Jante's law process

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Abstract

Fix some integers $d \geq 1$, $N \geq 3$, $1 \leq K < N$ and a $d-$dimensional random variable ζ. Define *an energy* of configuration of m points as the sum of all pairwise distances squared[1]. The process starts with initially N distinct points on $\mathbb{R}^d$. Next, of the total N points keep those $N-K$ which minimize the energy amongst all the subsets of size $N-K$, and replace thrown out points by K i.i.d. points sampled according to ζ, and of the total $N+K$ points keep those N which minimize the energy amongst all the subsets of size N. Repeat this process ad infinitum. We obtain various quite non-restrictive conditions under which the set of points converges to a certain limit. Observe that this is a very substantial generalization of the "Keynesian beauty contest process" introduced in [3] where $K=1$ and the distribution ζ was uniform on the unit cube.

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1 Introduction and main result

We study a generalization of the model presented in Grinfeld et al. [3]. Fix an integer $N \geq 3$ and some d-dimensional random variable ζ. Now arbitrary choose N distinct points on $\mathbb{R}^d$, $d \geq 1$. The process in [3], called there "Keynesian beauty contest process", is a discrete-time process with the following dynamics: given the configuration of N points we compute its center of mass μ and throw away the most distant from μ point; if there is more than one, we choose each one with equal probability. Then this point is replaced with a new point drawn independently each time from the distribution of ζ. In [3] it was shown that when ζ has a uniform distribution on a unit cube, then the configuration converges to some random point on $\mathbb{R}^d$, with the exception of the most distant point.

The aim of this paper is to remove the assumption on uniformity of ζ and obtain

[1] Please note that in physics this often corresponds to the moment of inertia; however, it can be viewed as "the energy" from the perspective of potential theory. For simplicity, we will use this term in the current paper.

some general sufficient conditions under which the similar convergence takes place. Additionally, it turns out we can naturally generalize the process by removing not just one but $K \geq 2$ points at the same time, and then replacing them with new K i.i.d. points sampled from ζ. We also give the process we introduce a different name, which we believe describes its essence much better. The "Law of Jante" is the concept that there is a pattern of group behaviour towards individuals within Scandinavian countries that criticises individual success and achievement as unworthy and inappropriate, in other words, it is better to be "like everyone else". The concept was created by Aksel Sandemose in [1], identified the Law of Jante as ten rules, and has been a very popular concept in Nordic countries since then.

We will use mostly the same notations as in [3]. Namely, let $\mathcal{X}_n = (x_1, x_2, \dots, x_n)$ for a vector of n points $x_i \in \mathbb{R}^d$; let $\mu_n(\mathcal{X}_n) := n^{-1} \sum_{i=1}^n x_i$ be the barycentre of $\mathcal{X}_n$. Denote by $\text{ord}(\mathcal{X}_n) = (x_{(1)}, x_{(2)}, \dots, x_{(n)})$ the barycentric order statistics of $x_1, \dots, x_n$, so that

$$\|x_{(1)} - \mu_n(\mathcal{X}_n)\| \leq \|x_{(2)} - \mu_n(\mathcal{X}_n)\| \leq \dots \leq \|x_{(n)} - \mu_n(\mathcal{X}_n)\|.$$

Here and throughout the paper $\|x\|$ denotes the Euclidean norm in $\mathbb{R}^d$, $x \cdot y$ is a dot product of two vectors $x, y \in \mathbb{R}^d$, and $B_r(x) = \{y \in \mathbb{R}^d : \|y - x\| < r\}$ is an open ball of radius r centred at x. As in [3] let us also define for $\mathcal{X}_n = (x_1, x_2, \dots, x_n) \in \mathbb{R}^{dn}$

$$G_n(\mathcal{X}_n) := G_n(x_1, \dots, x_n) := \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^{i-1} \|x_i - x_j\|^2 = \sum_{i=1}^n \|x_i - \mu_n(\mathcal{X}_n)\|^2 = \inf_{y \in \mathbb{R}^d} \sum_{i=1}^n \|x_i - y\|^2.$$

We can think of $G_n(\mathcal{X}_n)$ as of a measure of "diversity" among individuals with properties $x_1, \dots, x_n$.

In [3] where $K = 1$, the authors called $x_{(n)}$ the *extreme* point of $\mathcal{X}_n$, that is, a point of $x_1, \dots, x_n$ farthest from the barycentre, and the defined *core* of $\mathcal{X}_n$ as $\mathcal{X}'_n := (x_{(1)}, \dots, x_{(n-1)})$, the vector of $x_1, \dots, x_n$ with (one of) the extreme point removed. They also defined $F_n(\mathcal{X}_n) := G_{n-1}(\mathcal{X}'_n)$ and $F(t) = F_N(\mathcal{X}(t))$.

In our paper, when $K \geq 1$, we re-define the core as the subset of $x_1, \dots, x_N$ containing $N - K$ elements which minimizes the diversity of the remaining individuals, that is the subset which minimizes

$$\min_{\{y_1, \dots, y_{N-K}\} \subset \{x_1, \dots, x_N\}} G_{N-K}(y_1, \dots, y_{N-K}).$$

We will show below that, in fact, when $K = 1$ both definitions coincide.

The process runs as follows: Let $\mathcal{X}(t) = \{X_1(t), \dots, X_N(t)\}$ be distinct points in $\mathbb{R}^d$. Given $\mathcal{X}(t)$, let $\mathcal{X}'(t)$ be the core of $\mathcal{X}(t)$ and replace $\mathcal{X}(t) \setminus \mathcal{X}'(t)$ by K i.i.d. ζ-distributed random variables so that

$$\mathcal{X}(t+1) = \mathcal{X}'(t) \cup \{\zeta_{t+1;1}, \dots, \zeta_{t+1;K}\},$$

where $\zeta_{t;j}$, $t = 1, 2, \dots$, $j = 1, 2, \dots, K$, are i.i.d. random variables with a common distribution ζ. In case there is more than one element in the core, that is, a few configurations which minimize diversity, we chose any of it with equal probability, precisely as it was done in [3]. Now let $F(t) = G_{n-K}(\mathcal{X}'(t))$.

Finally, to finish specification of the process, we allow the initial configuration $\mathcal{X}(0)$ be arbitrary or random, with the only requirement that all the points of $\mathcal{X}(0)$ must lie in the support of ζ.

The following statement links the case $K = 1$ with the general $K \geq 1$.

Lemma 1. *If $K = 1$ then the only point not in the core is the one which is the furthermost from the center of mass of $\mathcal{X}$.*

Proof. Let $\mathcal{X} = (x_1, \dots, x_N)$. W.l.o.g. assume $\sum_{i=1}^{N} x_i = \mathbf{0} \in \mathbb{R}^d$ and thus the center of mass of $\mathcal{X}$ is located at $\mathbf{0}$. Here L consists of all subsets of $\{1, \dots, N\}$ containing just one element. If we throw away the l-th point, denoting $\mu_l = \frac{1}{N-1} \sum_{i \neq l} x_i = -\frac{x_l}{N-1}$ we get

$$\begin{aligned} G(l, \mathcal{X}) &= \sum_{i=1}^{N} \|x_i - \mu_l\|^2 - \|x_l - \mu_l\|^2 = \sum_{i=1}^{N} \|x_i\|^2 + N\|\mu_l\|^2 - 2\mu_l \cdot \sum_{i=1}^{N} x_i - \|x_l - \mu_l\|^2 \\ &= \sum_{i=1}^{N} \|x_i\|^2 + N\frac{\|x_l\|^2}{(N-1)^2} - \frac{\|x_l N\|^2}{(N-1)^2} = -\|x_l\|^2 \frac{N}{(N-1)^2} + \sum_{i=1}^{N} \|x_i\|^2. \end{aligned}$$

Therefore, the minimum of $G(l, \mathcal{X})$ is achieved by choosing an x_l with the largest $\|x_l\|$, that is, the furthermost from the centre of mass. □

Corollary 1. *For $K = 1$ Jante's law process coincides with the process studied in [3].*

The following statement is a trivial consequence of the definition of F.

Lemma 2. *For any $1 \leq K \leq N - 2$ and any distribution of ζ we have $F(t+1) \leq F(t)$.*

In case $K = 1$ the above statement coincides with Corollary 2.1 in [3].

Remark 1. *It is worth noting that throwing away $\mathcal{X}^*$ in general does not mean necessary throwing the K furthest points from the centre of mass of $\mathcal{X}$, unlike the case $K = 1$.*

Here's an example with $d = 1$, $N = 5$ and $K = 3$: set $\mathcal{X} = (-24, -19, -14, 28, 29)$. Then the centre of mass is at $\mu = 0$ and thus 28 and 29 have the largest and the second largest distance from μ, while it is clear that the energy is minimized by keeping exactly these two points in the core and throwing away the rest.

Finally, define *the range* of the configuration: for $n \geq 2$ and $x_1, \ldots, x_n \in \mathbb{R}^d$, write

$$D_n(x_1, \ldots, x_n) := \max_{1 \leq i,j \leq n} \|x_i - x_j\|.$$

The following statement is taken from [3] (Lemma 2.2.).

Lemma 3. *Let $n \geq 2$ and $x_1, \ldots, x_n \in \mathbb{R}^d$. Then*

$$\frac{1}{2} D_n(x_1, \ldots, x_n)^2 \leq G_n(x_1, \ldots, x_n) \leq \frac{1}{2}(n-1) D_n(x_1, \ldots, x_n)^2.$$

Let $D(t) = D_{N-K}(\mathcal{X}'(t))$. Then we have from Lemma 3

$$\sqrt{\frac{2}{N-K-1} \cdot F(t)} \leq D(t) \leq \sqrt{2F(t)}. \tag{1.1}$$

From Lemmas 2 and 3 it also follows immediately that

$$D(t+1) \leq \sqrt{2F(t)} \leq D(t)\sqrt{N-K-1}. \tag{1.2}$$

Let also $\mu'(t) = \mu_{N-K}(\mathcal{X}'(t))$ be the centre of mass of the core.

Assumption 1. $2K < N$.

Observe that if Assumption 1 is not fulfilled, then all the points of the points of the core can migrate large distances and that $F = 0$ does not necessarily imply that the configuration stops moving. For example, one can take $N = 4$, $K = 2$, and $\zeta \sim$ *Bernoulli*(p) to see that the core jumps from 0 to 1 and back infinitely often a.s.

In the other case, the new core must contain at least one point of the old core, and the following statement shows that if newly sampled points are far from the core, they immediately get rejected.

Lemma 4. *Under Assumption 1, if all the distances between K newly sampled points and the points of the core are more than $C := D\sqrt{N-K-1}$ then $\mathcal{X}'(t+1) = \mathcal{X}'(t)$.*

Proof. Since $N-2K \geq 1$ the new core $\mathcal{X}'(t+1)$ must contain at least one point of the old core $\mathcal{X}'(t)$. By (1.2) $D(t+1) \leq D(t)\sqrt{N-K-1}$ and therefore if one of the new points is in the new core, it should be no further than $D(t)\sqrt{N-K-1}$ from one of the points of the old core. □

Finally, we will use the following notations. For any two sets $A, B \subset \mathbb{R}^d$ let

$$\text{dist}(A,B) = \inf_{x \in A, y \in B} \|x-y\|.$$

If $d=1$ then write $\mathcal{X}'(t) \to +\infty$ if $\lim_{t\to\infty} \min\{x,\ x \in \mathcal{X}'(t)\} = \infty$ and similarly $\mathcal{X}'(t) \to -\infty$ if $\lim_{t\to\infty} \max\{x,\ x \in \mathcal{X}'(t)\} = -\infty$. If $d \geq 2$ we will write $\mathcal{X}'(t) \to \infty$ if $\min\{\|x\|,\ x \in \mathcal{X}'(t)\} = \text{dist}(\mathcal{X}'(t), 0) \to \infty$, otherwise we will write $\mathcal{X}'(t) \not\to \infty$. We will also write $\mathcal{X}'(t) \to \phi \in \mathbb{R}^d$ if all the coordinates of $\mathcal{X}'(t)$ converge to ϕ as $t \to \infty$.

2 Shrinking

Let ζ be *any* random variable on $\mathbb{R}^d$. As usual, define the support of this random variable as

$$\text{supp}\,\zeta = \overline{\{A \in \mathbb{R}^d :\ \mathbb{P}(\zeta \in A) > 0\}} = \{x \in \mathbb{R}^d :\ \forall \varepsilon > 0\ \ \mathbb{P}\left(\zeta \in B_\epsilon(x)\right) > 0\},$$

where the overline denotes set closure (see e.g. [5]). We also say that $\text{supp}\,\zeta$ is bounded in $\mathbb{R}^d$ if there is an $M > 0$ such that $\mathbb{P}(\|\zeta\| < M) = 1$.

It turns out that the following statement, which is probably known, is true.

Proposition 1. $\text{supp}\,\zeta$ *is bounded if and only if there exists some function* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *such that for any* $x \in \text{supp}\,\zeta$

$$\mathbb{P}\left(\zeta \in B_\delta(x)\right) \geq f(\delta)$$

for all $\delta > 0$.

Proof. Suppose such a function exists, but the support of ζ is not bounded. Fix any $\Delta > 0$. Then there must exist a infinite sequence of points $\{x_n\}_{n=1}^\infty \subseteq \text{supp}\,\zeta$, such that $\|x_i - x_j\| > 2\Delta$, whenever $i \neq j$. Since the sets $\{B_\Delta(x_n)\}$ are disjoint, this would imply that

$$\mathbb{P}\left(\zeta \in \mathbb{R}^d\right) \geq \mathbb{P}\left(\bigcup_{n=1}^\infty \{\zeta \in B_\Delta(x_n)\}\right) \geq \sum_{n=1}^\infty f(\Delta) = \infty$$

which is impossible.

Conversely, assume that $\operatorname{supp}\zeta$ is bounded. For all $\delta > 0$ define

$$f(\delta) = \inf_{x \in \operatorname{supp}\zeta} \mathbb{P}(\|\zeta - x\| \leq \delta).$$

We will show that $f(\delta) > 0$. Indeed, if not, there exists a sequence $\{x_n\}$ such that $\mathbb{P}(\|\zeta - x_n\| \leq \delta) \to 0$ as $n \to \infty$. Since the support of ζ is compact, $\{x_n\}$ must have a convergent subsequence; w.l.o.g. we can assume that it is $\{x_n\}$ itself and thus there is an x such that $x_n \to x$ and there exists N such that $\|x_n - x\| < \delta/2$ for all $n \geq N$. On the other hand, for these n

$$\mathbb{P}(\|\zeta - x\| \leq \delta/2) \leq \mathbb{P}(\|\zeta - x_n\| \leq \delta)$$

by the triangle inequality. Since the RHS converges to zero, this implies $\mathbb{P}(\|\zeta - x\| \leq \delta/2) = 0$ so $x \notin \operatorname{supp}\zeta$ which contradicts the fact that $x = \lim_{n\to\infty} x_n \in \operatorname{supp}\zeta$ by the definition of the support. □

Theorem 1. *Given any distribution ζ on $\mathbb{R}^d$, for any $N \geq 3$ and $1 \leq K \leq N-2$ we have*

$$\mathbb{P}\left(\{F(t) \to 0\} \bigcup \{\mathcal{X}'(t) \to \infty\}\right) = 1.$$

In particular if ζ has compact support, then $F(t) \to 0$ a.s.

Note that $F(t) \to 0$ is equivalent to $D(t) \to 0$.

Proof. We will first make use of the following lemma.

Lemma 5. *Suppose we are given a bounded set $S \in \mathbb{R}^d$ such that $\mathbb{P}(\zeta \in S) > 0$ and $N-K$ points $x_1, ..., x_{N-K}$ in $\operatorname{supp}(\zeta) \bigcap S$ satisfying $F(\{x_1, ..., x_{N-K}\}) > \varepsilon_1$. Let $\varepsilon_2 = \frac{\varepsilon_1}{2(N-K)^2}$. Then there exists a positive constant σ, only depending on ε_1, S, K and N, such that*

$$\mathbb{P}\left(F\left(\{\zeta_1, \ldots, \zeta_K, x_1, \ldots, x_{N-K}\}'\right) < F(\{x_1, \ldots, x_{N-K}\}) - \varepsilon_2\right) \geq \sigma.$$

Proof. We start with the case $K = 1$. Denote $D = \max_{1\leq i,j\leq N-K} \|x_i - x_j\|$, and $S_* = \{x : \operatorname{dist}(x, S) < D\sqrt{N-K-1}\}$, then the set $\overline{S_*}$ is a compact set such that $\{\zeta, x_1, \ldots, x_{N-1}\}' \in \overline{S_*}$ regardless of where the point ζ is sampled, by Lemma 4. Since $\overline{S_*}$ is compact it follows from Proposition 1 applied to $\zeta \cdot 1_{\{\zeta \in S\}}$ that there is an $f : \mathbb{R}^+ \to \mathbb{R}^+$, such that for any $x \in \operatorname{supp}\zeta \bigcap \overline{S_*}$, we have $\mathbb{P}(\zeta \in B_\delta(x)) \geq f(\delta)$. Assume that the core centre of mass $\mu' = 0$, and that (without loss of generality) $\|x_1\| \geq \|x_l\|, \forall 1 \leq l \leq N-1$. Let $\mu' = \frac{y+x_2+\ldots+x_{N-1}}{N-1}$ and consider the function

$$h(y) = \sum_{i=2}^{N-1} \|x_i - \mu'\|^2 + \|y - \mu'\|^2,$$

continuous in y. Pick a point x_j from $\{x_2, ..., x_{N-1}\}$ such that $\|x_1 - x_j\| \geq \frac{D}{2}$ – otherwise $\|x_i - x_j\| \leq \|x_1 - x_j\| + \|x_1 - x_i\| < D$, for all indices i, j, contradicting the definition of D.

Consider the configuration $\{x_j, x_2, ..., x_{N-1}\}$, where we have removed the point x_1 and replaced it with x_j. This configuration has centre of mass $\mu' = \frac{x_2+\ldots+x_{N-1}+x_j}{N-1} = \frac{x_j - x_1}{N-1}$. The Lyapunov function evaluated for this configuration is precisely $h(x_j)$. Denote $F_{\text{old}} = F(\{x_1, ..., x_{N-1}\})$, then

$$\begin{aligned}
h(x_j) &= \sum_{i=2}^{N-1} \|x_i - \mu'\|^2 + \|x_j - \mu'\|^2 = \sum_{i=1}^{N-1} \|x_i - \mu'\|^2 + \|x_j - \mu'\|^2 - \|x_1 - \mu'\|^2 \\
&= \sum_{i=1}^{N-1} \left(\|x_i\|^2 + \|\mu'\|^2 - 2x_i \cdot \mu'\right) + \|x_j\|^2 + \|\mu'\|^2 - 2x_j \cdot \mu' - \|x_1\|^2 - \|\mu'\|^2 + 2x_1 \cdot \mu' \\
&= \sum_{i=1}^{N-1} \|x_i\|^2 + (N-1)\|\mu'\|^2 + \|x_j\|^2 - \|x_1\|^2 - 2(x_j - x_1) \cdot \left(\frac{x_j - x_1}{N-1}\right) \\
&\leq F_{\text{old}} + \frac{\|x_j - x_1\|^2}{N-1} - 2\frac{\|x_j - x_1\|^2}{N-1} \leq F_{\text{old}} - \frac{D^2}{4(N-1)} \leq \left(1 - \frac{1}{2(N-1)^2}\right) F_{\text{old}},
\end{aligned}$$

where the last inequality follows from (1.1). Hence for some $\delta > 0$ if $\|y - x_j\| \leq \delta$ then $h(y) < \left(1 - \frac{1}{4(N-1)^2}\right) F_{\text{old}}$. So if ζ is sampled in $B_\delta(x_j)$ then we have a substantial decrease and this is with probability bounded below by $f(\delta)$, the result is thus proved for the case $K = 1$ with $\sigma = f(\delta)$.

The general case can be reduced to the case $K = 1$ as follows. Set $N' := N - K + 1$ and replace all N by N' in the arguments above. The decrease of F in this case will be at least by $\varepsilon_2(N')$. Indeed, since if at least one particle falls in the ball $\{y : \|y - x_j\| \leq \delta\}$ we could choose the sub-configuration where exactly one point falls in this ball while x_1 is removed, and since we are taking the minimum over all available configurations, the decrease has to be greater or equal than for this specific choice. □

Assume that $\mathbb{P}(\mathcal{X}'(t) \to \infty) < 1$, otherwise the theorem follows immediately. Recall that $B_r(0)$ is a ball of radius r centred at the origin and note that

$$\{\mathcal{X}'(t) \not\to \infty\} = \bigcup_{r=1}^{\infty} \{\mathcal{X}'(t) \in B_r(0) \ i.o.\} = \bigcup_{r=1}^{\infty} G_r \tag{2.3}$$

where $G_r = \bigcap_{k\geq 0} \{\tau_{k,r} < \infty\}$, $\tau_{k,r} = \inf\{t : t > \tau_{k-1,r}, \mathcal{X}'(t) \in B_r(0)\}$, $\quad k = 1, 2, \dots,$

with the convention that $\tau_{0,r} = 0$, $\inf \emptyset = +\infty$ and that if $\tau_{k,r} = +\infty$ then $\tau_{k',r} = +\infty$ for all $k' \geq k$.

By the monotonicity of F we have $F(t) \downarrow F_\infty \geq 0$. We will show that in fact

$$\mathbb{P}\left(\{\mathcal{X}'(t) \not\to \infty\} \bigcap \{F_\infty > 0\}\right) = 0 \tag{2.4}$$

which is equivalent to the statement of the Theorem.

Let n_0 be some integer larger than $4(N-K)^2$, this quantity being related to ε_2 from Lemma 5. Since

$$\{F_\infty > 0\} = \bigcup_{n=n_0}^{\infty} \left\{F_\infty > \frac{1}{n}\right\} = \bigcup_{n=n_0}^{\infty} \bigcup_{m=0}^{\infty} \{F_\infty \in I_{n,m}\}, \quad \text{where} \quad I_{n,m} = \left[\frac{1}{n} + \frac{m}{n^2}, \frac{1}{n} + \frac{m+1}{n^2}\right)$$

are disjoint sets for each fixed n. Consequently, taking into account (2.3), to establish (2.4) it suffices to show for each fixed n and m and r we have

$$\mathbb{P}\left(G_r \bigcap \{F_\infty \in I_{n,m}\}\right) = 0.$$

Let $A_k = \{F(\tau_{k,r} + 1) \in I_{n,m}\} \bigcap \{\tau_{k,r} < \infty\}$ then obviously

$$G_r \bigcap \{F_\infty \in I_{n,m}\} \subset \bigcup_{k_0 \geq 0} \bigcap_{k \geq k_0} A_k. \tag{2.5}$$

We will show now that for all k_0 we have $\mathbb{P}\left(\bigcap_{k\geq k_0} A_k\right) = 0$. which will imply that the probability of the RHS and hence that of the LHS of (2.5) is 0. Indeed, for any positive integer L

$$\mathbb{P}\left(\bigcap_{k\geq k_0} A_k\right) \leq \mathbb{P}\left(\bigcap_{k=k_0}^{k_0+L} A_k\right) = \mathbb{P}(A_{k_0}) \prod_{k=k_0+1}^{t_0+L} \mathbb{P}\left(A_k \mid \bigcap_{s=k_0}^{k-1} A_s\right).$$

We now proceed to calculate the conditional probabilities, $\mathbb{P}\left(A_k \mid \bigcap_{s=k_0}^{k-1} A_s\right)$. Setting $\varepsilon_1 = \frac{1}{n}$ and letting S be the ball of radius $\sqrt{2(1/n + (m+1)/n^2)}\left(1 + \sqrt{N-K-1}\right)$ centred at 0 in Lemma 5 and using the bound (1.1), we obtain

$$\varepsilon_2 = \frac{\varepsilon_1}{4(N-K)^2} = \frac{1}{4n(N-K)^2} > \frac{1}{n^2}$$

and thus with probability at least σ, given by Lemma 5, F will exit $I_{n,m}$, that is,

$$\mathbb{P}\left(F(\tau_{k,r} + 1) \in I_{n,m} \mid F(\tau_{k_0,r} + 1), F(\tau_{k_0+1,r} + 1), \dots, F(\tau_{k-1,r} + 1) \in I_{n,m}, \tau_{k,k} < \infty\right) \leq 1 - \sigma,$$

since $\zeta_{\tau_{k,r}+1;j}$ are all independent from $\mathcal{F}_{\tau_{k,r}}$ for $1 \le j \le K$.

From this we can conclude that, $\mathbb{P}\left(A_k \mid \bigcap_{s=k_0}^{k-1} A_s\right) \le 1-\sigma$ yielding $\mathbb{P}\left(\bigcap_{k\ge k_0} A_k\right) \le (1-\sigma)^L$ for all $L \ge 1$. Letting $L \to \infty$ shows that $\mathbb{P}\left(\bigcap_{k\ge k_0} A_k\right) = 0$, which in turn proves (2.4). □

Corollary 2. *Suppose Assumption 1 holds, $d = 1$, and ζ has a singular distribution. Then*

$$\mathbb{P}\left(\{\exists\phi:\ \mathcal{X}'(t) \to \phi\} \bigcup \{\mathcal{X}'(t) \to \infty\}\right) = 1.$$

Proof. Assume that $\mathcal{X}'(t) \not\to \infty$ occurs and for $a < b$ define

$$E_{a,b} = \{\liminf_{t\to\infty} x_{(k)}(t) < a\} \cap \{\limsup_{t\to\infty} x_{(k)}(t) > b\},$$

where $k \in \{1, 2, \ldots, N-K\}$ and $x_{(k)}$ is the $k-$th point of the core. By Theorem 1 $F(t) \to 0$ implying, in turn, that $D(t) \to 0$, and hence by Lemma 4

$$\mathsf{dist}(\mathcal{X}'(t), \mathcal{X}'(t+1)) := \max_{1\le i,j\le N-K} |x_{(i)}(t) - x_{(j)}(t+1)| \to 0 \tag{2.6}$$

as $t \to \infty$.

Since ζ is singular $\exists x \in (a, b)$ and $\epsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subseteq \text{supp}(\zeta)^c$. However, then

$$E_{a,b} \subseteq \text{dist}(\mathcal{X}'(t), \mathcal{X}'(t+1)) > 2\varepsilon \ \text{ i.o.}\}$$

implying from (2.6) that $\mathbb{P}(E_{a,b}) = 0$. Since this is true for all a and b, $\mathcal{X}'(t)$ must converge. □

2.1 Case $K = d = 1$.

In the case where $K = 1$ and the space is $\mathbb{R}^1$ we can obtain some more detailed results, given some further assumptions.

Assumption 2 (at most exponential oscillations in the tail)**.** *Suppose that there exist some $R_+, R_- \in \mathbb{R}$, a constant $C \ge 0$ such that given for any $a \ge R_+$ and $u > 0$ we have*

$$\mathbb{P}\left(a + u < \zeta \le a + 2u\right) \le C\,\mathbb{P}\left(a < \zeta \le a + u\right).$$

Similarly for all $a \le R_-$ and $u < 0$ we have

$$\mathbb{P}\left(a + 2u < \zeta \le a + u\right) \le C\,\mathbb{P}\left(a + u < \zeta \le a\right).$$

Remark 2. *Observe that nearly all common continuous distributions satisfy this assumption (exponential, normal, Pareto, etc.). An example of distribution for which the assumption is not fulfilled is e.g. one with the density*

$$f(x) = \begin{cases} \frac{1}{2}e^{-|x|}, & \lfloor x \rfloor \text{ is even,} \\ e^{-2|x|}, & \text{otherwise} \end{cases}$$

which has support on the whole $\mathbb{R}$.

By iterating the property in Assumption 2 for $a \geq R_+$ one attains that for $k = 1, 2, \ldots$

$$\mathbb{P}\left(\zeta \in (a + (k-1)u, a + ku]\right) \leq C^{k-1}\, \mathbb{P}\left(\zeta \in (a, a+u]\right).$$

It also follows that if we take $R_+ < a < b < c$ then

$$\mathbb{P}\left(\zeta \in (b, c]\right) \leq \mathbb{P}\left(\zeta \in \bigcup_{k=1}^{\lceil \frac{c-a}{b-a} \rceil} \left(a + (k-1)(b-a), k(b-a)\right]\right) \leq \sum_{k=1}^{\lceil \frac{c-a}{b-a} \rceil} C^{k-1}\, \mathbb{P}\left(\zeta \in (a, b]\right). \tag{2.7}$$

Using (2.7) one can compare the probabilities of selecting a new point in the intervals of different length and/or that are not consecutive; we see that in this case the upper bound we get is a polynomial in C.

Remark 3. *The assumption is somewhat related to the concept of O-regular variation (see [2], page 65) in the following sense: if we let* $g(x) := \mathbb{P}(R_+ < \zeta \leq R_+ + x)$ *for* $x > 0$ *then we see from* (2.7) *that* $\limsup_{x\to\infty} \frac{g(tx)}{g(x)} \leq \sum_{k=1}^{\lceil t \rceil} C^{k-1}$ *for* $t \geq 1$. *Therefore,* g *is an O-regularly varying function; moreover, if the support of* ζ *is* $\mathbb{R}^+$ *and* $R_+ = 0$ *then the distribution function of* ζ *itself is an O-regularly varying function.*

Assumption 2 immediately implies that the tail region is free of isolated atoms; moreover, it turns out that the tail region is free of atoms altogether.

Claim 1. *Suppose that Assumption 2 holds. Then* $\mathbb{P}(\zeta = x) = 0$ *for every* $x \in (-\infty, R_-) \cup (R_+, \infty)$.

Proof. Assume to the contrary that $\exists x \in (-\infty, R_-) \cup (R_+, \infty)$ such that $\mathbb{P}(\zeta = x) > 0$. Since $\mathbb{P}(\zeta = x) = \mathbb{P}\left(\bigcap_{n=1}^{\infty}\{\zeta \in (x - \frac{1}{n}, x]\}\right)$, by continuity of probability it follows that $\exists N$ such that $\mathbb{P}(\zeta \in (x - \frac{1}{N}, x]) \leq \left(\frac{1}{2C} + 1\right)\mathbb{P}(\zeta = x)$ which implies that $\mathbb{P}(\zeta \in (x - \frac{1}{N}, x)) \leq \frac{1}{2C}\,\mathbb{P}(\zeta = x)$. Therefore we have

$$\mathbb{P}\left(\zeta \in \left(x - \frac{1}{2N}, x - \frac{1}{N}\right]\right) \leq \mathbb{P}(\zeta \in (x - \frac{1}{N}, x)) \leq \frac{1}{2C}\,\mathbb{P}(\zeta = x) \leq \frac{1}{2C}\,\mathbb{P}\left(\zeta \in (x - \frac{1}{2N}, x]\right),$$

which contradicts Assumption 2. □

Theorem 2. *Suppose $K = 1$ and ζ satisfies Assumption 2 for some R_+ and R_-. Then*

(a) $\mathcal{X}' \not\to \infty$ *a.s. and consequently by Theorem 1 we have* $F(t) \to 0$ *a.s..*

(b)

$$\left\{\liminf_{t\to\infty} x_{(1)}(t) > R_+\right\} \bigcup \left\{\limsup_{t\to\infty} x_{(N-1)}(t) < R_-\right\} \subseteq \{\exists\phi :\ \mathcal{X}'(t) \to \phi\}$$

except perhaps a set of measure 0.

(c) Assuming $\mathbb{E}\,|\zeta| < \infty$, *then if* $\operatorname{supp}\zeta = [R_+, \infty)$ *or if* $\operatorname{supp}\zeta = (-\infty, R_-]$ *then the limits* $\lim_{t\to\infty} \mathbb{E}\, x_{(k)}(t)$ *and* $\lim_{t\to\infty} \mathbb{E}\,\mu'(t)$ *are both well defined for* $1 \le k \le N-1$ *($x_{(k)}(t)$ denotes the k:th barycentric order statistic) and moreover*

$$\lim_{t\to\infty} \mathbb{E}\, x_{(k)}(t) = \lim_{t\to\infty} \mathbb{E}\,\mu'(t).$$

(d) If $R_- > R_+$ *then* $\mathbb{P}\left(\exists\phi :\ \mathcal{X}'(t) \to \phi\right) = 1$.

Remark 4. *The last part of the theorem above applies to many distributions for which* $\operatorname{supp}\zeta = \mathbb{R}$, *e.g. to normal, Laplace or Cauchy distribution (one can take* $R_+ = -1$ *and* $R_- = +1$*).*

Proof. We begin with the proof of (a). Given some $L \ge 1$, from now on assume that $A_L = \left\{\sqrt{2F(0)} < \frac{L}{2}, |\zeta_{0;k}| < L, k = 1 \dots N\right\}$ occurs, this will imply that $D(t) \le \frac{L}{2}$ for all t. Notice that since the distance between any two points in the core at time t is bounded by $D(t)$ it follows that if one core point diverges to $+\infty$ so must all the other points, similarly if one of the points diverges to $-\infty$ so must all of the rest. Therefore it is enough to show that $\mathbb{P}\left(\{\mathcal{X}'(t) \to +\infty\} \bigcup \{\mathcal{X}'(t) \to -\infty\}\right) = 0$. We shall prove now that $\mathcal{X}'(t) \not\to +\infty$ a.s.; the proof that $\mathcal{X}'(t) \not\to -\infty$ a.s. is completely analogous.

Let $\pi_a = \inf\{t : \sqrt{2F(t)} < \frac{a}{2}\}$, $\eta_{1,a} = \tau_{1,a} = \pi_a$ and for $k > 1$ let

$$\begin{aligned}
\tau_{k,a} &= \inf\left\{t > \eta_{k-1,a} :\ x_{(1)}(t) > R_+ + a\right\},\\
\eta_{k,a} &= \inf\left\{t > \tau_{k,a} :\quad x_{(1)}(t) < R_+ + a\right\},\\
\gamma_{k,t,a} &= \min\left(\eta_{k,a}, \tau_{k,a} + t\right),
\end{aligned}$$

where $x_{(1)}(t)$ denotes the left-most point of the core at time t. If $\tau_{k,a} = \infty$ for some k then we set $\eta_{m,a} = \tau_{m,a} = \infty$ for all $m \ge k$. It is obvious that on A_L, $\pi_L = 0$.

Furthermore

$$\{\tau_{k,L}=\infty\}\cap\{\eta_{k-1,L}<\infty\}\subseteq\{\limsup_{t\to\infty}x_{(1)}(t)\le R_{+}+a\}\subseteq\{\mathcal{X}'(t)\not\to+\infty\}.$$

Let $C_k=\{\eta_{k,L}<\infty\}$ and note

$$\left(\bigcap_{k=2}^{\infty}C_k\right)\subseteq\{\mathcal{X}'(t)\subseteq B_{R_{+}+2L}(0)\ i.o.\}\subseteq\{\mathcal{X}'(t)\not\to+\infty\}.$$

Since $(\bigcap_{k=1}^{\infty}C_k)\subseteq\{\mathcal{X}'(t)\not\to+\infty\}$, if we could also show that

$$\mathbb{P}\left(\left(\bigcap_{k=2}^{\infty}C_k\right)^{c}\setminus\{\mathcal{X}'(t)\not\to+\infty\}\right)=\mathbb{P}\left(\left(\bigcup_{k=2}^{\infty}\{\eta_{k,L}=\infty\}\right)\bigcap\{\mathcal{X}'(t)\to+\infty\}\right)=0, \tag{2.8}$$

then it would follow that $\mathbb{P}\left(A_L\bigcap\{\mathcal{X}'(t)\not\to+\infty\}\right)=\mathbb{P}(A_L)$ and since $\mathbb{P}\left(\bigcup_{L=1}^{\infty}A_L\right)=1$ it would then follow from continuity of probability that $\mathbb{P}\left(\mathcal{X}'(t)\to+\infty\right)=0$.

Now we will show that $\mathbb{P}\left(\{\eta_{k,L}=\infty\}\bigcap\{\mathcal{X}'(t)\to+\infty\}\right)=0$ for every $k>1$ which will establish (2.8). For this purpose (and for the purpose of showing the other statements of the theorem) we will need the following lemma

Lemma 6. *For some fixed $k>1$ and $a>0$ let*

$$h_c(s)=\left(\sqrt{F(s)}+c\left[\mu'(s)+\max(0,-R_{+})\right]\right)I_{A_L}.$$

Then there exists $c>0$ such that $\lim_{t\to\infty}h_c(\gamma_{k,t,a})$ exists a.s. on $\tau_{k,a}<\infty$.

Proof of Lemma 6. We will show that $h_c(\gamma_{k,t,a})$ is a non-negative supermartingale with respect to $\{\mathcal{F}_{\gamma_{k,t,a}}\}_{t\ge0}$, and then the result will follow from the supermartingale convergence theorem. In order to make notations less cluttered from now on we set $\gamma_t:=\gamma_{k,t,a}$ throughout the proof of this lemma. First, observe that the positivity of $h_c(\gamma_t)$ is ensured by the term $c\max(0,-R_{+})$, and by the definition of γ_t and π_a. Therefore, from now on we can assume that $R_{+}\ge0$ without loss of generality. We have

$$\mathbb{E}\,|h_c(s)|\le\mathbb{E}\left[\left(\sqrt{F(s)}+c|\mu'(s)|\right)I_{A_L}\right]\le$$
$$\mathbb{E}\left[\left(\sqrt{F(0)}+c\left(|\mu'(0)|+\sum_{l=1}^{s}|\mu'(l)-\mu'(l-1)|\right)\right)I_{A_L}\right]$$
$$\le\mathbb{E}\left[\left(\frac{L}{2\sqrt{2}}+c\left(|\mu'(0)|+\sum_{l=1}^{s}D(l)\right)\right)I_{A_L}\right]\le\mathbb{E}\left[\left(\frac{L}{2\sqrt{2}}+c\left(L+\sum_{l=1}^{s}\sqrt{2F(l)}\right)\right)I_{A_L}\right]$$

$$\leq \mathbb{E}\left[\left(L+c\left(L+s\sqrt{2F(0)}\right)\right)I_{A_L}\right] \leq L\left(1+c\left(1+s/2\right)\right) < \infty,$$

where we used Lemma 3, the fact that $|\mu'(0)| \leq \max_{x\in\mathcal{X}'(0)}|x| \leq L$, $|\mu'(s+1)-\mu'(s)| \leq D(s+1), s \geq 0$, and that F is non-increasing. Hence $\mathbb{E}\,|h_c(s)| < \infty$.

Since $\{\gamma_t < \eta_{k,a}\} \in \mathcal{F}_{\gamma_t}$ we have

$$\begin{aligned}
&\mathbb{E}\left[h_c(\gamma_{t+1}) - h_c(\gamma_t)\,|\,\mathcal{F}_{\gamma_t}\right] = \mathbb{E}\left[\left(h_c(\gamma_{t+1}) - h_c(\gamma_t)\right)\left(I_{\gamma_t=\eta_{k,a}} + I_{\gamma_t<\eta_{k,a}}\right)\,|\,\mathcal{F}_{\gamma_t}\right] \\
&= \mathbb{E}\left[\left(h_c(\gamma_t+1) - h_c(\gamma_t)\right)I_{\gamma_t<\eta_{k,a}}\,|\,\mathcal{F}_{\gamma_t}\right] = \mathbb{E}\left[h_c(\gamma_t+1) - h_c(\gamma_t)\,|\,\mathcal{F}_{\gamma_t}\right]I_{\gamma_t<\eta_{k,a}} \\
&\leq \max\left(0, \mathbb{E}\left[\left(h_c(\gamma_t+1) - h_c(\gamma_t)\right)\,|\,\mathcal{F}_{\gamma_t}\right]\right)I_{\gamma_t<\eta_{k,a}} \\
&\leq \max\left(0, \mathbb{E}\left[\left(h_c(\gamma_t+1) - h_c(\gamma_t)\right)\,|\,\mathcal{F}_{\gamma_t}\right]\right).
\end{aligned}$$

It will suffice now to show that $\mathbb{E}(h(\gamma_t+1) - h(\gamma_t)\,|\,\mathcal{F}_{\gamma_t}) \leq 0$ a.s. Since $\gamma_t \leq \eta_{k,a}$ we can deduce

$$x_{(1)}(\gamma_t) \geq x_{(1)}(\eta_{k,a}) \geq x_{(1)}(\eta_{k,a}-1) - D(\eta_{k,a}-1) > R_+ + a - \sqrt{2F(\pi_a)} > R_+ + \frac{a}{2}. \tag{2.9}$$

The above inequalities show that all the core points lie to the right of R_+ at time γ_t, since this region is free of atoms we can conclude that $D(\gamma_t) > 0$ a.s.. Recall that the points of the core at time γ_t are ordered as $x_{(1)}(\gamma_t) \leq ... \leq x_{(N-1)}(\gamma_t)$, and let $\zeta = \zeta_{\gamma_t+1}$.

Let us introduce some new variables where we drop the time indices for the sake of brevity:

$$\begin{aligned}
D \quad &= D(\gamma_t), & \mathcal{F} \quad &= \mathcal{F}_{\gamma_t}, \\
y_k \quad &= \frac{x_{(k)}(\gamma_t) - x_{(1)}(\gamma_t)}{D}, & \zeta' \quad &= \frac{\zeta - x_{(1)}(\gamma_t)}{D}, \\
F_o \quad &= \sqrt{F\left(\{y_1,\cdots,y_{N-1}\}\right)}, & F_n \quad &= \sqrt{F\left(\{y_1,\cdots,y_{N-1},\zeta'\}'\right)}, \\
\mu_o' \quad &= \mu\left(\{y_1,\cdots,y_{N-1}\}\right), & \mu_n' \quad &= \mu\left(\{y_1,\cdots,y_{N-1},\zeta'\}'\right).
\end{aligned}$$

At time γ_t the transformed core consists of the new points $(y_1,\ldots,y_k)$ such that $0 = y_1 \leq \cdots \leq y_{N-1} = 1$. Notice that we will always reject ζ' if $\zeta' < -1$ but this is equivalent to $\zeta < x_{(1)}(\gamma_t) - D$ which is bounded below by $x_{(1)}(\gamma_t) - \frac{a}{2}$, by (2.9) this is strictly larger than R_+ so we can conclude that ζ is accepted into the core only if it lies to the right of R_+. Furthermore if $a > -1$ then since ζ is independent of $\mathcal{F}$ it follows that

$$\mathbb{P}\left(\zeta' \in (a+u, a+2u]\right) = \mathbb{P}\left(\zeta \in \left((a+u)D + x_{(1)}(\gamma_t), (a+2u)D + x_{(1)}(\gamma_t)\right]\right)$$

$$\leq C\,\mathbb{P}\left(\zeta \in \left(aD + x_{(1)}(\gamma_t), (a+u)D + x_{(1)}(\gamma_t)\right]\right) = C\,\mathbb{P}\left(\zeta' \in (a, a+u]\right), \qquad (2.10)$$

hence Assumption 2 translates to ζ'. If we combine (2.10) with the same type of argument as in (2.7) we see that if $-1 < a < b < c$, then

$$\mathbb{P}\left(\zeta' \in (b,c]\right) \leq \sum_{k=1}^{\lceil \frac{c-a}{b-a} \rceil} C^{k-1}\,\mathbb{P}\left(\zeta' \in (a,b]\right). \qquad (2.11)$$

Due to the translation invariance of $\sqrt{F}$ and μ we have

$$\mu'(\gamma_t + 1) - \mu'(\gamma_t) = D\left(\mu'_n - \mu'_o\right),$$
$$F\left(\gamma_t + 1\right) - F\left(\gamma_t\right) = D\left(\sqrt{F_n} - \sqrt{F_o}\right),$$

implying

$$\frac{1}{D}\left(h(\gamma_t + 1) - h(\gamma_t)\right) = \sqrt{F_n} - \sqrt{F_o} + c\left(\mu'_n - \mu'_o\right).$$

Denote $\Delta h = \sqrt{F_n} - \sqrt{F_o} + c\left(\mu'_n - \mu'_o\right)$; since $D > 0$ a.s. it follows that

$$\mathbb{E}\left[\left(h(\gamma_{t+1}) - h(\gamma_t)\right) \mid \mathcal{F}\right] \leq 0 \quad \Longleftrightarrow \quad \mathbb{E}\left[\Delta h \mid \mathcal{F}\right] \leq 0.$$

When the new point ζ is sampled then either 0,1 or ζ' is eliminated in the next step. There are 4 different cases, either $\zeta' < 0$, $\zeta' \in (0,1)$, $\zeta' > 1$ (recall that ζ has no atoms under Assumption 2). The new centre of mass for the whole configuration is thus

$$\mu_n = \frac{\zeta' + M\mu'_o}{M+1}, \qquad \text{where } M := N - 1.$$

If the point 0 is eliminated then centre of mass of the new core is $\mu'_n = \frac{\zeta'}{M} + u'_o$, and if the point 1 is eliminated then $\mu'_n = \frac{\zeta'-1}{M} + \mu'_o$. Note that by Claim 1 our probability measure is non-atomic to the right of R_+ and therefore the probability of a tie between which point should be eliminated is zero; consequently, we can disregard these events.

- In the case $\zeta' < 0$, only ζ' or 1 can be eliminated. The point 1 is eliminated if and only if $\mu_n - \zeta' < 1 - \mu_n$. This is equivalent to $\zeta' > \frac{M(2\mu'_o - 1) - 1}{M-1}$. So in this case the point 1 is eliminated if and only if $\zeta' \in \left(\frac{M(2\mu'_o-1)-1}{M-1}, 0\right)$. Denote this event by
$$L_1 = \left\{\min\left(\frac{M(2\mu'_o - 1) - 1}{M-1}, 0\right) < \zeta' < 0\right\}.$$

- In the case $\zeta' \in (0,1)$, ζ' is never eliminated, but one of the points 0 or 1 must be. The point 0 is eliminated iff $\mu_n > 1-\mu_n$, which is equivalent to $\zeta' > \frac{M+1}{2} - M\mu_o'$, hence $\zeta' \in \left(\min\left(\frac{M+1}{2} - M\mu_o', 1\right), 1\right)$. Let
$$B_0 = \left\{\min\left(\frac{M+1}{2} - M\mu_o', 1\right) < \zeta' < 1\right\}.$$
The point 1 is eliminated otherwise, in other words if $\zeta' \in (0,1) \setminus \left[\min\left(\frac{M+1}{2} - M\mu_o', 1\right), 1\right]$. Let
$$B_1 = \left\{0 < \zeta' < \min\left(\frac{M+1}{2} - M\mu_o', 1\right)\right\}.$$
- In the case $\zeta' > 1$ only ζ' or 0 can be eliminated. The point 0 will be eliminated if $\zeta' - \mu_n < \mu_n \iff \zeta' < \frac{2M\mu_o'}{M-1}$, that is if $\zeta' \in \left(1, \max\left(\frac{2M\mu_o'}{M-1}, 1\right)\right)$. Let
$$R_0 = \left\{1 < \zeta' < \max\left(\frac{2M\mu_o'}{M-1}, 1\right)\right\}.$$

We begin with the case $M = 2$. We have $\mu_o' = \frac{1}{2}$, $F_o = \frac{1}{2}$, $L_1 = \{-1 < \zeta' < 0\}$, $B_1 = \{0 < \zeta' < 1/2]\}$, $B_0 = \{1/2 < \zeta' < 1\}$, $R_0 = \{1 < \zeta' < 2\}$. When 1 is eliminated then $F_n = \frac{\zeta'^2}{2}$, moreover notice that in this case $\mu_o' - \mu_n'$ is non-positive. When 0 is eliminated then $\mu_n' = \frac{1+\zeta'}{2}$. We have

$$\begin{aligned}
\mathbb{E}(\Delta h \,|\, \mathcal{F}) &= \mathbb{E}\left[(\mu_n' - \mu_o') + c\,(F_n - F_o) \,|\, \mathcal{F}\right] \le c\,\mathbb{E}\left[(F_n - F_o)\, I_{L_1 \cup B_1} \,|\, \mathcal{F}\right] \\
&\quad + \mathbb{E}\left[(\mu_n' - \mu_o')\, I_{R_0 \cup B_0} \,|\, \mathcal{F}\right] \le \frac{c}{2}\,\mathbb{E}\left[\left(\zeta'^2 - 1\right) I_{B_1} \,|\, \mathcal{F}\right] + \frac{1}{2}\,\mathbb{E}\left[\zeta' I_{R_0 \cup B_0} \,|\, \mathcal{F}\right] \\
&\le \frac{c}{2}\left(\frac{1}{4} - 1\right)\mathbb{P}\left(0 < \zeta' < 1/2\right) + \frac{2}{2}\,\mathbb{P}\left(1/2 < \zeta' < 2\right) \\
&\le -\frac{3}{8}c\,\mathbb{P}\left(0 < \zeta' < 1/2\right) + \left(1 + C + C^2 + C^3\right)\mathbb{P}\left(0 < \zeta' < 1/2\right),
\end{aligned}$$

where we used (2.11) in the last inequality. It is obvious that the last expression can be made negative for large enough $c > 0$, as required.

Let us now consider the case $M \ge 3$. First we note that $\mu_o' \in \left(\frac{1}{M}, \frac{M-1}{M}\right)$ a.s., where the lower bound is approached as $y_2, ..., y_{M-1}$ all go to 0 while the upper bound is approached as $y_2, ..., y_{M-1}$ all go to 1. If we now denote by K_0 the event that 0 is eliminated, and K_1 the event that 1 is eliminated, then we have $K_0 = R_0 \cup B_0$ and $K_1 = L_1 \cup B_1$. Furthermore,
$$\mu_n' - \mu_o' = \frac{\zeta'}{M} I_{K_0} + \frac{\zeta' - 1}{M} I_{K_1}.$$

We also have

$$F_n = \left(F_o + \frac{M-1}{M}\zeta'^2 - 2\mu'_o\zeta'\right) I_{K_0}$$
$$+ \left(F_o + \frac{M-1}{M}\zeta'^2 - \frac{2(M\mu'_o - 1)}{M}\zeta' + \frac{2(M\mu'_o - 1)}{M} - \frac{M-1}{M}\right) I_{K_1}.$$

Observe that $\Delta h = h_0 I_{K_0} + h_1 I_{K_1}$, where

$$h_i = \sqrt{F_o + \Delta_i(\zeta', \mu'_o)} + c\frac{\zeta'}{M} - \sqrt{F_o} \quad i = 0, 1;$$
$$\Delta_i(x, y) = \frac{1}{M} \cdot \begin{cases} (M-1)x^2 - 2Mxy, & i = 0; \\ (M-1)(x^2-1) + 2(1-x)(My-1) & i = 1. \end{cases}$$

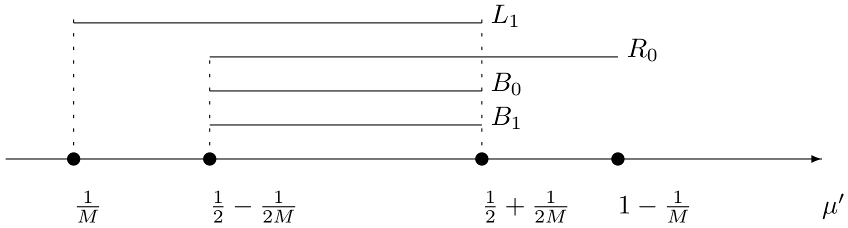
Using these notations we obtain

$$\begin{aligned} \mathbb{E}\left[\Delta h \,|\, \mathcal{F}\right] &= \mathbb{E}\left[h_0 I_{K_0} \,|\, \mathcal{F}\right] + \mathbb{E}\left[h_1 I_{K_1} \,|\, \mathcal{F}\right] \\ &= \mathbb{E}\left[h_0 I_{R_0} \,|\, \mathcal{F}\right] + \mathbb{E}\left[h_0 I_{B_0} \,|\, \mathcal{F}\right] + \mathbb{E}\left[h_1 I_{L_1} \,|\, \mathcal{F}\right] + \mathbb{E}\left[h_1 I_{B_1} \,|\, \mathcal{F}\right] \\ &= (I) + (II) + (III), \end{aligned}$$

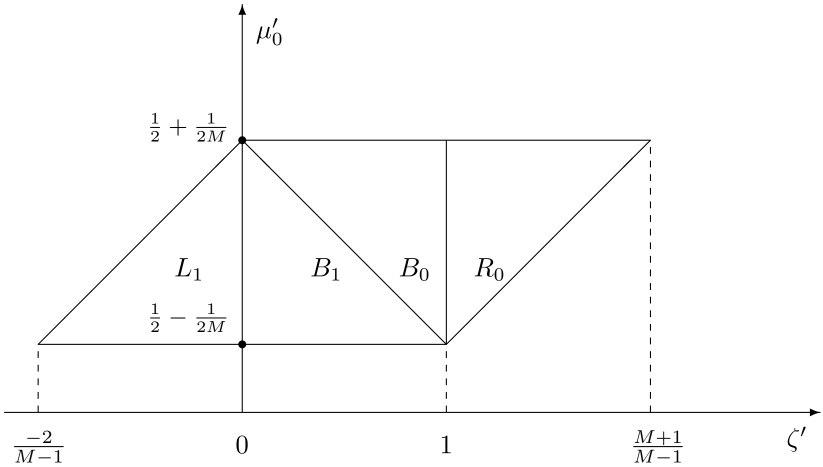
where

$$\begin{aligned} (I) &= \left(\mathbb{E}\left[h_1 I_{L_1} \,|\, \mathcal{F}\right]\right) I_{\mu'_o \in \left(\frac{1}{M}, \frac{M-1}{2M}\right)}, \\ (II) &= \left(\mathbb{E}\left[h_1 I_{L_1} \,|\, \mathcal{F}\right] + \mathbb{E}\left[h_1 I_{B_1} \,|\, \mathcal{F}\right] + \mathbb{E}\left[h_0 I_{R_0} \,|\, \mathcal{F}\right] + \mathbb{E}\left[h_0 I_{B_0} \,|\, \mathcal{F}\right]\right) I_{\mu'_o \in \left(\frac{M-1}{2M}, \frac{M+1}{2M}\right)}, \\ (III) &= \left(\mathbb{E}\left[h_0 I_{R_0} \,|\, \mathcal{F}\right]\right) I_{\mu'_o \in \left(\frac{M+1}{2M}, \frac{M-1}{M}.\right)} \end{aligned}$$

(Please see also the following diagram showing locations of ζ' for the events L_1, B_1, B_0 and R_0.)



It will suffice to show that all the three terms in the expression for $\mathbb{E}\left[\Delta h \,|\, \mathcal{F}\right]$ are non-positive. The fact that $(I) \leq 0$ is obvious, since if 1 is eliminated then the core centre of mass must move leftwards while F is always non-increasing. The second term (II) is a little more complicated and requires more careful study. We illustrate the possible combinations of ζ' and μ'_o on the following diagram.



We know present the following elementary statement.

Claim 2. *Let* $\Delta < 0$. *Then*

$$\sqrt{F_o + \Delta} - \sqrt{F_o} \leq -\frac{\Delta}{2M}.$$

Proof of Claim 2. The inequality follows from the fact that $\sqrt{F_0} \leq \sqrt{M/2} \leq M$ and the trivial inequality $\sqrt{x+y} - \sqrt{x} \leq \frac{y}{2\sqrt{x}}$ valid for all $x > 0$ and $x + y \geq 0$. □

Next, we find an upper bound for $\Delta_1(x, y)$ on the rectangle

$$A_1 = \left\{ (x, y) : \frac{M-1}{2M} \leq y \leq \frac{1}{2}, 0 \leq x \leq \frac{1}{2} \right\}.$$

Combining these estimates with Claim 2 we get that for $\frac{M-1}{2M} \leq \mu'_o \leq \frac{1}{2}$ and $0 \leq \zeta' \leq \frac{1}{2}$ (which is a subset of $B_1 \cap \{\frac{M-1}{2M} \leq \mu'_o \leq \frac{1}{2}\}$)

$$\sqrt{F_o + \Delta_1(\zeta', \mu'_o)} - \sqrt{F_o} \leq -\frac{1}{2M^2}. \tag{2.12}$$

On the other hand, if $\mu'_o \geq 1/2$ and $0 \leq \zeta' \leq 1$ then $\Delta_0(\zeta', \mu'_o) \leq ((M-1)/M - 2\mu'_o)\,\zeta' \leq -\zeta'/M$ and therefore by Claim 2

$$\sqrt{F_o + \Delta_0(\zeta', \mu'_o)} - \sqrt{F_o} \leq -\frac{\zeta'}{2M^2}. \tag{2.13}$$

Our next task is to find an upper bound for $\Delta_0(x, y)$ on the rectangle

$$A_2 := \left\{ (x, y) : \frac{1}{2} \leq y \leq \frac{M+1}{2M}, 1 \leq x \leq \frac{2M-1}{2M-2} \right\}.$$

As a result, we conclude that $\Delta_0 \leq -\frac{1}{4M}$ on A_2. Combining this with Claim 2 we get that when $\frac{1}{2} \leq \mu'_o \leq \frac{M+1}{2M}$ and $1 \leq \zeta' \leq \frac{2M-1}{2(M-1)}$ (this is a subset of $R_0 \cap \{\frac{1}{2} \leq \mu'_o \leq \frac{M+1}{2M}\}$)

$$\sqrt{F_o + \Delta_0(\zeta', \mu'_o)} - \sqrt{F_o} \leq -\frac{1}{8M^2}. \tag{2.14}$$

We will also again make use of the fact that by definition $h_1 I_{L_1} \leq 0$ and $h_1 I_{B_1} \leq 0$ so therefore,

$$\left(\mathbb{E}\left[h_1 I_{L_1} \,|\, \mathcal{F}\right] + \mathbb{E}\left[h_1 I_{B_1} \,|\, \mathcal{F}\right]\right) I_{\mu'_o \in \left(\frac{M-1}{2M}, \frac{M+1}{2M}\right)} \leq \mathbb{E}\left[h_1 I_{B_1} \,|\, \mathcal{F}\right] I_{\mu'_o \in \left(\frac{M-1}{2M}, \frac{1}{2}\right)}.$$

Now we make the following estimates:

$$\begin{aligned}
(II) \leq\ & \mathbb{E}\left[h_1 I_{B_1} \,|\, \mathcal{F}\right] I_{\mu'_o \in \left(\frac{M-1}{2M}, \frac{1}{2}\right)} + \left(\mathbb{E}\left[h_0 I_{R_0} \,|\, \mathcal{F}\right] + \mathbb{E}\left[h_0 I_{B_0} \,|\, \mathcal{F}\right]\right) I_{\mu'_o \in \left(\frac{M-1}{2M}, \frac{M+1}{2M}\right)} \\
\leq\ & \mathbb{E}\left[\left(\sqrt{F_o + \Delta_1(\zeta', \mu'_o)} - \sqrt{F_o}\right) I_{B_1} \,|\, \mathcal{F}\right] I_{\mu'_o \in \left(\frac{M-1}{2M}, \frac{1}{2}\right)} \\
+\ & \mathbb{E}\left[\left(\sqrt{F_o + \Delta_0(\zeta', \mu'_o)} - \sqrt{F_o}\right) \left(I_{B_0} + I_{R_0}\right) \,|\, \mathcal{F}\right] I_{\mu'_o \in \left(\frac{1}{2}, \frac{M+1}{2M}\right)} \\
+\ & \frac{c}{M} \left(\mathbb{E}\left[\zeta' I_{B_0} \,|\, \mathcal{F}\right] + \mathbb{E}\left[\zeta' I_{R_0} \,|\, \mathcal{F}\right]\right) I_{\mu'_o \in \left(\frac{M-1}{2M}, \frac{M+1}{2M}\right)} \\
\leq\ & \mathbb{E}\left[\left(\sqrt{F_o + \Delta_1(\zeta', \mu'_o)} - \sqrt{F_o}\right) I_{0 \leq \zeta' \leq \frac{1}{2}} \,|\, \mathcal{F}\right] I_{\mu'_o \in \left(\frac{M-1}{2M}, \frac{1}{2}\right)} \\
+\ & \mathbb{E}\left[\left(\sqrt{F_o + \Delta_0(\zeta', \mu'_o)} - \sqrt{F_o}\right) \left(I_{B_0} + I_{1 \leq \zeta' \leq \frac{2M-1}{2(M-1)}}\right) \,|\, \mathcal{F}\right] I_{\mu'_o \in \left(\frac{1}{2}, \frac{M+1}{2M}\right)} \\
+\ & \frac{c}{M} \left(\mathbb{E}\left[\zeta' I_{B_0} \,|\, \mathcal{F}\right] + \mathbb{E}\left[\zeta' I_{R_0} \,|\, \mathcal{F}\right]\right) I_{\mu'_o \in \left(\frac{M-1}{2M}, \frac{M+1}{2M}\right)},
\end{aligned} \tag{2.15}$$

where we used the fact that $\{0 < \zeta' < 1/2\} \cap \{\frac{M-1}{2M} < \mu'_o < 1/2\} \subseteq \{\frac{M-1}{2M} < \mu'_o < 1/2\} \cap B_1$, that $\{1 \leq \zeta' \leq \frac{2M-1}{2(M-1)}\} \cap \{\frac{1}{2} < \mu'_o < \frac{M+1}{2M}\} \subseteq \{\frac{1}{2} < \mu'_o < \frac{M+1}{2M}\} \cap R_0$, and that on B_1 we have that $h_1 \leq \sqrt{F_o + \Delta_1(\zeta', \mu'_o)} - \sqrt{F_o}$. Let us now study the terms in (2.15). Notice that the term in the last line of (2.15) (a.s.) equals

$$\frac{c}{M} \left(\mathbb{E}\left[\zeta' I_{B_0} \,|\, \mathcal{F}\right] + \mathbb{E}\left[\zeta' I_{R_0} \,|\, \mathcal{F}\right]\right) \left(I_{\mu'_o \in \left(\frac{M-1}{2M}, \frac{1}{2}\right)} + I_{\mu'_o \in \left(\frac{1}{2}, \frac{M+1}{2M}\right)}\right),$$

while it follows from (2.13) and (2.14) that

$$\begin{aligned}
& \mathbb{E}\left[\left(\sqrt{F_o + \Delta_0(\zeta', \mu'_o)} - \sqrt{F_o}\right) \left(I_{B_0} + I_{1 \leq \zeta' \leq \frac{2M-1}{2(M-1)}}\right) \,|\, \mathcal{F}\right] I_{\mu'_o \in \left(\frac{1}{2}, \frac{M+1}{2M}\right)} \\
& \leq \left(\mathbb{E}\left[-\frac{\zeta'}{2M^2} I_{B_0} \,|\, \mathcal{F}\right] - \frac{1}{8M^2} \mathbb{P}\left(1 \leq \zeta' \leq \frac{2M-1}{2(M-1)}\right)\right) I_{\mu'_o \in \left(\frac{1}{2}, \frac{M+1}{2M}\right)}.
\end{aligned}$$

From (2.12) it also follows that

$$\mathbb{E}\left[\left(\sqrt{F_o + \Delta_1(\zeta', \mu'_o)} - \sqrt{F_o}\right) I_{0 < \zeta' < \frac{1}{2}} \,|\, \mathcal{F}\right] I_{\mu'_o \in \left(\frac{M-1}{2M}, \frac{1}{2}\right)} \leq -\frac{1}{2M^2} \mathbb{P}\left(0 < \zeta' < \frac{1}{2}\right) I_{\mu'_o \in \left(\frac{M-1}{2M}, \frac{1}{2}\right)}.$$

Furthermore we note that $\mathbb{E}\left[\zeta' I_{B_0} \,|\, \mathcal{F}\right] \leq \mathbb{P}(B_0)$ and $\mathbb{E}\left[\zeta' I_{R_0} \,|\, \mathcal{F}\right] \leq \frac{M}{M-1}\,\mathbb{P}(R_0)$ for $\frac{M-1}{2M} < \mu_o' < \frac{1}{2}$ while $\mathbb{E}\left[\zeta' I_{R_0} \,|\, \mathcal{F}\right] \leq \frac{M+1}{M-1}\,\mathbb{P}(R_0)$ when $\mu_o' < \frac{M+1}{2M}$. We can now conclude

$$
\begin{aligned}
(II) &\leq \left[-\frac{1}{2M^2}\,\mathbb{P}\left(0<\zeta'<1/2\right) + \frac{c}{M}\left(\mathbb{P}(B_0) + \frac{M}{M-1}\,\mathbb{P}(R_0)\right)\right] I_{\mu_o'\in\left(\frac{M-1}{2M},\frac{1}{2}\right)} \\
&+ \mathbb{E}\left[\left(\frac{c\zeta'}{M} - \frac{\zeta'}{2M^2}\right) I_{B_0} \,|\, \mathcal{F}\right] I_{\mu_o'\in\left(\frac{1}{2},\frac{M+1}{2M}\right)} \\
&+ \left(-\frac{1}{8M^2}\,\mathbb{P}\left(1<\zeta'<\frac{2M-1}{2(M-1)}\right) + \frac{c}{M}\frac{M+1}{M-1}\,\mathbb{P}(R_0)\right) I_{\mu_o'\in\left(\frac{1}{2},\frac{M+1}{2M}\right)} \\
&\leq \left[\frac{c}{M}\left(C_1 + \frac{M}{M-1}C_2\right) - \frac{1}{2M^2}\right]\mathbb{P}\left(0<\zeta'<1/2\right) I_{\mu_o'\in\left(\frac{M-1}{2M},\frac{1}{2},\right)} \\
&+ \left[C_3\frac{c}{M}\frac{M+1}{M-1} - \frac{1}{8M^2}\right]\mathbb{P}\left(1<\zeta'<\frac{2M-1}{2(M-1)}\right) I_{\mu_o'\in\left(\frac{1}{2},\frac{M+1}{2M}\right)} \\
&+ \mathbb{E}\left[\zeta'\left(\frac{c}{M} - \frac{1}{2M^2}\right) I_{B_0} \,|\, \mathcal{F}\right] I_{\mu_o'\in\left(\frac{1}{2},\frac{M+1}{2M}\right)},
\end{aligned}
$$

where

$$
\begin{aligned}
C_1 &= \frac{\mathbb{P}\big(\zeta'\in(0,1)\big)}{\mathbb{P}(0<\zeta'<1/2)} \geq \frac{\mathbb{P}(B_0)}{\mathbb{P}(0<\zeta'<1/2)}, \qquad C_2 = \frac{\mathbb{P}\big(\zeta'\in(1,2)\big)}{\mathbb{P}(0<\zeta'<1/2)} \geq \frac{\mathbb{P}(R_0)}{\mathbb{P}(0<\zeta'<1/2)}, \\
C_3 &= \frac{\mathbb{P}\big(\zeta'\in(1,2)\big)}{\mathbb{P}\big(1<\zeta'<\frac{2M-1}{2(M-1)}\big)} \geq \frac{\mathbb{P}(R_0)}{\mathbb{P}\big(1<\zeta'<\frac{2M-1}{2(M-1)}\big)}.
\end{aligned}
$$

It follows from (2.11) that these constants are all bounded above by some polynomial in C whose power depends only on M; also note that $\zeta' \geq 0$ on $B_0 \cap \{\frac{1}{2} \leq \mu_0' \leq \frac{M+1}{2M}\}$. Therefore it is obvious that we can pick c small enough to make the first two terms in the last displayed inequality above non-positive, the last term is trivially non-positive since $\zeta' \geq 0$ on B_0.

Now we will show that $(III) \leq 0$. We begin by finding an upper bound for $\Delta_0(x,y)$ on the rectangle

$$
A_3 = \left\{(x,y) : \frac{M+1}{2M} \leq y \leq \frac{M-1}{M}, 1 \leq x \leq \frac{M}{M-1}\right\}.
$$

Hence $\Delta_0 \leq -\frac{1}{M}$ on A_3, and combining this with Claim 2 we obtain that if $\frac{M+1}{2M} \leq \mu_o' \leq \frac{M-1}{M}$ then

$$
\begin{aligned}
(III) = \mathbb{E}\left[h_0 I_{R_0} \,|\, \mathcal{F}\right] &\leq \mathbb{E}\left[\left(\sqrt{F_o + \Delta_0(\zeta',\mu_o')} - \sqrt{F_o}\right) I_{1\leq\zeta'\leq\frac{M}{M-1}} \,|\, \mathcal{F}\right] + \frac{c}{M}\,\mathbb{E}\left[\zeta' I_{R_0} \,|\, \mathcal{F}\right] \\
&\leq \mathbb{E}\left[\left(\sqrt{F_o - \frac{1}{M-1}} - \sqrt{F_o}\right) I_{1\leq\zeta'\leq\frac{M}{M-1}} \,|\, \mathcal{F}\right] + \frac{c}{M}\,\mathbb{E}\left[2I_{R_0}\right], \qquad (2.16)
\end{aligned}
$$

where we used the fact that $\{\frac{M+1}{2M} < \mu'_o < \frac{M-1}{M}\} \cap \{1 \le \zeta' \le \frac{M}{M-1}\} \subseteq \{\frac{M+1}{2M} < \mu'_o < \frac{M-1}{M}\} \cap R_0$ for the first term and that $\zeta' < 2$ on R_0 (since $\mu'_0 < \frac{M-1}{M}$) for the second term. If we apply Claim 2 to the first term in (2.16) and again apply the fact that $\zeta' < 2$ on R_0 for the second term then we see that it is less or equal to

$$\begin{aligned}
&\le \left(\sqrt{F_o - \frac{1}{M-1}} - \sqrt{F_o}\right) \mathbb{P}\left(\zeta' \in \left(1, \frac{M}{M-1}\right)\right) + 2\frac{c}{M}\,\mathbb{P}\left(\zeta' \in (1,2)\right) \\
&\le \left(-\frac{1}{2M(M-1)} + 2\frac{c}{M}C_4\right) \mathbb{P}\left(\zeta' \in \left(1, \frac{M}{M-1}\right)\right),
\end{aligned}$$

where $C_4 = \frac{\mathbb{P}(1<\zeta'<2)}{\mathbb{P}\left(1<\zeta'<\frac{M}{M-1}\right)}$, which again is by bounded above by some polynomial in C according to (2.11). For this reason it is clear that we can again pick c small enough to make also this term non-positive, which proves that that $\mathbb{E}[\Delta h \,|\, \mathcal{F}] \le 0$ and hence h_k is a non-negative supermartingale. □

Now we continue with the proof of (a) of Theorem 2. Fix k and $a := L$, and let c be defined by Lemma 6. If we denote by h_∞ the a.s. limit of $h_c(\gamma_{k,t,L})$ as $t \to \infty$ on $\{\tau_{k,L} < \infty\} \cap \{\eta_{k,L} = \infty\}$ then

$$h_\infty = \lim_{t\to\infty} \left(\sqrt{F(\tau_{k,L}+t)} + c\mu'(\tau_{k,L}+t)\right) I_{A_L} = \left(\sqrt{F_\infty} + \lim_{t\to\infty} c\mu'(t)\right) I_{A_L},$$

that is $\exists \lim_{t\to\infty} \mu'(t) \in \mathbb{R}$ on A_L, implying $\mathcal{X}'(t) \not\to +\infty$.

We will now prove (b). Notice that we have just proved that $F(t) \to 0$ a.s., and hence $\pi_{1/n} < \infty$ a.s., $\forall n > 0$. First, we will show that

$$\mathbb{P}\left(\left\{\liminf_{t\to\infty} x_{(1)}(t) > R_+\right\} \setminus \{\exists \phi :\ \mathcal{X}'(t) \to \phi\}\right) = 0. \tag{2.17}$$

Indeed, let $E_n = \left\{\liminf_{t\to\infty} x_{(1)}(t) \ge R_+ + \frac{1}{n}\right\}$, then $\left\{\liminf_{t\to\infty} x_{(1)}(t) > R_+\right\} = \bigcup_{n=1}^{\infty} E_n$ and it suffices to prove that $\mathbb{P}\left(E_n \setminus \{\exists \phi :\ \mathcal{X}'(t) \to \phi\}\right) = 0$. Notice that $E_n \subseteq \bigcup_{k=1}^{\infty} \left(\{\eta_{k,1/n} = \infty\} \cap \{\tau_{k,1/n} < \infty\}\right) \subseteq \bigcup_{k=1}^{\infty} \{\lim_t \gamma_{k,t,1/n} = \infty\}$. By Lemma 6 $h_c(\gamma_{k,t,1/n})$ has an a.s. limit for some $c > 0$ on $\{\eta_{k,1/n} = \infty\} \cap \{\tau_{k,1/n} < \infty\} \cap A_L$, thus

$$\mathbb{P}\left(A_L \cap \left(\{\eta_{k,1/n} = \infty\} \cap \{\tau_{k,1/n} < \infty\}\right) \setminus \{\exists \lim_{t\to\infty} \mu'(t)\}\right) = 0.$$

Using continuity of probability again, applied to the sets A_L, $L \to \infty$, we can get rid of the term A_L in the expression above. Since $F(t) \to 0$ a.s. from the first part of the theorem, we have $\{\exists \lim_{t\to\infty} \mu'(t)\} = \{\exists \phi :\ \mathcal{X}'(t) \to \phi\}$ except perhaps a set of measure zero, therefore

$$\mathbb{P}\left(E_n \setminus \{\exists \lim_{t\to\infty} \mathcal{X}'(t)\}\right) = \mathbb{P}\left(E_n \setminus \{\exists \lim_{t\to\infty} \mu'(t)\}\right) \leq$$
$$\mathbb{P}\left((\{\eta_{k,1/n} = \infty\} \cap \{\tau_{k,1/n} < \infty\}) \setminus \{\exists \lim_{t\to\infty} \mu'(t)\}\right) = 0.$$

Noting that $E_n \subseteq E_{n+1}$, (2.17) follows from continuity of probability; the proof of the respective statement for lim sup is completely analogous, and they together are equivalent to the second statement of the theorem.

We will now prove (c). Assume that $R_+ \geq 0$ and $\operatorname{supp}\zeta \subseteq [R_+, \infty)$, the case $R_- \leq 0$, $\operatorname{supp}\zeta = (-\infty, R_-]$ is analogous. If $R_+ \geq 0$ and $\operatorname{supp}\zeta \subseteq [R_+, \infty)$ then consider

$$h(t) = c\mu'(t) + \sqrt{F(t)} \geq 0,$$

for all $t \geq 0$ and some $c > 0$ to be chosen later. Notice that compared to h in Lemma 6 the restrictions to the sets A_L will no longer be necessary, neither will we need the stopping time construction introduced in the beginning of the proof of this theorem since our configuration will always stay in an area where Assumption 2 is valid. We can now make the following estimate,

$$\mathbb{E}\, h(t) \leq \mathbb{E}\left[\sqrt{F(0)} + c\left(|\mu'(0)| + \sum_{l=1}^{t} |\mu'(s) - \mu'(s-1)|\right)\right]$$
$$\leq c\,\mathbb{E}\,|\mu'(0)| + (1 + tc\sqrt{2})\,\mathbb{E}\left[\sqrt{F(0)}\right] \leq \left(c + 1 + tc\sqrt{2}(N-1)\right)\mathbb{E}\,|\zeta| < +\infty,$$

where we skipped a few steps which are analogous to those in the beginning of the proof of Lemma 6. Since the left most point of the core always lies to the right of R_+, calculations analogous to those in the proof of Lemma 6 will show that $\mathbb{E}[h(t+1) \,|\, \mathcal{F}_t] \leq h(t)$ a.s. for some $c > 0$ and we will assume that c is chosen in this way from now on. We conclude that for $t \geq 0$, then $\mathbb{E}\, h(t+1) \leq \mathbb{E}\, h(t)$ which leads to,

$$\exists \lim_{t\to\infty} \mathbb{E}\left[c\mu'(t) + \sqrt{F(t)}\right].$$

Since $\mathbb{E}\sqrt{F(0)} \leq 2(N-1)\,\mathbb{E}\,|\zeta|$ is finite by assumption on ζ and since $F(t) \to 0$ a.s. (by the previous part of this theorem), the dominated convergence theorem (since $F(t) \leq F(0)$, for $t \geq 0$) implies that $\lim_{t\to\infty} \mathbb{E}\sqrt{F(t)} = 0$. This implies that $\exists \lim_{t\to\infty} \mathbb{E}\,\mu'(t)$ (although this limit might be $+\infty$). For $1 \leq k \leq N-1$ we have $|x_{(k)}(t) - \mu'(t)| \leq D(t) \leq \sqrt{2F(t)}$ and therefore,

$$\liminf_{t\to\infty} \mathbb{E}\, x_{(k)}(t) \geq \liminf_{t\to\infty} \mathbb{E}\, \mu'(t) - \limsup_{t\to\infty} \mathbb{E}\, |x_{(k)}(t) - \mu'(t)|$$
$$\geq \lim_{t\to\infty} \mathbb{E}\, \mu'(t) - \lim_{t\to\infty} \sqrt{2F(t)} = \lim_{t\to\infty} \mathbb{E}\, \mu'(t).$$

Similarly

$$\limsup_{t\to\infty} \mathbb{E}\, x_{(k)}(t) \leq \limsup_{t\to\infty} \mathbb{E}\, \mu'(t) + \limsup_{t\to\infty} \mathbb{E}\, |x_{(k)}(t) - \mu'(t)| = \lim_{t\to\infty} \mathbb{E}\, \mu'(t),$$

and so $\lim_{t\to\infty} \mathbb{E}\, x_{(k)}(t) = \lim_{t\to\infty} \mathbb{E}\, \mu'(t)$.

We now prove (d). Assume that $R_+ < R_-$ in Assumption 2. Let $u = \liminf_{t\to\infty} x_{(1)}(t)$, $v = \limsup_{t\to\infty} x_{(N-1)}(t)$. Consider the event $A_{a,b} = \{u < a\} \cap \{v > b\}$ for some $a < b$. If $b \leq R_-$ or $a \geq R_+$ we have already showed that we have convergence, so suppose that $b > R_-$ and $a < R_+$. We now make the observation that the interval $[R_+, R_-]$ is regular with parameters $\delta = \frac{2}{3}$, $r = \frac{1}{2C}$ (see Definition 2 in the next Section) and in the event of $A_{a,b}$ we cross the interval $\left(a + \frac{b-a}{2}, b - \frac{b-a}{2}\right)$ i.o., however since this interval also inherits the regularity property, this would contradict Proposition 2 which states that a regular interval cannot be visited i.o. a.s. and so $P(A_{a,b}) = 0$. From this we can conclude that

$$\mathbb{P}\left(\{\exists \phi,\ s.t.\ \mathcal{X}'(t) \to \phi\}^c\right) = \mathbb{P}\left(\bigcup_{a,b\in\mathbb{Q}, a<R_+, b>R_-} A_{a,b} \right) = 0,$$

i.e. the core converges to a point a.s. □

3 Convergence of the core

Definition 1. *A subset* $A \subseteq \operatorname{supp}(\zeta)$ *is regular with parameters* $\delta_A \in (0,1), \sigma_A > 0, r_A > 0$ *if*

$$\mathbb{P}(\zeta \in B_{r\delta_A}(x) \,|\, \zeta \in B_r(x)) \geq \sigma_A \tag{3.18}$$

for any $x \in A$ *and* $r \leq r_A$.

Assumption 3. *For any* $x \in supp(\zeta)$ *there exists some* $\gamma = \gamma(x)$ *such that the set* $B_\gamma(x) \cap \operatorname{supp}(\zeta)$ *is regular.*

Remark 5. *We can iterate the inequality (3.18) to establish that*

$$\mathbb{P}(\zeta \in B_{r\delta_A^k}(x) \,|\, \zeta \in B_r(x)) \geq \sigma_A^k, \quad k \geq 2.$$

Hence it is not hard to check that if Definition 1 holds for some $\delta_A \in (0,1)$ *it holds for all* $\delta \in (0,1)$*, albeit possibly with a smaller* σ_A*.*

Lemma 7. *Under Assumption 3, for any compact subset* $A \subset \operatorname{supp}(\zeta)$ *and* $\delta \in (0,1)$ *there exists* r_A *and* σ_A *such that* A *is regular with parameters* δ, σ_A, r_A*.*

Proof. The union $\bigcup_{x\in A} B_{\gamma(x)}(x)$ is an open covering of A, where $B_{\gamma_x}(x)$ is the regular ball centred in x given to us by Assumption 3. Since A is compact it follows that there is a finite subcover of A. In other words there exist sequences

$$\{x_k\}_{k=1}^M \subseteq A, \qquad \{\sigma_k\}_{k=1}^M, \{r_k\}_{k=1}^M, \{\delta_k\}_{k=1}^M, \{\gamma_k\}_{k=1}^M \subseteq \mathbb{R}^+$$

such that $A \subseteq \bigcup_{k=1}^M B_{\gamma_k}(x_k)$ and $\mathbb{P}(\zeta \in B_{r\delta_k}(x) \,|\, \zeta \in B_r(x)) \geq \sigma_k$ for $x \in B_{\gamma_k}(x_k)$ and $r \leq r_k$. Now let $\sigma' = \min_{1\leq k\leq M} \sigma_k$, $\delta' = \max_{1\leq k\leq M} \delta_k$,$r' = \min_{1\leq k\leq M} r_k$. It follows that for any $x \in A$

$$\mathbb{P}(\zeta \in B_{r\delta'}(x) \,|\, \zeta \in B_r(x)) \geq \sigma',$$

when $r \leq r'$. We conclude by noting that by Remark 5 there exists σ_A such that for each $x \in A$

$$\mathbb{P}(\zeta \in B_{r\delta}(x) \,|\, \zeta \in B_r(x)) \geq \sigma_A.$$

□

Theorem 3. *Under Assumptions 1 and 3*

$$\mathbb{P}\left(\{\exists \phi \in \mathbb{R}^d : \; \mathcal{X}'(t) \to \phi\} \cup \{\mathcal{X}'(t) \to \infty\}\right) = 1.$$

Proof. Firstly, $\mathbb{P}\left(\{\exists \lim_t \mu'(t)\} \Delta \{\, \exists \phi, \; s.t. \; \mathcal{X}'(t) \to \phi\}\right) = 0$, since if $\mu'(t)$ converges then $\mathcal{X}'(t) \not\to \infty$ which implies $D(t) \to 0$ by Theorem 1, yielding convergence of the core to the same point.

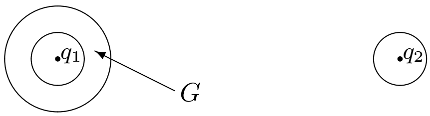
From an elementary calculus it follows that if neither the centre of mass converges to a finite point nor the configurations goes to infinity, then there must exist two arbitrarily small non-overlapping balls (w.l.o.g. centred at rational points) which are visited by μ' infinitely often, that is

$$\{\nexists \lim_t \mu'(t)\} \cap \{\mathcal{X}'(t) \not\to \infty\} = \bigcup_{n=1}^{\infty} \bigcup_{\substack{q_1, q_2 \in \mathbb{Q}^d, \\ \|q_1 - q_2\| \geq 6/n}} E_{q_1,q_2,n}, \tag{3.19}$$

$$\text{where } E_{q_1,q_2,n} = \left\{\mu'(t) \in B_{\frac{2}{n}}(q_1) \text{ i.o.}\right\} \bigcap \left\{\mu'(t) \in B_{\frac{2}{n}}(q_2) \text{ i.o.}\right\}.$$

To show (3.19), note that $\{\nexists \lim_t \mu'(t)\} \cap \{\mathcal{X}'(t) \not\to \infty\}$ is equivalent to existence of at least two distinct points in the set of accumulation points of $\{\mu'(t)\}_{t=1}^{\infty}$, say x_1 and x_2. Now take $q_1, q_2 \in \mathbb{Q}^d$ such that $\|q_j - x_j\| \le \frac{1}{n}$, $j = 1, 2$, then $\mu' \in B_{\frac{1}{n}}(x_j) \subseteq B_{\frac{2}{n}}(q_j)$, $j = 1, 2$, infinitely often; moreover $\|q_1 - q_2\| \ge \frac{8}{n} - \frac{1}{n} - \frac{1}{n} = \frac{6}{n}$ as required. Thus it suffices to prove that $\mathbb{P}(E_{q_1,q_2,n}) = 0$ for all $n \in \mathbb{N}$ and $q_1, q_2 \in \mathbb{Q}^d$ such that $\|q_1 - q_2\| \ge \frac{6}{n}$ to show that the LHS of (3.19) has measure zero, and then the Theorem will follow.

For simplicity w.l.o.g. assume that $q_1 = 0$ and denote $E := E_{0,q_2,n}$, $R = 2/n$, and $G = \text{supp}(\zeta) \cap (B_{2R}(0) \setminus B_R(0))$. Since every path from $B_{\frac{2}{n}}(0)$ to $B_{\frac{2}{n}}(q_2)$ must cross G, on E the shell G must be crossed infinitely often (by this we mean that $\|\mu'(t)\| > 2R$ i.o. and $\|\mu'(t)\| < R$ i.o.) – please see the illustration.



Since $\mathcal{X}'(t) \not\to \infty$ on E it follows from Theorem 1 that $F(t) \to 0$ a.s. on E and therefore additionally $\mathcal{X}'(t) \subset G$ i.o. (the core points cannot jump over the set G once the spread is sufficiently small); moreover the set G is regular by Lemma 7. We have also the following result.

Lemma 8. *Under Assumption 3, given $N - K$ points $x_1, \cdots, x_{N-K}$ in G, there are constants $a, \sigma \in (0, 1)$ depending on N, K and σ_G only, such that*

$$\mathbb{P}\left(\{F\left(\{\zeta_1, \ldots, \zeta_K, x_1, \ldots, x_{N-K}\}'\right) \le aF(\{x_1, \ldots, x_{N-K}\})\}\right) \ge \sigma.$$

(Remark the similarity of this statement with Lemma 5; the difference here, however, comes from the fact that the probability of decay σ, does not depend on the value of F, thanks to Assumption 3.)

Proof. We start with the case $K = 1$. Due to the translation invariance of F we can assume w.l.o.g. that $\sum_{i=1}^{N-1} x_i = 0$. Let $D = \max_{i,j \in \{1,\cdots,N-1\}} \|x_i - x_j\|$ and assume furthermore that $\|x_1\| \ge \|x_k\|, \forall k$ and take x_j such that $\|x_1 - x_j\| \ge \frac{D}{2}$. Let $\mu' = \frac{x_2 + \cdots + x_{N-1} + \zeta}{N-1} = \frac{\zeta - x_1}{N-1}$ and $F_{old} = F(\{x_1, \cdots, x_{N-1}\})$. If we take $\zeta \in B_{\frac{1}{8}\sqrt{\frac{F_{old}}{N}}}(x_1)$ then

$$\|\zeta - x_1\| \ge \|x_1 - x_j\| - \|\zeta - x_j\| \ge \frac{D}{2} - \frac{1}{8}\sqrt{\frac{F_{old}}{N}}.$$

From this we can deduce that $\|\zeta - x_1\|^2 \geq \frac{D^2}{8} \geq \frac{F_{old}}{4(N-1)}$. for some fixed $a \in (0,1)$ (which is only a function of N and K). By Lemma 4 the event $\left\{\zeta \notin B_{H\sqrt{2F_{old}}}(x_j)\right\}$, where $H = \sqrt{N-K-1}$, implies that $\{\zeta_1, x_1, \cdots, x_{N-1}\}' = \{x_1, \cdots, x_{N-1}\}$ (i.e. ζ is eliminated) and by Lemma 7 we can assume that δ and σ are chosen such that

$$\mathbb{P}\left(\zeta \in B_{\frac{1}{8}\sqrt{\frac{F_{old}}{N}}}(x_j) \,|\, \zeta \in B_{H\sqrt{2F_{old}}}(x_j)\right) \geq \sigma.$$

Skipping the first few steps that are identical to those in Lemma 5, we obtain the following estimate

$$F\left(\{\zeta, x_2, \cdots, x_{N-K}\}\right) = \sum_{i=2}^{N-1} \|x_i - \mu'\|^2 + \|\zeta - \mu'\|^2 \leq \left(1 - \frac{1}{4(N-1)^2}\right) F_{old}.$$

Since $F\left(\{\zeta, x_2, \cdots, x_{N-K}\}\right) < F_{old}$ one of the points $x_1, \cdots, x_{N-1}$ must be discarded. So in the case $K = 1$ we can pick $a = 1 - \frac{1}{4(N-1)^2}$. For general K one can make an argument analogue to the one made at the end of the proof of Lemma 5. □

Define for $t \geq 0$,

$$\eta(t) = \inf\{s \geq t+1 : \ \mathcal{X}'(s) \neq \mathcal{X}'(s-1) \ \text{ or } F(s) = 0\}.$$

(Notice that by definition if $F(\eta(t)) = 0$, i.e. all the points of the core have converged to a single point, then $\eta(t+1) = \eta(t) + 1$. So from now we assume that this is not the case.) Fix some large $M \geq 5$ such that

$$a^{\sigma M} \leq \frac{1}{16},$$

and define $\tau_0 = \tau_0^{(M)}$ such that

$$\mathcal{X}'(\tau_0) \subseteq B_{\frac{7}{4}R}(0) \setminus B_{\frac{5}{4}R}(0), \quad F(\tau_0) \leq \frac{R^2}{M^2\, 4^M}$$

and set also $\tau_i = \eta(\tau_{i-1}), i = 1, 2, \ldots$ (that is, the next time the core changes). Since $F(t) \to 0$ on E and we cross G infinitely often, we must visit the region $B_{\frac{7}{4}R}(0) \setminus B_{\frac{5}{4}R}(0)$ infinitely often as well, therefore $E \subseteq A_M := \{\tau_0^{(M)} < \infty\}$ for all $M \in \mathbb{N}$.

For $m \geq 0$ define

$$\begin{aligned}
A'_m &= A'_{m,M} = \left\{F(\tau_{(m+M)^2}) \leq \frac{R^2}{M^2 4^{2m+M}}\right\}, \\
A''_m &= A''_{m,M} = \left\{\mathcal{X}'(\tau_{(m+M)^2}) \subseteq B_{[2-2^{-m-2}]R}(0) \setminus B_{[1+2^{-m-2}]R}(0)\right\}, \qquad (3.20)\\
A_m &= A_{m,M} = A_{m-1} \cap \left(A'_m \cap A''_m\right).
\end{aligned}$$

Note that the definition is even consistent for $m = 0$ if we define $A_{-1} := \{\tau_0 < \infty\}$ and that in principle A_m, A'_m and A''_m also depend on M, but we omit the second index where this does not create a confusion.

Lemma 9. $\mathbb{P}\left(A_{m+1} \,|\, A_m\right) \geq 1 - e^{-\sigma^2(m+M)}$, $m = 0, 1, 2, \ldots$.

Proof. First, note that $A_m \subseteq A''_{m+1}$. Indeed, since $2K < N$, in the core of the new configuration we must have at least one point from the previous core (this is not true in general if $2K \geq N$), so

$$\min_{x\in\mathcal{X}'(t+1)} \|x\| \geq \min_{x\in\mathcal{X}'(t)} \|x\| - D(t+1)$$

and as a result on A_m we have

$$\begin{aligned}
\operatorname{dist}\left(\mathcal{X}'(\tau_{(m+M+1)^2}), B_R(0)\right) &= \min_{x\in\mathcal{X}'(\tau_{(m+M+1)^2})} \|x\| - R \\
&\geq \min_{x\in\mathcal{X}'(\tau_{(m+M)^2})} \|x\| - R - \sum_{t=\tau_{(m+M)^2+1}}^{\tau_{(m+M+1)^2}} D(t) \\
&\geq \min_{x\in\mathcal{X}'(\tau_{(m+M)^2})} \|x\| - R - [2(m+M)+1]\sqrt{2F(\tau_{(m+M)^2})} \\
&\geq \left(1 + \frac{1}{2^{m+2}} - 1 - \frac{2(m+M)+1}{\sqrt{M^2 4^{2m+M}}}\right) R \\
&\geq \left(\frac{1}{2^{m+2}} - \frac{1}{2^{m+3}}\frac{2(m+M)+1}{M\,2^{M+m-3}}\right) R \geq \frac{R}{2^{m+3}}
\end{aligned}$$

since for all $j \geq 0$ we have $D(t+j) \leq \sqrt{2F(t)}$ by Lemmas 2 and 3, and $\frac{2(m+M)+1}{M\,2^{M+m-3}} < 1$ for all $m \geq 0$ as long as $M \geq 5$. By a similar argument

$$\operatorname{dist}\left(\mathcal{X}'(\tau_{(m+M+1)^2}), (B_{2R}(0))^c\right) = 2R - \max_{x\in\mathcal{X}'(\tau_{(m+M+1)^2})} \|x\| \geq \frac{R}{2^{m+3}},$$

and hence A''_{m+1} occurs.

Consequently, when A_m occurs then $\mathcal{X}'(t) \subseteq G$ for all $t \in \left(\tau_{(m+M)^2}, \tau_{(m+1+M)^2}\right)$. At the same time the core undergoes $N = 2(m+M)+1$ changes between the times $\tau_{(m+M)^2}$ and $\tau_{(m+M+1)^2}$. During each of these changes the function F does not increase, and with probability at least σ decreases by a factor at least $a < 1$ regardless of the past, by Lemma 8 . Hence

$$\mathbb{P}\left(F(\tau_{(m+M+1)^2}) > a^{\sigma N/2} F(\tau_{(m+M)^2})\right) \leq \mathbb{P}(Y_1 + \cdots + Y_N < \sigma N/2),$$

where Y_i are i.i.d. Bernoulli(σ) random variables. It suffices now to get a bound on the RHS since $a^{\sigma N/2} \leq a^{\sigma(m+M)} \leq a^{\sigma M} \leq \frac{1}{16}$. However, the bound for the sum of Y_i follows from the Hoeffding inequality [4]:

$$\mathbb{P}(Y_1 + \cdots + Y_N < \sigma N/2) \leq \exp(-\sigma^2 N/2) \leq \exp(-\sigma^2(m+M)).$$

Consequently, A'_{m+1} and hence A_{m+1} also occur, with probability at least $\exp(-\sigma^2(m+M))$. □

Note that for a fixed M, $A_{m,M}$ is a decreasing sequence of events. Let $\bar{A}_M = \bigcap_{m=0}^{\infty} A_{m,M}$. Lemma 9 implies by induction on m that

$$\begin{aligned}\mathbb{P}\left(\bar{A}_M\right) &= \mathbb{P}\left(A_{0,M}\right) \prod_{m=1}^{\infty} \mathbb{P}\left(A_{m,M} \mid A_{m-1,M}\right) \geq \mathbb{P}\left(A_{0,M}\right) \prod_{m=1}^{\infty} \left(1 - e^{-\sigma^2(M+m)}\right) \\ &\geq \mathbb{P}(A_{0,M}) \left[1 - \sum_{m=1}^{\infty} e^{-\sigma^2(M+m)}\right] \geq \mathbb{P}(A_{0,M}) \left[1 - \sigma^{-2} e^{-\sigma^2 M}\right].\end{aligned}$$

It is easy to see that on $\bar{A}_M$ the points of the core $\mathcal{X}'(t)$ do not ever leave the set G after time τ_0, hence $\sup_{t>\tau_0} \|\mu'(t)\| < \frac{3R}{4}$ on $\bar{A}_M$. At the same time on E we must visit $B_{2/n}(q_2)$ which lies outside of the convex hull of G, thus $\sup_{t>\tau_0} \|\mu'(t)\| > \frac{3R}{4}$, therefore $E \cap \bar{A}_M = \emptyset$. Since $E \subseteq A_{0,M}$ and $\bar{A}_M \subseteq A_{0,M}$ we have

$$\mathbb{P}(E) = \mathbb{P}(E \setminus \bar{A}_M) \leq \mathbb{P}\left(A_{0,M} \setminus \bar{A}_M\right) = \mathbb{P}(A_{0,M}) - \mathbb{P}(\bar{A}_M) \leq \sigma^{-2} e^{-\sigma^2 M}\, \mathbb{P}\left(A_{0,M}\right) \leq \sigma^{-2} e^{-\sigma^2 M}$$

for any $M \geq 0$. Since M can be arbitrarily large we see that $\mathbb{P}(E) = 0$, finishing the proof. □

3.1 Convergence in $\mathbb{R}^1$

In case $d = 1$ we can obtain stronger results than for the general case $\zeta \in \mathbb{R}^d$, $d \geq 1$. For any interval $(a, b) \subset \mathbb{R}$ and any $\delta \in (0, 1)$ let us define a δ-*truncation* of (a, b) as

$$(a, b)_\delta = \left(a + \frac{\delta}{2}(b - a), b - \frac{\delta}{2}(b - a)\right).$$

Definition 2. *The interval* (a_1, b_1) *is called* regular, *if there are* $\delta, r \in (0, 1)$ *such that for any* $(a_2, b_2) \subseteq (a_1, b_1)$ *we have*

$$\mathbb{P}(\zeta \in (a_2, b_2)_\delta \mid \zeta \in (a_2, b_2)) \geq r. \tag{3.21}$$

Remark 6. *We can iterate the inequality (3.21) to establish that*

$$\mathbb{P}(\zeta \in \underbrace{(\dots(a_2,b_2)_\delta)\dots)_\delta}_{k \text{ times}} \mid \zeta \in (a_2,b_2)) \geq r^k, \quad k \geq 2$$

and the iteration of δ-truncation eventually shrinks an interval to a point while r^k is still $\in (0,1)$. Hence it is not hard to check that if Definition 2 holds for some $\delta \in (0,1)$ it holds for all δ in this interval.

Assumption 4 ("matryoshka" property)**.** *Suppose that any interval (a,b) such that $\mathbb{P}(\zeta \in (a,b)) > 0$ contains a regular interval.*

Remark 7. *The property above seems to hold for all common distribution; we were not able, in fact, to construct even a single counterexample, nor, unfortunately, to show that none exists.*

Theorem 4. *Under Assumptions 1 and 4, $\mathcal{X}'(t) \to \phi \in [-\infty, +\infty]$ a.s.*

The proof of this theorem immediately follows from the next proposition, since if $\{\mathcal{X}'(t) \not\to \pm\infty\} = \{\mu'(t) \not\to \pm\infty\}$ occurs then $\mu'(t)$ either converges to a finite number or crosses some interval infinitely often. However, every interval contains some regular interval by Assumption 4 and by Theorem 1 $D(t) \to 0$ a.s. if $\mu'(t) \not\to \pm\infty$, so the core must converge in this case.

Proposition 2. *For any a, b such that $a < b$, with probability one $\mu'(t)$ cannot cross the interval (a,b) infinitely many times.*

Proof. Suppose the contrary. From Assumption 4 it follows that (a,b) contains some regular interval, say (a_1,b_1) which also must be crossed infinitely often. Now the rest of the proof is almost the same as that of Theorem 3 so we will only highlight the differences, which lie in how Assumption 4 is used (in place of the stronger Assumption 3) when we define our "absorbing" region G. Here we let $G = (a_1, b_1)$ and assume w.l.o.g. that $a_1 = 0, b_1 = R$. Let $\zeta(t)$ and M satisfy the conditions of Theorem 3 and define $\tau_0 = \tau_0^{(M)}$ such that

$$\mathcal{X}'(\tau_0) \subseteq \left[\frac{1}{4}R, \frac{3}{4}R\right], \quad F(\tau_0) \leq \frac{R^2}{M^2\, 4^M}.$$

Let us define the events A'_m, A''_m, A_m for $m = 1, 2, \dots$ as in (3.20) with the only change that

$$A''_m = A''_{m,M} = \left\{\mathcal{X}'(\tau_{(m+M)^2}) \subseteq \left(2^{-(m+2)}R, \left[1 - 2^{-(m+2)}\right] R\right)\right\}.$$

We note that since G is regular so Lemma 8 can still be applied. The rest of the proof is identical to that of Theorem 3. $\square$

Corollary 3. *Suppose that* $\operatorname{supp}\zeta$ *is bounded. Then under Assumptions 1 and 4 we have* $\mathcal{X}'(t) \to \phi \in \mathbb{R}$ *a.s.*

Corollary 4. *Suppose that* $K = 1$ *and that Assumption 4 is valid in some interval* $[a, b]$ *and that in addition Assumption 2 is valid for some* $R_- \geq a$ *and* $R_+ \leq b$*. Then* $\mathcal{X}'(t) \to \phi \in \mathbb{R}$ *a.s.*

Proof. Let $u = \liminf_{t\to\infty} x_{(1)}(t)$, $v = \limsup_{t\to\infty} x_{(N-1)}(t)$. Consider the event

$$A_{c,d} = \{u \leq c\} \cap \{v \geq d\}$$

for some $c < d$. If $d < R_-$ or $c > R_+$ we already know from Theorem 2 that we have convergence, so suppose that both $c, d \in [a, b]$. In this case the interval $\left(c + \frac{d-c}{2}, d - \frac{d-c}{2}\right) \subset [c, d]$ is visited i.o. but since this interval inherits the property of Assumption 4 it follows from Proposition 2 that $\mathbb{P}(A_{c,d}) = 0$. From this it follows that

$$\mathbb{P}\left(\nexists\phi :\ \mathcal{X}'(t) \to \phi\right) = \mathbb{P}\left(\bigcup_{c,d\in\mathbb{Q},d<b,c>a} A_{c,d}\right) = 0,$$

i.e. the core converges to a point a.s. $\square$

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5 Bibliography

[1] Sandemose, Aksel (1936). A fugitive crosses his tracks. translated by Eugene Gay-Tifft. New York: A. A. Knopf.

[2] Bingham, N. H.; Goldie, C. M.; Teugels, J. L. Regular variation. Encyclopedia of Mathematics and its Applications, 27. Cambridge University Press, Cambridge, (1987).

[3] Grinfeld, Michael; Volkov, Stanislav; Wade, Andrew R. Convergence in a multidimensional randomized Keynesian beauty contest. Adv. in Appl. Probab. 47 (2015), no. 1, 57–82.

[4] Hoeffding, Wassily. Probability Inequalities for Sums of Bounded Random Variables. Journal of the American Statistical Association **58**, (1963), 13–30.

[5] Parthasarathy, K. R. (2005). Probability measures on metric spaces. AMS Chelsea Publishing, Providence, RI.