The p-contest with $p \neq 1$

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Abstract

We study the asymptotic properties of a Markov system of $N \geq 3$ points in $[0, 1]$ in which, at each step in discrete time, the point farthest from the current centre of mass times $p > 0$ is removed and replaced by an independent ζ-distributed point; the problem was posed in [4] when $\zeta \sim U[0, 1]$. In the present paper we obtain various criteria for the convergences of the system, both for $p < 1$ and $p > 1$.

In particular, when $p < 1$ and $\zeta \sim U[0, 1]$, we show that the limiting configuration converges to zero. When $p > 1$ (except a finite set of values of p depending on N), we show that the configuration must converge to either zero or one, and we present an example where both outcomes are possible. Finally, when $p > 1$, $N = 3$ and $\zeta \sim U[0, 1]$, we prove that the configuration can only converge to one a.s.

Our paper extends the results of [3, 5] where it was assumed that $p = 1$. It turns out that one can no longer use the Lyapunov function based just on the radius of gyration; when $0 < p < 1$ one has to find a much finer tuned function which turns out to be a supermartingale; the proof of this fact constitutes a large portion of the present paper.

Keywords: Keynesian beauty contest; Jante's law, rank-driven process.

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1 Introduction

This paper extends the results of [3] and [5] on the so-called *Keynesian beauty contest*, or, as it was called in [5], *Jante's law process.* Following [3], we remind that in the Keynesian beauty contest, we have N players guessing a number, and the person who guesses closest to the mean of all the N guesses wins; see [6, Ch. 12, §V]. The formal version, suggested by Moulin [7, p. 72], assumes that this game is played by choosing numbers on the interval $[0, 1]$, the "p-beauty contest", in which the target is the mean value, multiplied by a constant $p > 0$. For the applications of the p-contest in the game theory, we refer the reader to e.g. [1]; see also [2] and [3] and references therein for further applications and other relevant papers.

The version of the p-contest with $p \equiv 1$ was studied in [3, 5]. In [3] it was shown that in the model where at each unit of time the point furthermost from the center of mass is replaced by a point chosen uniformly on $[0, 1]$, then eventually all (but one) points converge almost surely to some random limit with the support, whose closure is the whole interval; many of the results were extended for the version of the model on $\mathbb{R}^d$, $d \geq 2$. The results of [3] were further generalized in [5], by removing the assumption that a new point is chosen uniformly on $[0, 1]$, as well as by allowing to remove more than one point at once, these points are being chosen in such a way that the moment of inertia of the resulting configuration is minimized. However, the case $p \neq 1$ was not addressed in either of these two papers.

Let us now formally define the model. The notations will be similar to those in [3, 5]. Let $\mathcal{X} = \{x_1, x_2, \dots, x_N\}$ be a set of N points $x_i \in \mathbb{R}$; let . Let $(x_{(1)}, x_{(2)}, \dots, x_{(N)})$ be the set $\mathcal{X}$ put in the non-decreasing order, $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(N)}$. As in [3, 5] let us define the barycentre as

$$\mu_N(x_1, \dots, x_N) := N^{-1} \sum_{i=1}^{N} x_i \quad \text{(the barycentre of } \mathcal{X}\text{)}.$$

Define also $p-$*centre of mass* as $p\mu_N(x_1, \dots, x_N)$ for some fixed $p > 0$.

The point, farthermost from the $p-$centre of mass, is called the *extreme* point of $\mathcal{X}$, and it can be either $x_{(1)}$ or $x_{(N)}$, and *the core* of $\mathcal{X}$, denoted by $\mathcal{X}'$, is constructed from $\mathcal{X}$ by removing (one of) the extreme point(s). In case of a tie (between the left-most and the right-most point) we chose either of them with equal probability (same as in [3, 5]).

Our process runs as follows. Let $\mathcal{X}(t) = \{X_1(t), \dots, X_N(t)\}$ be a set of distinct points in $\mathbb{R}$ at time $t = 0, 1, 2, \dots$. Given $\mathcal{X}(t)$, let $\mathcal{X}'(t)$ be the core of $\mathcal{X}(t)$ and replace $\mathcal{X}(t) \setminus \mathcal{X}'(t)$ by a ζ-distributed random variables so that

$$\mathcal{X}(t+1) = \mathcal{X}'(t) \cup \{\zeta_{t+1}\},$$

where ζ_t, $t = 1, 2, \dots$, are i.i.d. random variables with a common distribution ζ.

Finally, to finish specification of the process, we allow the initial configuration $\mathcal{X}(0)$ be arbitrary or random, with the only requirement that all the points of $\mathcal{X}(0)$ must lie in the support of ζ.

Throughout the paper we will use the notation $A \underset{\text{a.s.}}{\Longrightarrow} B$ for two events A and B, whenever $\mathbb{P}(A \cap B^c) = 0$, that is, when $A \subseteq B$ up to a set of measure 0.

2 The case $p < 1$

Throughout this Section we assume that $0 < p < 1$ and that $\operatorname{supp}\zeta = [0,1]$. Because of the scaling invariance, our results may be trivially extended to the case when $\operatorname{supp}\zeta = [0, A]$, $A \in (0,\infty)$; some of them are even true when $A = \infty$; however, to simplify the presentation from now on we will deal only with the case $A = 1$. Observe that throughout the paper we require that ζ has a *full support* on $[0,1]$, that is, $\mathbb{P}(\zeta \in (a,b)) > 0$ for all a, b such that $0 \le a < b \le 1$.

First, we present a general statement; more precise results will follow in case where $\zeta \sim U[0,1]$.

Proposition 1. *Let $\mathcal{X}'(t) = (x_{(1)}(t), \dots, x_{(N-1)}(t))$, where $x_{(i)}(t)$ are in the increasing order w.r.t. $i = 1, 2, \dots, N-1$. Then*

(a) $\liminf_{t\to\infty} x_{(N-1)}(t) = 0$;

(b) $\mathbb{P}\left(\exists \lim_{t\to\infty} \mathcal{X}'(t) \in (0,1]\right) = 0$;

(c) if $p < \frac{1}{2} + \frac{1}{2(N-1)}$ *then* $\mathbb{P}\left(\lim_{t\to\infty} \mathcal{X}'(t) = 0\right) = 1$.

(d) if $p \ge \frac{1}{2} + \frac{1}{2(N-1)}$ *then* $\{x_{(1)}(t) \to 0\} \underset{a.s.}{\Longrightarrow} \{\mathcal{X}'(t) \to 0\}$

Proof. (a) Since ζ has full support on $[0,1]$ it follows that (see [5], Proposition 1) there exists a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\mathbb{P}(\zeta \in (a,b)) \ge f(b-a) > 0 \qquad \text{for all } 0 \le a < b \le 1, \tag{2.1}$$

Also let $\mu = \mu(\mathcal{X}(t))$ throughout the proof.

Fix a small positive ε such that $p + 2\varepsilon < 1$. Suppose that for some t we have $x_{(N-1)}(t) \le b \le 1$. We will show that with a positive probability $x_{(N-1)}(t) \le b(1-\varepsilon)$. Assume that we have $\zeta_{t+1}, \dots, \zeta_{t+N-1} \in (pb, (p+\varepsilon)b) \subset (pb, b)$; this happens with probability no less than $[f(p\varepsilon b)]^{N-1}$. We claim that by the time $t+N$ we have $x_{(N-1)}(t+N-1) < (p+\varepsilon)b$. Indeed, $p\mu \le pb$ always lies to the left of the newly sampled points, therefore either there are no more points to the right of $(p+\varepsilon)b$ at some time $s \in [t, t+N-1]$ (which implies that there will be no points there at time $t+N$ due to the sampling range of the new points), or one of the older points, i.e. present at time t, gets removed (it can be possibly one to the left of pb). Since we have eventually to replace all the $N-1$ old points, this will imply the claim.

Fix a $\delta > 0$ and find M so large that $(1-\varepsilon)^M < \delta$. Let $B(s) = \{x_{(N-1)}(s) < \delta\}$. By iterating the above argument, we get that $\mathbb{P}(B(t+NM)|\mathcal{F}_t) \geq [f(p\varepsilon b)]^{NM}$, since at time t we can set $b = 1$. Therefore, $\sum_t \mathbb{P}(B(t+NM)|\mathcal{F}_t) = \infty$ and by Levy's extension to the Borel-Cantelly lemma infinitely many $B(s)$ occur. Since $\delta > 0$ is arbitrary, we get $\liminf_{t\to\infty x_{(N-1)}}(t) = 0$.

(b) If the core of the process converges it implies that the core centre of mass must converge so it suffices to show that core centre of mass cannot converge to any point besides 0. Given $q \in (0,1]$ we will show that $\mathbb{P}(\mu'(t) \in B_\epsilon(q), \quad t \geq T) = 0$ for some $\epsilon > 0$ and every $T \geq 0$. Let $A^T_{\epsilon,q} = \{\mu'(t) \in B_\epsilon(q), \quad t \geq T\}$ and

$$\tau_0 = \inf\{t \geq T : \mathcal{X}'(t) \in B_\epsilon(q)\},$$

$$\tau_k = \inf\{t > \tau_{k-1} : \mathcal{X}'(t) \in B_\epsilon(q)\}, \quad k \geq 1.$$

Notice that $A^T_{\epsilon,q} \underset{\text{a.s.}}{\Longrightarrow} \bigcap_{k=0}^\infty \{\tau_k < \infty\}$ and that,

$$p\mu'(\tau_k) \leq px_{(N-1)}(\tau_k) < p(q+\epsilon).$$

If $\Delta := q - \epsilon - p(q+\epsilon)$ and we let $\epsilon < q\frac{p+1}{N+p}$ (it is easily seen that this will imply that $\Delta > 0$) then if $\zeta_{\tau_k+1} \in \left(q-\epsilon-\frac{3}{2}\Delta, q-\epsilon-\Delta\right)$ we have that $x_{(N-1)}(\tau_k)$ will be rejected from the core at time $\tau_k + 1$ and furthermore

$$\begin{aligned} p\mu'(\tau_k+1) &< p\mu'(\tau_k) + \frac{\zeta_{\tau_k+1} - x_{(N-1)}(\tau_k)}{N-1} < q + \epsilon + \frac{q-\epsilon-\Delta-(q-\epsilon)}{N-1} \\ &= q + \epsilon - \frac{\Delta}{N-1} < q - \epsilon \end{aligned}$$

Consider $C_k = \{\zeta_{\tau_k+1} \in \left(q-\epsilon-\frac{3}{2}\Delta, q-\epsilon-\Delta\right)\}$, then $\mathbb{P}(C_k|\mathcal{F}_{\tau_k}) \geq f(\Delta/2)$ a.s. on $\{\tau_k < \infty\}$. Obviously $A^T_{\epsilon,q} \subseteq C^c_k$ for any $k \geq 0$ and $\{C_k\}_k$ (and therefore $\{C^c_k\}_k$) are all independent and this leads to,

$$\mathbb{P}\left(A^T_{\epsilon,q}\right) \leq \mathbb{P}\left(\bigcap_{k=0}^\infty C^c_k\right) = \prod_{k=0}^\infty \mathbb{P}(C^c_k) \leq \prod_{k=0}^\infty (1 - f(\Delta/2)) = 0.$$

Now we define $A_{\epsilon,q} = \bigcup_{T=1}^\infty A^T_{\epsilon,q}$ and notice that $\{\exists \lim_t \mu'(t) \in B_\epsilon(q)\} \subseteq A_{\epsilon,q}$. Since $A^T_{\epsilon,q} \subseteq A^{T+1}_{\epsilon,q}$ it follows from continuity of probability that $\mathbb{P}(A_{\epsilon,q}) = 0$ and therefore

$$\mathbb{P}\left(\exists \lim_t \mu'(t) \in (0,1]\right) \leq \sum_{q\in\mathbb{Q}\cap(0,1]} \mathbb{P}(A_{\epsilon,q}) = 0.$$

(c) First, we will show that it is the right-most point of the configuration which should be always removed; note that it suffices to check this only when $x_{(N)} > 0$. Indeed, by the assumption on p we have

$$\mu \le \frac{(N-1)x_{(1)} + (N-1)x_{(N)}}{N} = \frac{2p(N-1)}{N} \cdot \frac{x_{(1)} + x_{(N)}}{2p} < \frac{x_{(1)} + x_{(N)}}{2p}$$

implying

$$x_{(N)} - p\mu > p\mu - x_{(1)} \iff x_{(N)} - p\mu > |p\mu - x_{(1)}|$$

Therefore, $x_{(N)}$ is the furthermost point from the $p-$centre of mass. The result now easily follows from part (a).

(d) See Remark 6 of the alternative proof of Theorem 2 in the Appendix. □

Now comes the main result of this Section.

Theorem 1. *Suppose that $\zeta \sim U[0,1]$. Then $\mathcal{X}'(t) \to 0$ a.s.*

Proof. Lemma 1 (b) implies that we now only need to consider the case $p \ge \frac{N}{2(N-1)}$.

We now introduce a modification of this process on $[0, +\infty)$ which we will call the *borderless p-contest*; it is essentially the same process as the one in Section 3.4 of [3]. In order to do this, we need the following

Lemma 1. *Suppose that $x_1, \dots, x_{N-1} \ge 0$. Then there exists an $R = R(x_{(N-1)}) \ge 0$ such that x is the furthermost point from $p\mu = \frac{p}{N}(x_1 + \dots + x_{N-1} + x)$ whenever $x > R$.*

Proof of Lemma 1. Set $R = 6x_{(N-1)}$. Then $x > x_{(1)}$ is further from the centre of mass than $x_{(1)}$ iff

$$x - p\mu > |p\mu - x_{(1)}| \iff x - p\mu > p\mu - x_{(1)} \iff x\left(1 - \frac{2p}{N}\right) > 2p\frac{x_1 + \dots + x_{N-1}}{N} - x_{(1)}$$

This is true, due to the fact that $x > R$ and

$$x\left(1 - \frac{2p}{N}\right) > \frac{x}{3} > 2x_{(N-1)} > 2px_{(N-1)} > 2p\frac{x_1 + \dots + x_{N-1}}{N}$$

since $p < 1$ and $N \ge 3$. □

The borderless process is constructed as follows. Our core configuration starts as before in $[0,1]$, and we use the same rejection/acceptance criteria for new points. However, we will now allow points to be generated to the right of 1 as well. Let $R_t = R(x_{(N-1)}(t))$ where R is taken from Lemma 1. Then a new point is sampled uniformly and independently of the past on the interval $[0, R_t]$; formally, it is given by $R_t U_t$ where U_t are i.i.d. uniform $[0,1]$ random variables independent of anything. Observe that if we consider the embedded process only at the times when the core configuration changes, then the exact form of the function $R(\cdot)$ is irrelevant, due to the fact that the uniform distribution conditioned on a subinterval is also uniform on that subinterval.

Next, for $x = \{x_1, \ldots, x_{N-1}\}$ define the function

$$h(x) = F(x) + k\mu(x)^2, \tag{2.2}$$

where

$$F(x) = \sum_{i=1}^{N-1} (x_i - \mu(x))^2, \quad \mu(x) = \frac{1}{N-1} \sum_{i=1}^{N-1} x_i, \quad k = \frac{(N-1)^2(1-p)}{N-2}.$$

We continue with the following

Lemma 2. *For the borderless p-contest the sequence of random variables h_t where $h_t = h(\mathcal{X}'(t)) \geq 0$ has a finite limit a.s. as $t \to \infty$.*

Remark 1. *Note that the function $F(\cdot)$ defined above is a Lyapunov function for the process in [3]; this is no longer the case as long as $p \neq 1$; that's why we have to use a carefully chosen "correction" factor which involves the barycentre of the configuration.*

Proof of Lemma 2. We will show that h_t is a positive supermartingale, so the statement will immediately follow from the supermartingale convergence theorem.

Assume that $x_{(N-1)}(t) > 0$ (otherwise the process has stopped and the result is trivial). The inequality we are after is $\mathbb{E}[h_{t+1} - h_t | \mathcal{F}_t] \leq 0$, which is equivalent to

$$\mathbb{E}[h_{t+1} - h_t | x(t) = y] \leq 0$$

where $y = \{y_1, \ldots, y_{N-1}\}$. Note that the function $h(x)$ is homogeneous of degree 2 in x, therefore w.l.o.g. we can assume that $\max y \equiv 1$.

For simplicity let $M = N - 1 \geq 2$, and let

$$z = 6U_t \text{ (the newly sampled point)}, \quad a = \min y < 1 \text{ (the leftmost point)}$$

Note also that

$$p \geq \frac{N}{2(N-1)} = \frac{M+1}{2M} = \frac{1}{2} + \frac{1}{2M}. \tag{2.3}$$

Define

$$\begin{aligned} F_{old} &= F(y), & F_{new} &= F\left((y \cup \{z\})'\right) \\ \mu'_{old} &= \mu(y), & \mu'_{new} &= \mu\left((y \cup \{z\})'\right), \\ h_{old} &= F_{old} + k\mu'^2_{old}, & h_{new} &= F_{new} + k\mu'^2_{new} \end{aligned}$$

Thus we need to establish

$$\mathbb{E}[h_{new} - h_{old} | \mathcal{F}_t] \leq 0. \tag{2.4}$$

First of all, observe that if $\tilde{y} = (y \setminus \{y_i\}) \cup \{z\}$, that is, $\tilde{y}$ is obtained from y by replacing y_i with y_0, then

$$\begin{aligned} F(\tilde{y}) - F(y) &= \frac{z - y_i}{M}\left[(M-1)z + (M+1)y_i - 2M\mu(y)\right] \\ \mu(\tilde{y})^2 - \mu(y)^2 &= \frac{z - y_i}{M^2}\left[z - y_i + 2M\mu(y)\right] \end{aligned}$$

In particular, if we replace point a by the new point z, then

$$\Delta_a(z) := h_{new} - h_{old} = \frac{z-a}{M}\left[(M-1)z + (M+1)a - 2M\mu(y) + \frac{k}{M}(z - a + 2M\mu(y))\right]$$

and if we replace point 1, then

$$\Delta_1(z) := h_{new} - h_{old} = \frac{z-1}{M}\left[(M-1)z + (M+1) - 2M\mu(y) + \frac{k}{M}(z - 1 + 2M\mu(y))\right]$$

Note that both Δ_a and Δ_z depend only on four variables (a, z, μ, M) but not the whole configuration. Let us also define

$$m(z) = p \cdot \frac{y_1 + \cdots + y_M + z}{M+1} = p \cdot \frac{M\mu + z}{M+1},$$

the $p-$centre of mass of the old core and the newly sampled point.

There are three different cases that can occur: either (a) the point a is removed, (b) 1, the rightmost point of the previous core, is removed, or (c) the newly sampled point z is removed. In the third case the core remains unchanged, and the change in the value of the function h is trivially zero. The point a can only be removed if $z > a$; the point 1 can only be removed if $z < 1$; the point z can be possibly removed only if $z \in (0, a)$ or $z \in (1, \infty)$. Let us compute the critical values for z, for which there is a tie between the furthermost points.

Which point to remove?

$*$ Suppose $\boxed{z < a}$. Then there is a tie between z and 1 iff $m(z) = \frac{z+1}{2}$, that is if

$$z = t_{z1} := \frac{M(2p\mu - 1) - 1}{M + 1 - 2p} \in \begin{cases} (-\infty, 0) & \text{if } p < p_1 := \frac{M+1}{2M\mu} \\ (0, a) & \text{if } p_1 < p < p_2 := \frac{(M+1)(a+1)}{2M\mu+2a} \\ (a, +\infty) & \text{if } p > p_2. \end{cases}$$

Thus, we have:

- when $p < p_1$, point 1 is removed;
- when $p_1 < p < p_2$, if $z < t_{z1}$ then z is removed; if $z > t_{z1}$ point 1 is removed;
- when $p > p_2$, point z is removed.

$*$ Suppose $\boxed{a < z < 1}$. There is a tie between a and 1 iff $m(z) = \frac{a+1}{2}$, that is if

$$z = t_{a1} := \frac{(M+1)(a+1) - 2M\mu p}{2p} \in \begin{cases} (1, +\infty) & \text{if } p < p_3 := \frac{(M+1)(a+1)}{2M\mu+2}, \\ (a, 1) & \text{if } p_3 < p < p_2, \\ (-\infty, a) & \text{if } p > p_2. \end{cases}$$

Thus, we have:

- when $p < p_3$, point 1 is removed;
- when $p_3 < p < p_2$, if $z < t_{a1}$ then 1 is removed; if $z > t_{a1}$ then point a is removed;
- when $p > p_2$, point a is removed.

$*$ Suppose $\boxed{z > 1}$. There is a tie between z and a iff $m(z) = \frac{z+a}{2}$, that is if

$$z = t_{za} := \frac{2M\mu p - (M+1)a}{M + 1 - 2p} \in \begin{cases} (-\infty, 1) & \text{if } p < p_3, \\ (1, +\infty) & \text{if } p > p_3. \end{cases}$$

Thus, we have:

- when $p < p_3$, point z is removed;
- when $p > p_3$, if $z < t_{za}$ then a is removed; if $z > t_{za}$ then point z is removed.

We always have $p_1 < p_2$, $p_3 < p_2$ since

$$p_2 - p_1 = \frac{a(M+1)(M\mu - 1)}{2M\mu(M\mu + a)} = \frac{a(M+1)(a + (M-2)f)}{2M\mu(M\mu + a)} > 0,$$

$$p_2 - p_3 = \frac{(1-a)^2(M+1)}{2(M\mu+1)(M\mu+a)} > 0,$$

while

$$p_1 < p_3 \iff Ma\mu > 1 \iff f > \frac{1 - a - a^2(M-1)}{a(M-2)(1-a)} \text{ (when } M > 2)$$

The final observation is that $t_{za} < 6$, to there is indeed no need to sample the new point outside of the range $(0, 6)$; this holds since $M \geq 2$ and

$$6 - t_{za} = \frac{-2p(M\mu + 6) + Ma + 6M + a + 6}{M + 1 - 2p} > \frac{-2M\mu + Ma + 6M + a - 6}{M + 1 - 2p}$$

$$> \frac{-2M\mu + 6M - 6}{M + 1 - 2p} = \frac{2M(1-\mu) + 4M - 6}{M + 1 - 2p} > \frac{2}{M + 1 - 2p} > 0.$$

The five cases:

- $p < \min\{p_1, p_3\}$:
  - when $z < 1$, point 1 is removed
  - when $z > 1$, point z is removed
- $p > p_2$:
  - when $z < a$ or $z > t_{za} \in (1, \infty)$ point z is removed
  - when $a < z < t_{za}$, point a is removed
- $\max\{p_1, p_3\} < p < p_2$
  - when $z < t_{z1} \in (0, a)$ or $t > t_{za} \in (1, +\infty)$, point z is removed
  - when $t_{z1} < z < t_{a1} \in (a, 1)$, point 1 is removed
  - when $t_{a1} < z < t_{za}$, point a is removed
- $p_1 < p < p_3$ $(< p_2)$:
  - when $z < t_{z1} \in (0, a)$ or $z > 1$, point z is removed
  - when $t_{z1} < z < 1$, point 1 is removed

- $p_3 < p < p_1$ $(< p_2)$:
  - when $z < t_{a1} \in (a, 1)$, point 1 is removed
  - when $t_{a1} < z < t_{za} \in (1, +\infty)$, point a is removed
  - when $z > t_{za}$, point z is removed

Let

$$X_1 = p - p_1 = \frac{M(2\mu p - 1) - 1}{2M\mu},$$
$$X_2 = p - p_2 = \frac{2ap - a - 1 + (2\mu p - a - 1)M}{2(M\mu + a)},$$
$$X_3 = p - p_3 = \frac{2p - a - 1 + (2\mu p - a - 1)M}{2(M\mu + 1)}.$$

Define

$$\tilde{\mathbf{I}}_1 = \mathbb{E}(h_{t+1} - h_t | x(t) = y) \cdot 1_{X_1<0} \cdot 1_{X_3<0}$$
$$\tilde{\mathbf{I}}_2 = \mathbb{E}(h_{t+1} - h_t | x(t) = y) \cdot 1_{X_2>0}$$
$$\tilde{\mathbf{I}}_3 = \mathbb{E}(h_{t+1} - h_t | x(t) = y) \cdot 1_{X_2<0} \cdot 1_{X_1>0} \cdot 1_{X_3>0}$$
$$\tilde{\mathbf{I}}_4 = \mathbb{E}(h_{t+1} - h_t | x(t) = y) \cdot 1_{X_1>0} \cdot 1_{X_3<0}$$
$$\tilde{\mathbf{I}}_5 = \mathbb{E}(h_{t+1} - h_t | x(t) = y) \cdot 1_{X_1<0} \cdot 1_{X_3>0}$$

Because of the comment on the restriction of the uniform distribution on a subinterval, we have $\tilde{\mathbf{I}}_j = c_j \mathbf{I}_j$, $j = 1, 2, 3, 4, 5$, for some positive constants c_j where

$$\mathbf{I}_1 = \int_0^1 \Delta_1 dz \cdot 1_{X_1<0} \cdot 1_{X_3<0} =: \mathbf{A}_1 \cdot 1_{X_1<0} \cdot 1_{X_3<0}$$
$$\mathbf{I}_2 = \int_a^{t_{za}} \Delta_a dz \cdot 1_{X_2>0} =: \mathbf{A}_2 \cdot 1_{X_2>0}$$
$$\mathbf{I}_3 = \left[\int_{t_{z1}}^{t_{a1}} \Delta_1 dz + \int_{t_{a1}}^{t_{za}} \Delta_a dz\right] \cdot 1_{X_2<0} \cdot 1_{X_1>0} \cdot 1_{X_3>0} =: \mathbf{A}_3 \cdot 1_{X_2<0} \cdot 1_{X_1>0} \cdot 1_{X_3>0}$$
$$\mathbf{I}_4 = \int_{t_{z1}}^1 \Delta_1 dz \cdot 1_{X_1>0} \cdot 1_{X_3<0} =: \mathbf{A}_4 \cdot 1_{X_1>0} \cdot 1_{X_3<0}$$
$$\mathbf{I}_5 = \left[\int_0^{t_{a1}} \Delta_1 dz + \int_{t_{a1}}^{t_{za}} \Delta_a dz\right] \cdot 1_{X_1<0} \cdot 1_{X_3>0} =: \mathbf{A}_5 \cdot 1_{X_1<0} \cdot 1_{X_3>0}$$

where $\mathbf{A}_j$ are the respective integrals without multiplying them by the indicator functions. Thus to establish (2.4), it suffices to show that $\mathbf{I}_j \leq 0$ for each $j = 1, 2, 3, 4, 5$. This is done by very laborious calculations, which can be found in the Appendix. □

Let $(x_1^*(t),\dots,x_N^*(t))$ be a borderless p-contest and

$$A_L=\left\{\limsup_{t\to\infty} x^*_{(N-1)}(t)<L\right\},\quad L=1,2,\dots.$$

On A_L the path of the borderless p-contest can be coupled with a usual p-contest defined on $[0,L]$, so by Lemma 1 we have $\liminf_{t\to\infty} x^*_{(N-1)}(t)=0$ a.s. on A_L. At the same time it follows from Lemma 1 that $\limsup_{t\to\infty} x^*_{(N-1)}(t)<\infty$ a.s. and hence $\mathbb{P}(\cup_L A_L)=1$, consequently $\mathbb{P}\left(\liminf_{t\to\infty} x^*_{(N-1)}(t)=0\right)=1$.

We now return to our original p-contest process $\{x_1(t),\dots,x_{N-1}(t)\}$ and let

$$\begin{aligned}\tau_0^{(L)}&=\inf\{t>0:\ x_{N-1}(t)<1/L\},\quad \text{and}\\ \tau_s^{(L)}&=\inf\left\{t>\tau_{s-1}^{(L)}:\ \mathcal{X}'(t)\neq\mathcal{X}'\left(\tau_{s-1}^{(L)}\right)\right\}\quad s=1,2,\dots\end{aligned}$$

which are all a.s. finite for every $L\ge 1$ by Lemma 1. Let $W(s)=\{w_1(s),\dots,w_{N-1}(s)\}$ be a borderless p-contest with $W(0)=\mathcal{X}'(\tau_0^{(L)})$ and letting $U_s^*=U_{\tau_s^{(L)}}$ be the i.i.d. sequence that generates the new points for this process. Let $B_L=\{\limsup_t w_{N-1}(s)<1\}$ (The L-dependence comes from the stopping time $\tau^{(L)}$). Since

$$w_{N-1}(s)\le (N-1)\mu'(s)\le (N-1)\sqrt{h(s)},$$

we get from Doob's inequality that

$$\begin{aligned}\mathbb{P}(B_L^c)&\le\mathbb{P}\left(\limsup_k h(s)\ge (N-1)^2\right)\le (N-1)^2\mathbb{E}[h(0)]\\ &\le (N-1)^3\frac{(k+(N-1))}{L^2},\end{aligned}$$

so that $\mathbb{P}(B_L)>1-(N-1)^3\frac{(k+(N-1))}{L^2}$. Since $\tau_L\le\tau_{L+1}$ it follows that $B_L\subseteq B_{L+1}$ and so from continuity of probability it follows that $P(\cup_L B_L)=1$. Now make note of the fact that on B_L the borderless process $W(s)$ will coincide with the the regular p-contest process and since $W'(s)\to 0$ a.s. on B_L we therefore have that $\mathcal{X}'(t)\to 0$ a.s. on B_L, combining this with the fact that $P(\cup_L B_L)=1$ shows that $\mathcal{X}'(t)\to 0$ a.s.. $\square$

3 The case $p>1$

Throughout this section we suppose that ζ has a full support on $[0,1]$, and, unless explicitly stated otherwise, that $p>1$.

Theorem 2. *(a)* $\mathbb{P}(\{\mathcal{X}'(t)\to 0\}\cup\{\mathcal{X}'(t)\to 1\})=1$;

(b) if $x_{(1)}(0) \geq 1/p$ then $\mathbb{P}(\mathcal{X}'(t) \to 1) = 1$;

(c) if $x_{(k)}(0) > 0$, where k satisfies

$$\{2p(N-k) > N-2p\} \Longleftrightarrow \left\{k < N - \frac{N}{2p} + 1\right\}, \tag{3.5}$$

then $\mathbb{P}(\mathcal{X}'(t) \to 1) > 0$.

Remark 2. *In general, both convergences can have a positive probability. Let $N = 3$, $p \in (1, 3/2)$, and*

$$\zeta = \begin{cases} U, & \textit{with probability } 1/3; \\ 0, & \textit{with probability } 1/3; \\ 1, & \textit{with probability } 1/3, \end{cases}$$

where $U \in U[0,1]$ (so ζ has full support). Suppose we sample the points of $\mathcal{X}(0)$ from ζ. If they all start off in 0, then $p\mu \leq p/3 < 1/2$, so they cannot escape from 0. On the other hand, there is a positive probability they all start in $(1/p, 1]$, and then Theorem 2(b) says that they converge to 1.

The key idea behind the proof of Theorem 2 is that one can actually find the "ruling" order statistic of the core; namely, there exists some non-random $k = k(N, p) \in \{1, 2, \ldots, N-1\}$ such that $x_{(k)}(t) \to 0$ implies $\mathcal{X}'(t) \xrightarrow{\text{a.s.}} 0$, while $x_{(k)}(t) \not\to 0$ implies that $\mathcal{X}'(t) \xrightarrow{\text{a.s.}} 1$.

Proof. We start with the following two results, which tells us that there is *an absorbing area* $[\frac{1}{p}, 1]$ for the process, such that, once the core enters this area, it will never leave it, and moreover the core will keep moving to the right.

Claim 1. *Suppose that $x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_N \leq 1$ and $x_2 \geq p^{-1}$. Then $\{x_1, \cdots, x_N\}' = \{x_2, \cdots, x_N\}$*

Proof. Let $\mu = \frac{x_1 + \cdots x_N}{N}$. If $p\mu \geq x_N$ then the claim follows immediately; assume instead that $p\mu < x_N$. We need to check if $p\mu - x_1 > x_N - p\mu$, that is, if

$$2p(x_2 + \cdots + x_{N-1}) > (N-2p)(x_1 + x_N) \tag{3.6}$$

However, since $x_i \geq x_2$ for $i = 3, \ldots, N-1$ we have

$$2p(x_2 + \cdots + x_{N-1}) \geq 2px_2(N-2) \geq 2(N-2)$$

while $(N-2p)(x_1 + x_N) \leq 2(N-2p) < 2(N-2)$. Hence (3.6) follows. $\square$

Lemma 3. *If $x_{(1)}(t_0) \geq 1/p$ for some t_0, then $\mathcal{X}'(t) \to 1$ a.s.*

Proof. If $x_{(1)}(t_0) \geq 1/p$, then any point that lands in $[0, 1/p)$ is extreme, so $x_{(2)}(t) \geq 1/p$ for all $t \geq t_0$. Choose any positive $\varepsilon < 1 - \frac{1}{p}$, and let $A_t = \{\zeta_{t+1}, \dots, \zeta_{t+N-1} \in (1-\varepsilon, 1]\}$. Then if A_t happens for $s > t_0$, any point in $[0, 1-\varepsilon]$ is removed in preference to any of the new points coming in, so $x_{(2)}(s + N - 1) > 1 - \varepsilon$. As a result, by Claim 1 we get that $\mathcal{X}'(t) \in [0, 1-\varepsilon]$ for all $t \geq s$.

On the other hand, $\mathbb{P}(A_t) \geq [f(\varepsilon)]^{N-1} > 0$ (see (2.1)) for any t, and the events $A_t, A_{t+N}, A_{t+2N}, \dots$ are independent. Hence, eventually with probability 1, one of the A_t's must happen for some $t > t_0$, so a.s. $\mathcal{X}'(t) \in [0, 1-\varepsilon]$ for all large t. Since ε can be chosen arbitrary small, we get the result. □

The next two results show that if the is some $\varepsilon > 0$ such that infinitely often the core does not have any points in $[0, \varepsilon)$, then it must, in fact, converge to 1.

Lemma 4. *If $x_{(1)}(t_0) \geq \varepsilon$ for some t_0 and $\varepsilon > 0$, then $\mathbb{P}(x_{(1)}(t_0 + \ell) \geq p^{-1} | \mathcal{F}_t) \geq \delta$ for some $\ell = \ell(\varepsilon)$ and $\delta = \delta(\varepsilon) > 0$.*

Proof. Suppose that for some t we have $x_{(1)}(t) \geq \varepsilon$. We claim that it is possible to move $x_{(1)}$ to the right of $\frac{1+p}{2}\varepsilon$ in at most $N-1$ steps with positive probability, depending only on p and ε. Indeed, if $x_{(1)}(t) > \frac{1+p}{2}\varepsilon$ then we are already done. Otherwise, if the new point ζ_{t+1} is sampled in $\left(\frac{1+p}{2}\varepsilon, p\varepsilon\right] \subset [0,1]$ it cannot be rejected. If at this stage $x_{(1)}(t+1) > \frac{1+p}{2}\varepsilon$, then we are done. If not, we proceed again by sampling $\zeta_{t+2} \in \left(\frac{1+p}{2}\varepsilon, p\varepsilon\right]$, etc. After at most $N-1$ steps of sampling new points in $\left(\frac{1+p}{2}\varepsilon, p\varepsilon\right]$, the leftmost point $x_{(1)}$ will have moved to the right of $\frac{1+p}{2}\varepsilon$.

Thus, in no more than $N-1$ steps, with probability no less than $\left[f\left(\frac{p-1}{2}\varepsilon\right)\right]^{N-1} > 0$, $x_{(1)}$ is to the right of $\frac{1+p}{2}\varepsilon$. By iterating this argument at most m times, where $m \in \mathbb{N}$ is chosen such that $\left[\frac{1+p}{2}\right]^m \varepsilon > 1/p$, we achieve that $x_{(1)}$ is to the right of $1/p$ (for definiteness, one can chose $\ell = (N-1)m$ and $\delta = \left[f\left(\frac{p-1}{2}\varepsilon\right)\right]^{(N-1)m}$.) □

Lemma 5. *Let $\varepsilon \in (0,1)$, and define $B(\varepsilon) := \{x_{(1)}(t) \geq \varepsilon \text{ i.o.}\}$ Then $B(\varepsilon) \underset{a.s.}{\Longrightarrow} \{\mathcal{X}'(t) \to 1\}$.*

Corollary 1. *We have $\left\{\liminf_{t\to\infty} x_{(1)}(t) > 0\right\} \underset{a.s.}{\Longrightarrow} \{\mathcal{X}'(t) \to 1\}$.*

Proof of Lemma 5. Assume that $\varepsilon < \frac{1}{p}$ (otherwise the result immediately follows from Lemma 3). Also suppose that $\mathbb{P}(B(\varepsilon)) > 0$, since otherwise the result is trivial. Let ℓ and δ be the quantities from Lemma 4.

Define

$$\tau_0 = \inf\{t > 0 : \; x_{(1)}(t) > \varepsilon\},$$
$$\tau_k = \inf\{t > \tau_{k-1} + \ell : \; x_{(1)}(t) > \varepsilon\}, \quad k \geq 1,$$

with the convention that if $\tau_k = \infty$ then $\tau_m = \infty$ for all $m > k$. Notice that $B(\varepsilon) = \bigcap_{k=0}^{\infty}\{\tau_k < \infty\}$. On $B(\varepsilon)$ we can also define $D_{\tau_k} = \{x_{(1)}(\tau_k + \ell) \geq 1/p\}$. Since $\tau_k - \tau_{k-1} > \ell$ whenever both are finite, we have from Lemma 4 we have $\mathbb{P}(D_{\tau_{k+1}}|\mathcal{F}_{\tau_k}) \geq \delta$. Therefore,

$$B(\varepsilon) \underset{\text{a.s.}}{\Longrightarrow} \left\{ \sum_{k\geq 0} \mathbb{P}(D_{\tau_{k+1}}|\mathcal{F}_{\tau_k}) = \infty \right\}$$

hence by Lévy's extension of the Borel-Cantelli lemma it follows that a.s. on $B(\varepsilon)$) infinitely many (and hence at least one) of D_{τ_k} occur, that is, $x_{(1)}(\tau_k + \ell) \geq 1/p$. Now the result follows from Lemma 3. □

Assume for now that $p < \frac{N}{2}$; in this case $N - \frac{N}{2p} + 1 < N$ (see (3.5)). The case $p \geq \frac{N}{2}$ will be dealt with separately.

Claim 2. *Suppose* $0 \leq x_1 \leq \cdots \leq x_N$ *and* k *is such that*

$$k \in \{2, \ldots, N-1\}, \qquad N > 2p(N-k). \tag{3.7}$$

Let $\beta = \frac{2p(k-1)}{N-2p(N-k)} = 1 + \frac{(p-1)N+p(N-2)}{N-2p(N-k)} > 1$. *If* $x_N > \beta x_k$ *then* $\{x_1, \ldots, x_N\}' = \{x_1, \ldots, x_{N-1}\}$.

Proof. $x_N > \beta x_k$ implies

$$0 < [N - 2p(N-k)]x_N - 2p(k-1)x_k = Nx_N - 2p[(k-1)x_k + (N-k)x_N]$$
$$\leq Nx_N - 2p[x_2 + \cdots + x_N] \leq Nx_N + Nx_1 - 2p[x_1 + \cdots + x_N] = 2N \cdot \left[\frac{x_1 + x_N}{2} - p\mu\right],$$

since $N - 2p > 0$, hence $|x_N - p\mu| \geq x_N - p\mu > p\mu - x_1 = p|\mu - x_1|$ and thus x_N is the furthermost point from the p-centre of mass. □

Lemma 6. *Let* k *satisfy the conditions* (3.7) *and* β *is defined in Claim 2. Then a.s. there is a time* $\tau \geq 0$ *such that for all* $t \geq \tau$ *either* $x_{(k)}(t) = 0$, *or* $x_{(N-1)}(t) \leq \beta x_{(k)}(t)$.

Moreover, $\left\{x_{(k)}(t) \to 0\right\} \underset{a.s.}{\Longrightarrow} \{\mathcal{X}'(t) \to 0\}$.

Proof. Let

$$\tau = \inf\{t \geq 0 : \ x_{(N)}(t) \leq \beta x_{(k)}(t) \text{ or } x_{(k)}(t) = 0\}.$$

If $x_{(k)}(\tau) = 0$, then by Claim 2 for $t \geq \tau$ it is always the rightmost point of $\mathcal{X}(t)$ which is removed; the assumption of the full support on $[0, 1]$ ensures thus that $x_{(N-1)}(t) \downarrow 0$ and hence $\mathcal{X}'(t) \to 0$.

On the other hand, if $x_{(k)}(t) > 0$ for all t, it still follows from Claim 2 that $\tau < \infty$ a.s., since whenever $x_{(N)} > \beta x_{(k)} > 0$ every new points sampled to the right of $x_{(N)}$ is removed immediately, and at the same time there is a positive probability of sampling the new point in $[0, \beta x_{(k)}]$.

Finally, after time τ, we have $x_{(N-1)}(t) \leq \beta x_{(k)}(t)$ for all $t \geq \tau$, since any newly sampled point to the right of $\beta x_{(k)}$ gets rejected by Claim 2. Hence $x_{(N-1)}(t) \to 0$ if $x_{(k)}(t) \to 0$. □

The following statement shows that if all the points to the right of $x_{(k)}$ lie very near each other, while the left-most one lies near zero, then it is to be removed.

Claim 3. *Let $a \in (0, 1]$ and suppose that $k \in \{2, \ldots, N-1\}$ satisfies (3.5). Then there exist small $\delta, \Delta > 0$, depending on N, k, p, a such that if*

$$\begin{aligned} 0 \leq x_1 \leq \delta; \\ x_1 \leq x_i \leq x_N \quad \textit{for } i = 2, \ldots, N-1; \\ x_k, x_{k+1}, \ldots, x_N \in [a(1-\Delta), a) \end{aligned}$$

then $\{x_1, \ldots, x_N\}' = \{x_2, \ldots, x_N\}$.

Proof. The condition to remove the leftmost point is $p\mu - \frac{x_1+x_N}{2} > 0$ where $\mu = (x_1 + \cdots + x_N)/N$. However,

$$\begin{aligned} 2N\left(p\mu - \frac{x_1 + x_N}{2}\right) &= 2p(x_2 + \cdots + x_{N-1}) - (N-2p)x_1 - (N-2p)x_N \\ &\geq 2p(x_k + \cdots + x_{N-1}) - (N-2p)\delta - (N-2p)a \\ &\geq 2p(N-k)a(1-\Delta) - (N-2p)\delta - (N-2p)a \\ &= a\left[2p(N-k)(1-\Delta) - (N-2p)\right] - (N-2p)\delta \end{aligned}$$

The RHS is linear in δ and Δ, and when $\delta = \Delta = 0$ it is strictly positive by the assumption on k; hence it can also be made positive, by allowing $\delta > 0$ and $\Delta > 0$ to be sufficiently small. □

Corollary 2. *Suppose that $\mathcal{X}(t) = \{x_1, \dots, x_N\}$ satisfies the conditions of Claim 3 for some a and k. Let δ be the quantity from this claim. Then*

$$\mathbb{P}(x_{(1)}(t+j) > \delta \text{ for some } 1 \le j \le k | \mathcal{F}_t) \ge c = c_{a\Delta} > 0.$$

Proof. The probability to sample a new point $\zeta \in (a(1-\Delta), a]$ is bounded below by $f(a\Delta)$ where f is the same function as in (2.1). On the other hand, if the new point is sampled in $(a(1-\Delta), a]$ then $\mathcal{X}(t+1)$ continues to satisfy the conditions of Claim 3 as long as the leftmost point is in $[0, \delta]$. By repeating this argument at most k times and using the induction, we get the result with $c = [f(a\Delta)]^k > 0$. □

Lemma 7. *Let $k \in \mathbb{N}$ satisfy (3.5). Then*

$$\left\{x_{(k)}(t) \not\to 0\right\} \underset{a.s.}{\Longrightarrow} \{\mathcal{X}'(t) \to 1\}.$$

Proof. Note that by Lemma 5, it suffices to show that $\left\{x_{(k)}(t) \not\to 0\right\} \underset{\text{a.s.}}{\Longrightarrow} \left\{x_{(1)}(t) \not\to 0\right\}$.

If $x_{(k)}(t) \not\to 0$, there exists an $a > 0$ such that $x_{(k)}(t) \ge a$ for infinitely many t's. Let s be such a time. Now suppose that $\zeta_{s+i} \in I := (a(1-\Delta), a]$ for $i = 0, 1, \dots, N-1$ where Δ is defined in Claim 3; the probability of this event is strictly positive and depends only on a and δ (see (2.1)). As long as there are points of $\mathcal{X}(s+i)$ on *both* sides of the interval I, none of the points inside I can be removed; hence, for some $u \in \{s, s+1, \dots, s+N-1\}$ we have that either $\min \mathcal{X}(u) > a(1-\Delta)$ or $\max \mathcal{X}(u) \le a$. In the first case, $x_{(1)}(u) > a(1-\Delta)$.

In the latter case, both $x_{(N)}(u) \in I$ and $x_{(k)}(u) \in I$, since every time we replaced a point, the number of points to the left of I did not increase (and there were initially at most $k-1$ of them). As a result

$$a(1-\Delta) \le x_{(k)}(u) \le x_{(k+1)}(u) \le \dots \le x_{(N)}(u) \le a.$$

Together with Corollary 2, this yields

$$\{x_{(k)}(t) \ge a \text{ i.o.}\} \underset{\text{a.s.}}{\Longrightarrow} \{x_{(1)}(t) \ge \min\{a(1-\Delta), \delta\} \text{ i.o.}\} \underset{\text{a.s.}}{\Longrightarrow} \{x_{(1)}(t) \not\to 0\}$$

which proves Lemma 7. □

So far we have shown that if $k > N(1 - \frac{1}{2p})$ then $\{x_{(k)}(t) \to 0\} \underset{\text{a.s.}}{\Longrightarrow} \{\mathcal{X}'(t) \to 0\}$ and if $k < N(1 - \frac{1}{2p}) + 1$ then $\{x_{(k)}(t) \not\to 0\} \underset{\text{a.s.}}{\Longrightarrow} \{\mathcal{X}'(t) \to 1\}$. From this we may

conclude that if $N(1-\frac{1}{2p}) < k < N(1-\frac{1}{2p})+1$ (this is possible if $\frac{N}{2p}$ is not an integer) then $\mathcal{X}'(t)$ must converge either to 0 or 1. It remains to consider the case when $\frac{N}{2p}$ *is* an integer and we then choose $k = N(1-\frac{1}{2p})$ and proceed to show that $\{x_{(k)}(t) \to 0\} \underset{\text{a.s.}}{\Longrightarrow} \{\mathcal{X}'(t) \to 0\}$, while $\{x_{(k)}(t) \not\to 0\} \underset{\text{a.s.}}{\Longrightarrow} \{\mathcal{X}'(t) \to 1\}$. We start by showing $\{x_{(k)}(t) \to 0\} \underset{\text{a.s.}}{\Longrightarrow} \{\mathcal{X}'(t) \to 0\}$.
For this purpose let $a \in (0,1)$ and define $I_a^0 = \left[0, \frac{a}{2N^N}\right]$, $I_a^1 = \left(\frac{a}{2N^N}, \frac{a}{N^N}\right)$, $I_a = I_a^0 \cup I_a^1$ and

$$D_a(s) := \{\sup_{t\geq s} x_{(k)}(t) < \frac{a}{2N^N}\},$$

$$C_a(s) = \{\zeta_{s+1}, \dots, \zeta_{s+(N-k)} \in I_a^1\},$$

$$E_a := \{x_{(k)}(t) \to 0, \quad x_{(k+1)}(t) \geq a \quad i.o.\}$$

Lemma 8. *Suppose that $k = N(1-\frac{1}{2p})$ then $\mathbb{P}(E_a) = 0$ for any $a > 0$.*

Proof. Fix $a > 0$. Clearly $E_a \subseteq \{x_{(k)}(t) \to 0\} \subseteq \bigcup_{s=1}^{\infty} D_a(s)$. Let

$$\tau_0 = \inf\{t \geq 0 : x_k(t) < \frac{a}{2N^N}, \quad x_{k+1}(t) \geq a\},$$

while for $l \geq 1$

$$\tau_l = \inf\{t > \tau_{l-1} + N - 1 : x_k(t) < \frac{a}{2N^N}, \quad x_{k+1}(t) \geq a\}$$

and note that $\tau_l < \infty$ a.s. on E_a, for each $l \geq 1$. If at time $\tau_l + 1$, ζ_{τ_l+1} is sampled in I_a^1 then, using $\frac{k+2}{2N^{2N}} < 1$ (since $N \geq 2$) and by also plugging in $p = \frac{N}{2(N-k)}$ we find that,

$$\begin{aligned} p\mu(\tau_l+1) &= p\frac{(N-1)\mu'(\tau_l) + \zeta_{\tau_l+1}}{N} \leq p\frac{\left(k\frac{a}{2N^N} + (N-k-1)x_{(N-1)}(\tau_l)\right) + \frac{a}{N^N}}{N} \\ &\leq p\frac{a\left(\frac{k+2}{2N^N} - 1\right)}{N} + p\frac{N-k}{N}x_{(N-1)}(\tau_l) = \frac{a\left(\frac{k+2}{2N^N} - 1\right)}{2(N-k)} + \frac{x_{(N-1)}(\tau_l)}{2} < \frac{x_{(N-1)}(\tau_l)}{2}, \end{aligned}$$

which means that the right-most point gets rejected. Similarly if ζ_{τ_l+j} is sampled in I_a^1 for $j = 1...N-k$ (i.e. if $C_a(\tau_l)$ occurs) then

$$p\mu(\tau_l+j) \leq p\frac{a\left(\frac{k+2j}{2N^N} - j\right) + (N-k)x_{(N-1)}(\tau_l+j-1)}{N} < \frac{x_{(N-1)}(\tau_l+j-1)}{2},$$

for $j = 1, \dots, N-k$, implying that all the $N-k$ new points get accepted (and thereby rejecting all points to the right of a). If $C_a(\tau_l) \cap D_a(s(l))$ occurs then for $t \geq s(l) := \tau_l + N - k - 1$, the number of points in I_a^0 must be at least k. Consider the interval $I = I_a^0 \cup I_a^1$ and note that at time $s(l)$ all points of the core will lie in I. We will establish through the next claim that if at some time we have $N-1$ points in I, it will be impossible for $x_{(N-1)}(t)$ to ever reach above a while still keeping at least k points in I.

Claim 4. *Fix some $l \in \mathbb{N}$. On the event $C_a(\tau_l) \cap D_a(s(l))$ we have that if there are $0 \leq j \leq N-k-1$ points of the core in I^c at any $t \geq s(l)$ then $x_{(N-1)}(t) \leq N^j \frac{a}{N^N} = \frac{a}{N^{N-j}}$. In particular $x_{(N-1)}(t) < a$ on $C_a(\tau_l) \cap D_a(s(l))$ for all $t \geq s(l)$.*

Proof. We prove this by induction. When $j = 0$ this is true (on $C_a(\tau_l) \cap D_a(s(l))$) since at time $s(l)$ we have all core points in I and if we do not move any points into I^c then the right most point is to the left of $\frac{a}{N^n}$. Assume the claim is true for $j = J$. Since at time $s(l)$ there are no points in in I^c then if there are ever to be $J+1$ points in I^c there exists a time $t > s(l)$ when we go from J points in I^c to $J+1$ points in I^c at time $t+1$.

At time $t+1$, ζ_{t+1} will be rejected if $\zeta_{t+1} > \max(p\mu(t+1), x_{(N-1)}(t))$. Since $\zeta_{t+1} > p\mu(t+1)$ if and only if (by plugging in p) $\zeta_{t+1} > \frac{N-1}{N-k-1}\mu'(t)$, but $\frac{N-1}{N-k-1}\mu'(t) \leq Nx_{(N-1)}(t)$, so we may conclude that ζ_{t+1} will be rejected if $\zeta_{t+1} > Nx_{(N-1)}(t)$ regardless of how the $N-1-L$ points in I and the remaining J points in I^c are distributed. We obtain that

$$x_{(N-1)}(t+1) \leq Nx_{(N-1)}(t) \leq N^{J+1}\frac{a}{N^N} = \frac{a}{N^{N-(J+1)}},$$

by the induction hypothesis and this proves the claim. □

It follows directly from Claim 4 that

$$E_a \cap D_a(s(l)) \subseteq C_a(\tau_{l+k})^c, \quad \forall l, k \in \mathbb{N}_0,$$

as a consequence, since $\mathbb{P}\left(C_a(\tau_l)|\mathcal{F}_{\tau_l}\right) \geq f\left(|I_a^1|\right)^{N-l}$ on $\{\tau_l < \infty\}$ and since $\{C_a(\tau_{l+k})^c\}_k$ are all independent for every l by the construction of τ_l

$$\mathbb{P}\left(E_a \cap D_a(s(l))\right) \leq \mathbb{P}\left(\bigcap_{k=0}^{\infty} C_a(\tau_{l+k})^c\right) \leq \prod_{k=0}^{\infty}\left(1 - f(|I_a^1|)\right) = 0.$$

Using $E_a \subseteq \bigcup_{l=0}^{\infty} D_a(s(l))$, we conclude

$$\mathbb{P}\left(E_a\right) = \mathbb{P}\left(\bigcup_{l=1}^{\infty}\left(E_a \cap D_a(s(l))\right)\right) \leq \sum_{l=0}^{\infty}\mathbb{P}\left(E_a \cap D_a(s(l))\right) = 0.$$

□

Lemma 8 implies that when $k = N(1 - \frac{1}{2p})$ then

$$\mathbb{P}\left(x_{(k)}(t) \to 0, x_{(k+1)}(t) \not\to 0\right) = \mathbb{P}\left(\bigcup_{n \geq 1} E_{1/n}\right) \leq \sum_{n=1}^{\infty}\mathbb{P}(E_{1/n}) = 0,$$

i.e. $\{x_{(k)}(t) \to 0\} \underset{\text{a.s.}}{\Longrightarrow} \{x_{(k+1)}(t) \to 0\}$, but $k+1 > N(1-\frac{1}{2p})$ and so by Lemma 6, $\{x_{(k)}(t) \to 0\} \underset{\text{a.s.}}{\Longrightarrow} \{\mathcal{X}'(t) \to 0\}$. Note that since $k = N(1-\frac{1}{2p})$ then k obviously satisfies (3.5) so Lemma 7 implies that $\{x_{(k)}(t) \not\to 0\} \underset{\text{a.s.}}{\Longrightarrow} \{\mathcal{X}'(t) \to 1\}$ which completes the proof when $p < \frac{N}{2}$. For the case $p \geq \frac{N}{2}$ we have

Lemma 9. *If $p \geq \frac{N}{2}$ then $\mathcal{X}'(t) \to 1$ a.s.*

Proof. The case $p > \frac{N}{2}$ is easy: unless $x_{(N)} = 0$ we have

$$p \cdot \frac{x_{(1)} + \ldots x_{(N)}}{N} > \frac{x_{(1)} + \cdots + x_{(N)}}{2} \geq \frac{x_{(1)} + x_{(N)}}{2}$$

hence it is the left-most point which is always removed. For the case $p = \frac{N}{2}$ we notice that at each moment of time we either have a tie (between the left-most and right-most point) or remove the left-most point. At time t we can only have a tie if $x_{(1)}(t) = ... = x_{(N-1)}(t) = 0$ and if this is true then eventually the right-most point will be kept and the process becomes monotone after this (the left-most point will always be rejected). □

To prove part (c), note that unless $x_{(1)}(0) > 0$ already, by repeating the arguments from the beginning of the proof of Lemma 7, with a positive probability we can "drag" the whole configuration in at most $N-1$ steps to the right of zero, that is, there is $0 \leq t_0 \leq N-1$ such that $\mathbb{P}(\min \mathcal{X}'(t_0) > 0) > 0$. Now we can apply Lemma 4 and then Lemma 3.

This concludes the proof □

Remark 3. *For an alternative proof see section two of the Appendix.*

4 Non-convergence to zero for $p > 1$ and $N = 3$

In this section we prove the following

Theorem 3. *Suppose that $N = 3$, $p > 1$ and ζ, restricted to some neighbourhood of zero, is a continuous random variable with a non-decreasing density (e.g. uniformly distributed). Assume also that the initial points are all i.i.d. ζ-distributed. Then $\mathcal{X}'(t) \to 1$ as $t \to \infty$ a.s.*

Remark 4.

- *In case $p \geq 3/2$ we already know that $\mathcal{X}'(t) \to 1$ for any initial configuration and any distribution (see Lemma 9), so we have to prove the theorem only for $p \in (1, 3/2)$.*

- *Simulations suggest that the statement of Theorem 3 holds, in fact, for a much more general class of distributions ζ.*

Proof of Theorem 3. Let $\varepsilon \in (0, 1/2)$ be such that ζ conditioned[1] on $\{\zeta \leq \varepsilon\}$ has a non-decreasing density; according to the statement of the Theorem 3 such an ε must exist. Furthermore we can also assume that the denisty of ζ is bounded on $[0, 2\epsilon]$ by choosing ϵ small enough, indeed since the density is non-decreasing and is integrable it follows that it can have at most a single integrable singularity which then must be located at it's right-most point of definition. Let us fix this ε from now on. As before, denote by $x_1, \ldots, x_N$ N distinct points on $[0, 1]$, and let $x_{(1)}, \ldots, x_{(N)}$ be this unordered N-tuple sorted in the increasing order. Let

$$\{y_1, \ldots, y_{N-1}\} = \{x_1, \ldots, x_N\}'_p$$

be the unordered N-tuple $\{x_1, \ldots, x_N\}$ with the farthest point from p-centre of mass removed; w.l.o.g. assume that y_i are already in the increasing order.

Lemma 10. *The operation $\{\ldots\}'_p$ is monotone in p, that is, if $\hat{p} \geq \tilde{p}$ and*

$$\{\hat{y}_1, \ldots, \hat{y}_{N-1}\} = \{x_1, \ldots, x_N\}'_{\hat{p}},$$
$$\{\tilde{y}_1, \ldots, \tilde{y}_{N-1}\} = \{x_1, \ldots, x_N\}'_{\tilde{p}}$$

then $\hat{y}_i \geq \tilde{y}_i$, $i = 1, \ldots, N-1$.

Proof. Assume w.l.o.g. $x_1 \leq \ldots \leq x_N$, and let $\mu = \mu(\{x_1, \ldots, x_N\})$. Notice that, regardless of the value of p, the only points which can possibly be removed are x_1 or x_N (since they are the two extreme points). Therefore, it suffices to show that $\{x_1, \ldots, x_N\}'_{\tilde{p}} = \{x_2, \ldots, x_N\}$ implies $\{x_1, \ldots, x_N\}'_{\hat{p}} = \{x_2, \ldots, x_N\}$. Note also that $|x_1 - p\mu| = p\mu - x_1$ for all $p \geq 1$.

If $\tilde{p}\mu - x_1 > |\tilde{p}\mu - x_N|$ and $\tilde{p}\mu - x_N > 0$, that is, the $p-$centre of mass lies to the right of x_N, then $\hat{p}\mu > \tilde{p}\mu > x_N$ as well, and hence x_1 is discarded.

[1]note that the full support assumption ensures that the probability of this event is positive

On the other hand, if $\tilde{p}\mu - x_1 > |\tilde{p}\mu - x_N|$ and $\tilde{p}\mu < x_N$ then either $\hat{p}\mu < x_N$, or $\hat{p}\mu \geq x_N$. In the first case,

$$\hat{p}\mu - x_1 > \tilde{p}\mu - x_1 > |\tilde{p}\mu - x_N| = x_N - \tilde{p}\mu > x_N - \hat{p}\mu = |x_N - \hat{p}\mu|$$

so x_1 is discarded. In the second case, $p-$centre of mass lies to the right of x_N and so x_1 is also discarded. □

Lemma 11. *Let h be a real-valued function on the sets of N real numbers. Suppose that h is non-increasing in each of its arguments, namely*

$$h\left(x_1, x_2, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_N\right) \leq h\left(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_N\right)$$

whenever $x_i' \geq x_i$. Let $\mathcal{E}_t$ be some $\mathcal{F}_t$-measurable event, and suppose that

$$\mathbb{E}\left(h(\mathcal{X}'(t+1))|\mathcal{F}_t\right) \leq h(\mathcal{X}'(t)) \text{ on } \mathcal{E}_t \tag{4.8}$$

for $p = 1$. Then (4.8) *holds for $p > 1$ as well.*

Proof. Let

$$G_p(\mathcal{X}'(t), \zeta_{t+1}) = \{x_{(1)}(t), x_{(2)}(t), \ldots, x_{(N-1)}(t), \zeta_{t+1}\}_p'$$

be the new core after the new point ζ_{t+1} is sampled and the farthest point from the $p-$centre of mass is removed; note that $\mathcal{X}'(t+1) = G_p(\mathcal{X}'(t), \zeta_{t+1})$. Then on $\mathcal{E}_t$

$$\mathbb{E}(h(\mathcal{X}'(t+1))|\mathcal{F}_t) = \mathbb{E}(h(G_p(\mathcal{X}'(t), \zeta_{t+1}))|\mathcal{F}_t) \leq \mathbb{E}(h(G_1(\mathcal{X}'(t), \zeta_{t+1}))|\mathcal{F}_t) \leq h(\mathcal{X}'(t))$$

since the operation $\{\ldots\}_p'$ is monotone in p by Lemma 10 and h is decreasing in each argument. □

From now on assume $N = 3$ and $p = 1$. Denote $x_{(1)}(t) = a$, $x_{(2)}(t) = b$ and consider the events

$$\begin{aligned} L_b &= \left\{\zeta_{t+1} \in ((2a-b)^+, a)\right\}, & R_a &= \left\{\zeta_{t+1} \in (b, 2b-a)\right\}, \\ B_b &= \left\{\zeta_{t+1} \in \left(a, \tfrac{a+b}{2}\right)\right\}, & B_a &= \left\{\zeta_{t+1} \in \left(\tfrac{a+b}{2}, b\right)\right\} \end{aligned}$$

(we assume that b is smaller than $1/2$, yielding $2b - a < 1$.) If $x_{(2)}(t) \leq \varepsilon$ then $\mathcal{X}'(t+1) \neq \mathcal{X}'(t)$ implies that one of the events L_b, B_b, B_a or R_a occurs (i.e. all points sampled outside of $((2a-b)^+, 2b-a)$ are rejected at time $t+1$). Let us study the core $\mathcal{X}'(t+1) = \{\zeta_{t+1}, a, b\}'$ on these events: on L_b and B_b we have $\mathcal{X}'(t+1) = \{x, a\}$, while on B_a and R_a we have $\mathcal{X}'(t+1) = \{x, b\}$.

We have, assuming $x_{(1)}(t) = a$ and $x_{(2)}(t) = b$,

$$\mathbb{E}(h(\mathcal{X}'(t+1)) - h(\mathcal{X}'(t))|\mathcal{F}_t) = \mathbb{E}(h(\{\zeta, a, b\}') - h(a, b)).$$

When $0 \le a \le b \le \varepsilon$ we have $2b - a \le 2\varepsilon$. Define

$$g(x) = h(\{x, a, b\}') - h(a, b) = \begin{cases} h(x, a) - h(a, b), & \text{if } x \in ((2a-b)^+, a); \\ h(a, x) - h(a, b), & \text{if } x \in (a, (a+b)/2); \\ h(x, b) - h(a, b), & \text{if } x \in ((a+b)/2, b); \\ h(b, x) - h(a, b), & \text{if } x \in (b, 2b-a) \\ 0, & \text{otherwise,} \end{cases}$$

which is positive in the first two cases, and negative in the next two. Let $\varphi(x)$ be the density of ζ conditioned on $\{\zeta \in [0, 2\varepsilon]\}$. By the monotonicity of φ and the positivity (negativity resp.) of g on the first (second resp.) interval,

$$\begin{aligned} \Delta(a, b) := \mathbb{E}\left[g(\zeta) 1_{\zeta \in [0, 2\varepsilon]}\right] &= \int_{(2a-b)^+}^{\frac{a+b}{2}} g(x)\varphi(x)\mathrm{d}x + \int_{\frac{a+b}{2}}^{2b-a} g(x)\varphi(x)\mathrm{d}x \\ &\le \varphi\left(\frac{a+b}{2}\right) \int_{(2a-b)^+}^{\frac{a+b}{2}} g(x)\mathrm{d}x + \varphi\left(\frac{a+b}{2}\right) \int_{\frac{a+b}{2}}^{2b-a} g(x)\mathrm{d}x = \varphi\left(\frac{a+b}{2}\right) \cdot \Lambda, \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} \Lambda = \Lambda(a, b) = &\int_{(2a-b)^+}^{a} (h(x, a) - h(a, b))\mathrm{d}x + \int_{a}^{\frac{a+b}{2}} (h(a, x) - h(a, b))\mathrm{d}x \\ &+ \int_{\frac{a+b}{2}}^{b} (h(x, b) - h(a, b))\mathrm{d}x + \int_{b}^{2b-a} (h(b, x) - h(a, b))\mathrm{d}x. \end{aligned}$$

So if we can establish that $\Lambda \le 0$ for a suitable function h, then indeed $\Delta(a, b) \le 0$, and the supermartingale property follows.

Remark 5. *Notice that the method of proof, presented here, could possibly work for $N > 3$ as well; that is, if one can find a function $h(x_1, \dots, x_{N-1})$, which is positive and decreasing in each of its arguments, and $h(\mathcal{X}'(t))$ is a supermartingale provided $\max \mathcal{X}'(t) < \varepsilon$ for some $\varepsilon > 0$. Unfortunately, however, we were not able to find such a function.*

Set

$$h(x, y) = -2\log\left(\max\left\{x, \frac{y}{2}\right\}\right) \ge 0; \tag{4.10}$$

it is easy to check h is indeed monotone in each of its arguments as long as $x, y \in (0, 1]$. Let us now compute the integrals in the expression for Λ. We have

$$\Lambda = \begin{cases} 3(a-b)\ln 2 - 3a + 2b, & \text{if } a \le \frac{b}{3}; \\ (a+b)\ln(a+b) - (a+b)\ln a + (a-5b)\ln 2 + b, & \text{if } \frac{b}{3} < a \le \frac{b}{2}; \\ (a+b)\ln(a+b) + (2a-4b)\ln b + 3(b-a)\ln a + (b-5a)\ln 2 + b, & \text{if } \frac{b}{2} < a \le \frac{2b}{3}; \\ (a+b)\ln(a+b) + (2a-4b)\ln b + (5b-7a)\ln a - (a+b)\ln 2 & \\ \quad +3(b-a) + (4a-2b)\ln(2a-b), & \text{if } \frac{2b}{3} < a \le b. \end{cases}$$

It turns out that $h(\mathcal{X}'(t))$ indeed has a non-positive drift, provided $0 < a \le b \le \varepsilon$, as is shown by the following

Lemma 12. $\Lambda \le 0$ *for* $0 < a \le b \le 1$.

Proof. Substitute $a = b\nu$ in the expression for Λ. Then for $\nu \le 1/3$ we easily obtain

$$\Lambda = -b\left[3\nu(1-\ln 2) + \ln 8 - 2\right] \le 0.$$

For $1/3 < \nu \le 1/2$ we have $2\Lambda = -bC_1(\nu) \le 0$ where

$$C_1(\nu) = (1+\nu)\ln\frac{\nu}{1+\nu} + (5-\nu)\ln 2 - 1 > 0,$$

since $\frac{\partial^2 C_1(\nu)}{\partial^2 \nu} = -\frac{1}{\nu^2(1+\nu)} < 0$ and hence $\min_{1/3 \le \nu \le 1/2} C_1(\nu)$ is achieved at one of the endpoints $\nu = 1/3$ or $\nu = 1/2$; the values there are $C_1(1/3) = \ln(4) - 1 > 0$ and $C(1/2) = \frac{1}{2}\ln\left(\frac{512}{27}\right) - 1 > 0$ respectively.

For $1/2 < \nu \le 2/3$ we have $\Lambda = -bC_2(\nu) \le 0$ where

$$C_2(\nu) = -(1+\nu)\ln(1+\nu) + (3\nu - 3)\ln\nu - 1 + (5\nu - 1)\ln 2 > 0,$$

since $\frac{\partial^2 C_2(\nu)}{\partial^2 \nu} = \frac{2\nu^2+6\nu+3}{\nu^2(1+\nu)} > 0$ and $\left.\frac{\partial C_2(\nu)}{\partial \nu}\right|_{\nu=2/3} = \ln\left(\frac{256}{45}\right) - \frac{5}{2} < 0$ implies that $\frac{\partial C_2(\nu)}{\partial \nu} < 0$ for all $\nu \in [1/2, 2/3]$ and hence $\min_{1/2 \le \nu \le 2/3} C_2(\nu) = C_2(2/3) = \frac{1}{3}\ln\left(\frac{104976}{3125}\right) - 1 > 0$.

Finally, for $2/3 < \nu \le 1$ we have $\Lambda = -bC_3(\nu) \le 0$, where

$$C_3(\nu) = \nu \log\frac{2\nu^7}{(2\nu-1)^4(\nu+1)} + \log\frac{2(2\nu-1)^2}{\nu^5(\nu+1)} + 3(\nu - 1) > 0$$

since

$$\frac{\mathrm{d}^2 C_3(\nu)}{\mathrm{d}\nu^2} = \frac{(2\nu+5)(2\nu^2-1)}{(2\nu-1)\nu^2(\nu+1))}$$

changes its sign from $-$ to $+$ at $1/\sqrt{2} \in (2/3, 1)$ and therefore $\frac{\partial C_3(\nu)}{\partial \nu}$ achieves its maximum at the endpoints of the interval; thus

$$\max_{2/3 \leq \nu \leq 1} \frac{\partial C_3(\nu)}{\partial \nu} = \max_{\nu = 2/3, 1} \frac{\partial C_3(\nu)}{\partial \nu} = \max \left\{ -\frac{5}{2} + \ln\left(\frac{256}{45}\right), 0 \right\} = 0$$

Therefore, $C_3(\nu)$ is decreasing and hence $\min_{2/3 \leq \nu \leq 1} C_3(\nu) = C_3(1) = 0$. $\square$

Choose $\tau_0 = 0$, and for $k = 1, 2, \ldots$, define the sequence of stopping times

$$\begin{aligned} \eta_k &= \inf\left\{t > \tau_{k-1} : \quad x_{(2)}(t) < \epsilon\right\}, \\ \tau_k &= \inf\left\{t > \eta_k : \ x_{(2)}(t) > \epsilon\right\}, \\ \gamma_{k,t} &= \min\left(\eta_k + t, \tau_k\right), \end{aligned}$$

so that $\tau_0 < \eta_1 < \gamma_{1,t} < \tau_1 < \eta_2 < \gamma_{2,t} < \tau_2 < \ldots$ for all $t \geq 0$, with the usual conventions that $\inf \emptyset = +\infty$ and that if one of the stopping times is $+\infty$ then the subsequent ones are also $+\infty$. Note that $\{\mathcal{X}'(t) \to 0\} \subseteq \bigcup_{k=1}^{\infty}(\{\tau_k = \infty\} \cap \{\eta_k < \infty\})$ so it suffices to show that $\mathbb{P}\left(\{\tau_k = \infty\} \cap \{\eta_k < \infty\} \cap \{\mathcal{X}'(t) \to 0\}\right) = 0$ for all $k \geq 1$.

We will show that with h as in (4.10) then $\lim_{t\to\infty} h(\mathcal{X}'(\gamma_{k,t}))$ exists and is finite a.s. on $\{\tau_k = \infty\} \cap \{\eta_k < \infty\}$. Since $\lim_{b \downarrow 0} h(a, b) = +\infty$ this implies $\liminf_{t\to\infty} x_{(1)}(\gamma_{k,t}) > 0$ a.s. (since otherwise $\mathbb{P}\left(\lim_{t\to\infty} h(\mathcal{X}'(\gamma_{k,t})) = +\infty\right) > 0$). If we can show that $h(\mathcal{X}'(\gamma_{k,t}))$ is a positive supermartingale on $\{\tau_k = \infty\} \cap \{\eta_k < \infty\}$ then we are done. From now on fix $k \geq 1$ so that we may denote $\gamma_t = \gamma_{k,t}$ without loss of generality.

The positivity of $h(\mathcal{X}'(\gamma_t))$ follows by the definitions of h. Letting $\xi_t = h(\mathcal{X}'(t))$ (which is always non-negative) then by (4.9) and Lemma 12 it follows that

$$\mathbb{E}\left[(\xi_{t+1} - \xi_t) 1_{\zeta_{t+1} < 2\epsilon} | \mathcal{F}_t\right] \leq 0. \tag{4.11}$$

Note that if $\zeta_{t+1} \geq 2\epsilon$ and $x_{(1)}(t) < 2\epsilon$ then if ζ_{t+1} is accepted into the core, $\mathcal{X}'(t+1) = \{x_{(2)}(t), \zeta_{t+1}\}$ and if it is not accepted into the core then $\xi_{t+1} = \xi_t$. Since

$$\max\left(x_{(1)}(t), x_{(2)}(t)/2\right) \leq \max\left((x_{(2)}(t) \wedge \zeta_{t+1}, (x_{(2)}(t) \vee \zeta_{t+1})/2\right),$$

it follows that

$$\mathbb{E}\left[(\xi_{t+1} - \xi_t) 1_{\zeta_{t+1} \in [2\epsilon, 1]} 1_{x_{(1)}(t) < 2\epsilon} | \mathcal{F}_t\right] \leq 0. \tag{4.12}$$

If $\zeta_{t+1} \geq 2\epsilon$ and $x_{(1)}(t) \geq 2\epsilon$ then $\xi_t \leq -2\log(\epsilon)$ and $\xi_{t+1} \leq -2\log(\epsilon)$ which implies

$$\mathbb{E}\left[(\xi_{t+1} - \xi_t) 1_{\zeta_{t+1} \in [2\epsilon, 1]} 1_{x_{(1)}(t) \geq 2\epsilon} | \mathcal{F}_t\right] \leq -4\log(\epsilon). \tag{4.13}$$

We can conclude

$$\mathbb{E}\left[(\xi_{t+1}-\xi_t)1_{\zeta_{t+1}\in[2\epsilon,1]}|\mathcal{F}_t\right] \le -4\log(\epsilon) := c. \tag{4.14}$$

Combining (4.11) and (4.14) gives us $\mathbb{E}\left[(\xi_{t+1}-\xi_t)|\mathcal{F}_t\right] \le c$ taking expectations and iterating over t, we find that $\mathbb{E}\left[\xi_t\right] \le ct + \mathbb{E}[\xi_0]$ so $\mathbb{E}\left[\xi_t\right] < \infty$ if $\mathbb{E}[\xi_0] < \infty$. Now

$$\begin{aligned}\mathbb{E}[\xi_0] &\le -2\mathbb{E}[\log(\max(\zeta_1/2,\zeta_2/2))] \le -2\mathbb{E}[\log(\zeta/2)] \le -2\mathbb{E}\left[\log(\zeta/2)1_{\zeta<\epsilon}\right] - 2\log(\epsilon/2)\\ &\le -2\left(\varphi(\epsilon)\int_0^\epsilon \log(x)dx + \log(\epsilon)\right) = -2\left(\varphi(\epsilon)\epsilon\left(\log(\epsilon)-1\right)+\log(\epsilon/2)\right),\end{aligned}$$

which is finite. Now it suffices to show $\mathbb{E}\left[h(\mathcal{X}'(\gamma_t+1))|\mathcal{F}_{\gamma_t}\right] \le h(\mathcal{X}'(\gamma_t))$ since this will in fact imply $\mathbb{E}\left[h(\mathcal{X}'(\gamma_{t+1}))|\mathcal{F}_{\gamma_t}\right] \le h(\mathcal{X}'(\gamma_t))$ (see the proof of Theorem 2 in [5] for details). Note that $x_{(2)}(\gamma_t+1) \le px_{(2)}(\gamma_t) \le p\cdot\epsilon < 2\epsilon$, therefore it follows from Lemma 12 that $\mathbb{E}\left[h(\mathcal{X}'(\gamma_t+1))|\mathcal{F}_{\gamma_t}\right] \le h(\mathcal{X}'(\gamma_t))$.

We have thus showed that $h(\mathcal{X}'(\gamma_t))$ is a positive supermartingale and this concludes the proof. □

5 Appendix: The calculations for the proof of Lemma 2.

Observe that all expressions for $\mathbf{A}_j$ are fractions of the polynomials in (a, f, p, M); moreover, their denominators

$3M(M-1)$	(for $\mathbf{A}_1$),
$3M(M-1)(M+1-2p)^3$	(for $\mathbf{A}_2$ and $\mathbf{A}_4$),
$12M(M-1)(M+1-2p)^3p^3$	(for $\mathbf{A}_3$ and $\mathbf{A}_5$)

are always positive. Throughout the rest of the proof let $\mathsf{n}(w)$ denote the numerator of such a fraction w.

Case 1: $\mathbf{I}_1 \le 0$

Observe that

$$\mathsf{n}(\mathbf{A}_1) = -2M^2 - 3M\mu + 2M + 1 + [3M\mu - 1]Mp$$

and the term in the square brackets is positive as $M\mu \ge 1$, so the maximum of $\mathsf{n}(\mathbf{A}_1)$ is achieved at the highest possible value of p. However, in this case we have $p \le p_1$, hence

$$\mathsf{n}(\mathbf{A}_1)1_{X_1\le 0} \le \mathsf{n}(\mathbf{A}_1)|_{p=p_1} = -\frac{s_1}{2\mu}$$

where

$$s_1 = (M^2-2)\mu + (1-6\mu)(1-\mu)M + 1 = \begin{cases} 3(2\mu-1)^2, \text{ if } M=2;\\ 4\mu^2 + 1/2 + 14(\mu-1/2)^2, \text{ if } M=3;\\ (M-3)[(M-4)\mu + 6\mu^2 + 1] + s_1|_{M=3}, \text{ if } M \ge 4\end{cases}$$

Hence $s_1 \ge 0$ for $M = 2, 3, \ldots$ and thus $\mathbf{I}_1 \le 0$.

Case 2: $\mathbf{I}_2 \leq 0$

Here

$$\mathsf{n}(\mathbf{A}_2) = -4\left[a(M-p+1) - M\mu p\right]^2 s_2$$

where

$$s_2 = M^3\mu p - 4M^2\mu p^2 - M^3 a + 2M^2 ap + 5M^2\mu p + 2Map^2 - 3M^2\mu - 6Ma * p + 4M\mu p \\ + 3Ma - 3M\mu - 2ap + 2a,$$

and we need to show that $s_2 \geq 0$.

Assume first $M = 2$. Then (using the fact that $\mu = (1+a)/2$)

$$X_2 \geq 0 \Longleftrightarrow p \geq \frac{3a+3}{4a+2} \geq 1$$

which is impossible; so from now on $M \geq 3$.

To establish $\mathbf{I}_2 \leq 0$, it will suffice to demonstrate that

$$s_3 := 2Ms_2 - 2M^3(M\mu + a)X_2 \geq 0$$

as $\mathbf{I}_2$ has a factor $1_{X_2 \geq 0}$, and $s_2 1_{X_2\geq 0} \geq \frac{s_3}{2M} 1_{X_2 \geq 0}$. Substituting

$$p = \left[\frac{1}{2} + \frac{1}{2M}\right] + \left[\frac{1}{2} - \frac{1}{2M}\right] w$$

where $w \in [0,1)$ corresponding to the condition (2.3), we get

$$s_3 = M\left(-2M^2\mu w^2 + M^2\mu w + 4M\mu w^2 + M^3 - 3M^2\mu - M\mu w - 2\mu w^2 + M^2 - M\mu + 2\mu\right) \\ - a\cdot(M-1)\left[M\left((M-1)^2 - (w-2)^2\right) + (1-w)\left(M^2 - w - 1\right)\right]$$

The expression in the square brakets is non-negative for $M \geq 3$, so the minimum of s_3 is achieved when $a = 1$; i.e.

$$s_3 \geq s_3|_{a=1} = -2M^3\mu w^2 + M^3\mu w + 4M^2\mu w^2 - 3M^3\mu + M^3 w - M^2\mu w + M^2 w^2 \\ - 2M\mu w^2 + 3M^3 - M^2\mu - 5M^2 w - 2Mw^2 + 2M^2 + 2M\mu + 4Mw + w^2 - 2M - 1 =: s_4$$

But

$$\frac{\partial s_4}{\partial \mu} = -M\left((3-w)M^2 + (1+w)M - 2 + 2(M-1)^2 w^2\right) < 0$$

so

$$s_4 \geq s_4|_{\mu=1} = (1-w)(M-1)(wM(2M-3) + M + w + 1) \geq 0.$$

Case 3: $\mathbf{I}_3 \leq 0$

Here

$$\mathsf{n}(\mathbf{A}_3) = -(M+1)(1-a)s_5$$

and it suffices to show that $s_5 \geq 0$. If $M = 2$, then $\mu = (a+1)/2$ and $p \geq 3/4$, so

$$s_5 = 3(3-2p)\left[(1-a)^2(8p-5) + (32(1-a)^2 + 144a)(1-p)^4\right.$$

$$+12(1-p)^2(4p+a(4ap+10p-3))\big] \geq 0.$$

For $M \geq 3$, let $M = 3+\delta$, $\delta = 0, 1, \ldots$. Then $s_5 = \sum_{i=0}^{5} e_{i+1}\delta^i$ where we will show that all $e_i \geq 0$. Indeed, we have

$$\begin{aligned}
e_1 &= -432a\mu p^5 - 1296\mu^2p^5 + 288a^2p^4 + 2736a\mu p^4 - 144ap^5 + 432p^4\mu^2 - 432\mu p^5 - 1632p^3a^2 \\
&\quad - 2880a\mu p^3 + 1200ap^4 + 4320\mu^2p^3 + 2736\mu p^4 + 2624p^2a^2 - 1152a\mu p^2 - 3744ap^3 - 1728\mu^2p^2 \\
&\quad - 2880\mu p^3 + 288p^4 - 1536pa^2 + 4928p^2a - 1152p^2\mu - 1632p^3 + 768a^2 - 3072ap + 2624p^2 \\
&\quad + 1536a - 1536p + 768 \\
e_2 &= -288a\mu p^5 - 1296\mu^2p^5 + 168a^2p^4 + 2208a\mu p^4 - 48ap^5 - 360p^4\mu^2 - 288\mu p^5 - 1160p^3a^2 \\
&\quad - 1392a\mu p^3 + 600ap^4 + 6984\mu^2p^3 + 2208\mu p^4 + 1760p^2a^2 - 3840a\mu p^2 - 2264ap^3 - 2016\mu^2p^2 \\
&\quad - 1392\mu p^3 + 168p^4 - 576pa^2 + 2720p^2a - 3840p^2\mu - 1160p^3 + 768a^2 - 1152ap + 1760p^2 \\
&\quad + 1536a - 576p + 768 \\
e_3 &= -48a\mu p^5 - 432\mu^2p^5 + 24a^2p^4 + 576a\mu p^4 - 600p^4\mu^2 - 48\mu p^5 - 268p^3a^2 + 216a\mu p^3 + 72ap^4 \\
&\quad + 4404\mu^2p^3 + 576\mu p^4 + 324p^2a^2 - 3240a\mu p^2 - 412ap^3 - 876\mu^2p^2 + 216\mu p^3 + 24p^4 + 336pa^2 \\
&\quad + 180p^2a - 3240p^2\mu - 268p^3 + 288a^2 + 672ap + 324p^2 + 576a + 336p + 288 \\
e_4 &= -48\mu^2p^5 + 48a\mu p^4 - 216p^4\mu^2 - 20p^3a^2 + 192a\mu p^3 + 1356\mu^2p^3 + 48\mu p^4 - 4p^2a^2 - 1164a\mu p^2 \\
&\quad - 20ap^3 - 168\mu^2p^2 + 192\mu p^3 + 228pa^2 - 112p^2a - 1164p^2\mu - 20p^3 + 48a^2 + 456ap - 4p^2 \\
&\quad + 96a + 228p + 48 \\
e_5 &= -24p^4\mu^2 + 24a\mu p^3 + 204\mu^2p^3 - 4p^2a^2 - 192a\mu p^2 - 12\mu^2p^2 + 24\mu p^3 + 45pa^2 - 16p^2a \\
&\quad - 192p^2\mu + 3a^2 + 90ap - 4p^2 + 6a + 45p + 3 \\
e_6 &= 360p(a+1-2\mu p)^2 \geq 0.
\end{aligned}$$

The fact that $e_6 \geq 0$ is trivial; we will prove separately that $e_1, \ldots, e_5 \geq 0$ below. In what follows, we substitute $p = \frac{1+\nu}{2}$, where $\nu \in (0,1)$.

Proof that $e_1 \geq 0$

We have

$$\frac{\partial^2 e_1}{\partial a^2} = 4[9\nu^4 - 66\nu^3 + 76\nu^2 + 2\nu + 235] > 0,$$

hence e_1 achieves its minimum at

$$a_{cr} = \frac{9\nu^5 - 105\nu^4 + 426\nu^3 - 46nu^2 + 397\nu - 1669 + 9\mu(1+\nu)^2(3\nu^3 - 29\nu^2 + 13\nu + 109)}{8[9\nu^4 - 66\nu^3 + 76\nu^2 + 2\nu + 235]}$$

which solves $\frac{\partial e_1}{\partial a} = 0$. Note that it may happen $a_{cr} \notin [0,1]$. However, in any case,

$$e_1 \geq e_1|_{a=a_{cr}} = \frac{1}{32} \cdot \frac{3(1+\nu)^2 c_1}{9\nu^4 - 66\nu^3 + 76\nu^2 + 2\nu + 235}$$

so it will suffice to show that

$$\begin{aligned}
c_1 &= 16(1-\nu)^2 c_{1a} + 3(1-\mu)c_{1b}, \quad \text{where} \\
c_{1a} &= -27\nu^6 + 144\nu^5 - 102\nu^4 + 1620\nu^3 - 9883\nu^2 + 12484\nu + 1732 \\
c_{1b} &= 81\mu\nu^8 - 108\mu\nu^7 + 135\nu^8 - 1260\mu\nu^6 - 828\nu^7 - 12276\mu\nu^5 + 276\nu^6 + 84774\mu\nu^4
\end{aligned}$$

$$- 4404\nu^5 - 157140\mu\nu^3 + 69170\nu^4 + 152628\mu\nu^2 - 198372\nu^3 - 156108\mu\nu + 182084\nu^2 + 27969\mu - 60588\nu + 73967$$

is positive. We have

$$c_{1a} = 3\nu^3(540 - 9\nu^3 + 48\nu^2 - 34\nu) + \nu(12484 - 9883\nu) + 1732 > 0.$$

Similarly,

$$c_{1b} = 61440(1-\mu) + (1-\nu)[c_{1b1} + c_{1b2}\mu]$$

where

$$\begin{aligned} c_{1b1} &= (-135\nu^7 + 693\nu^6 + 417\nu^5 + 4821\nu^4) - 64349\nu^3 + 134023\nu^2 - 48061\nu + 12527 \\ &\geq -64349\nu^3 + 134023\nu^2 - 48061\nu + 12527 \geq 1000(-67\nu^3 + 134\nu^2 - 67\nu + 12) \\ &= \frac{1000}{27}\left[56 + 67(4-3\nu)(1-3\nu)^2\right] > 0 \end{aligned}$$

and

$$\begin{aligned} c_{1b2} &= (-81\nu^7 + 27\nu^6 + 1287\nu^5 + 13563\nu^4) - 71211\nu^3 + 85929\nu^2 - 66699\nu + 894 \\ &\geq -71211\nu^3 + 85929\nu^2 - 66699\nu + 89409 > 80000(-\nu^3 + \nu^2 - \nu + 1) \geq 0. \end{aligned}$$

So, $c_{1b1}, c_{1b2} > 0 \implies c_{1b} > 0$ and since $c_{1a} > 0$ we have $c_1 \geq 0$ and thus $e_1 \geq 0$.

Proof that $e_2 \geq 0$

We have

$$\frac{\partial^2 e_2}{\partial a^2} = 21\nu^4 - 206\nu^3 + 136\nu^2 + 398\nu + 1571 > 0$$

so similarly to the previous case

$$e_2 \geq e_2|_{a=a_{cr}} = \frac{3(1+\nu^2)[582912(1-\mu)^2 + (1-\nu)c_2]}{8[21\nu^4 - 206\nu^3 + 136\nu^2 + 398\nu + 1571]}$$

where

$$a_{cr} = \frac{3\nu^5 - 60\nu^4 + 296\nu^3 - 82\nu^2 - 155\nu - 2786 + \left(18\nu^3 - 222\nu^2 - 150\nu + 2010\right)(1+\nu)^2\mu}{2[21\nu^4 - 206\nu^3 + 136\nu^2 + 398\nu + 1571]}$$

solves $\frac{\partial e_2}{\partial a} = 0$ and

$$\begin{aligned} c_2 &= 3\nu^7 - 123\nu^6 + 1330\nu^5 - 1918\nu^4 - 28897\nu^3 + 65177\nu^2 + 93100\nu + 120544 \\ &\quad + \left(36\nu^7 - 624\nu^6 + 348\nu^5 + 25616\nu^4 - 7332\nu^3 - 272368\nu^2 - 134556\nu + 688784\right) \\ &\quad + \left(108\nu^7 - 72\nu^6 - 4848\nu^5 - 35916\nu^4 + 247548\nu^3 - 252720\nu^2 + 144456\nu - 647676\right)\mu^2 \end{aligned}$$

Now,

$$\frac{\partial^2 c_2}{\partial \mu^2} = \nu^4(216\nu^3 - 144\nu^2 - 9696\nu - 71832) + \nu^3(495096\nu - 505440) + (288912\nu - 1295352) < 0$$

hence the minimum of c_2 w.r.t. $\mu \in [0,1]$ can be achieved either at $\mu = 0$ or at $\mu = 1$. At the same time

$$\begin{aligned} c_2|_{\mu=0} &= 3\nu^7 + 1330\nu^5 + 65177\nu^2 + 93100\nu + (120544 - 123\nu^6 - 1918\nu^4 - 28897\nu^3) > 0, \\ c_2|_{\mu=1} &= (1-\nu)(161652 - 147\nu^6 + 672\nu^5 + 3842\nu^4 + 16060\nu^3 + (264652 - 195259\nu)\nu) \geq 0, \end{aligned}$$

so $c_2 \geq 0$ and hence $e_2 \geq 0$.

Proof that $e_3 \geq 0$

We have

$$\frac{\partial^2 e_3}{\partial a^2} = 3\nu^4 - 55\nu^3 - 21\nu^2 + 471\nu + 1010 > 0$$

so similarly to the previous case

$$e_3 \geq e_3|_{a=a_{cr}} = \frac{3(1+\nu)^2\left[(1-\nu)^2 c_{3a} + (1-\mu)c_{3b}\right]}{8(3\nu^4 - 55\nu^3 - 21\nu^2 + 471\nu + 1010)}$$

where

$$a_{cr} = \frac{-9\nu^4 + 67\nu^3 + 165\nu^2 - 579\nu - 1820 + 3(1+\nu)^2\mu(\nu^3 - 21\nu^2 - 63\nu + 499)}{2(3\nu^4 - 55\nu^3 - 21\nu^2 + 471\nu + 1010)}$$

solves $\frac{\partial e_3}{\partial a} = 0$ and

$$\begin{aligned}
c_{3a} &= -3\nu^6 + 12\nu^5 + 632\nu^4 + 1794\nu^3 - 37624\nu^2 + 65244\nu + 64877 > 0 \\
c_{3b} &= 2(1-\nu)(-3\nu^7 + 12\nu^6 + 652\nu^5 + 2417\nu^4 - 42561\nu^3 + 73864\nu^2 + 41336\nu + 91323) \\
&\quad + (1-\mu)(12\nu^7 - 3\nu^8 + 660\nu^6 + 2980\nu^5 - 45986\nu^4 + 79796\nu^3 + 1780\nu^2 + 218524\nu + 8477) \geq 0
\end{aligned}$$

Hence $e_3 \geq 0$.

Proof that $e_4 \geq 0$

We have

$$\frac{\partial^2 e_4}{\partial a^2} = 209\nu + 317 - 5\nu^3 - 17\nu^2 > 0$$

so similarly to the previous case

$$e_4 \geq e_4|_{a=a_{cr}} = \frac{3(1+\nu)^2[(1-\nu)^2 c_{4a} + 4(1-\mu)c_{4b}]}{8(209\nu + 317 - 5\nu^3 - 17\nu^2)}$$

where

$$a_{cr} = \frac{5\nu^3 + 71\nu^2 - 329\nu - 587 + 6\mu(1+\nu)^2(88 - 10\nu - \nu^2)}{2(209\nu + 317 - 5\nu^3 - 17\nu^2)}$$

solves $\frac{\partial e_4}{\partial a} = 0$ and

$$\begin{aligned}
c_{4a} &= 8\nu^4 + 40\nu^3 - 1395\nu^2 + 4354\nu + 4757 > 0 \\
c_{4b} &= 4(1-\nu)(4\nu^5 + 21\nu^4 - 712\nu^3 + 2011\nu^2 + 3102\nu + 3050) \\
&\quad + (1-\mu)(2\nu^6 + 11\nu^5 - 360\nu^4 + 912\nu^3 + 1705\nu^2 + 3655\nu + 543) \geq 0.
\end{aligned}$$

Hence $e_4 \geq 0$.

Proof that $e_5 \geq 0$

We have

$$\frac{\partial^2 e_5}{\partial a^2} = 49 + 41\nu - 2\nu^2 > 0$$

so similarly to the previous case

$$e_5 \geq e_5|_{a=a_{cr}} = \frac{3(1+\nu)^2[(1-\nu)^2 c_{5a} + (1-\mu)c_{5b}]}{2(49 + 41\nu - 2\nu^2)}$$

where

$$a_{cr} = \frac{4\nu^2 - 37\nu - 47 + 3\mu(15-\nu)(1+\nu)^2}{49 + 41\nu - 2\nu^2}$$

solves $\frac{\partial e_5}{\partial a} = 0$ and

$$c_{5a} = 15 - \nu^2 + 15\nu > 0$$

$$c_{5b} = 2(1-\nu)(14\nu^2 + 28\nu + 19 - \nu^3) + (1-\mu)(13\nu^3 + 40\nu^2 + 49\nu + 11 - \nu^4) \geq 0.$$

Hence $e_5 \geq 0$.

As a result, $s_5 \geq 0$ and thus $\mathbf{I}_3 \leq 0$.

Case 4: $\mathbf{I}_4 \leq 0$

Here

$$\mathsf{n}(\mathbf{A}_4) = -4(M\mu p - M + p - 1)^2 s_6$$

where

$$s_6 = 2p - 2 + (3\mu + 6p - 3 - 4\mu p - 2p^2)M + (4\mu p^2 - 5\mu p + 3\mu - 2p)M^2 + (1 - \mu p)M^3$$

Then with $M = 2 + \delta$

$$\frac{\partial s_6}{\partial \delta} = [5(1-\mu) + 2(1-p)(p + 2 + 10\mu - 8\mu p)] + [8(1-\mu) + 2(1-p)(2 + 7\mu - 4\mu p)]\delta \geq 0$$

and as a result for $\delta \geq 0$ we have

$$s_6 \geq s_6|_{\delta=0} = 2(3-2p)[p(1-\mu) + \mu(1-3p)] \geq 0.$$

Case 5: $\mathbf{I}_5 \leq 0$

Here

$$\mathsf{n}(\mathbf{A}_5) = -s_7.$$

We need to show that $s_7 \geq 0$ when $X_1 \leq 0$ and $X_3 \geq 0$.

Since $X_1 \leq 0$, we have $2Mp\mu \leq M + 1$. Together with $X_3 \geq 0$ this implies

$$0 \leq \mathsf{n}(X_3) = 2Mp\mu - (M+1) - a(M+1) + 2p \leq -a(M+1) + 2p$$

whence

$$a \leq \frac{2p}{M+1}.$$

Let us show that for this a we have $s_7 \geq 0$; we also substitute Let $a = b \cdot \frac{2p}{M+1}$, where $b \in [0,1]$.

First, **let** $M = 2$, then $\mu = \frac{1+a}{2}$, $p \in [3/4, 1)$, and $s_7 = \frac{3-2p}{27} s_8$ where

$$\begin{aligned} s_8 = {} & 512b^3p^8 - 2688b^3p^7 + 5760b^3p^6 + 3456b^2p^7 - 6912b^3p^5 - 12672b^2p^6 + 5184b^3p^4 \\ & + 16416b^2p^5 + 5184bp^6 - 1944b^3p^3 - 11664b^2p^4 - 10368bp^5 + 7776b^2p^3 + 1728p^5 - 2916b^2p^2 \\ & + 11664bp^3 - 11664bp^2 - 7776p^3 + 4374bp + 17496p^2 - 17496p + 6561 \end{aligned}$$

Note that we can write $s_8 = e_1 + e_2(1-p) + e_3(1-p)^2$, where

$$
\begin{aligned}
128e_1 &= (9-\nu^2-6\nu)(81-\nu^3-9\nu^2-63\nu)(\nu^3+15\nu^2+81-9\nu) > 0\\
128e_2 &= 3(9-\nu^2)(\nu^6+21\nu^5+168\nu^4+666\nu^3+81\nu^2+81\nu+486) > 0\\
64(e_1+e_3) &= [2\nu^8+33\nu^7+234\nu^6+783\nu^5] + [-648\nu^4-6561\nu^3+30618\nu^2-28431\nu+13122]\\
&\geq -648\nu^4-6561\nu^3+30618\nu^2-28431\nu+13122\\
&\geq -1000\nu^4-7000\nu^3+24000\nu^2-29000\nu+13000\\
&= 1000(1-\nu)(5+8(1-\nu)^2+\nu^3) \geq 0.
\end{aligned}
$$

with $p = \frac{3+\nu}{4}$, $\nu \in [0,1]$. Consequently, since $(1-p)^2 < 1$ and $e_1 > 0$,

$$s_8 = e_1 + e_2(1-p) + e_3(1-p)^2 \geq e_2(1-p) + (e_1+e_3)(1-p)^2 \geq 0$$

and thus $s_7 \geq 0$ as required.

For $M \geq 3$, set $M = 3+\delta$, $\delta \geq 0$. Then

$$s_7 = \sum_{i=0}^{9} e_{i+1}\delta^i$$

where

$$
\begin{aligned}
e_1 &= 196608 + (98304b - 393216)p + (-49152b^2 - 442368\mu^2 - 196608b - 737280\mu + 589824)p^2\\
&\quad + (-24576b^3 + 221184b\mu^2 + 98304b^2 + 1990656\mu^2 + 294912b + 663552\mu - 540672)p^3\\
&\quad + (49152b^3 + 73728b^2\mu - 552960b\mu^2 - 331776\mu^3 - 147456b^2 - 2322432\mu^2 - 270336b\\
&\qquad + 110592\mu + 233472)p^4\\
&\quad + (-83968b^3 + 184320b^2\mu - 55296b\mu^2 + 774144\mu^3 + 135168b^2 + 1050624\mu^2 + 116736b\\
&\qquad - 239616\mu - 36864)p^5\\
&\quad + (52224b^3 - 175104b^2\mu + 165888b\mu^2 - 331776\mu^3 - 58368b^2 - 165888\mu^2 - 18432b + 55296\mu)p^6\\
&\quad + (-9216b^3 + 27648b^2\mu + 9216b^2)p^7
\end{aligned}
$$

$$
\begin{aligned}
e_2 &= 393216 + (172032b - 540672)p + (-73728b^2 - 958464\mu^2 - 221184b - 2088960\mu + 835584)p^2\\
&\quad + (-30720b^3 + 423936b\mu^2 + 86016b^2 + 4589568\mu^2 + 344064b + 1898496\mu - 823296)p^3\\
&\quad + (30720b^3 + 282624b^2\mu - 1308672b\mu^2 - 663552\mu^3 - 135168b^2 - 5031936\mu^2 - 344064b\\
&\qquad - 18432\mu + 344064)p^4\\
&\quad + (-77312b^3 + 181248b^2\mu + 4608b\mu^2 + 1658880\mu^3 + 138240b^2 + 2068992\mu^2 + 142848b\\
&\qquad - 360960\mu - 49152)p^5\\
&\quad + (50176b^3 - 228864b^2\mu + 290304b\mu^2 - 691200\mu^3 - 56832b^2 - 290304\mu^2 - 19968b + 78336\mu)p^6\\
&\quad + (-7680b^3 + 32256b^2\mu + 7680b^2)p^7
\end{aligned}
$$

$$
\begin{aligned}
e_3 &= 344064 + (129024b - 208896)p + (-46080b^2 - 906240\mu^2 - 49152b - 2558976\mu + 430080)p^2\\
&\quad + (-15360b^3 + 347136b\mu^2 + 3072b^2 + 4718592\mu^2 + 129024b + 2217984\mu - 522240)p^3\\
&\quad + (-6144b^3 + 334848b^2\mu - 1337856b\mu^2 - 566784\mu^3 - 30720b^2 - 4778496\mu^2 - 175104b
\end{aligned}
$$

$$- 198144\mu + 210432)p^4$$
$$+ (-24448b^3 + 42240b^2\mu + 100992b\mu^2 + 1543680\mu^3 + 52992b^2 + 1744512\mu^2 + 69504b$$
$$- 223872\mu - 26112)p^5$$
$$+ (17856b^3 - 118464b^2\mu + 210816b\mu^2 - 615168\mu^3 - 20544b^2 - 210816\mu^2 - 8064b + 44160\mu)p^6$$
$$+ (-2112b^3 + 14016b^2\mu + 2112b^2)p^7$$

$$e_4 = 172032 + (53760b + 64512)p + (-15360b^2 - 488448\mu^2 + 44544b - 1790976\mu + 64512)p^2$$
$$+ (-3840b^3 + 157440b\mu^2 - 23040b^2 + 2843904\mu^2 + 1416960\mu - 176640)p^3$$
$$+ (-9984b^3 + 193536b^2\mu - 772608b\mu^2 - 268032\mu^3 + 7680b^2 - 2598912\mu^2 - 44544b$$
$$- 170496\mu + 68352)p^4$$
$$+ (93024b\mu^2 - 2464b^3 - 13632b^2\mu + 814336\mu^3 + 9024b^2 + 816480\mu^2 + 16800b - 73056\mu - 6912)p^5$$
$$+ (2784b^3 - 30336b^2\mu + 81312b\mu^2 - 303168\mu^3 - 3264b^2 - 81312\mu^2 - 1440b + 12384\mu)p^6$$
$$+ (-192b^3 + 2688b^2\mu + 192b^2)p^7$$

$$e_5 = 53760 + (13440b + 91392)p + (-2880b^2 - 164160\mu^2 + 34560b - 792768\mu - 26880)p^2$$
$$+ (-480b^3 + 42720b\mu^2 - 11520b^2 + 1109088\mu^2 - 13440b + 548640\mu - 33600)p^3$$
$$+ (62496b^2\mu - 3264b^3 - 275952b\mu^2 - 75792\mu^3 + 4800b^2 - 885744\mu^2 - 5664b - 67536\mu + 12432)p^4$$
$$+ (160b^3 - 8544b^2\mu + 38928b\mu^2 + 266192\mu^3 + 576b^2 + 229104\mu^2 + 2016b - 13200\mu - 912)p^5$$
$$+ (160b^3 - 3840b^2\mu + 17568b\mu^2 - 89344\mu^3 - 192b^2 - 17568\mu^2 - 96b + 1728\mu)p^6 + 192b^2\mu p^7$$

$$e_6 = 10752 + (2016b + 38976)p + (-288b^2 - 35232\mu^2 + 10848b - 230880\mu - 14784)p^2$$
$$+ (-24b^3 + 6936b\mu^2 - 2544b^2 + 290664\mu^2 - 4032b + 132888\mu - 3408)p^3$$
$$+ (11568b^2\mu - 456b^3 - 62472b\mu^2 - 12816\mu^3 + 816b^2 - 193752\mu^2 - 288b - 14328\mu + 1200)p^4$$
$$+ (32b^3 - 1536b^2\mu + 8688b\mu^2 + 55168\mu^3 + 38544\mu^2 + 96b - 1248\mu - 48)p^5$$
$$+ (-192b^2\mu + 2016b\mu^2 - 15744\mu^3 - 2016\mu^2 + 96\mu)p^6$$

$$e_7 = 1344 + (168b + 9072)p + (-12b^2 - 4716\mu^2 + 1824b - 44340\mu - 3024)p^2$$
$$+ (624b\mu^2 - 276b^2 + 51252\mu^2 - 504b + 19764\mu - 144)p^3$$
$$+ (-24b^3 + 1152b^2\mu - 8760b\mu^2 - 1200\mu^3 + 48b^2 - 26568\mu^2 - 1584\mu + 48)p^4$$
$$+ (-96b^2\mu + 1008b\mu^2 + 7072\mu^3 + 3600\mu^2 - 48\mu)p^5 + (96b\mu^2 - 1536\mu^3 - 96\mu^2)p^6$$

$$e_8 = 96 + (6b + 1236)p + (-360\mu^2 + 162b - 5424\mu - 300)p^2$$
$$+ (24b\mu^2 - 12b^2 + 5868\mu^2 - 24b + 1656\mu)p^3 + (48b^2\mu - 696b\mu^2 - 48\mu^3 - 2088\mu^2 - 72\mu)p^4$$
$$+ (48b\mu^2 + 512\mu^3 + 144\mu^2)p^5 - 64\mu^3p^6$$

and e_9 and e_{10} are given further.

Now we will show that $e_i \geq 0$ $i = 1, \ldots, 9$.

Proof that $e_1, \ldots, e_8 > 0$

It turns out that the easiest is to use a computer-assisted proof here; to this end we developed the method which we call *Box method*; perhaps it was known, but we do not know the proper reference, hence we describe it below.

First of all, we substitute

$$p = \frac{1+x_1}{2}, \; b = x_2, \mu = x_3; \quad x_i \in [0,1], \; i = 1,2,3.$$

Let $\mathrm{m} = \min_{a_i \le x_i \le b_i, i=1,2,3} f(x_1, x_2, x_3)$ where

$$f(x_1, x_2, x_3) = f_+(x_1, x_2, x_3) - f_-(x_1, x_2, x_3)$$

and f_+ and f_- are polynomials with non-negative coefficients. We want to show that $\mathrm{m} > 0$.

Let

$$G_{f;M} = \min_{i_1, i_2, i_3 = 0, \ldots, M-1} \left[f_+\left(\frac{i_1}{M}, \frac{i_2}{M}, \frac{i_3}{M}\right) - f_-\left(\frac{i_1+1}{M}, \frac{i_2+1}{M}, \frac{i_3+1}{M}\right)\right].$$

Since

$$\mathrm{m} \ge G_{f;M} \to \mathrm{m}$$

as $M \to \infty$, we conclude that $\mathrm{m} > 0$ if and only if $G_{f,M} \ge 0$ for some $M \ge 1$. Checking that $G_{f,M} \ge 0$ can be quite laborious for large M, however this could be easily done with the help of a computer; please not the results are still *completely rigorous*, unlike e.g. simulations. The results are presented in the following table:

$$\begin{array}{cccc} G_{e_1,2000} > 825, & G_{e_2,500} > 25, & G_{e_3,400} > 1860, & G_{e_4,300} > 2397, \\ G_{e_5,200} > 672, & G_{e_6,200} > 148, & G_{e_7,200} > 5, & G_{e_8,400} > 3. \end{array}$$

Consequently, $e_1 > 0, \ldots, e_8 > 0$.

Proof that $e_9 \ge 0$ and $e_{10} \ge 0$

The method for the previous section will not work for e_9 and e_{10} as these functions touch zero in the required area, and hence the minimum is, in fact, 0. Therefore, the box method introduced above wouldn't work and we have to handle these two cases analytically.

We have

$$e_9 = 4p^2\mu(4\mu^2p^3 - 18\mu p^2 + 99\mu p - 3 + 15p - 96) - 12p^2 + 93p + 3 + [6p^2(1-2\mu p)(2\mu p + 1)]b$$

hence the minimum is achieved either at $b = 0$ or $b = 1$.

For $\mu < 1/(2p)$ we have $e_9 \ge e_{9a}$, where

$$\begin{aligned} e_{9a} = e_9|_{b=0} &= 2s^3p^2 - 18p^2s^2 + 30sp^2 + 99s^2p - 12p^2 - 192ps - 3s^2 + 93p + 3 \\ &= 2p^2 + (1-s)[6(1-p) + (1-s)(99p + 2p^2s - 14p^2 - 3)] \ge 0 \end{aligned}$$

where $s = 2p\mu \in [0,1]$.

In case $\mu \geq 1/(2p)$ we have $e_9 \geq e_{9b}$, where

$$e_{9b} = e_9|_{b=1} = 16p^5s^3 - 24p^4s^3 - 72p^4s^2 + 12p^3s^3 + 468p^3s^2 - 2s^3p^2 - 24p^3s - 426p^2s^2 + 24sp^2 + 111s^2p + 2p^2 - 18ps - 3s^2 + 6s$$

where $\mu = \frac{1}{2p} + s\left(1 - \frac{1}{2p}\right)$, $s \in [0,1]$. Now,

$$\frac{\partial^2}{\partial s^2} e_{9b} = 6(2p-1)^2(14 + (2p-1)(2p^2 s - 3p + 15)) \geq 0$$

so the minimum of e_{9b} w.r.t. s is achieved where $\frac{\partial}{\partial s} e_{9b} = 0$, i.e.

$$s_{cr} = \frac{6p^2 - 33p + 1 + R}{2p^2(2p-1)}, \quad \text{where} \quad R = \sqrt{44p^4 - 400p^3 + 1105p^2 - 66p + 1}$$

and equals

$$\frac{3996p^5 - 284p^6 - 19956p^4 + 37329p^3 - 3291p^2 + 99p - 1 + (400p^3 - 44p^4 - 1105p^2 + 66p - 1)R}{2p^4}$$

$$\geq 22120.5 - 1576\sqrt{197} = 0.285896\dot{>}0$$

for $p \geq 1/2$.

Finally, trivially we have

$$e_{10} = 3p(2\mu p - 1)^2 \geq 0.$$

Consequently, $s_7 \geq 0$ and $\mathbf{I}_5 \leq 0$. Combining all of the above; it proves Lemma 2.

6 Alternative proof of Theorem 2

Proof. Assume for now that $p < \frac{N}{2}$; in this case $N - \frac{N}{2p} + 1 < N$ (see (3.5)). The case $p \geq \frac{N}{2}$ will be dealt with separately.

Claim 5. *Let* $A_i := \left\{x_{(i)}(t) \to 0\right\}$ *and suppose that for some* $1 \leq k \leq N-2$ *we have*

$$\{2p(N-k-1) < N\} \Longleftrightarrow \left\{k > N - \frac{N}{2p} - 1\right\}. \tag{6.15}$$

Then $A_k \subseteq \{\exists \lim_{t\to\infty} x_{(k+1)}(t)\}$.

Proof. Fix any $a > 0$. Let $\delta > 0$ be so small that

$$2pN\delta < [N - 2p(N-k-1)]a. \tag{6.16}$$

In the event A_k there exists a finite $\tau = \tau_\delta(\omega)$ such that

$$\left\{\sup_{t\geq\tau} x_{(k)}(t) \leq \delta\right\} \Longleftrightarrow \{\text{card } (\mathcal{X}'(t) \cap [0,\delta]) \geq k \text{ for all } t \geq \tau.\}$$

From now on assume that $t \geq \tau$. We will show below that $x_{(k+1)}(t+1) \leq \max\{x_{(k+1)}(t), a\}$.

To begin, let us prove that $x_{(k+1)}(t+1) \leq x_{(k+1)}(t)$ as long as $x_{(k+1)}(t) > \delta$. Indeed, if the new point ζ is sampled to the left of $x_{(k+1)}(t)$, then regardless of which point is to be removed, $x_{(k+1)}(t+1) \leq x_{(k+1)}(t)$. If the new point ζ is sampled to the right, then the farthest point from the $p-$centre of mass must be the rightmost one (and hence $x_{(k+1)}(t+1) = x_{(k+1)}(t)$) since there are exactly k points in $[0,\delta]$ and none of these can be removed by the definition of τ.

On the other hand, if $x_{(k+1)}(t) \leq \delta$ then either $x_{(k+2)}(t) \leq a$ or $x_{(k+2)}(t) > a$. In the first case, $x_{(k+1)}(t+1) \leq x_{(k+2)}(t) \leq a$ even if $x_{(1)}$ is removed. In the other case, when $x_{(k+2)}(t) > a$, we have $x_{(N-1)} > a$ as well, and

$$\begin{aligned}
p\mu(\mathcal{X}(t+1)) &\leq p\frac{(k+1)\delta + (N-k-1)x_{(N)}}{N} < \frac{2pN\delta - [N-2p(N-k-1)]x_{(N)} + Nx_N}{2N} \\
&\leq \frac{Nx_N - \{[N-2p(N-k-1)]a - 2pN\delta\})}{2N} < \frac{x_{(N)}}{2}
\end{aligned}$$

by (6.16), so $x_{(N)} = x_{(N)}(t)$ must be removed and thus $x_{(k+1)}(t+1) \leq x_{(k+1)}(t)$.

Consequently, we obtained

$$\begin{aligned}
A_k &\subseteq \bigcap_{t\geq\tau} \{x_{(k+1)}(t+1) \leq \max\{x_{(k+1)}(t), a\}\} \\
&\subseteq \left(\bigcup_{t\geq 0} \{x_{(k+1)}(s) \leq a \text{ for all } s \geq t\}\right) \cup \left(\bigcup_{t\geq 0} \{x_{(k+1)}(s) \leq x_{(k+1)}(s+1) \text{ for all } s \geq t\}\right) \\
&\subseteq \left\{\limsup_{t\to\infty} x_{(k+1)}(t) \leq a\right\} \cup \left\{\exists \lim_{t\to\infty} x_{(k+1)}(t)\right\}
\end{aligned}$$

since $a > 0$ is arbitrary, we get

$$A_k \subseteq \left\{\limsup_{t\to\infty} x_{(k+1)}(t) \leq 0\right\} \cup \left\{\exists \lim_{t\to\infty} x_{(k+1)}(t)\right\} = \left\{\exists \lim_{t\to\infty} x_{(k+1)}(t) \geq 0\right\}$$

□

Lemma 13. *Suppose that* (6.15) *holds for some* $1 \leq k \leq N-2$. *Then* $A_k \underset{a.s.}{\Longrightarrow} A_{k+1}$.

Proof. Let $\tilde{A}_{k+1}^{\geq a} := \left\{\lim_{t\to\infty} x_{(k+1)}(t) \geq a\right\}$ (the existence of this limit on A_k follows from Claim 5). It suffices to show that $\mathbb{P}\left(A_k \cap \tilde{A}_{k+1}^{\geq a}\right) = 0$ for all $a > 0$; then from the continuity of probability we get that $\mathbb{P}\left(A_k \cap \{\lim_{t\to\infty} x_{(k+1)}(t) > 0\}\right) = 0$ and hence $A_k \underset{\text{a.s.}}{\Longrightarrow} A_{k+1}$.

Fix an $a > 0$. Let

$$C_t = \left\{x_{(k)}(t) < \frac{a}{3} \text{ and } x_{(k+1)}(t) > \frac{2a}{3}\right\}, \qquad \bar{C}_T = \bigcap_{t\geq T} C_t,$$

then

$$A_k \cap \tilde{A}_{k+1}^{\geq a} \subseteq \bigcup_{T\geq 0} \bar{C}_T = \left\{\exists T > 0 :\ x_{(k)}(t) < \frac{a}{3} \text{ and } x_{(k+1)}(t) > \frac{2a}{3} \text{ for all } t \geq T\right\}.$$

If the probability of the LHS is positive, then, using the continuity of probability and the fact that $\bar{C}_T$ is an increasing sequence of events, we obtain that $\lim_{T\to\infty} \mathbb{P}(\bar{C}_T) > 0$. Consequently, there exists a *non-random* T_0 such that $\mathbb{P}(\bar{C}_{T_0}) > 0$.

This is, however, impossible, as at each time point t, with probability at least $f(a/3)$ (see (2.1)) the new point ζ_t is sampled in $B := \left(\frac{a}{3}, \frac{2a}{3}\right)$ and then either $x_{(k)}(t+1) \in B$ or $x_{(k+1)}(t+1) \in B$. Formally, this means that

$$\mathbb{P}(C_{t+1}|C_t, \mathcal{F}_t) \leq 1 - f(a/3) \quad \text{for all } t \geq 0.$$

By induction, for all $k \geq 1$,

$$\mathbb{P}(\bar{C}_{T_0}|\mathcal{F}_{T_0}) \leq \mathbb{P}\left(\bigcap_{T=T_0}^{T_0+k} C_t|\mathcal{F}_{T_0}\right) \leq [1 - f(a/3)]^k.$$

Since k is arbitrary, and $f(a/3) > 0$, by taking the expectation, we conclude that $\mathbb{P}(\bar{C}_{T_0}) = 0$ yielding a contradiction.

Hence the probability of the event $A_k \cap \tilde{A}_{k+1}^{\geq a}$ is zero. □

Corollary 3. *Suppose that* (6.15) *holds for some* $1 \leq k \leq N-2$. *Then*

$$\left\{x_{(k)}(t) \to 0\right\} \underset{a.s.}{\Longrightarrow} \{\mathcal{X}'(t) \to 0\}.$$

Proof. Observe that if k satisfies (6.15) then $k+1$ satisfies (6.15) as well. Thus by iterating Lemma 13 we obtain that $A_k \underset{\text{a.s.}}{\Longrightarrow} A_{k+1} \underset{\text{a.s.}}{\Longrightarrow} A_{k+2} \underset{\text{a.s.}}{\Longrightarrow} \dots \underset{\text{a.s.}}{\Longrightarrow} A_{N-1}$, i.e. $x_{(N-1)}(t) \to 0$, which is equivalent to the statement of Corollary. □

Remark 6. *Note that the condition* (6.15) *does not assume* $p > 1$*; hence the conclusion of Corollary 3 holds for the case* $0 < p \leq 1$ *as well.*

For the case $p \geq \frac{N}{2}$ we have

Part (b) follows from Lemma 3.

To prove part (c), note that unless $x_{(1)}(0) > 0$ already, by repeating the arguments from the beginning of the proof of Lemma 7, with a positive probability we can "drag" the whole configuration in at most $N-1$ steps to the right of zero, that is, there is $0 \leq t_0 \leq N-1$ such that $\mathbb{P}(\min \mathcal{X}'(t_0) > 0) > 0$. Now we can apply Lemma 4 and then Lemma 3.

Let us now prove part (a). First, assume $p < \frac{N}{2}$. It is always possible to find an integer k which satisfies both (3.5) and (6.15), so let k be such that

$$N - \frac{N}{2p} - 1 < k < N - \frac{N}{2p} + 1$$

(if $N/(2p) \in \mathbb{N}$ this k will be unique). Now the statement of the theorem follows from Corollary 3 and Lemma 7.

Finally, in case $p \geq \frac{N}{2}$ the theorem follows from Lemma 9. $\square$

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