11) \times 10^{-3}, \quad |\eta_{+-}| = 2.232(11) \times 10^{-3} \,. \tag{6.33}$$

Before specifying this further, let us take a short detour into isospin representations. Let us pretend for a moment that $m_u = m_d$, such that strong $SU(2)$ isospin is an exact symmetry. Then, the pions are in the adjoint rep of $SU(2)$, i.e. they are in the isospin-1 rep, with dimensionality 3. As such, with $|j, m\rangle$, the three projections are

$$|\pi^+\rangle = |1,1\rangle, \quad |\pi^0\rangle = |1,0\rangle, \quad |\pi^-\rangle = |1,-1\rangle. \tag{6.34}$$

If we want to consider a two-pion state, we hence have

$$\begin{aligned} &\text{Dimensionality:} \quad 3 \otimes 3 = 5 \oplus 3 \oplus 1, \\ &\text{Isospin:} \qquad\qquad 1 \otimes 1 = 2 \oplus 1 \oplus 0. \end{aligned} \tag{6.35}$$

Starting with the highest weight state and using ladder operators to obtain the remainder, with

$$\begin{aligned} T_\pm &= T_1 \pm iT_2, \quad [T_i, T_j] = i\epsilon_{ijk}T_k, \\ T_\pm|j,m\rangle &= \sqrt{j(j+1) - m(m\pm 1)}\,|j, m\pm 1\rangle, \end{aligned} \tag{6.36}$$

where T_i are the generators of $SU(2)$ isospin, the isospin-2 rep is given by

$$\begin{pmatrix} |2,\ 2\rangle \\ |2,\ 1\rangle \\ |2,\ 0\rangle \\ |2,-1\rangle \\ |2,-2\rangle \end{pmatrix} = \begin{pmatrix} |1,1;1,1\rangle \\ \frac{1}{\sqrt{2}}\left(|1,0;1,1\rangle + |1,1;1,0\rangle\right) \\ \frac{1}{\sqrt{6}}\left(|1,-1;1,1\rangle + 2|1,0;1,0\rangle + |1,1;1,-1\rangle\right) \\ \frac{1}{\sqrt{2}}\left(|1,0;1,-1\rangle + |1,-1;1,0\rangle\right) \\ |1,-1;1,-1\rangle \end{pmatrix}, \tag{6.37}$$

where $|j,m\rangle \otimes |j',m'\rangle \equiv |j,m;j',m'\rangle$. In a similar manner, but using also the orthogonality between $|2,0\rangle$ and $|1,0\rangle$, and $|2,1\rangle$ and $|1,1\rangle$, the isospin-1 rep is given by

$$\begin{pmatrix} |1,\ 1\rangle \\ |1,\ 0\rangle \\ |1,-1\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}\left(|1,0;1,1\rangle - |1,1;1,0\rangle\right) \\ \frac{1}{\sqrt{2}}\left(|1,-1;1,1\rangle - |1,1;1,-1\rangle\right) \\ \frac{1}{\sqrt{2}}\left(|1,0;1,-1\rangle - |1,-1;1,0\rangle\right) \end{pmatrix}, \tag{6.38}$$

and, finally, the isospin-0 rep by

$$|0,\ 0\rangle = \frac{1}{\sqrt{3}}\left(-|1,0;1,0\rangle + |1,-1;1,1\rangle + |1,1;1,-1\rangle\right). \tag{6.39}$$

With the two-pion states being totally symmetric in isospin space, and with them separately belonging to projections of the isospin-1 state, they must be represented by

$$|\pi^0\pi^0\rangle = |1,0;1,0\rangle, \quad |\pi^+\pi^-\rangle = \frac{1}{\sqrt{2}}\left(|1,-1;1,1\rangle + |1,1;1,-1\rangle\right), \tag{6.40}$$

in isospin space, such that, combined with eqs. (6.37-6.39), we obtain the Clebsch-Gordan decomposition

$$|\pi^0\pi^0\rangle = -\frac{1}{\sqrt{3}}\,|0,\ 0\rangle + \sqrt{\frac{2}{3}}\,|2,\ 0\rangle, \quad |\pi^+\pi^-\rangle = \sqrt{\frac{2}{3}}\,|0,\ 0\rangle + \frac{1}{\sqrt{3}}|2,\ 0\rangle. \tag{6.41}$$

In order words, there are four possible transition amplitudes in isospin space for Kaons decaying into two pions, namely

$$\begin{aligned} &\langle\pi\pi, I=0\,|\mathcal{H}_{eff}|K^0\rangle, \quad \langle\pi\pi, I=2\,|\mathcal{H}_{eff}|K^0\rangle, \\ &\langle\pi\pi, I=0\,|\mathcal{H}_{eff}|\bar{K}^0\rangle, \quad \langle\pi\pi, I=2\,|\mathcal{H}_{eff}|\bar{K}^0\rangle, \end{aligned} \tag{6.42}$$

with $|0,0\rangle \equiv |\pi\pi, I=0\rangle$ and $|2,0\rangle \equiv |\pi\pi, I=2\rangle$, and with $A(i\to f) \equiv \langle f|\mathcal{H}_{eff}|i\rangle$.

Now there is only one last thing to consider before we can specify the Kaon decays - namely that there can exist elastic scattering among the decay products. This is usually incorporated by including a so-called *scattering phase shift* δ for the two final states $|f\rangle$ and $|f'\rangle$. In short, the phase shift $\delta_\ell(k)$ is one possible way of parameterizing the scattering amplitude when written as a partial wave expansion. They are real functions of momenta k and the factor $e^{2i\delta_\ell(k)}$ gives the contribution to the ℓ^{th} partial wave. In our case, they are the S-wave phase shifts for $|2,0\rangle$ and $|0,0\rangle$, respectively. Combining this fact with unitarity and *CPT*-invariance leads to Watson's theorem [37], stating that

$$\left[e^{-i\delta}\langle \bar{f}'|\mathcal{H}_{\mathit{eff}}|K^0\rangle\right]^* = e^{-i\delta}\langle f|\mathcal{H}_{\mathit{eff}}|\bar{K}^0\rangle, \tag{6.43}$$

such that, by defining the following transition amplitudes in isospin space

$$A_0 \equiv e^{-i\delta_0}\langle \pi\pi, I=0\,|\mathcal{H}_{\mathit{eff}}|K^0\rangle, \;\; A_2 \equiv e^{-i\delta_2}\langle \pi\pi, I=2\,|\mathcal{H}_{\mathit{eff}}|K^0\rangle, \tag{6.44}$$

it immediately follows that

$$A_0^* \equiv e^{-i\delta_0}\langle \pi\pi, I=0\,|\mathcal{H}_{\mathit{eff}}|\bar{K}^0\rangle, \;\; A_2^* \equiv e^{-i\delta_2}\langle \pi\pi, I=2\,|\mathcal{H}_{\mathit{eff}}|\bar{K}^0\rangle, \tag{6.45}$$

and, hence, from eq. (6.41),

$$\begin{aligned} A(K^0 \to \pi^+\pi^-) &= \sqrt{\frac{2}{3}}\,A_0 e^{i\delta_0} + \frac{1}{\sqrt{3}}\,A_2 e^{i\delta_2}, \\ A(K^0 \to \pi^0\pi^0) &= -\frac{1}{\sqrt{3}}\,A_0 e^{i\delta_0} + \sqrt{\frac{2}{3}}\,A_2 e^{i\delta_2}. \end{aligned} \tag{6.46}$$

Combining these results with eq. (6.7), we find that

$$\eta_{00} \simeq \epsilon - 2\epsilon', \quad \eta_{+-} \simeq \epsilon + \epsilon', \tag{6.47}$$

with

$$\epsilon' \equiv \frac{i}{\sqrt{2}}\mathrm{Im}\left(\frac{A_2}{A_0}\right)e^{i(\delta_2-\delta_0)}, \quad \epsilon \equiv \frac{\langle 0,0|\mathcal{H}_{\mathit{eff}}|K_L\rangle}{\langle 0,0|\mathcal{H}_{\mathit{eff}}|K_S\rangle} = \bar{\epsilon} + \frac{i\mathrm{Im}(A_0)}{\mathrm{Re}(A_0)} \simeq \frac{\mathrm{Im}(M_{12})}{\sqrt{2}\Delta M_K}e^{i\pi/4}, \tag{6.48}$$

which is then to be compared with eq. (6.33), to establish whether NP is compatible with current measurements. For more details, see e.g. Ref. [37].

7 Extending the SM Gauge Sector

All four papers in this thesis involves extensions of the SM gauge sector. In this section we focus specifically on two limitations related to this – namely the effect it can have on the pole mass of the Z-boson and the possibility of gauge anomalies. These are both key restrictions, as the Z-boson pole mass is one of the most accurately measured EW observables, while gauge anomalies can induce unitarity violation.

7.1 Mixing between Z and Z'

When extending the SM gauge sector with an abelian symmetry, as in Paper III and IV, a mixing is introduced between Z and Z'. To generalize the discussion, let us consider the mixing in the context of having n scalar doublets, Φ_i, and m scalar singlets, S_i, charged under the new $U(1)'$ symmetry. Starting with obtaining the gauge boson bilinears, we have

$$\left|D_\mu \langle S_i\rangle\right|^2 + \sum_i \left|D_\mu \langle \Phi_i\rangle\right|^2 = \frac{1}{2} m^2_{ab} A^a_\mu A^{\mu b} + \sum_i^n \frac{v_i^2}{4} W^+_\mu W^{\mu -} \,, \tag{7.1}$$

with

$$m^2_{ab} \equiv \frac{1}{4}\begin{pmatrix} x_1 g_2^2 & -x_1 g_1 g_2 & -2x_2 g_2 g' \\ -x_1 g_1 g_2 & x_1 {g_1}^2 & 2x_2 g_1 g' \\ -2x_2 g_2 g' & 2x_2 g_1 g' & 4x_3 {g'}^2 \end{pmatrix}, \quad A^a_\mu \equiv \begin{pmatrix} W^3_\mu \\ B_\mu \\ \hat{Z}'_\mu \end{pmatrix}, \tag{7.2}$$

and with x_1, x_2 and x_3 defined as

$$x_1 \equiv \sum_{i=1}^n v_i^2, \quad x_2 \equiv \sum_{i=1}^n X_{\Phi_i} v_i^2, \quad x_3 \equiv \sum_{i=1}^n X^2_{\Phi_i} v_i^2 + \sum_{i=1}^m X^2_{S_i} w_i^2, \tag{7.3}$$

where v_i and X_{Φ_i} are the VEVs and $U(1)'$-charges of the Higgs doublets, and where w_i and X_{S_i} the VEVs and $U(1)'$-charges of the scalar singlets, respectively.

The move from the gauge flavor basis $(W^3_\mu, B_\mu, \hat{Z}'_\mu)$ to the mass eigenbasis (A_μ, Z_μ, Z'_μ), is then conveniently carried out via an intermediate basis which, in the limit of g' tending to zero, automatically identifies the photon and the Z boson with the SM ones. In other words, the intermediate basis, denoted by $(A_\mu, \hat{Z}_\mu, \hat{Z}'_\mu)$ is related to the gauge flavor basis and the mass eigenbasis via the transformations

$$\begin{pmatrix} W^3_\mu \\ B_\mu \\ \hat{Z}'_\mu \end{pmatrix} = \begin{pmatrix} s_W & c_W & 0 \\ c_W & -s_W & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_\mu \\ \hat{Z}_\mu \\ \hat{Z}'_\mu \end{pmatrix}, \quad \begin{pmatrix} A_\mu \\ Z_\mu \\ Z'_\mu \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_M & -s_M \\ 0 & s_M & c_M \end{pmatrix} \begin{pmatrix} A_\mu \\ \hat{Z}_\mu \\ \hat{Z}'_\mu \end{pmatrix}, \tag{7.4}$$

respectively, such that, starting with the mass matrix in eq. (7.2) and using the first basis transformation in eq. (7.4),

$$\begin{aligned} m^2_{ab} A^a_\mu A^{\mu b} &= \frac{1}{4} \begin{pmatrix} A_\mu & \hat{Z}_\mu & \hat{Z}'_\mu \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & x_1 (g_1^2 + g_2^2) & -2x_2 g' \sqrt{g_2^2 + g_2^2} \\ 0 & -2x_2 g' \sqrt{g_1^2 + g_2^2} & 4x_3 {g'}^2 \end{pmatrix} \begin{pmatrix} A_\mu \\ \hat{Z}_\mu \\ \hat{Z}'_\mu \end{pmatrix} \\ &\equiv \begin{pmatrix} A_\mu & \hat{Z}_\mu & \hat{Z}'_\mu \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \hat{M}_Z^2 & \delta \hat{M}^2_{ZZ'} \\ 0 & \delta \hat{M}^2_{ZZ'} & \hat{M}^2_{Z'} \end{pmatrix} \begin{pmatrix} A_\mu \\ \hat{Z}_\mu \\ \hat{Z}'_\mu \end{pmatrix}. \end{aligned} \tag{7.5}$$

From the intermediate basis, we then rotate to the mass basis using the second basis transformation in eq. (7.4)

$$m_{ab}^2 A_\mu^a A^{\mu b} = \begin{pmatrix} A_\mu & Z_\mu & Z'_\mu \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & m_Z^2 & 0 \\ 0 & 0 & m_{Z'}^2 \end{pmatrix} \begin{pmatrix} A_\mu \\ Z_\mu \\ Z'_\mu \end{pmatrix} , \tag{7.6}$$

by which the mass of Z- and Z' is given by

$$m_{Z^{(\prime)}}^2 = \frac{1}{2} \left(\hat{M}_Z^2 + \hat{M}_{Z'}^2 \mp \sqrt{(\hat{M}_Z^2 - \hat{M}_{Z'}^2)^2 + 4\delta \hat{M}_{ZZ'}^4} \right) , \tag{7.7}$$

respectively, resulting in the mixing angle

$$\tan(2\theta_M) = \frac{2\delta \hat{M}_{ZZ'}^2}{\hat{M}_{Z'}^2 - \hat{M}_Z^2} , \tag{7.8}$$

where $\hat{M}_Z^2$, $\hat{M}_{Z'}^2$ and $\delta \hat{M}_{ZZ'}^2$ are defined in terms of the VEVs, $U(1)'$ charges and gauge couplings, as specified in eq. (7.5).

7.2 Anomalies

While classical symmetries correspond to transformations leaving the action invariant, symmetries in a quantum theory must also preserve the path integral measure. If it does not, the symmetry is said to be anomalous, meaning that the associated current is not conserved. Anomalies are not necessarily problematic. Global anomalies, such violation of baryon number, are present in the Standard Model, and even necessary for baryogenesis. However, if the anomalous current happens to couple to a massless spin-1 particle, the corresponding Ward identity is violated and we end up with unphysical, longitudinal polarizations. Hence, when extending the SM gauge sector, a central part is to avoid gauge anomalies.

A natural starting point when wanting to relate classical conservation laws and QFT is, of course, to consider Ward identities. Hence, let us begin with the abelian case, by defining the Fourier transformed current three-point functions

$$\begin{aligned} T^{\mu\nu}(k,p,q) &\equiv \int \mathrm{d}^4x\, \mathrm{d}^4y\, \mathrm{d}^4z\; e^{i(k\cdot x + p\cdot y - q\cdot z)} \langle 0| T\{ j^\mu(x) j^\nu(y) P(z) \} |0\rangle, \\ T^{\mu\nu\lambda}(k,p,q) &\equiv \int \mathrm{d}^4x\, \mathrm{d}^4y\, \mathrm{d}^4z\; e^{i(k\cdot x + p\cdot y - q\cdot z)} \langle 0| T\left\{ j^\mu(x) j^\nu(y) j_5^\lambda(z) \right\} |0\rangle, \end{aligned} \tag{7.9}$$

where the axial-, vector- and pseudoscalar currents are given by

$$j_5^\mu \equiv \bar{\psi}\gamma^\mu\gamma^5\psi, \quad j^\mu \equiv \bar{\psi}\gamma^\mu\psi, \quad P \equiv \bar{\psi}\gamma^5\psi, \tag{7.10}$$

and where, in the classical theory, the axial current is conserved only in the massless limit

$$\begin{aligned} \partial_\mu j_5^\mu &= \left(\partial_\mu\bar{\psi}\right)\gamma^\mu\gamma^5\psi + \bar{\psi}\gamma^\mu\gamma^5\partial_\mu\psi = 2mi\bar{\psi}\gamma^5\psi \equiv 2miP, \\ \partial_\mu j^\mu &= \left(\partial_\mu\bar{\psi}\right)\gamma^\mu\psi + \bar{\psi}\gamma^\mu\partial_\mu\psi = im\bar{\psi}\psi - im\bar{\psi}\psi = 0, \end{aligned} \tag{7.11}$$

using that $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$, the equations of motion, and the fact that γ^5 and γ^μ anti-commute.

The axial Ward identity is then obtained by contracting $T^{\mu\nu\lambda}$ with q_λ, such that, up to a surface term and a Schwinger term,[13]

$$\begin{aligned} q_\lambda T^{\mu\nu\lambda} &= -i\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4z\; e^{i(k\cdot x+p\cdot y-q\cdot z)}\,\langle 0|T\left\{j^\mu(x)j^\nu(y)\left(\partial_\lambda^z j_5^\lambda(z)\right)\right\}|0\rangle \\ &= 2m\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4z\; e^{i(k\cdot x+p\cdot y-q\cdot z)}\,\langle 0|T\{j^\mu(x)j^\nu(y)\,P(z)\}|0\rangle \\ &= 2mT^{\mu\nu}, \end{aligned} \tag{7.12}$$

where, in the first step, we used that $q_\lambda e^{i(k\cdot x+p\cdot y-q\cdot z)} = i\partial_\lambda^z(e^{i(k\cdot x+p\cdot y-q\cdot z)})$ and integrated by parts. Similarly, the vector Ward identities, up to surface- and Schwinger terms, are given by

$$k_\mu T^{\mu\nu\lambda} = i\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4z\; e^{i(k\cdot x+p\cdot y-q\cdot z)}\,\langle 0|T\left\{\left(\partial_\mu^x j^\mu(x)\right)j^\nu(y)j_5^\lambda(z)\right\}|0\rangle = 0, \tag{7.13}$$

and equivalently $p_\nu T^{\mu\nu\lambda} = 0$, using eq. (7.11).

However, while Schwinger terms can safely be neglected [38], nothing ensures that the surface terms can be. In general, the relations are hence of the form

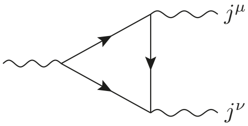
$$q_\lambda T^{\mu\nu\lambda} = 2mT^{\mu\nu} + \mathcal{A}_1^{\mu\nu}, \quad k_\mu T^{\mu\nu\lambda} = \mathcal{A}_2^{\nu\lambda}, \quad p_\nu T^{\mu\nu\lambda} = \mathcal{A}_3^{\mu\lambda}, \tag{7.14}$$

[13]After integrating by parts, we end up with having a derivative acting on a time-ordered correlation function. Writing the time-ordering in terms of Heaviside functions and using the product rule, the terms where the derivative acts on the Heaviside function are referred to as Schwinger terms [19].

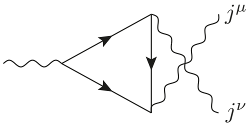
such that the axial current (with $m = 0$) and the vector currents are all still anomalous, unless some, or all, of the factors $\mathcal{A}_i$ can be set to zero.

To determine the factors $\mathcal{A}_i$, let us evaluate the amputated three-point function $T^{\mu\nu\lambda}$ perturbatively. Starting with the massless case, the lowest order diagrams (which also happens to be the only order contributing to the anomalies for QED [23]) are given by

$$T^{\mu\nu\lambda} = j_5^\lambda$$



$$+ \quad j_5^\lambda$$



$$= -\int \frac{d^4\ell}{(2\pi)^4} \mathrm{tr}\left[\gamma^\mu \frac{i\not{\ell}}{\ell^2+i\epsilon}\gamma^\lambda\gamma^5 \frac{i(\not{\ell}-\not{q})}{(\ell-q)^2+i\epsilon}\gamma^\nu \frac{i(\not{\ell}-\not{k})}{(\ell-k)^2+i\epsilon}\right] + \begin{Bmatrix} k \leftrightarrow p \\ \mu \leftrightarrow \nu \end{Bmatrix}, \tag{7.15}$$

where the overall minus sign comes from there being a closed fermion loop. Contracting with q_λ, using Feynman slash identities and evaluating the traces, we then have

$$q_\lambda T^{\mu\nu\lambda} = -4\epsilon^{\mu\nu\rho\sigma} \int \frac{d^4\ell}{(2\pi)^4} \left(\frac{\ell^\rho p^\sigma}{\ell^2(\ell+p)^2} + \frac{\ell^\rho k^\sigma}{\ell^2(\ell-k)^2} \right) + \begin{Bmatrix} k \leftrightarrow p \\ \mu \leftrightarrow \nu \end{Bmatrix}, \tag{7.16}$$

and, in a similar manner, the contraction with k_μ results in

$$k_\mu T^{\mu\nu\lambda} = 4\epsilon^{\nu\lambda\rho\sigma} \int \frac{d^4\ell}{(2\pi)^4} \left(\frac{(\ell-k)^\rho(\ell+p)^\sigma}{(\ell-k)^2(\ell+p)^2} \right) + \begin{Bmatrix} k \leftrightarrow p \\ \mu \leftrightarrow \nu \end{Bmatrix}. \tag{7.17}$$

At a first glance, it might appear as if both eq. (7.16) and eq. (7.17) can be made to vanish by a simple variable shift, as the Levi-Civita tensor is completely anti-symmetric. However, this is not the case. As the integral is linearly divergent, it is not invariant under a linear shift. In other words, after the change of variables $\ell^\mu \to \ell^\mu + a^\mu$ in eq. (7.16) and eq. (7.17), they each have one surviving term, given by

$$\Delta^{\mu\nu}(a^\sigma) = \int \frac{d^4\ell}{(2\pi)^4} \left(f^{\mu\nu}(\ell^\sigma + a^\sigma) - f^{\mu\nu}(\ell^\sigma)\right) \neq 0. \tag{7.18}$$

where $f^{\mu\nu}$ is used to denote the shifted integrand. To evaluate this one remaining term, we Wick rotate, Taylor expand $f^{\mu\nu}$ and use Gauss's theorem, such that, to the first non-vanishing order

$$\Delta^{\mu\nu}(a^\sigma) \simeq i \int_\Omega \frac{d^4\ell_E}{(2\pi)^4} a^\sigma \frac{\partial}{\partial \ell_E^\sigma} f^{\mu\nu}(\ell_E) = i a^\sigma \int_{\partial\Omega} \frac{d^3 S_\sigma}{(2\pi)^4} f^{\mu\nu}(\ell_E), \tag{7.19}$$

where $\partial\Omega$ is the surface of the 4-ball. Taking the limit of the radius $|k_E|$ going to infinity and considering a general shift $a^\mu = b_1 k^\mu + b_2 p^\mu$, we can then obtain [20]

$$\begin{aligned} q_\lambda T^{\mu\nu\lambda} &= \frac{i}{4\pi^2} \epsilon^{\mu\nu\rho\sigma} k_\rho p_\sigma (b_1 - b_2), \\ k_\mu T^{\mu\nu\lambda} &= \frac{i}{4\pi^2} \epsilon^{\nu\lambda\rho\sigma} k_\rho p_\sigma (1 - b_1 + b_2). \end{aligned} \tag{7.20}$$

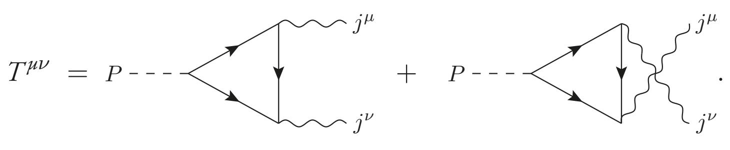
In other words, there is no choice of b_1 and b_2 for which both the vector- and axial currents are anomaly-free simultaneously. We either end up with anomalous vector currents or anomalous axial currents, and here the decision is simple. For massless QED, the vector symmetry is local and couples to a massless spin-1 particle, while the axial symmetry is global. Hence, we set $b_1 = b_2 + 1$, ending up with

$$q_\lambda T^{\mu\nu\lambda} = \mathcal{A}^{\mu\nu}, \quad k_\mu T^{\mu\nu\lambda} = p_\nu T^{\mu\nu\lambda} = 0. \tag{7.21}$$

where $\mathcal{A}^{\mu\nu}$ is the Adler-Bell-Jackiw (ABJ) anomaly [39, 40]

$$\mathcal{A}^{\mu\nu} = \frac{i}{4\pi^2} \epsilon^{\mu\nu\rho\sigma} k_\rho p_\sigma. \tag{7.22}$$

In the massive case, we get an identical result, but with an additional factor of $2mT^{\mu\nu}$ in the right-hand side of the first equation in eq. (7.21). As such, that calculation also involves evaluating diagrams with the axial current replaced by the pseudoscalar current, i.e.



(7.23)

7.3 Anomaly Conditions

Generalizing the above procedure to the non-abelian case, we instead have the three-point functions

$$
\begin{aligned}
T^{abc}_{\mu\nu}(k,p,q) &\equiv \int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4z\; e^{i(k\cdot x+p\cdot y-q\cdot z)}\,\langle 0|T\left\{j^a_\mu(x)\,j^b_\nu(y)\,P^c(z)\right\}|0\rangle,\\
T^{abc}_{\mu\nu\lambda}(k,p,q) &\equiv \int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4z\; e^{i(k\cdot x+p\cdot y-q\cdot z)}\,\langle 0|T\left\{j^a_\mu(x)\,j^b_\nu(y)\,j^{5c}_\lambda(z)\right\}|0\rangle,
\end{aligned}
\tag{7.24}
$$

with the axial-, vector- and pseudoscalar currents defined as $j^{5a}_\mu \equiv \bar{\psi}_i\gamma_\mu\gamma_5 T^a_{ij}\psi_j$, $j^a_\mu \equiv \bar{\psi}_i\gamma_\mu T^a_{ij}\psi_j$ and $P^a \equiv \bar{\psi}_i\gamma_5 T^a_{ij}\psi_j$, and where a, b, c are adjoint rep indices and i, j fundamental rep indices. Placing again the anomaly fully in the axial Ward identity, the end result is given by

$$
q^\lambda T^{abc}_{\mu\nu\lambda} = 2mT^{abc}_{\mu\nu} + \mathcal{A}^{abc}_{\mu\nu}, \qquad k^\mu T^{abc}_{\mu\nu\lambda} = p^\nu T^{abc}_{\mu\nu\lambda} = 0,
\tag{7.25}
$$

with the anomaly

$$
\mathcal{A}^{abc}_{\mu\nu} = \frac{i}{8\pi^2}\epsilon_{\mu\nu\rho\sigma}k^\rho p^\sigma \;\mathrm{tr}(\{T^a, T^b\}T^c).
\tag{7.26}
$$

In other words, the vector current is conserved, while the axial current is anomalous unless the trace in eq. (7.26) vanishes. As the axial symmetry may, in general, be gauged, it is of interest to investigate under which conditions the trace is zero. Let us use a subscript $X = 1, 2, 3$ for the generators in eq. (7.26), and define T^a_X as a generator of $SU(3)\times SU(2)\times U(1)$

$$
\begin{aligned}
T^a_3 &\equiv T^a_{SU(3)} \otimes \mathbb{I}_{SU(2)} \otimes \mathbb{I}_{U(1)},\\
T^a_2 &\equiv \mathbb{I}_{SU(3)} \otimes T^a_{SU(2)} \otimes \mathbb{I}_{U(1)},\\
T^a_1 &\equiv \mathbb{I}_{SU(3)} \otimes \mathbb{I}_{SU(2)} \otimes T^a_{U(1)},
\end{aligned}
\tag{7.27}
$$

where $T^a_{SU(3)}$ is a generator in the fundamental rep of $SU(3)$. Note that the index a runs from one to eight in the first equation, from one to three in the second equation, and not at all in the third equation.

Hence, if we want to evaluate the trace for e.g. $[SU(2)]^2U(1)$, we have that

$$
\begin{aligned}
\sum_{\text{species}} \mathrm{tr}\left(\{T^a_2, T^b_2\}T^c_1\right) &= \sum_{\text{species}} \mathrm{tr}\left(\mathbb{I}_{SU(3)} \otimes \left\{T^a_{SU(2)}, T^b_{SU(2)}\right\} \otimes T^c_{U(1)}\right)\\
&= \mathrm{tr}\left\{T^a_{SU(2)}, T^b_{SU(2)}\right\} \sum_{\text{species}} \mathrm{tr}\left(\mathbb{I}_{SU(3)}\right)\mathrm{tr}\left(T^c_{U(1)}\right),
\end{aligned}
\tag{7.28}
$$

where in the first step, we used eq. (7.27) and the relation $(a \otimes b)(c \otimes d) = (ac) \otimes (bd)$, while in the second step, we used that $\mathrm{tr}(a \otimes b) = \mathrm{tr}(a)\mathrm{tr}(b)$. To account for the possibility of having several copies of fermions in a certain rep, we sum over "species". If we consider e.g. the SM, we hence have that

$$\sum_{\text{species}} \mathrm{tr}\left(\{T_2^a, T_2^b\} T_1^c\right) = \mathrm{tr}\left\{T^a_{SU(2)}, T^b_{SU(2)}\right\}\left(3 \cdot \frac{1}{6} \cdot 3 + 1 \cdot \left(-\frac{1}{2}\right) \cdot 3\right) = 0, \quad (7.29)$$

using that the trace of $\mathbb{I}_{SU(3)}$ is three for the quark doublet q_i and one for the lepton doublet ℓ_i, while the (trace of the generator of) hypercharge is $1/6$ and $-1/2$, respectively. Here the number of species corresponds to the number of generations. Note that the anomaly would have cancelled also without summing over generations.

In Paper III and IV, where the SM gauge sector is extended by a gauged $U(1)'$ symmetry, anomaly cancellation hence constrains the possible $U(1)'$ charges, limiting the number of valid implementations. In appendix 1.A, we present an additional classification, namely all anomaly-free implementations of the gauged Branco-Grimus-Lavoura (BGL) model [41, 42] with three generations of right-handed neutrinos and a type-I seesaw mechanism.

8 Concluding Remarks

We have now covered some of the main topics required for putting Paper I-IV into context. To mention a few concepts of particular importance for Paper I and II – which treats the construction and evolution of a supersymmetric model from the GUT scale to the EW scale – we have the procedure of RGE evolution and manual alteration of beta functions described in Sec. 2.4, matching in Sec. 1.1 and 6.2, and supersymmetry in Sec. 5.

For Paper III and IV – which instead involves the classification and comparison of anomaly-free implementations in 2HDMs with a gauged abelian symmetry – the topics of central importance are the weak Hamiltonian covered in Sec. 1.1 , electroweak observables and the narrow-width approximation in Sec. 3, n-Higgs-doublet models and the alignment limit in Sec. 4, meson observables in Sec. 6 and, finally, the anomaly conditions in Sec. 7.3.

References

[1] M. Kobayashi and T. Maskawa, "CP-Violation in the Renormalizable Theory of Weak Interaction," *Progress of Theoretical Physics* **49** no. 2, (Feb, 1973) 652–657. https://doi.org/10.1143/PTP.49.652.

[2] E. D. Bloom *et al.*, "High-Energy Inelastic $e - p$ Scattering at 6° and 10°," *Phys. Rev. Lett.* **23** (Oct, 1969) 930–934. https://link.aps.org/doi/10.1103/PhysRevLett.23.930.

[3] M. Breidenbach *et al.*, "Observed Behavior of Highly Inelastic Electron-Proton Scattering," *Phys. Rev. Lett.* **23** (Oct, 1969) 935–939. https://link.aps.org/doi/10.1103/PhysRevLett.23.935.

[4] J. J. Aubert *et al.*, "Experimental Observation of a Heavy Particle *J*," *Phys. Rev. Lett.* **33** (Dec, 1974) 1404–1406. https://link.aps.org/doi/10.1103/PhysRevLett.33.1404.

[5] J. E. Augustin *et al.*, "Discovery of a Narrow Resonance in e^+e^- Annihilation," *Phys. Rev. Lett.* **33** (Dec, 1974) 1406–1408. https://link.aps.org/doi/10.1103/PhysRevLett.33.1406.

[6] S. W. Herb *et al.*, "Observation of a Dimuon Resonance at 9.5 GeV in 400-GeV Proton-Nucleus Collisions," *Phys. Rev. Lett.* **39** (Aug, 1977) 252–255. https://link.aps.org/doi/10.1103/PhysRevLett.39.252.

[7] G. Arnison *et al.*, "Experimental observation of isolated large transverse energy electrons with associated missing energy at s=540 GeV," *Physics Letters B* **122** no. 1, (1983) 103 – 116.

[8] M. Banner *et al.*, "Observation of single isolated electrons of high transverse momentum in events with missing transverse energy at the CERN pp collider," *Physics Letters B* **122** no. 5, (1983) 476 – 485.

[9] G. Arnison *et al.*, "Experimental observation of lepton pairs of invariant mass around 95 GeV/c^2 at the CERN SPS collider," *Physics Letters B* **126** no. 5, (1983) 398 – 410.

[10] R. Ansari *et al.*, "Measurement of W and Z production properties at the CERN pp collider," *Physics Letters B* **194** no. 1, (1987) 158 – 166.

[11] G. Taubes, *Nobel dreams*. Random House, 1986.

[12] **CDF** Collaboration, F. Abe *et al.*, "Observation of Top Quark Production in $\overline{p}p$ Collisions with the Collider Detector at Fermilab," *Phys. Rev. Lett.* **74** (Apr, 1995) 2626–2631. https://link.aps.org/doi/10.1103/PhysRevLett.74.2626.

[13] **D0** Collaboration, S. Abachi *et al.*, "Search for High Mass Top Quark Production in $p\overline{p}$ Collisions at $\sqrt{s} =$ 1.8 TeV," *Phys. Rev. Lett.* **74** (Mar, 1995) 2422–2426. https://link.aps.org/doi/10.1103/PhysRevLett.74.2422.

[14] S. Chatrchyan *et al.*, "Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC," *Physics Letters B* **716** no. 1, (2012) 30 – 61.

[15] G. Aad *et al.*, "Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC," *Physics Letters B* **716** no. 1, (2012) 1 – 29.

[16] H. Georgi, "Effective field theory," *Annual Review of Nuclear and Particle Science* **43** no. 1, (1993) 209–252, https://doi.org/10.1146/annurev.ns.43.120193.001233.

[17] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory*. Addison-Wesley, Reading, USA, 1995.

[18] G. Buchalla, A. J. Buras, and M. E. Lautenbacher, "Weak decays beyond leading logarithms," *Rev. Mod. Phys.* **68** (1996) 1125–1144, arXiv:hep-ph/9512380.

[19] S. Weinberg, *The Quantum Theory of Fields. Vol. 1: Foundations*. Cambridge University Press, 6, 2005.

[20] M. D. Schwartz, *Quantum Field Theory and the Standard Model*. Cambridge University Press, 3, 2014.

[21] M. Luo, H. Wang, and Y. Xiao, "Two-loop renormalization group equations in general gauge field theories," *Phys. Rev. D* **67** (Mar, 2003) 065019. https://link.aps.org/doi/10.1103/PhysRevD.67.065019.

[22] T. Appelquist and J. Carazzone, "Infrared Singularities and Massive Fields," *Phys. Rev. D* **11** (1975) 2856.

[23] S. Weinberg, "Effective Gauge Theories," *Phys. Lett. B* **91** (1980) 51–55.

[24] I. Brivio and M. Trott, "The Standard Model as an Effective Field Theory," *Phys. Rept.* **793** (2019) 1–98, `arXiv:1706.08945 [hep-ph]`.

[25] **Particle Data Group** Collaboration, M. Tanabashi *et al.*, "Review of Particle Physics," *Phys. Rev.* **D98** no. 3, (2018) 030001.

[26] B. Lee, "Perspectives on theory of weak interactions," in *16th International Conference on High-Energy Physics*, vol. IV, p. 266. 1972.

[27] H.-S. Tsao, "Higgs Boson Quantum Numbers and the Pell Equation," in *Guangzhou Conference on Theoretical Particle Physics*, vol. II, p. 1240. 1980.

[28] I. P. Ivanov, "Building and testing models with extended Higgs sectors," *Prog. Part. Nucl. Phys.* **95** (2017) 160–208, `arXiv:1702.03776 [hep-ph]`.

[29] G. Branco, P. Ferreira, L. Lavoura, M. Rebelo, M. Sher, and J. P. Silva, "Theory and phenomenology of two-Higgs-doublet models," *Phys. Rept.* **516** (2012) 1–102, `arXiv:1106.0034 [hep-ph]`.

[30] S. Weinberg, *The Quantum Theory of Fields. Vol. 3: Supersymmetry*. Cambridge University Press, 6, 2013.

[31] A. Salam and J. Strathdee, "Supergauge Transformations," *Nucl. Phys. B* **76** (1974) 477–482.

[32] J. Donoghue, E. Golowich, and B. R. Holstein, *Dynamics of the standard model*, vol. 2. CUP, 2014.

[33] H. Georgi, *Weak Interactions and Modern Particle Theory*. Dover, 2009.

[34] A. J. Buras, "Weak Hamiltonian, CP violation and rare decays," in *Les Houches Summer School in Theoretical Physics, Session 68: Probing the Standard Model of Particle Interactions*, pp. 281–539. 6, 1998. `arXiv:hep-ph/9806471`.

[35] A. Crivellin, A. Kokulu, and C. Greub, "Flavor-phenomenology of two-Higgs-doublet models with generic Yukawa structure," *Phys. Rev. D* **87** no. 9, (2013) 094031, `arXiv:1303.5877 [hep-ph]`.

[36] A. J. Buras, S. Jager, and J. Urban, "Master formulae for Delta F=2 NLO QCD factors in the Standard Model and beyond," *Nucl. Phys. B* **605** (2001) 600–624, `arXiv:hep-ph/0102316`.

[37] R. Belusevic, "Neutral kaons," *Springer Tracts Mod. Phys.* **153** (Mar, 1998) 1–182. http://cds.cern.ch/record/359341.

[38] R. Bertlmann, *Anomalies in quantum field theory*. Oxford University Press, 1996.

[39] S. L. Adler, "Axial-vector vertex in spinor electrodynamics," *Phys. Rev.* **177** (Jan, 1969) 2426–2438. https://link.aps.org/doi/10.1103/PhysRev.177.2426.

[40] J. S. Bell and R. Jackiw, "A PCAC puzzle: $\pi 0 \to \gamma\gamma$ in the σ-model," *Il Nuovo Cimento A (1965-1970)* **60** no. 1, (1969) 47–61. https://doi.org/10.1007/BF02823296.

[41] G. C. Branco, W. Grimus, and L. Lavoura, "Relating the scalar flavor changing neutral couplings to the CKM matrix," *Phys. Lett.* **B380** (1996) 119–126, arXiv:hep-ph/9601383 [hep-ph].

[42] A. Celis, J. Fuentes-Martin, M. Jung, and H. Serôdio, "Family nonuniversal Z' models with protected flavor-changing interactions," *Phys. Rev.* **D92** no. 1, (2015) 015007, arXiv:1505.03079 [hep-ph].

9 Publications

Paper I

José E. Camargo-Molina, António P. Morais, Astrid Ordell, Roman Pasechnik, Marco O. P. Sampaio, and Jonas Wessén: *Reviving trinification models through an E_6-extended supersymmetric GUT*, e-print: arXiv:1610.03642 [hep-ph]. Phys. Rev. D**95** (2017), 075031.

This article introduces the SHUT model, based on an idea from previous work by Roman, António, Eliel, Jonas and Marco. António, Roman and Eliel had a central part in finding the particular implementation, after which the majority of the results were obtained during my master thesis project, with calculations carried out by António and I (independently), under the supervision of Roman and Jonas. I took part in both the writing and editing process.

Paper II

José E. Camargo-Molina, António P. Morais, Astrid Ordell, Roman Pasechnik and Jonas Wessén: *Scale hierarchies, symmetry breaking and particle spectra in $SU(3)$-family extended SUSY trinification*, e-print: arXiv:1711.05199 [hep-ph]. Phys. Rev. D**99** (2019), 035041.

A more in-depth study of the SHUT model, with the majority of results obtained by António and I (independently), again under the supervision of Roman and Jonas. Parts of the results in Sec. III and IV were also obtained by Jonas and Eliel. I took part in either writing or major editing of all sections.

Paper III

Franz Nottensteiner, Astrid Ordell, Roman Pasechnik and Hugo Serôdio: *Classification of anomaly-free 2HDMs with a gauged U(1)' symmetry*, e-print: arXiv:1909.05548 [hep-ph]. Phys. Rev. **D100** (2019), 115038.

In this paper, we classified and compared all anomaly-free 2HDM-U(1)'s within the SM fermion content. The classification itself was initiated in a master thesis project by Franz in 2017, supervised by Hugo and Roman, and then verified and generalized by me in 2019. The remainder of the content in Paper III was worked out by Hugo and I. Hugo invented the classification method and was also in charge of the phenomenological analysis, while I performed analytical calculations required for the scan. For example, I calculated the 1-loop contributions to the Wilson coefficients for all $\Delta F = 2$ processes, and for $b \to s\gamma$.

Furthermore, the various calculations in Sec. III were carried out by both Hugo and I, independently.

The writing was split rather evenly between Hugo and I, with Hugo having the main responsibility for Sec. I, II-B, IV and parts of Sec. III and V, while I had the main responsibility for Sec. II-A, II-C, VI, the majority of III and parts of V. All authors took part in the editing process.

Paper IV

Astrid Ordell, Roman Pasechnik and Hugo Serôdio: *Anomaly-free 2HDMs with a gauged Abelian symmetry and two generations of right-handed neutrinos*. Preprint number: LU-TP 20-28. Accepted for publication in Phys. Rev. D.

In this work, we classified and compared all anomaly-free 2HDM-U(1):s in the case of having two generations of right-handed neutrinos and a type-I seesaw mechanism. The classification was carried out by Hugo and I, independently. Again, Hugo did the phenomenological analysis, while I performed analytical calculations going into it, such as the Wilson coefficients for $\ell \to \ell'\gamma$, and Wilson coefficients and branching ratio for $\ell \to 3\ell$. I also prepared the input cards for Hugo's code. Regarding the writing, Roman wrote Sec. I, Hugo most of Sec. V, while I wrote Sec. II, III, IV, VI, VII and parts of V. Again, all authors took part in the editing process.

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Appendix 1.A Anomaly-free Implementations of the Gauged BGL Model

The Branco-Grimus-Lavoura (BGL) model is, to this day, one of the most frequently used implementations of the two-Higgs-doublet model (2HDM). In this appendix, we classify all of its allowed instances, in the case of having a gauged abelian flavor symmetry and three generations of massive neutrinos, gaining their mass via a type-I seesaw mechanism. With this setting, there are a total of three valid implementations, out of which neither have been previously explored. The results presented below have been independently confirmed by Hugo Serôdio.

1.A.1 The Model

Below we extend the Standard Model (SM) gauge group with a gauged abelian flavor symmetry, $U(1)'$, and the SM particle content with the corresponding neutral gauge boson Z', three generations of right-handed neutrinos $\nu_R^{1,2,3}$, a scalar singlet S and an additional Higgs doublet Φ_2. In general, the entire particle content is charged under the new abelian symmetry, with the charges allowed to vary in between generations.[14]

In addition, we demand that the scalar singlet, and at least one of the scalar doublets, are charged under $U(1)'$. With this, a minimum of two vacuum expectation values (VEVs) enter into the breaking of the flavor symmetry. Without the scalar singlet, only the electroweak (EW) scale would enter the breaking, which almost completely eliminates the valid parameter space.

Using the definitions in Paper IV, the Yukawa interactions in the flavor eigenbasis are given by

$$\begin{aligned} -\mathcal{L}_{\text{Yukawa}} &= \overline{q_L^0}\Gamma_a\Phi_a d_R^0 + \overline{q_L^0}\Delta_a\tilde{\Phi}_a u_R^0 + \overline{\ell_L^0}\Pi_a\Phi_a e_R^0 + \overline{\ell_L^0}\Sigma_a\tilde{\Phi}_a\nu_R \\ &+ \frac{1}{2}\overline{\nu_R^c}\left(A + BS + CS^*\right)\nu_R + \text{H.c.}\,, \end{aligned} \tag{1.A.1}$$

with a running from one to two, $\tilde{\Phi} \equiv i\sigma_2\Phi^*$, and with the BGL quark Yukawa textures

[14] However, there are of course charge assignments under which several generations could end up with the same charge, or where a subset of the charges are zero.

given by

$$\Gamma_1 : \begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ 0 & 0 & 0 \end{pmatrix}, \quad \Gamma_2 : \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \times & \times & \times \end{pmatrix},$$
$$\Delta_1 : \begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Delta_2 : \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \times \end{pmatrix}, \tag{1.A.2}$$

while the textures of Π_a, Σ_a, A, B and C are, at this stage, completely arbitrary.

1.A.2 Method of Finding Anomaly-Free Implementations

The method presented in this section follows the same basic principle as introduced in Paper III and IV, but implemented in a new context. In short, we have that *if* the abelian flavor symmetry is a symmetry of the Lagrangian, then (for the quark Yukawa sector)

$$(\Gamma_a)_{ij} = e^{i\theta(X_{q_i} - X_{d_j} - X_{\Phi_a})}(\Gamma_a)_{ij}, \quad (\Delta_a)_{ij} = e^{i\theta(X_{q_i} - X_{u_j} + X_{\Phi_a})}(\Delta_a)_{ij}, \tag{1.A.3}$$

such that the Yukawa textures correspond to linear constraints on the corresponding charges

$$(\Gamma_a)_{ij} = \text{any} \ \text{ if } \ X_{q_i} - X_{d_j} = X_{\Phi_a},$$
$$(\Gamma_a)_{ij} = 0 \quad \text{ if } \ X_{q_i} - X_{d_j} \neq X_{\Phi_a}, \tag{1.A.4}$$

and similarly for the up-sector, where X_q, X_u, X_d, X_Φ are the $U(1)'$ charges of the left-handed quark doublets, the right-handed up-type quarks, the right-handed down-type quarks and the scalar doublets, respectively. As such, the BGL textures in eq. (1.A.2) correspond to the following 36 constraints

$$X_{q_{1,2}} - X_{d_{1,2,3}} = X_{\Phi_1}, \ X_{q_3} - X_{d_{1,2,3}} \neq X_{\Phi_1}, \ X_{q_3} - X_{d_{1,2,3}} = X_{\Phi_2},$$
$$X_{q_{1,2}} - X_{d_{1,2,3}} \neq X_{\Phi_2}, \ X_{q_{1,2}} - X_{u_3} \neq -X_{\Phi_1}, \ X_{q_3} - X_{u_3} = -X_{\Phi_2},$$
$$X_{q_{1,2}} - X_{u_{1,2}} = -X_{\Phi_1}, \ X_{q_3} - X_{u_{1,2,3}} \neq -X_{\Phi_1}, \ X_{q_{1,2}} - X_{u_{1,2,3}} \neq -X_{\Phi_2},$$
$$X_{q_3} - X_{u_{1,2}} \neq -X_{\Phi_2}. \tag{1.A.5}$$

Similarly, there will be (linear) conditions coming from the textures in the leptonic sectors, and (linear, quadratic and cubic) conditions coming from the anomaly constraints involving $U(1)'$,[15]

$$\left[U(1)'\right]^3, \ U(1)' \, [\text{Gravity}]^2, \ U(1)' \, [U(1)_\text{Y}]^2,$$
$$U(1)' \, [SU(2)_\text{L}]^2, \ U(1)' \, [SU(3)_\text{C}]^2, \ \left[U(1)'\right]^2 U(1)_\text{Y}, \tag{1.A.6}$$

[15] As the right-handed neutrinos are only charged under $U(1)'$, the only anomalies altered in comparison to Paper III are the ones coming from $[U(1)']^3$ and $U(1)' \, [\text{Gravity}]^2$. There, the right-handed neutrino charge will contribute with a term of the exact same form as that of the right-handed electron.

excluding the ones that anyway cancel trivially. In total, we hence have one large system of equations for the $U(1)'$ charges, with equations coming from the quark sector, the leptonic sector and the anomaly constraints. Any solution to this system, for which all charges are rational, is then classified as valid.

To find the linear constraints from the leptonic sectors, we must loop over all possible textures for the charged leptons, $\Pi_{1,2}$, the Majorana neutrinos A, B and C, and the Dirac neutrinos $\Sigma_{1,2}$. Starting out with the physical requirements, we have

(i) No massless charged leptons, $\det M_e \neq 0$;

(ii) Three generations of massive neutrinos, $\det M_\nu \neq 0$;[16]

(iii) A non-zero complex phase in the PMNS matrix, $\det [M_e M_e^\dagger, M_\nu M_\nu^\dagger] \neq 0$,

where the second condition is the only one differing from the conditions in Paper III and IV. For a type-I seesaw mechanism, the second condition translates into M_R and M_D both being 3×3 matrices with non-zero determinants.

Note also that any two models that are reachable from one another via the following permutations, are degenerate

$$\begin{aligned}&\Gamma'_{1,2} = \mathcal{P}_i^\mathrm{T} \Gamma_{1,2} \mathcal{P}_j, \quad \Delta'_{1,2} = \mathcal{P}_i^\mathrm{T} \Delta_{1,2} \mathcal{P}_k, \quad \Pi'_{1,2} = \mathcal{P}_l^\mathrm{T} \Pi_{1,2} \mathcal{P}_m \\ &\Sigma'_{1,2} = \mathcal{P}_l^\mathrm{T} \Sigma_{1,2} \mathcal{P}_n, \quad A' = \mathcal{P}_n^\mathrm{T} A \mathcal{P}_n, \quad B' = \mathcal{P}_n^\mathrm{T} B \mathcal{P}_n, \quad C' = \mathcal{P}_n^\mathrm{T} C \mathcal{P}_n,\end{aligned} \tag{1.A.7}$$

as it would simply correspond to a relabelling of flavor indices. Here, $\mathcal{P}$ is the three-dimensional representation of the permutation group S_3, with all indices running from one to six. For time efficiency, degenerate textures are excluded from the loop.

Majorana Neutrino Sector

In total, there are 11 minimal textures for A, B and C that fulfil the constraint of M_R being a 3×3 symmetric matrix with a non-zero determinant,

[16]Note that there exist no anomaly-free implementation of the BGL textures with two generations of massive neutrinos, as shown in Paper IV.

$$
\begin{aligned}
&(1) \quad A:\begin{pmatrix}\times & 0 & 0\\ 0 & 0 & \times\\ 0 & \times & 0\end{pmatrix}, \quad B:0, \quad C:0,\\
&(2) \quad A:\begin{pmatrix}\times & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad B:\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & \times\\ 0 & \times & 0\end{pmatrix}, \quad C:0,\\
&(3) \quad A:\begin{pmatrix}\times & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad B:\begin{pmatrix}0 & 0 & 0\\ 0 & \times & 0\\ 0 & 0 & 0\end{pmatrix}, \quad C:\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \times\end{pmatrix},\\
&(4) \quad A:\begin{pmatrix}0 & \times & 0\\ \times & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad B:\begin{pmatrix}0 & 0 & \times\\ 0 & 0 & 0\\ \times & 0 & 0\end{pmatrix}, \quad C:\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & \times\\ 0 & \times & 0\end{pmatrix},
\end{aligned}
\tag{1.A.8}
$$

where texture (1) and (2) come in three, respectively six, versions – all possible permutations of A, B and C. For texture (3) and (4), on the other hand, we only need to consider the presented texture, as permutations of rows and columns solely correspond to a relabelling of flavor indices.

Here, invariance under the flavor symmetry corresponds to

$$A_{ij} = e^{i\alpha(X_{\nu_i}+X_{\nu_j})}A_{ij}, \quad B_{ij} = e^{i\alpha(X_{\nu_i}+X_{\nu_j}+X_S)}B_{ij}, \tag{1.A.9}$$

and equivalently for C_{ij}, but with the sign in front of X_S flipped. Hence, texture (1) translates to

$$2X_{\nu_1} = 0, \quad X_{\nu_2} + X_{\nu_3} = 0, \tag{1.A.10}$$

and texture (4) to

$$X_{\nu_1} + X_{\nu_2} = 0, \quad X_{\nu_1} + X_{\nu_3} = X_S, \quad X_{\nu_2} + X_{\nu_3} = -X_S, \tag{1.A.11}$$

and so on.

Note that, in contrast to the quark sector, where the textures are known, we do not include conditions of the form $X_{\nu_i} + X_{\nu_j} \neq 0$. For example, with the condition in eq. (1.A.10), the texture of A will always be forced to have those three non-zero entires (the so-called *minimal* texture), but it is not limited to having only this. The solution to the system of equations may very well allow for one or several additional non-zero entries. As such, the 11 minimal textures in eq. (1.A.8) actually incorporate every single allowed texture imaginable, up to a relabelling of flavor indices.

In addition to this, from the phase-sensitive part of the scalar potential we have one of the following four conditions

$$X_S = \pm\left(X_{\Phi_1} - X_{\Phi_2}\right), \quad X_S = \pm\frac{1}{2}\left(X_{\Phi_1} - X_{\Phi_2}\right), \tag{1.A.12}$$

where either one is allowed in the case of having only non-zero textures in A. For more details, see Paper IV.

Charged Lepton Sector

For the charged lepton sector, we have the same minimal textures as in Paper III and IV. To avoid repetition, we will here present only the final result, which are the following four minimal textures

$$\begin{aligned}
&(1) \quad \Pi_1 : \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & \times \end{pmatrix}, \; \Pi_2 : \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
&(2) \quad \Pi_1 : \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; \Pi_2 : \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \times \end{pmatrix}, \\
&(3) \quad \Pi_1 : \begin{pmatrix} \times & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; \Pi_2 : \begin{pmatrix} 0 & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & \times \end{pmatrix}, \\
&(4) \quad \Pi_1 : \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; \Pi_2 : \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & \times \end{pmatrix}.
\end{aligned} \tag{1.A.13}$$

Dirac Neutrino Sector

For the Dirac neutrino textures, permutation of rows and columns are no longer independent from those in the charged lepton- and Majorana neutrino sectors. As a result, besides fulfilling the constraint of M_{D} having a non-zero determinant, we must now also consider textures that are equivalent up to permutations. In total, this amounts to six possible minimal combined textures,

$$1:\begin{pmatrix}\times & 0 & 0\\ 0 & \times & 0\\ 0 & 0 & \times\end{pmatrix},\ 2:\begin{pmatrix}\times & 0 & 0\\ 0 & 0 & \times\\ 0 & \times & 0\end{pmatrix},\ 3:\begin{pmatrix}0 & \times & 0\\ \times & 0 & 0\\ 0 & 0 & \times\end{pmatrix},$$
$$4:\begin{pmatrix}0 & \times & 0\\ 0 & 0 & \times\\ \times & 0 & 0\end{pmatrix},\ 5:\begin{pmatrix}0 & 0 & \times\\ \times & 0 & 0\\ 0 & \times & 0\end{pmatrix},\ 6:\begin{pmatrix}0 & 0 & \times\\ 0 & \times & 0\\ \times & 0 & 0\end{pmatrix},$$

which in turn corresponds to 48 possible textures for $\Sigma_{1,2}$ – eight for each of the textures displayed above; 111, 112, 121, 211, 122, 212, 221 and 222, where the numbers correspond to whether the non-zero texture appear in Σ_1 or Σ_2. As an example, the eight possibilities for texture number 2 are given by

$$(111)\qquad \Sigma_1:\begin{pmatrix}\times & 0 & 0\\ 0 & 0 & \times\\ 0 & \times & 0\end{pmatrix},\ \Sigma_2:\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$
$$(112)\qquad \Sigma_1:\begin{pmatrix}\times & 0 & 0\\ 0 & 0 & \times\\ 0 & 0 & 0\end{pmatrix},\ \Sigma_2:\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & \times & 0\end{pmatrix},$$
$$(121)\qquad \Sigma_1:\begin{pmatrix}\times & 0 & 0\\ 0 & 0 & 0\\ 0 & \times & 0\end{pmatrix},\ \Sigma_2:\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & \times\\ 0 & 0 & 0\end{pmatrix},$$
$$(211)\qquad \Sigma_1:\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & \times\\ 0 & \times & 0\end{pmatrix},\ \Sigma_2:\begin{pmatrix}\times & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

and so on for 122, 212, 221, 222.

1.A.3 Anomaly-Free Solutions for the BGL Model

Using the procedure described in Sec. 1.A.2, we find a total of three anomaly-free implementations for the BGL model with type-I seesaw, namely

ν**BGL-I Scenario**

$$\Pi_1, \Sigma_1, B = \begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Pi_2, \Sigma_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \times \end{pmatrix}, \\ A = 0, \quad C = \begin{pmatrix} 0 & 0 & \times \\ 0 & 0 & \times \\ \times & \times & 0 \end{pmatrix}. \tag{1.A.14}$$

ν**BGL-IIa Scenario**

$$\Pi_1, \Sigma_1 = \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & 0 \end{pmatrix}, \Pi_2 = \begin{pmatrix} 0 & 0 & 0 \\ \times & 0 & 0 \\ 0 & 0 & \times \end{pmatrix}, \Sigma_2 = \begin{pmatrix} 0 & \times & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \times \end{pmatrix}, \\ A = \begin{pmatrix} \times & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \times \\ 0 & \times & 0 \end{pmatrix}, \quad C = 0. \tag{1.A.15}$$

ν**BGL-IIb Scenario**

$$\Pi_1, \Sigma_1 = \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & 0 \end{pmatrix}, \Pi_2 = \begin{pmatrix} 0 & 0 & 0 \\ \times & 0 & 0 \\ 0 & 0 & \times \end{pmatrix}, \Sigma_2 = \begin{pmatrix} 0 & \times & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \times \end{pmatrix}, \\ A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \times \\ 0 & \times & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \times & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C = 0. \tag{1.A.16}$$

The corresponding charges for each model are presented in Tab. 1.1.

For the gauged BGL model with three generations of right-handed neutrinos and a type-I seesaw mechanism, there are hence only three valid implementations. Out of these three, model I has no flavor-changing interactions in the leptonic sector at tree-level, as Π_1 and Π_2 commute, and equivalently for Σ. For model IIa and IIb, on the other hand, there are tree-level flavor-changing interactions mediated by neutral NP scalars and Z'. The phenomenological validity of either of these models is yet to be determined.

Table 1.1: Allowed charges for the various models. For model νBGL-I and -IIa we have $x_{tL} = -7x + 2y$ and $x_{tR} = -16x + 5y$. Model νBGL-IIb has $x_{tL} = (-13x + 4y)/3$ and $x_{tR} = (-32x + 11y)/3$.

Charges	νBGL-I	νBGL-IIa	νBGL-IIb
q_L	$\begin{bmatrix} x \\ x \\ x_{tL} \end{bmatrix}$	$--$	$--$
u_R	$\begin{bmatrix} y \\ y \\ x_{tR} \end{bmatrix}$	$--$	$--$
d_R	$\begin{bmatrix} 2x-y \\ 2x-y \\ 2x-y \end{bmatrix}$	$--$	$--$
ℓ_L	$\begin{bmatrix} -3x \\ -3x \\ 21x-6y \end{bmatrix}$	$\begin{bmatrix} x-y \\ -7x+y \\ 21x-6y \end{bmatrix}$	$\frac{1}{3}\begin{bmatrix} -x-2y \\ -17x+2y \\ 39x-12y \end{bmatrix}$
e_R	$\begin{bmatrix} -2x-y \\ -2x-y \\ 30x-9y \end{bmatrix}$	$\begin{bmatrix} -2x-2y \\ -6x \\ 30x-9y \end{bmatrix}$	$\frac{1}{3}\begin{bmatrix} 2x-5y \\ -14x-y \\ 58x-19y \end{bmatrix}$
N_R	$\begin{bmatrix} -4x+y \\ -4x+y \\ 12x-3y \end{bmatrix}$	$\begin{bmatrix} 0 \\ -8x+2y \\ 12x-3y \end{bmatrix}$	$\frac{1}{3}\begin{bmatrix} -4x+y \\ -20x+5y \\ 20x-5y \end{bmatrix}$
Φ	$\begin{bmatrix} -x+y \\ -9x+3y \end{bmatrix}$	$\begin{bmatrix} -x+y \\ -9x+3y \end{bmatrix}$	$\frac{1}{3}\begin{bmatrix} 3(-x+y) \\ -19x+7y \end{bmatrix}$
S	$8x-2y$	$-4x+y$	$\frac{8x-2y}{3}$