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On Scalable H-infinity Control

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Abstract

Many of the classical approaches to controller synthesis do not scale well for large and complex systems. This is mainly due to computational complexity and the lack of distributed structure in the resulting controllers. It is important that limitations on the information given and processed by sensors and actuators can be incorporated into the design procedure. However, such constraints may greatly complicate controller synthesis. In this thesis, the need for scalability is addressed and a scalable as well as optimal control law is presented. The criteria on optimality is measured in the H_∞ norm, a norm that is fundamental in the theory of robust control and treats the objective of worst-case disturbance attenuation.

The optimal controller is a state feedback law applicable to linear and time-invariant systems with some symmetry in their structure. More specifically, the system has to be stable and have a state-space representation with a symmetric state matrix. Furthermore, the state and control inputs have to be penalized separately. An analog result is given for infinite-dimensional systems. In the infinite-dimensional case, the criteria on the system are essentially as in the finite-dimensional case, however, somewhat more involved.

Systems with the aforementioned property of symmetry have states that affect each other with equal rate coefficients. Such representations appear, for instance, in different types of transportation networks such as buffer systems. The heat equation is an infinite-dimensional system for which the result is applicable. This equation can model heat conduction systems as well as other types of diffusion, such as chemical diffusion. Examples are included to demonstrate the simplicity in synthesis as well as the performance of the control law.

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1

Introduction

Scalability is of uttermost importance in the control of large and complex systems, such as energy networks, chemical networks or socioeconomic networks. Generally, these systems are sparse in their structure, due to that communication among sensors and actuators is limited. One would like to take advantage of this sparsity in the controller design. Hopefully, this would render a scalable control law less rigid to changes in the dynamics or structure of the system. Ultimately, it should be possible to implement controllers that only depend upon a subset of the global information of the system. We will call such information local. Furthermore, the controllers should impact the system only locally, however, still meet the global objectives. Such controllers are called distributed controllers as they are distributed throughout the system, each controlling only a part of the system, based on local information. This is in contrast to many of the classical control methods. They often result in controllers that are global in the sense that they could require all the available information of the system and also impact it more than locally.

The aim of this thesis is to address the lack of scalability in the conventional control approaches. The results are given for H_∞ state feedback synthesis. H_∞ norm criteria are considered in robust control and treat worst-case disturbance attenuation.

This thesis consists of four introductory chapters and two papers. In the subsequent section, the main contributions of the two papers are described. Thereafter, an outline of the remaining chapters is given in Section 1.2.

1.1 Contributions

Paper I

Lidström, C. and A. Rantzer (2016). "Scalable H_∞ control for systems with symmetric state matrix". *Manuscript prepared for journal submission*.

This paper addresses H_∞ state feedback and gives a very simple form for an optimal control law applicable to stable finite-dimensional linear and time-invariant systems with symmetric state matrix. The control law as well as the minimal value of the norm can be expressed in terms of the matrices of the system's state-space representation given that the state and control inputs are penalized separately. Thus, the optimal control law is easy to synthesize, scalable and transparent.

Given a system with a compatible sparsity pattern, the scalability of the optimal control law is further improved and in some cases the control law is also distributed. Examples of such sparsity patterns are included in the paper. Furthermore, we identify a special class of systems, common in applications, for which the property of internal positivity is preserved in closed-loop with the optimal control law. Given an internally positive system, the state and output vectors are non-negative as long as the disturbance input and initial state of the system are non-negative.

The optimal control law is extended to incorporate coordination among a heterogeneous group of linear and time-invariant subsystems, with the aforementioned properties necessary for applicability. The extended control law is comprised of a decentralized and a centralized term, where the centralized term is identical for all subsystems. The decentralized term is only dependent on the given subsystem. Thus, this control law might be suitable for distributed control purposes as well.

The first author contributed with a conjecture giving the structure of optimal the control law as well as an initial version of the proof. The initial proof was, as is the final version given in Paper I, based on the linear matrix inequality statement for H_∞ state feedback. The first author derived the crucial step of the proof, giving the optimal choice of matrices to fulfil the linear matrix inequality. A. Rantzer contributed with revision of the proof and with ideas for and reviewing of the manuscript.

Paper I is based on and covers the results presented in the following paper, which is therefore omitted from this thesis.

Lidström, C. and A. Rantzer (2016). "Optimal H_∞ state feedback for systems with symmetric and Hurwitz state matrix". *Accepted to the 2016 American Control Conference*.

Paper II

Lidström, C., A. Rantzer, and K. Morris (2016). "H_∞ optimal control for infinite-dimensional systems with strictly negative generator". *Submitted to the 55th IEEE Conference on Decision and Control, 2016*.

This paper gives the infinite-dimensional analog of the result presented for finite-dimensional systems in Paper I. We consider linear and time-invariant infinite-dimensional systems with bounded input and output operators and give a simple form for an optimal state feedback law applicable to systems for which the operator corresponding to the state matrix fulfil essentially the requirements given in the finite-dimensional case, however, somewhat more involved. The state and control input have to be penalized separately for the result to hold, as in the finite-dimensional case. Moreover, the control law as well as the minimal value of the norm can be written in terms of the operators of the system's state-space representation.

The idea to extend the result in Paper I to infinite-dimensional systems was contributed by A. Rantzer and K. Morris. The first author formalized the theorem as well as the proof. A. Rantzer and K. Morris also revised the proof and reviewed the manuscript.

1.2 Outline of the introductory chapters

Chapter 2 reviews the theory on H_∞ state feedback required to state the contributions made in this area. Thereafter, in Chapter 3, the contributions are stated more in depth than they were in the previous section as well as compared with related work. Chapter 3 also includes examples to demonstrate the theory. Conclusions and directions for future research are given in Chapter 4.

2

H_∞ State Feedback Theory

The first section of this chapter gives notation. In the subsequent sections, some fundamental results in H_∞ state feedback theory for finite and infinite-dimensional linear time-invariant systems are reviewed.

2.1 Notation

The set of real numbers is denoted $\mathbb{R}$ and the space n-by-m real-valued matrices is denoted $\mathbb{R}^{n\times m}$. The set of nonnegative real numbers is denoted $\mathbb{R}_+$ and the space n-by-1 nonnegative vectors is denoted $\mathbb{R}_+^n$. The identity matrix is written as I when its size is clear from context, otherwise I_n to denote it is of size n-by-n. Similarly, a column vector of all ones is written $\mathbf{1}$ if its length is clear form context, otherwise $\mathbf{1}_n$ to denote it is of length n.

For a matrix M, the inequality $M \geq 0$ means that M is element-wise non-negative and $M \in \mathbb{R}^{n\times n}$ is said to be Hurwitz if all eigenvalues have negative real part. The matrix M is said to be Metzler if its off-diagonal elements are nonnegative and the spectral norm of M is denoted $\|M\|$. Furthermore, for a square symmetric matrix M, $M \prec 0$ ($M \preceq 0$) means that M is negative (semi)definite while $M \succ 0$ ($M \succeq 0$) means M is positive (semi)definite.

The H_∞ norm of a transfer function $G(s)$ is written as $\|G\|_\infty$. It is well known that this operator norm equals the induced 2-norm, that is

$$\|G\|_\infty = \sup_{v\neq 0} \frac{\|Gv\|_2}{\|v\|_2}.$$

2.2 The H_∞ control problem

It is difficult or even impossible to construct a model that exactly describes a physical system. Therefore, it is of great importance for a control system to be robust against uncertainties in the description of the system,

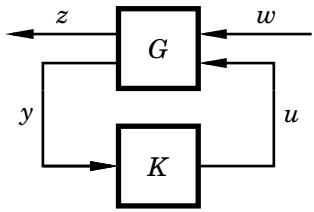

Figure 2.1 Feedback interconnection of system G and controller K. The signals z, y, u and w are the controlled output, measurements, control signal and disturbance, respectively.

such as unknown external disturbances. The theory that treats synthesis of controllers with such properties is called robust control. The robust synthesis problem is to find a controller such that the system behaves as intended in spite of the uncertainties. This problem is formulated via a norm constraint on the closed-loop system's response to the environment. When the performance of the system is measured in the H_∞-norm, the synthesis problem is also known as the H_∞ control problem.

H_∞ control became a major research area in the 1980's, see [Zames, 1981] where it was first formulated in an input-output setting. However, it is only the state-space based approaches to this problem for linear and time-invariant systems that will be reviewed here, stated in [Doyle et al., 1989]. The theory reviewed in this section and the subsequent ones is based on [Zhou et al., 1996], [Dullerud and Paganini, 2013] and [Rantzer, 1996] for the finite-dimensional case and [Curtain and Zwart, 2012], [Morris, 2010] and [Curtain, 1993] for the infinite-dimensional case. The references just stated are also references for further details.

The case with state feedback

Consider the feedback interconnection given in Figure 2.1. The upper block, denoted G, represents the system while the lower block, denoted K, represents the controller. The signals z, y, u and w are the controlled output, measurements, control signal and disturbance, respectively.

Consider G to be a linear and time-invariant (LTI) finite-dimensional system represented in state-space as follows

$$\begin{aligned} \dot{x} &= Ax + Bu + Hw, \quad x(0) = 0, \\ z &= Cx + Du \\ y &= Fx + Ew \end{aligned} \tag{2.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^q$, $z \in \mathbb{R}^l$ and $y \in \mathbb{R}^p$, and the matrices are real-valued and of appropriate dimensions. Since we are reducing the

response from the disturbance w in the H_∞ setting, the initial condition is set to zero. The feedback interconnection depicted in Figure 2.1 and the measurements y specified by (2.1) concern output feedback. For the case of state feedback we set $F = I$ and $E = 0$, i.e., $y = x$. It is shown in [Khargonekar et al., 1988] that the optimal H_∞ controller given state feedback can be made static without restriction. Therefore, from now on, the controller K is modelled as a static state feedback law $u = Kx$ where $K \in \mathbb{R}^{m\times n}$. System (2.1) in closed-loop with the control law $u = Kx$ is given in state-space by

$$\begin{aligned} \dot{x} &= (A + BK)x + Hw, \quad x(0) = 0 \\ z &= (C + DK)x \end{aligned} \tag{2.2}$$

and the corresponding transfer function is

$$G_K(s) = (C + DK)(sI - (A + BK))^{-1}H. \tag{2.3}$$

The objective in H_∞ state feedback control is, given a constant $\gamma > 0$, to find a matrix K such that the closed-loop system (2.2) is stable, i.e., $A + BK$ is Hurwitz, and $\|G_K\|_\infty < \gamma$, where G_K is defined in (2.3). That is, to find a controller K such that the maximum gain of the closed-loop system or the worst case effect of the disturbance w on z is bounded by γ in the L_2 sense. The H_∞-norm criteria on (2.3) is appropriate if we know little about the spectral characteristics of the disturbance w, in contrast to for instance H_2 control.

2.3 State-space based synthesis of H_∞ state feedback

Probably the most well-known state-space based approach to the H_∞ state feedback control problem is the algebraic Riccati equation (ARE) approach, see [Stoorvogel, 1992] for details. In [Dullerud and Paganini, 2013], an approach based on the K-Y-P-lemma, see [Rantzer, 1996], is treated. In this approach the frequency domain constraint given on the H_∞ norm of the closed-loop system's transfer function is equivalently written as a linear matrix inequality (LMI) constraint. Below, we state these equivalent statements for (2.3).

i) There exists a stabilizing controller $K \in \mathbb{R}^{m\times n}$ such that $\|G_K\|_\infty < \gamma$

ii) There exists matrices $X \succ 0$, $X \in \mathbb{R}^{n\times n}$, and $Y \in \mathbb{R}^{m\times n}$ such that

$$\begin{bmatrix} XA^T + AX + Y^TB^T + BY & H & XC^T + Y^TD^T \\ H^T & -\gamma^2 I & 0 \\ CX + DY & 0 & -I \end{bmatrix} \prec 0.$$

The two statements are related through $K = YX^{-1}$ and they are also equivalent to that there exists a solution to a certain ARE. However, the ARE statement is omitted here as the theory presented in Paper I and II is related to the LMI criterion given in (ii).

The equivalent statements (i) and (ii) concern suboptimal H_∞ state feedback. For optimal H_∞ state feedback one wants to find a matrix K such that $A + BK$ is Hurwitz and $\|G_K\|_\infty$ is minimized. Such optimal controllers can be computed via a bisection algorithm over for instance statement (ii).

2.4 H_∞ state feedback for infinite-dimensional systems

Infinite-dimensional models are often needed when the physical system of interest is both temporally and spatially distributed. For instance, heat conduction systems can be modelled by a parabolic partial differential equation known as the heat equation. Infinite-dimensional LTI systems can be described in state-space similarly to (2.1). However, the state now evolves on an infinite-dimensional Hilbert space and what before was the state matrix A is instead an operator acting on this infinite-dimensional Hilbert space instead of on $\mathbb{R}^n$. More specifically, the operator A with domain $D(A)$ generates a strongly continuous semigroup on the state-space. Furthermore, B is a linear and bounded operator that maps the control input space to the state space and correspondingly for operators C, D and H. See [Curtain and Zwart, 2012] for more details and definitions of these notions.

The definition of stability for finite-dimensional systems generalizes to infinite-dimensional systems. However, the Hurwitz criteria on the matrix A for stability in the finite-dimensional case is exchanged for a requirement on the semigroup generated by the operator A in the infinite-dimensional case. Statements similar to (i) and (ii) can be stated for infinite-dimensional systems as well. However, they are operator-valued and of course infinite-dimensional. Furthermore, the sought after controller K is a bounded and linear operator from the state space to the control input space.

In the infinite-dimensional case, closed-form solutions to the H_∞ state feedback problem are generally hard or not possible to obtain. Therefore, it is common to consider the state-space based synthesis problem for a finite-dimensional approximation of the original system. In this procedure one has to ensure that the controller synthesized for the approximated system fulfils the specifications for the original system as well. This can be problematic but there exists conditions under which this approach works, see [Ito and Morris, 1998].

3

Contributions, comparisons and examples

In the first section of this chapter, the contributions of this thesis are reviewed as well as compared with related work. Examples are given in the subsequent section to demonstrate the contributions.

3.1 Contributions and comparison with related work

In both Paper I and II, we consider the case of optimal H_∞ state feedback. In Paper I, we show that given the LTI system (2.1) with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $H \in \mathbb{R}^{n\times q}$, the state feedback controller $K_{\text{opt}} = B^T A^{-1}$ is optimal if the state matrix A is symmetric and Hurwitz. To arrive at this result we also have to assume that $C^T D = 0$ in (2.1), i.e., that the state and control inputs are penalized separately, and that $C^T C = I_n$ and $D^T D = I_m$. The most telling part of the proof of this result is where statement (ii) in Section 2.3 is written equivalently as follows: There exists matrices $X \succ 0$, $X \in \mathbb{R}^{n\times n}$, and $Y \in \mathbb{R}^{m\times n}$ such that

$$(X + A)^2 + (Y^T + B)(Y^T + B)^T - A^2 - BB^T + \gamma^{-2} HH^T \prec 0.$$

It is easy to see that the choice $X = -A$ and $Y = -B^T$ minimizes the smallest possible γ for which this inequality holds. Furthermore, as A is symmetric and Hurwitz, i.e., $-A \succ 0$, the stated choice for X is indeed feasible. Thus, $K = YX^{-1} = B^T A^{-1}$ is an optimal controller. In some cases, the latter assumptions stated for matrices C and D can be lifted to only require the matrices C and D to be injective. This alters the expression of the optimal controller K_{opt} somewhat. Further, the optimal control law is applicable to systems that can be represented in state-space by a symmetric state matrix after variable transformation, however, the penalized variables are then altered.

The classical state-space based approaches to the H_∞ state feedback problem, as the ones described in Section 2.3 of Chapter 2, involve solving an algebraic Riccati equation or inequality, see [Zhou et al., 1996] and [Rantzer, 1996]. For large systems, such an approach might get problematic, and it is generally difficult to directly relate the solution of the Riccati equation to the structure of the system. On the contrary, our optimal controller K_{opt} is clearly related to the matrices of the system's state-space representation. Furthermore, it is easy to synthesize given its explicit form and is computationally scalable as it only requires some fairly inexpensive matrix calculations in order to be computed. Thus, given this control law there is no trade-off between scalability and performance.

The result described above can in some cases be used for distributed control purposes even though the H_∞ control problem is formulated in a centralized way, i.e., with one physical system to be controlled by one single control unit. Consider that the system at hand is comprised of a collection of subsystems that are sparsely connected and where each have their own controller that base its decision on the locally available subset of the global information of the system. As earlier described, such controllers are distributed or even decentralized, dependent on if they exchange information or not. The optimal controller K_{opt} becomes distributed if the matrices A and B in (2.1) have compatible sparsity patterns. Consider for instance the matrix A to be diagonal. Then, whatever structure is in B also appears in the control law $u = K_{\text{opt}}x = B^T A^{-1}x$. In comparison, controllers derived by the ARE and LMI approaches are often dense and thus not distributed.

In classical control theory, sparsity of the controllers are generally not taken into consideration. In fact, structural constraints may greatly complicate the design procedure or even make it impossible to follow through. However, design is simplified in some cases, see e.g. [Rotkowitz and Lall, 2006] and [Tanaka and Langbort, 2011]. Our method results in a control law that is equal in performance to the central non-structured controller of the system. This is not the case in the methods previously mentioned. However, they treat more general classes of systems. Examples of compatible sparsity patterns are given in Paper I, e.g., diagonality or block-diagonality of the state matrix while the control input matrix is sparse.

The optimal control law $u = K_{\text{opt}}x$ is extended to incorporate coordination among a heterogeneous group of LTI subsystems. See Section 3.2 of this chapter for details. It is assumed that the state matrices of the subsystems are all symmetric and Hurwitz. Furthermore, the control input signals of the subsystems have to fulfil a linear coordination constraint. The resulting control law is comprised of a local term, only dependent upon the subsystem itself, and a global term, however equal for all subsystems. The latter term could therefore be calculated globally and then

distributed to the different subsystems. This structure of the control law might make it suitable for distributed control purposes. In [Madjidian and Mirkin, 2014] a similar control law has been derived, however, in the H_2 setting with a requirement of identical subsystems. In the H_∞ framework just described, and presented in Paper I, it is possible to consider heterogeneous subsystems. However, as given by the assumption, they all have to have symmetric and Hurwitz state matrices.

In [Tanaka and Langbort, 2011; Rantzer, 2015], the closed-loop system property of internal positivity is utilized for scalability of classical control approaches. A continuous-time LTI system is internally positive if its state matrix is Metzler and the remaining system matrices have nonnegative entries. Given an internally positive system, the state and output are nonnegative given a nonnegative input and initial state. One might ask if the assumption of internal positivity on the closed-loop system is restrictive. In Paper I, we identify a class LTI systems for which it is not. More specifically, the class consists of systems with diagonal and Hurwitz state matrix A and where the matrix product $-BB^T$ is Metzler. With these system properties, the closed-loop system given the optimal controller K_{opt}, from disturbance to state, is internally positive. In other words, the property of internal positivity is preserved in closed-loop. Moreover, controller K_{opt} is clearly distributed as long as the control input matrix B is sparse. Synthesis of structured H_∞ state feedback control is treated in both [Tanaka and Langbort, 2011] and [Rantzer, 2015]. It was a major breakthrough when this work on positive systems showed that distributed and scalable H_∞-optimal controllers can be derived. However, in their work, positivity is added as an extra requirement with a potentially negative effect on the H_∞ performance. In Paper I, an important class of problems is identified for which no such negative effects exist.

The result given in Paper II is the infinite-dimensional analog of the result stated in Paper I. The criteria of symmetry and Hurwitz stability of the finite-dimensional system's state matrix are exchanged for the requirements that the generator of the infinite-dimensional system is closed, densely defined, self-adjoint and strictly negative while the input and output operators are bounded. For these infinite-dimensional systems, an optimal H_∞ state feedback controller is given by $u = B^*A^{-1}x$ where B^* is the adjoint of the operator B, while the state and possibly control input are infinite-dimensional signals. As mentioned in Section 2.4 in the previous chapter, it is generally hard or even impossible to obtain closed-form expressions for H_∞ controllers for infinite-dimensional systems. Therefore, we indicate the possible benefits of having such a closed-form expression in the computation of H_∞ state feedback controllers for large-scale systems and propose that the optimal control law can be used in benchmarking of general purpose algorithms. Furthermore, the performance of

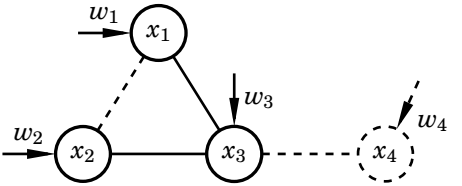

Figure 3.1 Graph of a transportation network where the links indicate transportation routes. The optimal controller K_{opt} scales well with the expansion of the system given by the dashed lines.

the optimal controller is demonstrated by means of a numerical example that considers the heat equation.

3.2 Examples

The optimal controllers presented in Paper I and Paper II are applicable to systems with some inherent structural symmetry. In the finite-dimensional case, systems for which the derived optimal control law is applicable have to have states that affect each other with equal rate coefficients. In the subsequent subsections, different features of the controllers are demonstrated by means of examples.

Linear transportation networks and scalability

Consider a dynamical system that is interconnected according to the graph given in solid lines in Figure 3.1 and described by the following LTI system

$$\begin{aligned} \dot{x}_1 &= -x_1 + u_{13} + w_1 \\ \dot{x}_2 &= -3x_2 + u_{23} + w_2 \\ \dot{x}_3 &= -2x_3 - u_{13} - u_{23} + w_3. \end{aligned} \tag{3.1}$$

The system (3.1) could for instance model a transportation network connecting three buffers where the content of each buffer is given by the state. Furthermore, the control inputs u_{ij} determine the transfer between buffer i and buffer j. The objective is to decide upon the the control inputs u_{13} and u_{23} such that disturbances w_1, w_2 and w_3 are optimally attenuated in the H_∞ norm sense. Some examples of transportation networks are irrigation systems, power systems, chemical dynamic systems, communication and computation networks, and production planning and logistics.

Rewrite (3.1) as follows

$$\dot{x} = -\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}}_{B} u + w,$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad u = \begin{bmatrix} u_{13} \\ u_{23} \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}.$$

Indeed, this system has a symmetric and Hurwitz state matrix A. Thus, considering penalized variables x and u, the static state feedback controller from Paper I, i.e.,

$$\begin{bmatrix} u_{13} \\ u_{23} \end{bmatrix} = K_{\text{opt}} x = B^T A^{-1} x = \begin{bmatrix} -1 & 0 & 1/2 \\ 0 & -1/3 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

is optimal in the H_∞ norm sense. System (3.1) in closed-loop with $u = K_{\text{opt}} x$ given above becomes an internally positive system from the disturbance to the state. Thus, this system structure is part of the class of systems for which internal positivity is preserved by the controller.

Now, consider that (3.1) is expanded according to the dashed lines in Figure 3.1 and that the dynamics of the entire system is given by

$$\begin{aligned} \dot{x}_1 &= -x_1 + u_{12} + u_{13} + w_1 \\ \dot{x}_2 &= -3x_2 - u_{12} + u_{23} + w_2 \\ \dot{x}_3 &= -2x_3 - u_{13} - u_{23} + u_{34} + w_3 \\ \dot{x}_4 &= -x_4 - u_{34} + w_4. \end{aligned} \tag{3.2}$$

The optimal control law from Paper I for this expanded system becomes

$$\begin{bmatrix} u_{12} \\ u_{13} \\ u_{23} \\ u_{34} \end{bmatrix} = \begin{bmatrix} -1 & 1/3 & 0 & 0 \\ -1 & 0 & 1/2 & 0 \\ 0 & -1/3 & 1/2 & 0 \\ 0 & 0 & -1/2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

Given the expression above, it becomes evident that neither u_{13} nor u_{23} are altered as the system changes from (3.1) to (3.2). Furthermore, each control input is only comprised of the states it affects via the control input matrix B, i.e., the nodes it connects in Figure 3.1. Thus, given a system of this structure, the control law is clearly distributed.

In the classical synthesis methods of H_∞ controllers, one has to solve a Riccati equation or inequality, as previously mentioned. This approach might not scale well for large systems. Thus, the controller K_{opt} is to recommend when treating systems for which it is applicable. For an initial study of the differences in the performance of K_{opt} and optimal ARE and LMI controllers consider Section 6 in Paper I.

Coordination in the H_∞ framework

Consider a group of ν LTI systems with uncoupled dynamics

$$\dot{x}_i = A_i x_i + B_i u_i + H_i w_i, \quad i = 1, \ldots, \nu,$$

where each state $x_i \in \mathbb{R}^n$ can be measured, $u_i \in \mathbb{R}^m$ and $w_i \in \mathbb{R}^q$. Furthermore, it is assumed that each A_i is symmetric and Hurwitz. The objective is to design the control inputs u_i such that the impact from the disturbances w_i on the state and control input is minimized, in the H_∞ norm sense, while the control inputs fulfil the following constraint

$$u_1 + u_2 + \cdots + u_\nu = 0.$$

Assume that the model is a description around some operating point. Then, the constraint given above can be thought of as that the collective behavior of the control inputs should be kept at the value of the operating point. If it models a network of buffers, these should be kept such as to meet a certain demand. However, there is freedom in distributing the effort to meet this criteria among the subsystems. The control law

$$u_i = B_i^T A_i^{-1} x_i - \frac{1}{\nu} \sum_{k=1}^{\nu} B_k^T A_k^{-1} x_k \quad \text{for } i = 1, \ldots, \nu$$

fulfils $\sum_{i=1}^{\nu} u_i = 0$ and minimizes the H_∞ norm of the closed-loop system from w to the penalized variables x and u. As previously mentioned, it is comprised of a decentralized and a centralized part, where the latter is equal for all subsystems. This specific structure favors scalability of the control law.

Temperature control in buildings and comparison to H_2 control

Consider the following model of the temperature dynamics in adjacent rooms in a building, governed by Fourier's law of thermal conduction,

$$\begin{aligned}
\dot{T}_1 &= r_1(T_{out} - T_1) + r_{12}(T_2 - T_1) + u_1 + w_1 \\
\dot{T}_2 &= r_2(T_{out} - T_2) + r_{12}(T_1 - T_2) + r_{23}(T_3 - T_2) + u_2 + w_2 \\
\dot{T}_3 &= r_3(T_{out} - T_3) + r_{23}(T_2 - T_3) + u_3 + w_3
\end{aligned} \tag{3.3}$$

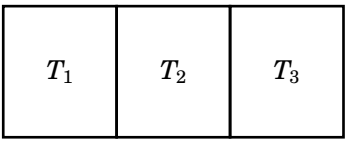

Figure 3.2 Schematic of a building with three rooms for which the temperature dynamics, see model (3.3), has the required symmetry property.

where T_i is the average temperature in room $i = 1$, 2 and 3. The rate coefficients $r_\bullet$ are constant, real-valued and strictly positive. In Figure 3.2, the three rooms are depicted and it is assumed that they have the same air mass. Furthermore, T_{out} is the outdoor temperature and disturbances specific for each room, such as when a window is opened, are modeled by disturbances w_i. It is assumed that the average temperatures can be measured as well as controlled, the latter through heating and cooling devices modeled by the control inputs u_i. The system (3.3) has a state matrix that is symmetric and Hurwitz and thus the optimal control law given in Paper I is applicable to this system.

In Paper I, the optimal H_∞ state feedback controller K_{opt} as well as the optimal H_2 controller are determined for (3.3), given a set of parameters. See Paper I for more details on the synthesis procedure. There is only a minimal difference in the performance of K_{opt} and the optimal H_2 controller. This could motivate the use of the H_∞ controller, despite the fact that an assumption of stochastic disturbances might seem more appropriate for this specific application. Especially so if one considers a model with a larger number of rooms than in (3.3), as K_{opt} is much more simple to synthesize. However, a more in depth study of the applicability K_{opt} to temperature control in buildings is left for future research.

Towards H_∞ control for large-scale systems

The applications presented in the previous subsections treated finite-dimensional systems. In the infinite-dimensional case, the fact that an explicit expression for an optimal H_∞ state feedback law is obtained, is beneficial for controller synthesis for more general systems. This is due to that the derived controller can be used for benchmarking in evaluation of general purpose algorithms.

In Paper II, an explicit form is derived for an optimal control law applicable to an H_∞ state feedback problem considering the heat equation. The heat equation is a parabolic partial differential equation that models heat transfer and describes the distribution of heat in a given region over time. Furthermore, this equation is also used for modeling other types of

diffusion, such as chemical diffusion. The theory developed in Paper II can thus be applied to several problems in heat transfer as well as other systems modeled by the heat equation.

4

Conclusions

This thesis addresses scalable H_∞ control and gives a simple form for an optimal H_∞ state feedback law applicable to systems with some symmetry in their structure. More specifically, in the finite-dimensional case, the state matrix of the system's state-space representation have to be symmetric and Hurwitz. In the infinite-dimensional case, where the state of the system evolves on a Hilbert space, the input and output operators have to be bounded while the generator is closed, densely defined, self-adjoint and strictly negative. The optimal control law is scalable, transparent and easy to synthesize. Its closed-form is clearly related to the matrices, or, more generally the operators, of the system's state-space representation. This fact favors scalability even more. Especially given systems with compatible sparsity structure as the resulting control law in these cases might be distributed. Extensions of the main result consider internally positive systems and coordination among heterogeneous groups of subsystems. In the infinite-dimensional case, it is rare to have a closed-form expression for an optimal H_∞ state feedback law. Therefore, this result can be beneficial in the evaluation and improvement of algorithms for synthesis of controllers for large-scale systems.

One natural extension of the theory presented in this thesis is towards output or measurement feedback. It could also be of interest to consider if similar criteria on the structure of the system can be utilized to obtain closed-form control laws given another norm criteria, such as the H_2 or L_1 setting. Similarily, this hypothesis could be tested in the frequency domain setting as well. Furthermore, the dual problem of optimal state-estimation can be considered and initial attempts, by the authors of Paper I, to derive the discrete-time analog of the result are promising but not entirely straightforward.

It would be insightful to derive bounds on the performance given deviations from the criteria on the system structure. This would possibly extend the class of systems for which it is applicable. It is possible to take

a robust control approach to this problem, in which tools for treatment of uncertainties are well known.

Another direction for future research is towards non-linear system. Either considering the same type of setup, however, in the non-linear setting or to incorporate non-linear phenomena such as control input saturation into the presented framework. The latter would be of interest as control input saturations is a common feature among the physical systems intended for its application.

In the continuation of the work in the infinite-dimensional case, the optimal norm bound could be valuable information in order to speed up algorithms for optimal actuator placement. Furthermore, it would be useful to compare the synthesis procedure as well as the performance of the control law presented here with a control law synthesized by any of the conventional, approximative methods. Furthermore, there are several areas of application to explore, e.g., systems in financial mathematics and problems in heat transfer, all modeled by the heat equation.

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Paper I

Scalable H_∞ Control for Systems with Symmetric State Matrix

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Abstract

Conventional synthesis methods for H_∞ state feedback can become very complex for large-scale systems. Furthermore, it is difficult to relate the structure of the controllers synthesized by these methods to the structure of the system. However, we give a simple form for an optimal H_∞ state feedback law clearly related to the system's structure. It is applicable to continuous-time linear and time invariant systems with symmetric and Hurwitz state matrix. More specifically, the control law as well as the minimal value of the norm can be expressed in terms of the matrices of the system's state space representation given that the state and control input are penalized separately. Thus, the control law is transparent, easy to synthesize and scalable. Given that the system possesses a compatible sparsity pattern, the optimal controller is also distributed. Examples of such sparsity patterns are included. Furthermore, we show that for a special class of linear and time-invariant systems, that commonly appears in applications, the closed-loop system with the optimal controller is internally positive. For these systems, the control law not only preserves sparsity but also positivity. Finally, we apply the optimal control law to a heterogeneous group of subsystems with a criterion on coordination. This results in an extension of the control law that might be suitable for distributed control purposes as well. This is due to that it is comprised of a decentralized term and a centralized term, where the latter is equal for all subsystems. Examples demonstrate the simplicity in synthesis and the performance of the optimal control law.

Manuscript prepared for journal submission.

1. Introduction

We consider H_∞ state feedback of linear and time-invariant (LTI) systems. The classical state-space based approach to this problem involves solving a Riccati equation or inequality, see [Zhou et al., 1996] and [Rantzer, 1996]. For large systems, this approach can get very complex and it might be difficult to directly relate the solution of the Riccati equation to the structure of the system. However, we show that for stable systems with symmetric state matrix, an optimal H_∞ state feedback law can be given on a very simple form that is clearly related to the matrices of the system's state-space representation.

Consider a state-space representation of a continuous-time LTI system subject to a disturbance w, i.e., $\dot{x} = Ax + Bu + w$. We show that, if the state matrix A is symmetric and Hurwitz, then the static state feedback law $u = B^T A^{-1} x$ minimizes the H_∞ norm of the closed-loop system from disturbance w to the penalized variables x and u. This controller is easy to synthesize and computationally scalable as it only requires some fairly inexpensive matrix calculations in order to be computed. Furthermore, it is transparent, which simplifies analysis of its structure, and distributed given certain sparsity patterns of the matrices A and B.

Given a system with sparse structure, one would like to take advantage of this sparsity in the controller design. Hopefully, it would render a scalable control law less rigid to changes in the dynamics or structure of the system. Ultimately, the controller should be distributed throughout the system and decisions should be made locally based on local information, while still meeting the global objectives. This is in contrast to a global controller that needs full information of the entire system.

Given compatible sparsity patterns of A and B, e.g., block diagonality, the control law $u = B^T A^{-1} x$ is sparse and also distributed. In comparison, controllers derived by the algebraic Riccati equation (ARE) approach are often dense, see [Zhou et al., 1996] for details on this method. This generally holds for H_∞ controllers derived by the linear matrix inequality (LMI) approach as well, see [Dullerud and Paganini, 2013]. In classical control theory, sparsity of the controllers are generally not taken into consideration. In fact, structural constraints may greatly complicate the design procedure or even make it impossible to follow through. However, design is simplified in some cases, see e.g. [Rotkowitz and Lall, 2006] that treats quadratic invariance and [Tanaka and Langbort, 2011] that treats internal positivity. Our method results in a control law that is equal in performance to the central non-structured controller of the system. This is not the case for the methods previously mentioned. However, they treat more general classes of systems.

H_∞ control became a major research area in the 1980's, see [Zames, 1981]. The state-space based solution approach enabled optimization tools to be used, e.g., see [Doyle et al., 1989]. Further, the H_∞ norm condition can be turned into a LMI by the Kalman-Yakubovich-Popov lemma [Gahinet and Apkarian, 1994], see Lemma 1 in Appendix for the version used in this paper. As the theory on H_∞ control emerged, a sub-optimal decentralized version took form, e.g., see [Zhai et al., 2001]. However, as previously mentioned, imposing general sparsity constraints on the controller might complicate the design procedure. See [Mahajan et al., 2012], and the references therein, for an extensive review of the literature on norm-optimal control subject to structural or communication constraints.

The H_∞ framework treats worst-case disturbances, as opposed to stochastic disturbances in the H_2 framework. Inspired by the H_2 coordination problem treated in [Madjidian and Mirkin, 2014], we show that the derived optimal control law can be extended to incorporate coordination in a system comprised of heterogeneous subsystems, given a linear coordination constraint. The coordinated control law is a superposition of a decentralized and a centralized part, where the latter is equal for all agents. Thus, it might be well suited for distributed control purposes.

If the state matrix A is diagonal and $-BB^T$ is Metzler, the closed-loop system with the optimal control law $u = B^T A^{-1}$, from disturbance to state, is internally positive. Thus, for such systems the property of internal positivity is preserved in closed-loop by the control law. In [Tanaka and Langbort, 2011] and [Rantzer, 2015], it is shown that the property of internal positivity can be used in order to derive distributed and scalable H_∞ optimal controllers. However, in this work, positivity is added as an extra requirement, with a potentially negative effect on the H_∞ performance whereas we point out an important class of problems, for which no such negative effects exist.

Systems with symmetric state matrices have states that affect each other with equal rate coefficients. Such representations appear, for instance, in buffer networks and models of temperature dynamics in buildings. In Section 6 we give a numerical study of the performance of the optimal control law given these types of systems. The main theorem stated in Section 2 was presented in [Lidström and Rantzer, 2016] but has been revised and made somewhat more general. Moreover, an extension of the results presented here, to infinite-dimensional systems is treated in [Lidström et al., 2016].

The outline of this paper is as follows. This section is ended with notation. In Section 2, the main result is stated and proved while Section 3 treats the scalability of the optimal control law. Section 4 gives the extension of the control law that incorporates coordination while Section 5

gives the result on internal positivity. In Section 6, the performance of our optimal control law is compared, by a numerical examples, to optimal controllers synthesized by the ARE and LMI approaches. It is also compared to the optimal H_2 controller. Concluding remarks and directions for future work are given in Section 7.

The set of real numbers is denoted $\mathbb{R}$ and the space n-by-m real-valued matrices is denoted $\mathbb{R}^{n\times m}$. The set of nonnegative real numbers is denoted $\mathbb{R}_+$ while the space n-by-1 nonnegative vectors is denoted $\mathbb{R}_+^n$. The identity matrix is written as I when its size is clear from context and otherwise I_n to denote it is of size n-by-n. Similarly, a column vector of all ones is written $\mathbf{1}$ if its length is clear form context and otherwise $\mathbf{1}_n$ to denote it is of length n.

For a matrix M, the inequality $M \geq 0$ means that M is element-wise non-negative and $M \in \mathbb{R}^{n\times n}$ is said to be Hurwitz if all eigenvalues have negative real part. The matrix M is said to be Metzler if its off-diagonal elements are nonnegative and the spectral norm of M is denoted $\|M\|$. Furthermore, for a square symmetric matrix M, $M \prec 0$ ($M \preceq 0$) means that M is negative (semi)definite while $M \succ 0$ ($M \succeq 0$) means M is positive (semi)definite.

The H_∞ norm of a transfer function $F(s)$ is written as $\|F(s)\|_\infty$. It is well known that this operator norm equals the induced 2-norm, that is

$$\|F\|_\infty = \sup_{v\neq 0} \frac{\|Fv\|_2}{\|v\|_2}.$$

2. An optimal H_∞ state feedback law

This section begins with a short review of the H_∞ state feedback problem. Then, the main result of this paper is stated, followed by remarks on generalizations. To begin with, consider a continuous-time LTI system

$$\begin{aligned} \dot{x} &= Ax + Bu + Hw \\ z &= \begin{bmatrix} x \\ u \end{bmatrix} \end{aligned} \tag{1}$$

where the state matrix $A \in \mathbb{R}^{n\times n}$ is symmetric and Hurwitz and the state $x \in \mathbb{R}^n$ can be measured. The control input $u \in \mathbb{R}^m$, disturbance $w \in \mathbb{R}^q$ and controlled output $z \in \mathbb{R}^{n+m}$. Moreover, matrices B and H are real-valued and of appropriate dimensions. Given (1), consider a static state feedback law $u := Kx$, where $K \in \mathbb{R}^{m\times n}$. The transfer function of the closed-loop system with K, from disturbance w to z, is given by

$$G_K(s) = \begin{bmatrix} I \\ K \end{bmatrix} (sI - (A + BK))^{-1} H. \tag{2}$$

We consider the H_∞ state feedback problem, for which the optimal controller can be made static without restriction, see [Khargonekar et al., 1988]. Thus, the considered problem is to find a matrix K, if there exists one, such that (2) is stable and $\|G_K\|_\infty < \gamma$ for some pre-specified constant $\gamma > 0$. In fact, we consider synthesis of an *optimal* H_∞ state feedback law, i.e., a matrix K such that the closed-loop system is stable and $\|G_K\|_\infty$ is minimized.

We show that for (1) with symmetric and Hurwitz state matrix A, an optimal H_∞ state feedback controller K can be given explicitly in the matrices A and B. This is the main result of this paper and it is stated in the following theorem. Comments regarding the result given a more general controlled output z are made after the theorem.

THEOREM 1
Consider the system (1) with $A \in \mathbb{R}^{n\times n}$ symmetric and Hurwitz, $B \in \mathbb{R}^{n\times m}$ and $H \in \mathbb{R}^{n\times q}$. Then, the norm $\|G_K\|_\infty$ is minimized by the static state feedback controller $K_{\text{opt}} = B^T A^{-1}$. The minimal value of the norm is $\|H^T(A^2 + BB^T)^{-1}H\|^{\frac{1}{2}}$. Moreover, $\|G_{K_{\text{opt}}}\|_\infty = \|G_{K_{\text{opt}}}(0)\|$. □

Proof Given $\gamma > 0$, the following statements are equivalent.

(i) There exists a stabilizing controller $K \in \mathbb{R}^{m\times n}$ such that

$$\|G_K\|_\infty = \sup_{\omega\in\mathbb{R}} \left\| \begin{bmatrix} I \\ K \end{bmatrix} (i\omega I - A - BK)^{-1} H \right\| < \gamma.$$

(ii) There exist matrices $P \succ 0$, $P \in \mathbb{R}^{n\times n}$, and $K \in \mathbb{R}^{m\times n}$ such that

$$\begin{bmatrix} (A+BK)^T P + P(A+BK) & PH & [I \;\; K^T] \\ H^T P & -\gamma^2 I & 0 \\ [I \;\; K^T]^T & 0 & -I \end{bmatrix} \prec 0.$$

(iii) There exist matrices $X \succ 0$, $X \in \mathbb{R}^{n\times n}$, and $Y \in \mathbb{R}^{m\times n}$ such that

$$\begin{bmatrix} AX + XA + BY + Y^T B^T & H & [X \;\; Y^T] \\ H^T & -\gamma^2 I & 0 \\ [X \;\; Y^T]^T & 0 & -I \end{bmatrix} \prec 0.$$

(iv) There exist matrices $X \succ 0$, $X \in \mathbb{R}^{n\times n}$, and $Y \in \mathbb{R}^{m\times n}$ such that

$$(X + A)^2 + \left(Y + B^T\right)^T \left(Y + B^T\right) - A^2 - BB^T + \gamma^{-2} HH^T \prec 0.$$

(v)

$$-A^2 - BB^T + \gamma^{-2}HH^T \prec 0.$$

(vi)

$$\|H^T \left(A^2 + BB^T\right)^{-1} H\|^{\frac{1}{2}} < \gamma.$$

The equivalence between (i) and (ii) is given by the K-Y-P-lemma, see Lemma 1 given in Appendix. Statement (ii) can be equivalently written as (iii) after right- and left-multiplication with $\mathrm{diag}(P^{-1}, I, I)$ and change of variables $(P^{-1}, KP^{-1}) \to (X, Y)$. The equivalence between (iii) and (iv) is obtained by applying Schur's complement lemma and completion of squares to the inequality in (iii). Choosing $X = -A$ and $Y = -B^T$ shows equivalence between (iv) and (v). It is possible to choose $X = -A$ as A is symmetric and Hurwitz, i.e., $A \prec 0$. Finally, notice that $A^2 + BB^T \succ 0$. Thus, $\left(A^2 + BB^T\right)^{-1} \succ 0$ and

$$\text{(v)} \iff HH^T \prec \gamma^2 \left(A^2 + BB^T\right) \iff \text{(vi)}.$$

Given $X = -A$ and $Y = -B^T$, γ is minimized and $K_{\text{opt}} = YX^{-1} = B^T A^{-1}$ minimizes the norm in (i). Thus, we have proven that

$$\|G_{K_{\text{opt}}}\|_\infty = \|H^T \left(A^2 + BB^T\right)^{-1} H\|^{\frac{1}{2}}.$$

Now, we will prove the last statement given in the theorem, i.e., the equality $\|G_{K_{\text{opt}}}\|_\infty = \|G_{K_{\text{opt}}}(0)\|$. That is, we want to prove that $\|G_{K_{\text{opt}}}(0)\|$ is equal to the norm-expression stated above. Consider (2) with $K = K_{\text{opt}} = B^T A^{-1}$ at $s = 0$, i.e.,

$$G_{K_{\text{opt}}}(0) = -\begin{bmatrix} I \\ B^T A^{-1} \end{bmatrix} (A + BB^T A^{-1})^{-1} H.$$

Now, the following holds

$$\|G_{K_{\text{opt}}}\| = \sqrt{\lambda_{\max}(G_{K_{\text{opt}}}(0)^T G_{K_{\text{opt}}}(0))} = \sigma_{\max}(G_{K_{\text{opt}}}(0))$$

where $\lambda_{\max}$ and $\sigma_{\max}$ denote the largest eigenvalue and largest singular value, respectively. However, as the matrix $G_{K_{\text{opt}}}(0)^T G_{K_{\text{opt}}}(0)$ is square, real-valued, symmetric and positive definite

$$\lambda_{\max}(G_{K_{\text{opt}}}(0)^T G_{K_{\text{opt}}}(0)) = \|G_{K_{\text{opt}}}(0)^T G_{K_{\text{opt}}}(0)\|.$$

Then as

$$\begin{aligned} G_{K_{\text{opt}}}(0)^T G_{K_{\text{opt}}}(0) \\ &= H^T(A + BB^TA^{-1})^{-T}(I + A^{-1}BB^TA^{-1})(A + BB^TA^{-1})^{-1}H \\ &= H^T(A^2 + BB^T)^{-1}A(I + A^{-1}BB^TA^{-1})A(A^2 + BB^T)^{-1}H \\ &= H^T(A^2 + BB^T)^{-1}H. \end{aligned}$$

it becomes clear that

$$\|G_{K_{\text{opt}}}(0)\| = \|G_{K_{\text{opt}}}(0)^T G_{K_{\text{opt}}}(0)\| = \|H^T(A^2 + BB^T)^{-1}H\|^{\frac{1}{2}} = \|G_{K_{\text{opt}}}\|_\infty$$

and the proof is done. □

The result stated in Theorem 1 can be made more general. Instead of z defined by (1) consider

$$z = \begin{bmatrix} x \\ Du \end{bmatrix} \tag{3}$$

where $z \in \mathbb{R}^{n+l}$ and $D \in \mathbb{R}^{l \times m}$. If D^TD is invertible the optimal controller is given by the following remark.

Remark 1
Consider the cost (3) with D^TD invertible. Then, the control law becomes $K_{\text{opt}} = (D^TD)^{-1}B^TA^{-1}$ and the norm is given by

$$\|H^T(A^2 + B(D^TD)^{-1}B^T)^{-1}H\|^{\frac{1}{2}}.$$

□

It is also possible to consider an altered cost on the state x, for instance, to multiply it by a scalar. However, thorough investigation of the expression for the resulting control law given more sophisticated alterations to the cost is left for future research. Moreover, for systems written on descriptor form, i.e., $E\dot{x} = Ax + Bu + Hw$, with both $-E$ and A symmetric and Hurwitz, there exists a variable transformation such that the system can be written on state-space form with a diagonal state matrix. This is known as simultaneous diagonalization, see [Roger and Charles, 1985, Th 7.6.6, p. 466], and would render a system for which Theorem 1 is applicable. In general, the optimal control law is applicable to systems that can be represented in state-space by a symmetric state matrix after variable transformation. However, note that the penalized variables might then have been altered.

The closed-form expression for the optimal control law given by Theorem 1, i.e., $K_{\text{opt}} = B^TA^{-1}$, is very simple. In comparison with the ARE

approach or general LMI setup for synthesis of a state feedback controller, controller K_{opt} is computationally cheaper to synthesize. It only requires some relatively inexpensive matrix calculations. Moreover, given K_{opt} it is easy to relate the structure of the controller to the structure of the matrices of the system's state space representation, which often is not the case with other synthesis methods. This transparency simplifies analysis of the controller's structure and enables scalability. These features will be exploited in Section 3. Moreover, notice that K_{opt} is independent of the matrix H in (1) as remarked below.

REMARK 2
The optimal control law $K_{\text{opt}} = B^T A^{-1}$ given by Theorem 1 is not dependent upon the matrix H in (1). However, the minimal value of the norm is. Notice that if H is a column vector, the expression $H^T(A^2 + BB^T)^{-1}H$ present in the value of the norm, is a scalar. □

There is a clear relation between the optimal control law given by Theorem 1 and an optimal controller derived by bisecting over an ARE constraint. The latter is given by $L = -B^T P$ where P is the solution to the ARE, see [Stoorvogel, 1992]. Another interesting feature of the optimal control law in Theorem 1 is the property stated last in the theorem and commented on in the following remark.

REMARK 3
The equality $\|G_{K_{\text{opt}}}\|_\infty = \|G_{K_{\text{opt}}}(0)\|$ given by Theorem 1 reveals that static disturbances are the worst-case disturbances for (2) with $K = K_{\text{opt}}$. □

Even if the result in Theorem 1 is limited to the class of systems with symmetric and Hurwitz state matrix A, it still gives some insights for more general systems. Mainly, statement (iv) in the proof of Theorem 1 reveals a lower bound on the optimal performance given any LTI system. Without the symmetry assumption on A, statement (iv) is as follows; There exists matrices $X \succ 0$ and Y such that

$$(X+A)(X+A)^T + (Y+B^T)^T(Y+B^T) - AA^T - BB^T + \gamma^{-2}HH^T \prec 0. \quad (4)$$

The first two terms in (4) are positive semidefinite regardless of the choices of X and Y. Thus, the lowest possible bound on γ is achieved when they can be made zero. Furthermore, notice that if A is not Hurwitz, the matrix $(X + A)(X + A)^T$ can not be made zero. This is summarized in the following remark.

REMARK 4
The following lower bound holds for any controller K, regardless of the properties of A,

$$\|G_K\|_\infty \geq \|H^T(AA^T + BB^T)^{-1}H\|^{\frac{1}{2}}.$$

Theorem 1 is applicable to systems with some inherent symmetry structure. The criteria on stability might be seen as less restrictive due to that the system at hand could be stabilized by a precompensator, as long as the symmetry property is preserved, before constructing the optimal controller. However, when this is possible and how to construct such a compensator is left for future research.

3. Scalability of the control law

Scalable control methods are of uttermost importance for large-scale and complex systems. This is due to the fact that they often have many sensors and actuators, with limited communication. In much of the classical control theory, including H_∞ control, it is assumed that controllers have access to the same measurements. In fact, both sensors and controllers are generally assumed to share the global information of the entire system.

In this section, we will demonstrate by means of an example that the optimal state feedback controller $K_{\text{opt}} = B^T A^{-1}$ given by Theorem 1 is clearly scalable and also distributed given a system with a compatible sparsity pattern. Furthermore, the example demonstrates the simplicity in synthesis as well as the transparency of the optimal controller. However, it is worthwhile to first comment on the case of diagonal state matrix A. The structure or sparsity pattern of $K_{\text{opt}} = B^T A^{-1}$ is then only dependent on the sparsity pattern of B. Thus, in such a case, the control law is distributed according to B^T, i.e., each control input will only be constructed by the states that it affects through B. This case is treated further in Section 5.

Example 1

Consider the following LTI system, which is comprised of three finite-dimensional subsystems S_1, S_2 and S_3,

$$\begin{aligned} S_1: &\quad \dot{x}_1 = A_1 x_1 + B_1 u_1 + w_1, \\ S_2: &\quad \dot{x}_2 = A_2 x_2 + B_2 u_1 + B_3 u_2 + w_2, \\ S_3: &\quad \dot{x}_3 = A_3 x_3 + B_4 u_2 + w_3. \end{aligned} \tag{5}$$

The matrices A_1, A_2 and A_3 are assumed to be symmetric and Hurwitz. Then, Theorem 1 is applicable to (5) and the optimal control law is given by

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} B_1^T A_1^{-1} & B_2^T A_2^{-1} & 0 \\ 0 & B_3^T A_2^{-1} & B_4^T A_3^{-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \tag{6}$$

In Figure 1 on the following page, system (5) is depicted by the graph in solid lines. Each subsystem is represented by a circular node while the

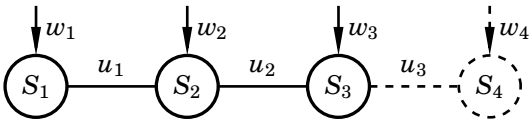

Figure 1. Graphical representation of (5) in solid lines. Additional subsystem S_4 and control input u_3 in dashed lines.

control inputs are given by lines drawn in between the nodes they affect through the control input matrices in (5). Similarly, each disturbance w_i is drawn as an arrow that points toward the subsystem it affects in (5). The control law (6) is clearly distributed as each control input vector u_i is only constructed from the states it affects in (5), i.e., the nodes it connects in Figure 1. Consider the fourth subsystem depicted in dashed lines in Figure 1 and denoted S_4. The dynamics of this additional subsystem and the altered dynamics of subsystem S_3 are given by

$$\begin{aligned} S_3: &\quad \dot{x}_3 = A_3 x_3 + B_4 u_2 + B_5 u_3 + w_3, \\ S_4: &\quad \dot{x}_4 = A_4 x_4 + B_6 u_3 + w_4, \end{aligned}$$

where matrix A_4 is also assumed to be symmetric and Hurwitz. Now, Theorem 1 is still applicable and the extended optimal control law becomes

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} B_1^T A_1^{-1} & B_2^T A_2^{-1} & 0 & 0 \\ 0 & B_3^T A_2^{-1} & B_4^T A_3^{-1} & 0 \\ 0 & 0 & B_5^T A_3^{-1} & B_6^T A_4^{-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}. \tag{7}$$

Notice that the control law scales well for this type of system as control inputs u_1 and u_2 do not change from (6) to (7). Moreover, the control law is still distributed as the additional control input u_3 is only constructed from states x_3 and x_4. □

The structure of the controller $K_{\text{opt}} = B^T A^{-1}$ is clearly dependent on the structure of the matrices A and B, as was clear in Example 1. The control law becomes distributed and scales more easily if the system has a compatible sparsity pattern. In particular, matrices A and B should have sparsity patterns such that sparsity is preserved when K_{opt} is constructed. Existence of such structural properties in the controller are thus limited by the structural features of the considered system. This is different from most methods found in literature where sparsity is generally imposed upon the controller. As previously mentioned, imposing such sparsity constraints might complicate the design procedure. In the cases when design is simplified, e.g., see [Rotkowitz and Lall, 2006; Tanaka and

Langbort, 2011; Rantzer, 2015; Rantzer, 2016], the imposed sparsity constraints might have a potentially negative effect on the H_∞ performance. This is not the case with the controller given by Theorem 1. However, the previously mentioned approaches can treat more general systems.

4. Coordination in the H_∞ framework

In this section, we apply the control law given by Theorem 1 to control a heterogeneous group of subsystems that have to cooperate in order to fulfil a linear coordination constraint. The resulting controller is comprised of a decentralized and a centralized term, where the latter is equal for all agents. Thus, the derived control law is scalable and might be suitable for distributed control purposes as well.

Consider a continuous-time LTI system comprised of a heterogeneous group of ν subsystems

$$\dot{x}_i = A_i x_i + B_i u_i + H_i w_i, \quad i = 1, \dots, \nu \tag{8}$$

where A_i, for $i = 1, \dots, \nu$, is symmetric and Hurwitz, state $x_i \in \mathbb{R}^n$ is measurable, $u_i \in \mathbb{R}^m$ and $w_i \in \mathbb{R}^q$. With penalized variables $x = (x_i)$ and $u = (u_i)$, the optimal control law given by Theorem 1 is $u_i = B_i^T A_i^{-1} x_i$. That is, each control input only considers the state of the subsystem it affects in (8) and the subsystems are completely decoupled.

Now, consider that the control inputs have to coordinate in order to fulfil a common goal. In particular, consider that the they have to fulfil the following constraint

$$u_1 + u_2 + \cdots + u_\nu = 0. \tag{9}$$

Assume that (8) is the system description around some operating point. Then, the constraint in (9) describes that the collective behavior of the control inputs should be kept at the value of the operating point. If it models a buffer of some quantity, this should be kept such as to meet a certain demand. However, there is freedom in distributing the production of this quantity between the subsystems.

In Corollary 1 stated below, we give an H_∞ optimal state feedback law for (8) that fulfills the coordination constraint (9).

Corollary 1
Consider system (8) where each subsystem $i = 1, \dots, \nu$ has a symmetric and Hurwitz state matrix. Then,

$$u_i = B_i^T A_i^{-1} x_i - \frac{1}{\nu} \sum_{k=1}^{\nu} B_k^T A_k^{-1} x_k \quad \text{for } i = 1, \dots, \nu$$

fulfils $\sum_{i=1}^{\nu} u_i = 0$ and minimizes the norm of the closed-loop system from w to the penalized variables x and u. The minimal value of the norm is given by

$$\left\| H^T \left(A^2 + B(I - \tfrac{1}{\nu}\mathbf{1}\mathbf{1}^T)B^T\right)^{-1} H \right\|^{\frac{1}{2}}.$$

□

Proof Rewrite u_1 through (9) as follows

$$u_1 = -u_2 - u_3 \cdots - u_\nu, \tag{10}$$

and define $\tilde{u} = [u_2, u_3, \ldots, u_\nu]^T$. Then,

$$u = \underbrace{\begin{bmatrix} -\mathbf{1}_{\nu-1}^T \\ I_{\nu-1} \end{bmatrix}}_{D} \tilde{u}$$

and the overall system of (8) can be written as

$$\dot{x} = \underbrace{\operatorname{diag}(A_1, \ldots, A_\nu)}_{A} x + \underbrace{\operatorname{diag}(B_1, \ldots, B_\nu)}_{B} D\tilde{u} + w.$$

The penalized variable u can now be written in terms of $\tilde{u}$, i.e., $u = D\tilde{u}$. Define $R = D^T D = I + \mathbf{1}\mathbf{1}^T$, which has dimension $(\nu - 1) \times (\nu - 1)$ and notice that $R^{-1} = I_{\nu-1} - \frac{1}{\nu}(\mathbf{1}\mathbf{1}^T)_{\nu-1}$. Now we can apply Theorem 1, see also Remark 1. The optimal control law is

$$\begin{aligned}
\tilde{u} &= R^{-1} D^T B^T A^{-1} x \\
&= \left(I_{\nu-1} - \frac{1}{\nu}\mathbf{1}_{\nu-1}\mathbf{1}_{\nu-1}^T \right) \begin{bmatrix} -\mathbf{1}_{\nu-1}^T \\ I_{\nu-1} \end{bmatrix}^T B^T A^{-1} x \\
&= \left(\begin{bmatrix} 0 & I_{\nu-1} \end{bmatrix} - \frac{1}{\nu}\mathbf{1}_{\nu-1}\mathbf{1}_{\nu}^T \right) B^T A^{-1} x.
\end{aligned}$$

Thus, u_i for $i = 2, \ldots, \nu$, i.e., the elements in $\tilde{u}$, is

$$u_i = B_i^T A_i^{-1} x_i - \frac{1}{\nu} \sum_{k=1}^{\nu} B_k^T A_k^{-1} x_k. \tag{11}$$

Now, consider u_1 again,

$$\begin{aligned}
u_1 &\overset{(10)}{=} -\sum_{i=2}^{\nu} u_i \overset{(11)}{=} -\sum_{i=2}^{\nu} \left(B_i^T A_i^{-1} x_i - \frac{1}{\nu} \sum_{k=1}^{\nu} B_k^T A_k^{-1} x_k \right) \\
&= -\left(\sum_{k=1}^{\nu} B_k^T A_k^{-1} x_k - B_1^T A_1^{-1} x_1 - \frac{\nu - 1}{\nu} \sum_{k=1}^{\nu} B_k^T A_k^{-1} x_k \right) \\
&= B_1^T A_1^{-1} x_1 - \frac{1}{\nu} \sum_{k=1}^{\nu} B_k^T A_k^{-1} x_k.
\end{aligned}$$

Control input u_1 has the same structure as (11), as expected by symmetry in the indices. Thus, the optimal control law is given by

$$u_i = B_i^T A_i^{-1} x_i - \frac{1}{\nu} \sum_{k=1}^{\nu} B_k^T A_k^{-1} x_k$$

for each subsystem $i = 1, \ldots, \nu$ in (8). The minimal value of the norm is given by

$$\|H^T (A^2 + BD(D^T D)^{-1} D^T B^T)^{-1} H\|^{\frac{1}{2}} \\ = \|H^T \left(A^2 + B(I - \tfrac{1}{\nu}\mathbf{1}\mathbf{1}^T)B^T\right)^{-1} H\|^{\frac{1}{2}}$$

by Remark 1. □

REMARK 5
The first term of u_i in the control law given by Corollary 1 is a local term, only dependent upon subsystem i, while the second term is dependent on global information of the overall system. However, as this term is equal for all control inputs u_i, the control law might still be appropriate for distributed control use. □

In [Madjidian and Mirkin, 2014], a similar type of problem is considered, however in the H_2 framework with stochastic disturbances and the necessity of homogeneous subsystems. The optimal control law derived in [Madjidian and Mirkin, 2014] and the control law in Corollary 1 are similar in structure. However, our approach can treat heterogeneous systems in addition to homogeneous ones. On the contrary, it is only applicable to subsystems with symmetric and Hurwitz state matrices, properties that are not necessary in [Madjidian and Mirkin, 2014]. Notice also that it is possible to consider differently sized subsystems and disturbances, i.e., $x_i \in \mathbb{R}^{n_i}$ and $w_i \in \mathbb{R}^{q_i}$.

Consider (8) with $\nu = 2$ and $H_i = I$ in closed-loop with the control law given by Corollary 1, i.e.,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\frac{1}{2} \begin{bmatrix} B_1 B_1^T A_1^{-1} & -B_1 B_2^T A_2^{-1} \\ -B_2 B_1^T A_1^{-1} & B_2 B_2^T A_2^{-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{u} + Iw. \quad (12)$$

where $w = \begin{bmatrix} w_1 & w_2 \end{bmatrix}^T$ and we denote $x = \begin{bmatrix} w_1 & w_2 \end{bmatrix}^T$. If $A_1 = A_2$ and $B_1 = B_2$, that is the subsystems are identical, then $u = 0$ if $x_1 = x_2$. Consider for instance $A_1 = A_2 = -1$ and $B_1 = B_2 = 1$. Then the state matrix of the closed-loop system, denoted A_{cl}, becomes

$$A_{\text{cl}} = \frac{1}{2} \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix},$$

which has the slowest pole, or smallest absolute eigenvalue, -1 with corresponding eigenvector $\mathbf{1}$. This is the worst-case direction in which the state could be disturbed and clearly u in (12) becomes zero in this direction. Due to the homogeneity of the system we lose controllability in one direction. However, this is not generally the case for heterogeneous subsystems.

5. Systems for which K_{opt} preserves internal positivity

Consider system (1) in Section 2, however, with somewhat more restrictive assumptions on the matrices A, B and H. In particular, that A is diagonal and Hurwitz, $-BB^T$ is Metzler and H is element-wise non-negative. In this section, we show that such a system in closed-loop with the controller given by Theorem 1 results in an internally positive system, see Definition 1 and Lemma 2 in Appendix, where the latter states criteria for a system to be internally positive. Thus, internal positivity is non restrictive on the closed-loop performance. This is in contrast to the methods in [Tanaka and Langbort, 2011; Rantzer, 2015; Rantzer, 2016], where internal positivity is imposed as a constraint on the closed-loop system which might degrade the closed-loop performance. Corollary 2 below states the aforementioned result which is thereafter demonstrated by an example. Internally positive systems are shown to be suitable models when describing systems with network structure for which the components of the state, input and output vectors are naturally positive and the signal flows are directed. See, e.g., the system treated in Example 2.

Corollary 2
Consider (1) with A diagonal and Hurwitz while $-BB^T$ is Metzler and $H \geq 0$. Then, $\|G_K\|_\infty$ defined by (2) is minimized by $K_{\text{opt}} = B^T A^{-1}$ and the closed-loop system $\dot{x} = (A + BK_{\text{opt}})x + Hw$ is internally positive. □

Proof As A is Hurwitz and diagonal, Theorem 1 is applicable. Thus, $K_{\text{opt}} = B^T A^{-1}$ minimizes $\|G_K\|_\infty$. The closed-loop system from w to x with K_{opt} is given by

$$\dot{x} = (A + BK_{\text{opt}})x + Hw.$$

This system is internally positive, by Lemma 2 in Appendix. In particular,

$$A + BK_{\text{opt}} = A + BB^T A^{-1}$$

is Metzler given that $-BB^T$ is Metzler and A is diagonal and Hurwitz. Furthermore, H is element-wise nonnegative by assumption. □

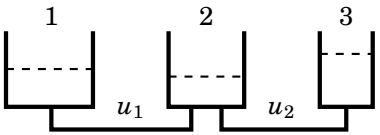

Figure 2. Three buffers denoted 1, 2 and 3 connected via links with flow u_1 and u_2, respectively, as described by (13). The dashed lines represent some steady state of the system.

REMARK 6
Notice that the property of internal positivity is preserved in closed-loop with K_{opt} as the open-loop system from disturbance w to state x, i.e., $\dot{x} = Ax + Hw$, is internally positive. □

EXAMPLE 2
Consider three buffers connected via links with flows u_1 and u_2, as depicted in Figure 2. The dynamics of the levels in the buffers, around some steady state depicted by the dashed lines in Figure 2, is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = -\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}}_{B} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + w. \tag{13}$$

State x_i corresponds to the level in buffer i =1, 2 and 3, respectively. Each buffer has some internal dynamics dependent on its own state, as given by matrix A. However, with different rate coefficients.

The objective is to construct a control law that minimizes the impact from the disturbance w to the penalized variables x and u in the H_∞ norm sense. That is, we want to keep the system at its steady state, i.e, $x_i = 0$ for all i, with the least possible effort.

Given the matrix B in (13), the matrix product $-BB^T$ is Metzler. Thus, by Corollary 2, system (13) in closed-loop with $u = K_{\text{opt}}x$, where

$$K_{\text{opt}} = \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1/2 & -1/4 \end{bmatrix},$$

is internally positive. This implies that, the states of the closed-loop will always be nonnegative given nonnegative disturbance. To get some further intuition of what the control law $u = K_{\text{opt}}x$ does, consider for instance control input u_1. It is given by $u_1 = x_1 - x_2/2$. Thus, u_1 is strictly positive if $x_1 > x_2/2$ and the controller K_{opt} redistributes the quantity of buffer 1 and buffer 2 relative to their internal rate coefficients. As in Example 1, K_{opt} has the same sparsity pattern as B^T and thus each control input only considers local information, i.e., from the buffers it connects. □

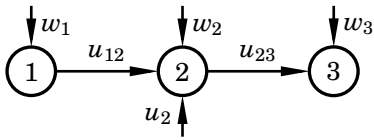

Figure 3. Associated graph of (14).

In Example 2 above, the B-matrix is an incidence matrix, see [Newman, 2010] for a formal definition. This is a special case of B-matrix for which $-BB^T$ is Metzler. However, often seen in applications.

6. Numerical study of the performance of the controller

In this section, we investigate the performance of the optimal control law by means of numerical examples. Due to the non-uniqueness of optimal H_∞ controllers it is worthwhile to compare the controller given by Theorem 1, i.e., K_{opt}, with optimal H_∞ controllers derived via other synthesis methods. Furthermore, we compare K_{opt} with the optimal H_2 controller, see [Zhou et al., 1996].

6.1 Comparison of K_{opt}, ARE, LMI and optimal H_2 controllers

Consider the following LTI system, and depicted in Figure 3,

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = -\underbrace{\begin{bmatrix}a_1 & 0 & 0\\ 0 & a_2 & 0\\ 0 & 0 & a_3\end{bmatrix}}_{A}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \underbrace{\begin{bmatrix}-b_1 & 0 & 0\\ b_2 & b_3 & -b_4\\ 0 & 0 & b_5\end{bmatrix}}_{B}\begin{bmatrix}u_{12}\\ u_2\\ u_{23}\end{bmatrix} + \begin{bmatrix}w_1\\ w_2\\ w_3\end{bmatrix} \tag{14}$$

where $a_i > 0$, for $i = 1$, 2 and 3, and $b_j > 0$, for $j = 1, \ldots, 5$.

We derive optimal controllers for instances of (14), where the parameters a_i and b_i are randomly generated in $(0.1, 5]$, by means of Theorem 1, the ARE-approach, the LMI approach and optimal H_2 controller synthesis. The computations are performed in MATLAB, see [MATLAB, 2012]. The controller from Theorem 1 is denoted K_{opt}, while ARE, LMI and H_2-control stands for the controllers derived by bisecting over the ARE and LMI constraints and by the H_2-optimal control approach, respectively.

Figure 4 shows the singular values of the closed-loop systems' transfer functions given the different controllers, for the following set of parameters

$$a_1 = 2.63,\ a_2 = 3.54,\ a_3 = 0.85,$$

$$b_1 = 4.77,\ b_2 = 2.75,\ b_3 = 3.43,\ b_4 = 0.28 \text{ and } b_5 = 4.1.$$

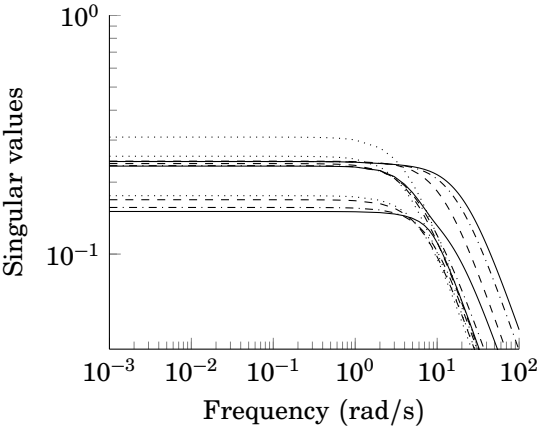

Figure 4. Singular values for closed-loop systems with K_{opt} (solid lines), ARE (dashed lines), LMI (dashdotted) and H_2 (dotted).

However, the results were similar for other instances of (14). In Figure 4, it becomes evident that the singular values of the closed-loop system with K_{opt} are less than the singular values of the other closed-loop systems, in the frequency range of the worst-case disturbance, i.e., up to 3 rad/s. That is, the controller K_{opt} is better at attenuating not only the worst-case disturbance, but also other disturbances in this frequency range. However, it has the worst performance at higher frequencies.

In Table 1, the H_2 and H_∞ norm values of the closed-loop systems are given, still considering the previously stated set of parameters. Of course, the H_∞ norm value of the closed-loop systems with K_{opt}, ARE and LMI controllers are identical. Controller K_{opt} results in the closed-loop system with the highest H_2 norm value. The ARE approach does slightly better than the LMI approach in this norm.

System (14) in closed-loop with K_{opt} will always become internally positive from disturbance to state given the range of values for the parameters stated above, as given by Corollary 2. This is generally not the case for the closed-loop systems with the ARE and LMI controllers. However, the optimal H_2 controllers preserved the property of internal positivity for

Table 1. H_2 and H_∞ norm values of closed-loop systems.

Norm	K_{opt}	ARE	LMI	H_2-control
H_2	0.96	0.77	0.87	0.66
H_∞	0.24	0.24	0.24	0.31

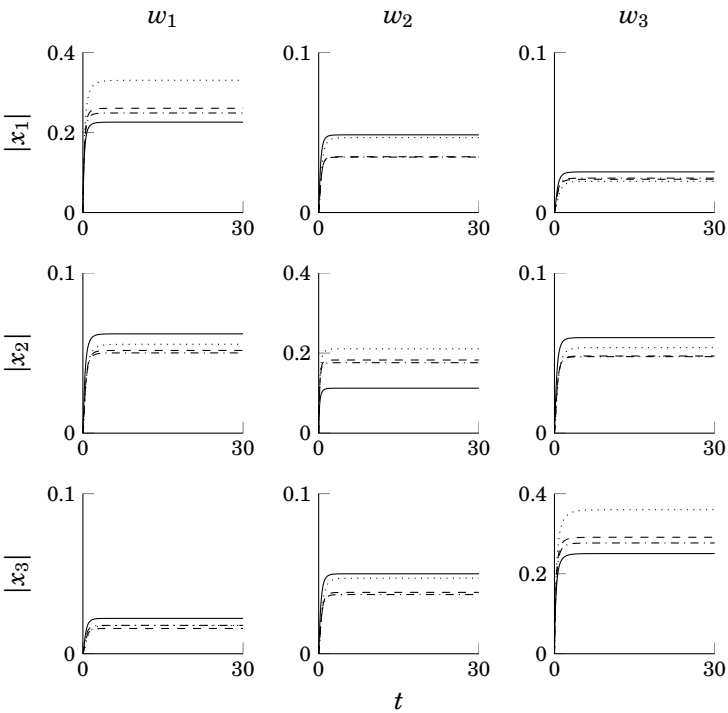

Figure 5. Average step response for states x_1, x_2 and x_3 for closed-loop systems with controller K_{opt} (solid lines), ARE (dashed lines), LMI (dash-dotted) and H_2 (dotted).

most of the instances of (14) considered in this study.

Figure 5 shows the step responses of the closed-loop systems, averaged over 100 instances. Again, the parameters a_i and b_i for the system instances are randomly generated in $(0.1, 5]$. To clarify, we average over the absolute value of the step response at each time instance given a unit disturbance.

In Figure 5, it seems as if controller K_{opt} is better at attenuating local disturbances than any of the other controllers, however, at the expense of less disturbance attenuation non-locally. With local disturbances we mean the disturbance that points towards the state in Figure 3. For instance, consider disturbance w_1. Its impact on state x_1 is lower for controller K_{opt} than the other controllers while the opposite can be said about its impact on the remaining states. However, overall the H_∞ and H_2 controllers are comparable in performance.

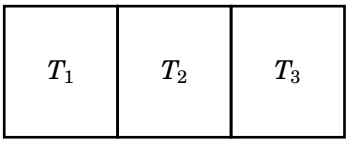

Figure 6. Schematic of a building with three rooms. The temperature dynamics of the rooms is given by (15) and fulfil the required symmetry property.

6.2 Temperature control in buildings and further comparison to H_2 control

In this subsection, the control law given by Theorem 1 is applied to control the temperature in adjacent rooms in a building. Consider a building with three rooms as depicted in Figure 6. The following model describes the temperature dynamics in the three room as governed by Fourier's law of thermal conduction. The average temperature T_i in each room $i = 1$, 2 and 3, is given by

$$\begin{aligned} M_1 c\dot{T}_1 &= R_1(T_{out} - T_1) + R_{12}(T_2 - T_1) + u_1 + w_1 \\ M_2 c\dot{T}_2 &= R_2(T_{out} - T_2) + R_{12}(T_1 - T_2) + R_{23}(T_3 - T_2) + u_2 + w_2 \\ M_3 c\dot{T}_3 &= R_3(T_{out} - T_3) + R_{23}(T_2 - T_3) + u_3 + w_3 \end{aligned} \tag{15}$$

where M_i is the air mass in room i and c is the specific heat capacity of air. The parameters $R_\bullet$ are constant, real-valued and strictly positive. They are the rate coefficients of the system, in other words they represent the heat conduction through the walls of the rooms. For instance, R_{12} is the rate coefficient of the heat transfer through the wall between room 1 and 2. Furthermore, T_{out} is the outside temperature and disturbances specific for each room, such as when a window is opened, are modeled by disturbances w_i. It is assumed that the average temperatures can be measured as well as controlled, the latter through heating and cooling devices modelled by the control inputs u_i.

Consider the rooms to have equal air masses, i.e., that $M_1 = M_2 = M_3$, and assume that the outside temperature can be disregarded as an unknown disturbance. Then (15) can be written on the familiar state-space form (1). It is then easy to see that the state matrix A is symmetric and Hurwitz and thus Theorem 1 is applicable.

Now, given a set of parameters for (15), we will use the optimal controller K_{opt} for disturbance attenuation as well as reference tracking, i.e., $u = K_{opt}x + L_r r$ where r is a piece-wise constant reference signal and L_r is determined such that the state x follows the reference r without stationary error. Figure 7 shows the response from the closed-loop sys-

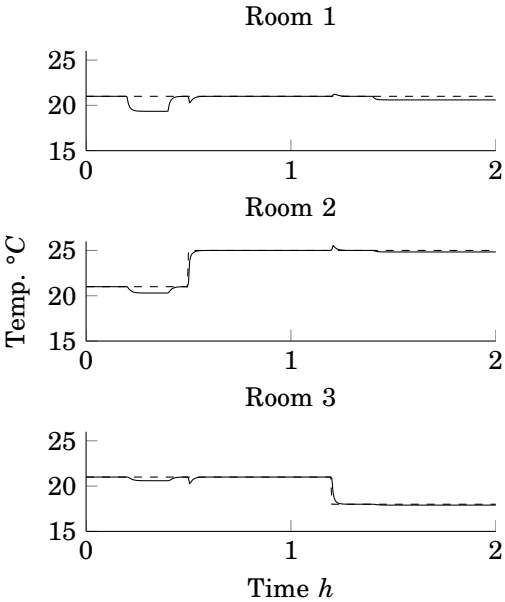

Figure 7. Temperature T_1 (top), T_2 (middle) and T_3 (bottom) in solid lines and reference-values in dashed lines, over time given in hours (h). Disturbance in Room 1 between 0.2-0.4 h. Change in reference in Room 2 from 21 °C to 25 °C at 0.5 h. Change in reference in Room 3 from 21 °C to 18 °C at 1.2 h. Disturbance in Room 1 at 1.4 h and onwards which give rise to stationary errors.

tem given certain piece-wise constant reference and disturbance signals. See the caption of Figure 7 for more details. Given a disturbance, the controller K_{opt} tries to keep the temperatures as close to the reference values as possible while minimizing the cost that comes with heating and cooling. However, it is not able to remove static errors due to the lack of integral action.

Given the considered application, it might be more motivated to consider the disturbances to be stochastic, i.e, to consider the H_2 norm instead of the H_∞ norm in synthesis. However, when comparing the optimal H_2 controller for the example just stated with the derived H_∞ control law, the disturbance attenuation of the two closed-loop systems were close to identical. If this is still true given a system of larger dimension, it would motivate the use K_{opt} as it is simpler to compute. However, to answer this question as well as to carry out a more in depth study of the applicability of K_{opt} to temperature control in buildings is left for future research.

7. Conclusions

We give a simple form for an optimal H_∞ state feedback law applicable to LTI systems with symmetric and Hurwitz state matrix given separable cost on state and control input. More specifically, the simple form is given in terms of the matrices of the system's state space representation which makes the structure of the controller transparent. It also simplifies synthesis and enables scalability of the control law, especially given systems with sparse matrices. Furthermore, we demonstrate different sparsity patterns for which the controller becomes distributed. The examples we give consider diagonal or block diagonal state matrices and somewhat more general sparsity patterns of the remaining system matrices. Further, we identify a class of systems for which internal positivity is preserved in closed-loop by the optimal control law. The optimal control law is also extended to incorporate coordination among heterogeneous subsystems given by a linear coordination constraint. The resulting coordinated control law has a similar structure for all subsystems. More specifically, for each subsystem, it is a superposition of a local term and an averaged centralized term where the latter is equal for all subsystems involved in the coordination. Thus, this coordination control law might be well suited for distributed control purposes.

The performance of the optimal controller is compared with controllers synthesized by classical state-space based approaches to H_∞ control as well as H_2 control. The numerical evaluation shows that the different control laws have comparable overall performance. However, the derived optimal control law seems to perform better in the frequency range of the worst-case disturbance. However, the opposite can be said for the other frequencies. Moreover, the optimal control law also seems to be better at attenuating local disturbances, at the expense of attenuation on a distance. These observations are worth to be investigated further and are one direction for future work.

We have shown that the structural property of symmetry can be exploited in H_∞ state feedback. It is natural to ask if this also holds in the case of output feedback. That is, if it is possible to exploit some structural property of the system in order to construct a closed-form expression for an optimal output feedback control law. Further directions for extensions of this work include to consider saturation constraints on the optimal control law as such are common in the systems intended for its application. Moreover, another direction for future research is to investigate if similar closed-form expressions can be derived given that the objective is stated with another norm or that the considered system is non-linear.

Appendix

LEMMA 1— THE KALMAN-YAKUBOVICH-POPOV LEMMA
Given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $M = M^T \in \mathbb{R}^{(n+m)\times(n+m)}$, with $\det(j\omega I - A) \neq 0$ for $\omega \in \mathbb{R}$ and (A, B) controllable, the following two statements are equivalent:

(i)

$$\begin{bmatrix} (j\omega I - A)^{-1} B \\ I \end{bmatrix}^* M \begin{bmatrix} (j\omega I - A)^{-1} B \\ I \end{bmatrix} \preceq 0$$

$\forall \omega \in \mathbb{R} \cup \{\infty\}$.

(ii) There exists a matrix $P \in \mathbb{R}^{n\times n}$ such that $P = P^T$ and

$$M + \begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} \preceq 0$$

The corresponding equivalence for strict inequalities holds even if (A, B) is not controllable. □

Proof See [Rantzer, 1996]. □

REMARK 7
If the upper left corner of M is positive semidefinite, it follows from statement (ii) in Lemma 1 and Hurwitz stability of A that $P \succeq 0$ [Rantzer, 1996]. □

DEFINITION 1— [KACZOREK, 2001]

$$\begin{aligned} \dot{x} &= Ax + Bv, \quad x(0) = x_0 \\ y &= Cx + Dv, \end{aligned}$$

is called internally positive if for every $x_0 \in \mathbb{R}^n_+$ and all $u(t) \in \mathbb{R}^m_+$ the state vector $x(t) \in \mathbb{R}^n_+$ and $y(t) \in \mathbb{R}^q_+$ for $t \geq 0$. □

LEMMA 2
The LTI system

$$\begin{aligned} \dot{x} &= Ax + Bv \\ y &= Cx + Dv \end{aligned}$$

is internally positive if and only if

i A is Metzler, and

ii $B \geq 0$, $C \geq 0$ and $D \geq 0$.

□

Proof See [Kaczorek, 2001]. □

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Paper II

H_∞ Optimal Control for Infinite-Dimensional Systems with Strictly Negative Generator

Carolina Lidström, Anders Rantzer and Kirsten Morris

Abstract

We address H_∞-control of linear time-invariant infinite-dimensional systems, where the state evolves on a separable Hilbert space, and give a simple form for an optimal state feedback law applicable to systems with bounded input and output operators and closed, densely defined, self-adjoint and strictly negative state operator. That is, the state operator generates an exponentially stable strongly continuous semi-group on the considered Hilbert space. We penalize the state and control input separately. Furthermore, we give a closed form expression for the L_2-gain of the closed-loop system given this optimal controller. The result is an extension of the finite-dimensional case, derived by the first two authors. Examples demonstrate the simplicity in synthesis as well as the performance of the control law.

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1. Introduction

Infinite-dimensional models are often needed when the physical system of interest is both temporally and spatially distributed. For instance, heat conduction systems can be modelled by a parabolic partial differential equation known as the heat equation, see [Renardy and Rogers, 2006] for details on this equation. We consider H_∞ state feedback control of linear and time-invariant infinite-dimensional systems. The H_∞ control problem was first formulated for finite-dimensional systems, see [Zhou et al., 1996] and the references therein. There are both state-space based and frequency domain based solutions to the H_∞ control problem for infinite-dimensional systems, as in the finite-dimensional case. In the frequency domain approach, see [Foias et al., 1996], one needs to determine the transfer function of the system, which in general can be hard. In the state-space based approach to this problem, the synthesis involves solving an infinite-dimensional operator-valued Riccati equation or inequality, see [Bensoussan and Bernhard, 1993] and [Van Keulen, 2012]. Closed-form solutions are generally hard or not possible to obtain. However, we show that for certain infinite-dimensional systems, it is not only possible to give an analytic solution to the infinite-dimensional operator-valued Riccati inequality, but also the resulting controller has a very simple form.

We consider infinite-dimensional systems with bounded input and output operators and where the state evolves on a separable Hilbert space. Moreover, the state operator is closed, densely defined, self-adjoint and strictly negative. Thus, it generates an exponentially stable strongly continuous semigroup. See [Curtain and Zwart, 2012] for further details. We give a simple form for an optimal H_∞ state feedback law applicable to these systems, given that the state and control input are penalized separately. More specifically, the control law is given by the product of the adjoint of the control input operator and the inverse of the state operator. Furthermore, we provide a closed-form expression for the L_2-gain of the closed-loop system's transfer function. The result the analog to the result for finite-dimensional systems derived by the first two authors in [Lidström and Rantzer, 2016]. The heat equation is an example of a system to which the derived control law is applicable. Examples on the heat equation are given in Section 4 to show the simplicity in synthesis and the performance of the control law.

As mentioned earlier, closed-form solutions of the operator-valued Riccati equation are generally hard or not possible to obtain. Therefore, one common approach is to consider the state-space based synthesis problem for a finite-dimensional approximation of the original system. In this procedure one has to ensure that the controller synthesized for the approximated system fulfils the specifications for the original system as well.

This can be problematic but there exists conditions under which this approach works, see [Ito and Morris, 1998]. In general, if it is possible to obtain the optimal H_∞ controller for an infinite-dimensional system, it is itself also infinite-dimensional. It can be problematic to implement such controllers due to memory requirements and long real-time computations. Thus, finite-dimensional approximations are constructed. Of course, also in this scenario, one has to make sure that the approximated controller stabilises the original system and that the performance level is as desired.

We do not cover any approximation procedure for the optimal control law that we derive. See [Morris, 2010] for how it could be performed. However, our result may be used for benchmarking in evaluation of general purpose algorithms such as the different approaches mentioned above. Furthermore, given a system with the properties that we consider, the fact that we have a closed form expression for the optimal control law could speed up the synthesis procedure for large systems, e.g., the procedure in [Kasinathan et al., 2014]. However, an investigation on this matter is outside the scope of the work presented here.

The outline of this paper is as follows. Section 2 gives some mathematical preliminaries and the notation used. The main theorem is stated in Section 3 together with its proof. In Section 4, we illustrate the simplicity in synthesis and the performance of the derived control law by means of an example. Section 4 also includes some further discussion. Concluding remarks are given in Section 5.

2. Mathematical preliminaries

The notations $\mathbb{R}$ and $\mathbb{C}$ stand for the set of real and complex numbers, respectively while the set of nonnegative real numbers is denoted $\mathbb{R}_+$. The notation $\mathrm{Re}(x)$ where $x \in \mathbb{C}$ denotes the real part of x. We will only consider linear operators on separable Hilbert spaces, where we denote the inner product and norm by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively.

The domain of an operator T is denoted by $D(T)$, the adjoint of T is denoted by T^* and the inverse of T, if it exists, is denoted by T^{-1}. An operator T is called self-adjoint if $T^* = T$ and $D(T^*) = D(T)$. The set of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$ is denoted $\mathcal{L}(\mathcal{X}, \mathcal{Y})$, and $\mathcal{L}(\mathcal{X}) = \mathcal{L}(\mathcal{X}, \mathcal{X})$. The norm of an operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is defined as follows

$$\|T\| = \sup_{\substack{x \in D(T) \\ x \neq 0}} \frac{\|Tx\|_{\mathcal{Y}}}{\|x\|_{\mathcal{X}}}.$$

DEFINITION 1—[CURTAIN AND ZWART, 2012, P. 606, DEF. A.3.71]
A self-adjoint operator A on the Hilbert space $\mathcal{Z}$ is nonnegative if $\langle Az, z\rangle \geq 0$ for all $z \in D(A)$, A is positive if $\langle Az, z\rangle > 0$ for all nonzero $z \in D(A)$ and A is strictly positive (coercive) if there exists an $m > 0$ such that

$$\langle Az, z\rangle \geq m\|z\|^2 \text{ for all } z \in D(A).$$

□

We will use the notation $A \succ 0$ for strict positivity of the self-adjoint operator A. We will use the terminology strictly negative denoted $A \prec 0$ when $-A \succ 0$.

REMARK 1
Let $\mathcal{Z}$ be a Hilbert space and consider a self-adjoint strictly negative operator A. It is clear from the definition of strict negativity that A is injective, thus A^{-1} exists. Furthermore, it can be shown that it is bounded, positive and $A^{-1} \in \mathcal{L}(\mathcal{Z})$. See [Curtain and Zwart, 2012, Ex. A.4.2] for details on this. □

DEFINITION 2— [CURTAIN AND ZWART, 2012, P. 15, DEF. 2.1.2]
A strongly continuous semigroup is an operator-valued function $S(t)$ from $\mathbb{R}_+$ to $\mathcal{L}(\mathcal{Z})$ that satisfies the following properties

1. $S(0) = I$,
2. $S(t+\tau) = S(t)S(\tau)$ for $t, \tau \geq 0$,
3. $\lim_{t\to 0, t>0} S(t)z = z$ for all $z \in \mathcal{Z}$.

□

DEFINITION 3— [CURTAIN AND ZWART, 2012, P. 215, DEF. 5.1.1]
A strongly continuous semigroup, $S(t)$, on a Hilbert space Z is *exponentially stable* if there exist constants $M, \alpha > 0$ such that $\|T(t)\| \leq Me^{-\alpha t}$ for all $t \geq 0$. □

DEFINITION 4—[CURTAIN AND ZWART, 2012, P. 20, DEF. 2.1.8]
The generator $A : D(A) \to \mathcal{Z}$ of a strongly continuous semigroup $S(t)$ on a Hilbert space Z is defined by

$$D(A) = \{z \in \mathcal{X} \mid \lim_{\substack{t\to 0\\ t>0}} \frac{S(t)z - z}{t} \text{ exists}\}$$

$$Az = \lim_{\substack{t\to 0\\ t>0}} \frac{S(t)z - z}{t} \quad \text{for all } z \in D(A).$$

□

Remark 2
If A is the generator of a strongly continuous semigroup as in Definition 4, then the domain of A, i.e., $D(A)$, is dense in $\mathcal{Z}$ and A is a closed operator, see [Curtain and Zwart, 2012, p. 21, Th. 2.1.10] □

Lemma 1— [Curtain and Zwart, 2012, p. 33, Cor. 2.2.3]
Sufficient conditions for a closed, densely defined operator on a Hilbert space to be the infinitesimal generator of a strongly continuous semigroup satisfying $\|S(t)\| \leq e^{wt}$ are:

$$\begin{aligned} \mathrm{Re}(\langle Az, z\rangle) &\leq w\|z\|^2 \quad \text{for } z \in D(A), \\ \mathrm{Re}(\langle A^* z, z\rangle) &\leq w\|z\|^2 \quad \text{for } z \in D(A^*). \end{aligned}$$

□

Remark 3
If A is self-adjoint, then the sufficient condition becomes $\langle Az, z\rangle \leq w\|z\|^2$ for $z \in D(A)$. Furthermore, if A is strictly negative by Definition 1, the condition clearly holds for some $w < 0$. Thus, by Definition 3, $S(t)$ is exponentially stable. Hence, A is the generator of an exponentially stable strongly continuous semigroup. □

If A is the generator of a strongly continuous semigroup $S(t)$ on the Hilbert space $\mathcal{Z}$, then for all $z_0 \in D(A)$, the differential equation on $\mathcal{Z}$

$$\frac{dz(t)}{dt} = Az(t), \quad z(0) = z_0,$$

has the unique solution $z(t) = S(t)z_0$. Consider an input $u \in L_2(0, t; \mathcal{U})$, where $\mathcal{U}$ is a Hilbert space and $L_p(\Omega; \mathcal{X})$ is the class of Lebesque measurable $\mathcal{X}$-valued functions f with

$$\int_\Omega |f(t)|^p dt < \infty, \quad p \in [0, \infty].$$

Given u and an operator $B \in \mathcal{L}(\mathcal{U}, \mathcal{Z})$, the differential equation

$$\frac{dz(t)}{dt} = Az(t) + Bu(t), \quad z(0) = z_0,$$

has the following solution at any time t

$$z(t) = S(t)z_0 + \int_0^t S(t-s)Bu(s)ds.$$

If we consider an output signal

$$y(t) = Cz(t) + Du(t)$$

where $C \in \mathcal{L}(\mathcal{Z},\mathcal{Y})$ and $D \in \mathcal{L}(\mathcal{U},\mathcal{Y})$, the output at any time t given an input u is

$$y(t) = CS(t)z_0 + C\int_0^t S(t-\tau)Bu(\tau)d\tau + Du(t).$$

The Laplace transform of $y(t)$ given $z_0 = 0$ yields the transfer function of the system, denoted G, as follows

$$\hat{y}(s) = G(s)\hat{u}(s).$$

In what follows, the considered systems are assumed to be causal.

Definition 5— [Morris, 2010, p. 10, Def. 2.5]
A system is *externally stable* or *L_2-stable* if for every input $u \in L_2(0,\infty;\mathcal{U})$, the output $y \in L_2(0,\infty;\mathcal{Y})$. If a system is externally stable, the maximum ratio between the norm of the input and the norm of the output is called the *L_2-gain*. □

Define

$$H_\infty = \{G : \mathbb{C}_0^+ \to \mathbb{C} \,|\, G \text{ analytic and } \sup_{\operatorname{Re} s>0} |G(s)| < \infty\},$$

where $\mathbb{C}_0^+$ are all complex number with real part larger than zero, with norm

$$\|G\|_\infty = \sup_{\operatorname{Re} s>0} \|G(s)\|.$$

The lemma below is stated for systems with finite-dimensional input and output spaces, e.g., $\mathcal{U}$ and $\mathcal{Y}$ are $\mathbb{R}$, but it generalises to infinite-dimensional ones. The notation $M(H_\infty)$ stands for matrices with entries in H_∞.

Lemma 2— [Morris, 2010, p. 10, Def. 2.6]
A linear system is externally stable if and only if its transfer function matrix $G \in M(H_\infty)$. In this case, $\|G\|_\infty$ is the *L_2-gain* of the system and we say that G is a *stable transfer function*. □

Definition 6—[Morris, 2010, p. 10, Def.2.9]
The pair (A,B) is exponentially stabilizable if there exists a $K \in \mathcal{L}(\mathcal{Z},\mathcal{U})$ such that $A + BK$ generates an exponentially stable strongly continuous semigroup. □

3. Main theorem

Consider a linear time-invariant infinite-dimensional system

$$\frac{dz(t)}{dt} = Az(t) + Bu(t) + Hd(t) \tag{1}$$

where the state $z(t) \in \mathcal{Z}$, where $\mathcal{Z}$ is a separable Hilbert space. The operator A is closed, densely defined, self-adjoint and strictly negative. Then by Lemma 1, a version of the Lumer-Philips Theorem, A is the generator of an exponentially stable strongly continuous semigroup on $\mathcal{Z}$. See Remark 3 for further comments on this statement. The state $z(t)$ is assumed to be measurable with initial condition $z(0) = 0$. Furthermore, the control signal $u(t) \in \mathcal{U}$ and the disturbance $d(t) \in L_2(0, \infty; \mathcal{V})$, where $\mathcal{U}$ and $\mathcal{V}$ are Hilbert spaces, and $B \in \mathcal{L}(\mathcal{U}, \mathcal{Z})$ and $H \in \mathcal{L}(\mathcal{V}, \mathcal{Z})$.

Consider H_∞ state feedback of (1) given unit cost on the state $z(t)$ and control input $u(t)$, separately, i.e., the cost function is given by

$$\zeta(t) = \begin{bmatrix} z(t) \\ u(t) \end{bmatrix}.$$

Given a stabilizing static state feedback controller $K \in \mathcal{L}(\mathcal{Z}, \mathcal{U})$, i.e., $u(t) = Kz(t)$, the closed-loop system from the disturbance $d(t)$ to the controlled output $\zeta(t)$ is given by

$$\begin{aligned} \frac{dz(t)}{dt} &= (A + BK)z(t) + Hd(t) \\ \zeta(t) &= \begin{bmatrix} I \\ K \end{bmatrix} z(t) \end{aligned} \tag{2}$$

where $A+BK$ generates an exponentially stable strongly continuous semigroup. We denote the Laplace transform of the closed-loop system given a controller K by G_K, i.e.,

$$\hat{\zeta}(s) = G_K(s)\hat{d}(s).$$

In the following theorem, we give a closed-form expression for a state feedback controller K that minimizes the L_2-gain of G_K. The optimal control law can be considered to be constant without restriction, see [Morris, 2010] for further details to this statement.

THEOREM 1
Consider the system (1) where A is closed, densely defined, self-adjoint and strictly negative, $B \in \mathcal{L}(\mathcal{U}, \mathcal{Z})$ and $H \in \mathcal{L}(\mathcal{V}, \mathcal{Z})$, where $\mathcal{Z}$, $\mathcal{U}$ and $\mathcal{V}$ are Hilbert spaces. Then, $\|G_K\|_\infty$ is minimized by the state feedback controller $K_{\text{opt}} = B^* A^{-1}$ and the minimal value of the norm is given by $\|H^*(A^2 + BB^*)^{-1}H\|^{\frac{1}{2}}$. □

Proof The proof is divided into two parts. In the first part we show that

$$\|G_{K_{\text{opt}}}\| \leq \|H^*(A^2 + BB^*)^{-1}H\|^{\frac{1}{2}}.$$

In the second part of the proof, we show that no controller can achieve strict inequality. Hence, equality holds. In both parts of the proof, we use the following equivalence given by the strict bounded real lemma in infinite dimensions, see [Curtain, 1993, Theorem 1.1], applied to (2): Given $\gamma > 0$ and a controller $K \in \mathcal{L}(\mathcal{Z}, \mathcal{U})$, the following two statements are equivalent

(i) $A+BK$ generates an exponentially stable strongly continuous semigroup $T(t)$ on the Hilbert space $\mathcal{Z}$ and

$$\|G_K\|_\infty < \gamma.$$

(ii) There exists a self-adjoint, *nonnegative* operator $\tilde{P} \in \mathcal{L}(\mathcal{Z})$ such that

$$(A + BK)^*\tilde{P} + \tilde{P}(A + BK) + I + K^*K + \gamma^{-2}\tilde{P}HH^*\tilde{P} \prec 0. \tag{3}$$

First, as A is closed, densely defined, self-adjoint and strictly negative then by Lemma 1, A is the generator of an exponentially stable strongly continuous semigroup on $\mathcal{Z}$, denoted $S(t)$. Furthermore, we know that (A, B) is exponentially stabilizable as $S(t)$ is exponentially stable. The domain of $A + BK$, i.e., $D(A + BK)$, is equal to the domain of A as $BK \in \mathcal{L}(\mathcal{Z})$.

For the first part of the proof consider (ii) and set $\tilde{P} = -A^{-1}$, $K = K_{\text{opt}} = B^*A^{-1}$ and take any γ with

$$\|H^*(A^2 + BB^*)^{-1}H\|^{\frac{1}{2}} < \gamma.$$

It is possible to set $\tilde{P} = -A^{-1}$ as A is self-adjoint and strictly negative, thus $-A^{-1}$ is self-adjoint, nonnegative and $A^{-1} \in \mathcal{L}(\mathcal{Z})$, see Remark 1. Now, we will prove that $\|G_{K_{\text{opt}}}\|_\infty < \gamma$ by the equivalence between (ii) and (i). First, notice that

$$\tilde{P}(A + BK) = -A^{-1}(A + BB^*A^{-1}) = -I - K^*K.$$

Thus, (3) can be equivalently written as

$$-I - K^*K + \gamma^{-2}A^{-1}HH^*A^{-1} \prec 0. \tag{4}$$

Inequality (4) holds if and only if

$$\begin{bmatrix} I + K^*K & -A^{-1}H \\ -H^*A^{-1} & \gamma^2 I \end{bmatrix} \succ 0 \tag{5}$$

by the Schur Complement Lemma for bounded linear operators, see [Dritschel and Rovnyak, 2010, Def. 3.1 and Lem. A.1]. Again, by the same Lemma, inequality (5) is equivalent to

$$\gamma^2 I - H^*(A^2 + BB^*)H \succ 0. \tag{6}$$

where we have used that

$$\gamma^2 I - H^* A^{-1}(I + K^*K)^{-1}A^{-1}H = \gamma^2 I - H^*(A^2 + BB^*)H.$$

Inequality (6) is true by the definition of γ. Hence, $\|G_{K_{\text{opt}}}\| < \gamma$ by the equivalence between (ii) and (i).

For the second part of the proof, consider again (3). Given a self-adjoint, nonnegative operator $\tilde{P}$ that solves (3), we can construct a self-adjoint, strictly positive operator $P_\epsilon \succ 0$ by $P_\epsilon = \tilde{P} + \epsilon I$, where $\epsilon > 0$ is some small real number. Then, we can define

$$M_\epsilon = (A + BK)^* P_\epsilon + P_\epsilon(A + BK) + I + K^*K + \gamma^{-2} P_\epsilon HH^* P_\epsilon$$

and we know that $M_0 \prec 0$. Furthermore,

$$M_\epsilon = M_0 + \epsilon(A^* + A) + \epsilon(K^*B^* + BK) + I + K^*K + \gamma^{-2}(P_\epsilon HH^* P_\epsilon - P_0 HH^* P_0).$$

The right-hand side of this equality is negative for small ϵ as $2A \prec 0$ and K, B, P and H are bounded. Thus, $M_\epsilon \prec 0$, i.e., the following holds

$$(A + BK)^* P + P(A + BK) + I + K^*K + \gamma^{-2} PHH^* P \prec 0$$

for some $P \succ 0$. This P is invertible and we can rewrite the inequality further as

$$P^{-1}(A + BK)^* + (A + BK)P^{-1} + P^{-2} + P^{-1}K^*KP^{-1} + \gamma^{-2}HH^* \prec 0.$$

We perform the change of variables

$$(P^{-1}, KP^{-1}) \to (X, Y),$$

thus $X \in \mathcal{L}(\mathcal{Z})$ and $Y \in \mathcal{L}(\mathcal{U}, \mathcal{Z})$, and sum of squares to write the inequality as follows

$$(X + A)^2 + (Y^* + B)(Y^* + B)^* - A^2 - BB^* + \gamma^{-2}HH^* \prec 0.$$

The first two terms of the operator expression are always non-negative and thus no controller can satisfy a bound γ smaller than $\|H^*(A^2 + BB^*)^{-1}H\|^{\frac{1}{2}}$. Hence the controller constructed in the first part is optimal and the proof is complete. □

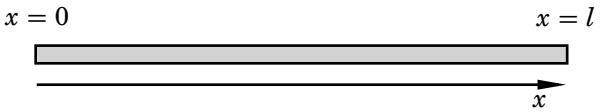

Figure 1. Rod of length l with one-dimensional spatial coordinate x.

4. Control of the heat equation

In this section, we illustrate the simplicity in synthesis of the control law given by Theorem 1. The example concerns control of the heat equation, see (7) below, that describes the distribution of heat, or variation in temperature, in a region over time. The equation also describes other types of diffusion, such as chemical diffusion.

Consider the following partial differential equation that models heat propagation in a rod of length l

$$\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t) \quad 0 < x < l,\, t \geq 0. \tag{7}$$

The temperature at time t at position x is $z(x,t) \in \mathcal{Z} = L_2(0,l)$. See Figure 1 for a depiction of the rod.

To fully determine the temperature of the rod, the initial temperature profile as well as the boundary conditions have to be specified. As we consider H_∞ control, the initial temperature is set to zero. We will consider Dirichlet boundary conditions, i.e.,

$$z(0,t) = 0, \quad z(l,t) = 0.$$

Define the operator A as

$$A = \frac{d^2 z}{dx^2}$$

with domain

$$D(A) = \{z \in L_2(0,l)\,|z,\ \frac{dz}{dt} \text{ locally absolutely continuous,}$$
$$\frac{d^2 z}{dx^2} \in L_2(0,l) \text{ with } z(0) = 0,\, z(l) = 0\}.$$

This operator fulfils the requirements for Theorem 1, i.e., it is closed, densely defined, self-adjoint and strictly negative. For a proof of this see [Tucsnak and Weiss, 2009, pp. 92-94]. Thus, by Lemma 1, A generates an

exponentially stable strongly continuous semigroup $S(t)$ on $L_2(0,l)$, the state z evolves on the space $L_2(0,l)$ and we can write (7) as

$$\dot{z}(t) = Az(t), \quad z(x,0) = 0.$$

Now, suppose the temperature is controlled by an input $u(t)$ and affected by a disturbance $d(t)$ as follows

$$\dot{z}(t) = Az(t) + Bu(t) + Hd(t), \quad z(x,0) = 0,$$

where $B, H \in \mathcal{L}(\mathbb{R}, L_2(0,l))$, $u \in L_2(0,\infty;\mathbb{R})$ and the disturbance $d \in L_2(0,\infty;\mathbb{R})$. Given the properties stated for the system, Theorem 1 is applicable. We will now, given some explicit examples of operators B and H, write down the closed-form expression for the control law given by Theorem 1.

As can be seen in Theorem 1, the structure of the optimal control law, i.e., $K_{\text{opt}} = B^*A^{-1}$ is not dependent upon the operator H. We will only consider

$$(Hd)(x) = d(t) \text{ for all } 0 < x < l.$$

In other words, the disturbance is uniformly distributed along the entire rod. We will treat operators B defined by

$$Bu = \chi_{[0,\alpha]}(x)u(t) \tag{8}$$

where $0 < \alpha \leq l$ and

$$\chi_{[0,\alpha]}(x) = \begin{cases} 1 & \text{if } 0 < x < \alpha \\ 0 & \text{otherwise.} \end{cases}$$

Thus, for $\alpha = l$ the control input is uniformly distributed along the entire rod while for instance for $\alpha = l/2$ it is only distributed in $0 < x < l/2$ while it is zero for the remaining part of the rod. The adjoint of operator B defined in (8) is

$$B^*y(x,t) = \int_0^\alpha y(x,t)dx \text{ for } y \in L_2(0,l).$$

Consider the following equality, as a step towards explicitly stating the optimal control law $u(t) = K_{\text{opt}}z(x,t) = B^*A^{-1}z(x,t)$,

$$z(x,t) = Ay(x,t), \quad y \in D(A).$$

The function $y(x,t)$ can be written as

$$y(x,t) = \int_0^l G(x,s)z(s,t)ds$$

where

$$G(x,s) = \begin{cases} \frac{(s-l)}{l}x & \text{if } 0 < x < s \\ \frac{s}{l}(x-l) & \text{if } s < x < l \end{cases}$$

is the Green's function of A. Note that $G(x,s)$ is piece-wise linear in x with $G(0,s) = G(l,s) = 0$. Now, if $\alpha = 1$ in (8), then

$$\begin{aligned} u(t) = B^* A^{-1} z(x,t) &= \int_0^l \int_0^l G(x,s) z(s,t)\, ds\, dx \\ &= \int_0^l \underbrace{\left[\int_0^l G(x,s)\, dx \right]}_{:=f(s)} z(s,t)\, ds \end{aligned}$$

where

$$f(s) = \frac{s(s-l)}{2}.$$

The control input is thus a weighted integral of the deviation in temperature along the spatial coordinate. The quadratic weight $f(s)$ determines the scalar signal for controlling the temperature profile, as a compromise between the deviation in temperature from zero and the cost for changing the temperature. The general form of the control signal, i.e., without any specific value on α, is similarly given by

$$\begin{aligned} u(t) &= \int_0^l \left(\int_0^\alpha G(x,s)\, dx \right) z(s,t)\, ds \\ &= \int_0^\alpha f_1(s) z(s,t)\, ds + \int_\alpha^l f_2(s) z(s,t)\, ds \end{aligned}$$

where

$$f_1(s) = \frac{s(s-l)}{2} + \frac{s(l-\alpha)^2}{2l} \text{ and } f_2(s) = \frac{\alpha^2}{2l}(s-l).$$

The weighting function is altered dependent on if the spatial coordinate is less than or larger than α, to account for the asymmetry in B.

Given a constant disturbance $d(t) = 1$ for $t \geq 0$, the state $z(x,t)$ is determined numerically in MATLAB, see [MATLAB, 2012], by the finite element method for 200 time steps with interval length 0.01 and spatial segments of length 0.1, with $l = 3$. The integrals in the expression of the control law are approximated numerically by the trapezoidal rule.

In Figure 2a, the time trajectory of the temperature at the midpoint, i.e., $x = l/2$, is shown for the control input operators B defined by (8)

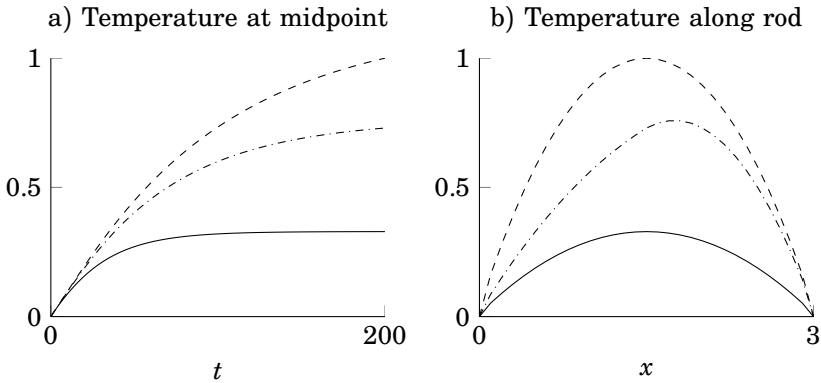

Figure 2. Response from unit disturbance for system with $B = 0$ given by the dashed lines and with B defined in (8) with $\alpha = l$ given by the solid lines and with $\alpha = l/2$ given by the dashed dotted lines. a) Temperature at $x = l/2$ over time, b) temperature along the rod at $t = 200$.

given $\alpha = l$ and $\alpha = l/2$ as well as $B := 0$. Clearly, when $\alpha = l$ we get the best disturbance attenuation as shown by the solid line. When $\alpha = l/2$, the controller is not able to attenuate the disturbance as effectively and of course with $B = 0$ the system evolves only according to the heat equation with a disturbance. In Figure 2b we show the temperature distribution of the rod at the final time $t = 200$. Here one can see that the temperature distribution given with $\alpha = l/2$ is not symmetric along x. This is due to that the control input operator B in this case is asymmetric in x. The temperature distributions are normalized such that $z(200, l/2)$ given $\alpha = 0$ is equal to 1.

5. Conclusions

We have given a closed form expression for an optimal H_∞ state feedback controller applicable to systems with bounded input and output operators and closed, densely defined, self-adjoint and strictly negative state operator. We have shown by an example, the simplicity in synthesis of the control law and how it performs. The control law may be used in benchmarking of general purpose algorithms for H_∞-controller synthesis. Future work include comparison of a finite-dimensional approximation of the optimal controller to a controller derived by the early lumping approximation scheme for H_∞-control. Possible benefits of having a closed form expression for an optimal controller in the controller synthesis for large scale systems, with the given properties, will also be investigated.

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