Barycentric Markov processes and stability of stochastic integrators

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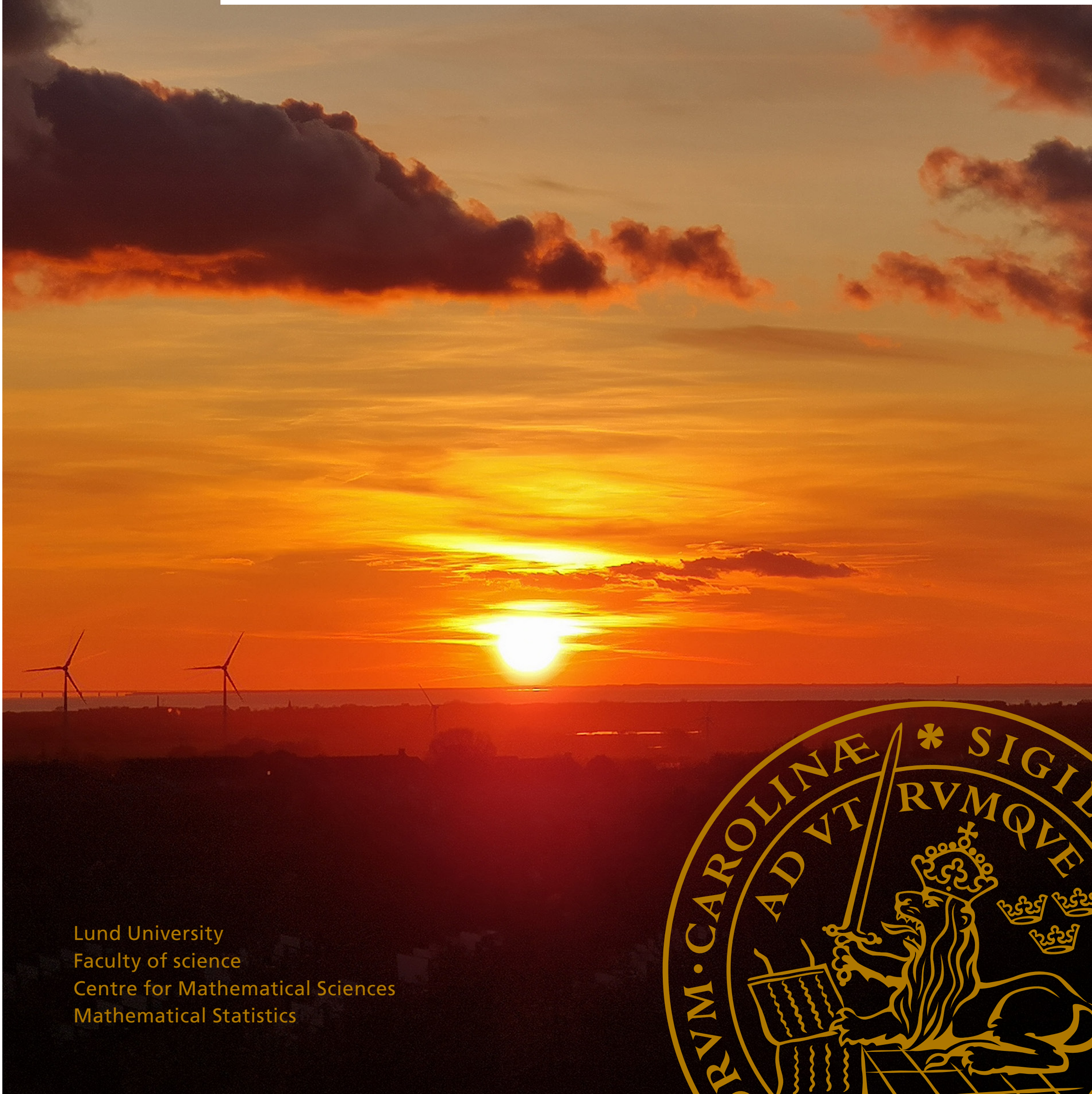
PHILIP KENNERBERG



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Barycentric Markov processes and stability of stochastic integrators

av

Philip Kennerberg

Lund University

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Title and subtitle

Barycentric Markov processes and stability of stochastic integrators

This thesis consists of four papers that broadly concerns two different topics. The first topic is so called barycentric Markov processes. By a barycentric Markov process we mean a process that consists of a point/particle system evolving in (discrete) time, whose evolution depends in some way on mean value of the current points in the system. In common for the three first papers which are on this topic is that we study how all the points of the so called core (a certain subset of points in the system) of the system converges to the same point.

The first article concerns how an N-point system behaves when we reject the K<N/2 points that minimize the sample variance of the remaining N-K points (the core). We then replace the rejected points with K new points which follow some fixed distribution and which are all independent from the past points. When K=1 this is equivalent to rejecting the point which is furthest from the center of mass. We prove that under rather weak assumptions on the sampling distribution, the points of the core converge to the same point as well as that regardless of any assumptions on our sampling distribution, the sampling variance of the core converges to zero or the core "drifts off to infinity".

The second article concerns a similar problem as the first one. We once again consider an N-point system but at each time step we reject the point furthest from the center of mass multiplied by a positive number p and replace it with a point from a fixed distribution with full support on [0,1], which is independent from all past points. If p=1 we obtain a special case of the previous article. If p\not=1 it turns out that this process behaves very differently from the process in the first article, the stationary distribution to which the core points converge turns out to be a Bernoulli distribution.

The third article studies yet another N-point system but now on a discrete circle. During each time step we compute the distances for each point from the mean of its two neighbors and reject the one with largest such distance (thereby obtaining our core) and replace it with a new point independent from past points. Two different cases are considered, the first is with uniformly distributed points in [0,1] and the other is with a discrete uniform distribution (i.e. uniformly distributed on an equally spaced grid).

The fourth and last article is on the topic of stochastic calculus. The main objective is to study "stability" of integrators for stochastic integrals. We examine how converging sequences of processes in the role of integrators retain their convergence properties for their corresponding integrals when the integrators are transformed under certain classes of functions. The convergence is on one hand in the uniform (over compact time intervals) in L^p-sense and on the other hand in the UCP-sense (uniform convergence in probability on compact time intervals). We examine processes with quadratic variations (along some refining sequence) transformed by absolutely continuous functions as well as Dirichlet processes transformed by C^1 functions.

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List of papers

A Jante's law process

Kennerberg, P., Volkov, S. (2018). Advances in Applied Probability, 50(2), 414-439. doi:10.1017/apr.2018.20

B Convergence in the p-Contest

Kennerberg, P., Volkov, S. Convergence in the p-Contest. J Stat Phys 178, 1096–1125 (2020).

C A local barycentric version of the Bak-Sneppen model

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D Some stability results for stochastic integrators

Working paper

Abstract

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The first article concerns how an N-point system behaves when we reject the $K < N/2$ points that minimize the sample variance of the remaining $N - K$ points (the core). We then replace the rejected points with K new points which follow some fixed distribution and which are all independent from the past points. When $K = 1$ this is equivalent to rejecting the point which is furthest from the center of mass. We prove that under rather weak assumptions on the sampling distribution, the points of the core converge to the same point as well as that regardless of any assumptions on our sampling distribution, the sampling variance of the core converges to zero or the core "drifts off to infinity".

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The fourth and last article is on the topic of stochastic calculus. The main objective is to study "stability" of integrators for stochastic integrals. We examine how converging

sequences of processes in the role of integrators retain their convergence properties for their corresponding integrals when the integrators are transformed under certain classes of functions. The convergence is on one hand in the uniform (over compact time intervals) in L^p-sense and on the other hand in the UCP-sense (uniform convergence in probability on compact time intervals). We examine processes with quadratic variations (along some refining sequence) transformed by absolutely continuous functions as well as Dirichlet processes transformed by C^1 functions.

Populärvetenskaplig sammanfattning

Denna avhandling behandlar i huvudsak två olika ämnen. Det första ämnet är, i en bred mening, så kallade barycentriska Markovprocesser. Med barycentriska Markovprocesser åsyftas här, processer som består av en punkt/partikeluppsättning som utvecklas under (diskret) tid och vars utveckling på något sätt styrs av dem ingående partiklarnas medelpunkt. Gemensamt för alla tre artiklar som avhandlar detta ämne är att vi studerar hur punkterna i systemets så kallade kärna (en viss sorts delmängd av systemet punkter) alla konvergerar till en och samma punkt.

Den första artikeln avhandlar hur ett N-punktssystem beter sig när vi förkastar $K < N/2$ punkter som minimerar sampelvariansen av de $N - K$ återstående punkterna (kärnan). Sedan ersätter vi de förkastade punkterna med K nya punkter som följer någon fix fördelning och som är oberoende av tidigare punkter, detta är vad som sker vid ett tidssteg. När $K = 1$ så svarar detta mot att förkasta den punkt som ligger längst ifrån systemets medelpunkt. Vi bevisar att under ganska så svaga antaganden på fördelningen som de nya punkterna slumpas utifrån så konvergerar systemets punkter mot en och samma punkt, när antalet tidssteg går mot oändligheten samt att utan några fördelningsantaganden alls så måste sampelvariansen av kärnan konvergera till noll eller så drar kärnan ”iväg mot oändligheten”.

Den andra artikeln avhandlar ett snarlikt problem som den första artikeln. Vi betraktar återigen ett N-punktssystem men vid varje tidssteg så förkastar vi den här gången den punkt som ligger längst ifrån medelpunkten multiplicerad med en positiv konstant p och ersätter den med en oberoende punkt vars fördelning har fullt stöd på intervallet $[0, 1]$. Om $p = 1$ så har vi ett specialfall av processen som studerades i den första artikeln. Om $p \neq 1$ så visar det sig att denna process beter sig väldigt annorlunda från processen i den föregående artikeln, den stationära fördelningen mot vilken kärnans punkter närmar sig visar sig alltid vara Bernoullifördelad.

Den tredje artikeln studerar återigen ett N-punktssystem men nu på en diskret cirkel. Under ett tidssteg så beräknar vi avstånden till medelpunkten mellan varje punkts två grannar och förkastar den punkt med högst sådant avstånd (detta är vår kärna) och ersätter sedan denna med en ny punkt som är obereonde av tidigare punkter. Vi låter antalet tidssteg gå mot oändligheten och studerar konvergens av kärnans punkter. Två fall behandlas i denna artikel, dels likformigt fördelade punkter på intervallet $[0, 1]$, samt diskret likformigt fördelade punkter (dvs likformigt fördelade över en grid där punkterna är ekvidistanta).

Den fjärde och sista artikeln är inom ämnet stokastisk kalkyl. Huvudsyftet med denna artikel är att studera ”stabilitet” av integratorer för stokastiska integraler. Vi undersöker hur konvergerande följder av processer i egenskap av integratorer bibehåller sin konvergensegenskap för tillhörande integraler när integratorerna transformeras av olika klasser av funktioner. Konvergensen är dels i likformig (över kompakta tidsintervall) L^p-mening och dels i UCP-mening (likformig konvergens i sannolikhet över kompakta tidsintervall). Vi undersöker dels processer med kvadratisk variation (längs någon given förfiningssekvens) transformerade under absolutkontinuerliga funktioner samt Dirichletprocesser transformerade under C^1 funktioner.

Chapter 1

Introduction

1 Semimartingales

In this section we give a *very* minimal introduction to the concept of semimartingales. No proofs of the results are given here, they can all be found in Chapter two of [3]. Semimartingales can be defined in two equivalent manners. The most straightforward definition is to define it as a sum of a local martingale and a processes of finite variation. The second definition, which has more of a functional analysis flare is the following. Let $\mathbf{S}$ denote the space of so called simple predictable processes which are of the form

$$H_t = H_0 + \sum_{k=1}^{n} H_k I_{(T_k, T_{k+1}]}(t), \tag{1.1}$$

for some finite sequence of stopping times $T_1 \leq ... \leq T_{n+1} < \infty$ and where $H_k \in \mathcal{F}_{T_k}$. Given a continuous-time stochastic processes X and a simple predictable process $H \in \mathbf{S}$, we define the linear operator $I_X : \mathbf{S} \to \mathbf{L}^0$ by

$$I_X(H) = H_0 X_0 + \sum_{k=1}^{n} H_k \left(X_{T_{k+1}} - X_{T_k} \right),$$

when H has the form of (1.1).

Definition 1. *An adapted càdlàg process X is called a semimartingale if for each $t \in \mathbb{R}^+$, the map $I_{X^t} : \mathbf{S} \to \mathbf{L}^0$ is continuous in the sense that $\sup_{t,\omega} |H^n_\omega(t) - H_\omega(t)| \to 0$ implies $I_{X^t}(H^n) \xrightarrow{\mathbb{P}} I_{X^t}(H)$.*

The fact that these two definitions are equivalent is known as the Bichteler-Dellacherie Theorem.

Definition 2. *A sequence of processes* $\{X^n\}_n$ *is said to converge in the topology of uniform convergence in probability (ucp) to a* X *if* $(X^n - X)_t^* = \sup_{s \leq t} |X_s^n - X_S| \xrightarrow{\mathbb{P}} 0$.

Theorem 1. *The space* $\mathbf{S}$ *is dense in* $\mathbb{L}$ *(the space of caglad processes) under the ucp topology.*

Definition 3. *Let* $H \in \mathbf{S}$ *and* X *be a cadlag process, we define the (linear) map* $J_X : \mathbf{S} \to \mathbb{D}$*, called the stochastic integral of* H *with respect to* X *by*

$$J_X(H) = H_0 X_0 + \sum_{i=1}^{n} H_i \left(X^{T_{i+1}} - X^{T_i} \right)$$

with H *of the form in* (1.1)*.*

Theorem 2. *The map* $J_X : \mathbf{S}_{ucp} \to \mathbb{D}_{ucp}$ *is continuous (here* $mathbfS_{ucp}$ *and* $\mathbb{D}_{ucp}$ *denotes* $mathbfS$ *and* $\mathbb{D}$ *equipped with the ucp-topology respectively.*

From Theorem 1 and Theorem 2 it follows that $J_X : \mathbb{L}_{ucp} \to \mathbb{D}_{ucp}$ is continuous.

2 Random measures

This section gives a brief introduction to the concept of random measures and their corresponding integrals. All results and definition are taken either from chapter 2 of [12] or chapter 3 of [7], all proofs are omitted but can be found in these textbooks.

Definition 4. *A random measure on* $\mathbb{R}^+ \times \mathbb{R}^d$ *is a family* $\{\mu_\omega(.,.), \omega \in \Omega\}$ *of non-negative measures on* $(\mathbb{R}^+ \times \mathbb{R}^d, \mathbb{B}(\mathbb{R}^+) \times \mathbb{B}(\mathbb{R}^d))$ *with the property that* $\mu_\omega(\{0\}, A) = 0$ *for every* $A \in \mathbb{B}(\mathbb{R}^d)$ *(i.e. no point mass at time zero).*

Let $\Omega' := \Omega \times \mathbb{R}^+ \times \mathbb{R}^d$, $\mathcal{O}' := \mathcal{O} \times \mathbb{B}(\mathbb{R}^d)$ and $\mathcal{P}' := \mathcal{P} \times \mathbb{B}(\mathbb{R}^d)$, where $\mathcal{O}$ denotes the optional sigma algebra (the sigma algebra generated by mappings of the type $(\omega, t) \to f(\omega, t)$ where $f(\omega, .) \in \mathbb{D}$, the space of cadlag functions) and $\mathcal{P}$ denotes the predictable sigma algebra (the sigma algebra generated by mappings of the type $(\omega, t) \to f(\omega, t)$ where $f(\omega, .)$ is a caglad function). We say a function on Ω' is optional if it is $\mathcal{O}'$-measurable and say it is predictable if it is $\mathcal{P}'$-measurable. Let $W(\omega, t, x) : \Omega' \to \mathbb{R}$

be an optional function on Ω', since $W(\omega,.,.)$ is $\mathbb{B}(\mathbb{R}^+)\times\mathbb{B}(\mathbb{R}^d)$)-measurable for every $\omega\in\Omega$, we may define the integral process $(W\cdot\mu)_t$ as $\int_{[0,t]\times\mathbb{B}(\mathbb{R}^d)} W(\omega,s,x)\mu_\omega(ds,dx)$ when $\int_{[0,t]\times\mathbb{B}(\mathbb{R}^d)} |W(\omega,s,x)|\mu_\omega(ds,dx)<\infty$ and as $+\infty$ otherwise. We say that μ is optional if $W\cdot\mu$ is an optional process for every optional function W. Similarly we say that μ is predictable if $W\cdot\mu$ is predictable for every predictable W.

Definition 5. *An optional measure μ is called $\mathcal{P}'$-σ-finite if there exists a strictly positive predictable function V on Ω' such that* $\mathbb{E}\left[\int_{\mathbb{R}^+\times\mathbb{R}^d} V(\omega,s,x)\mu_\omega(ds,dx)\right]<\infty$

We shall say that nonnegative process is *integrable* if it has an a.s. limit as $t\to\infty$ and this limit has a finite expected value. Also we will use the term *transition kernel* (as is done in [12]) of a measurable space $(A,\mathcal{A})$ into another measurable space $(B,\mathcal{B})$ we will mean a family $\{\alpha(a,.): a\in A\}$ of non-negative measures on $(B,\mathcal{B})$ such that $\alpha(.,C)$ is $\mathcal{A}$-measurable for each $C\in\mathcal{B}$. Recall (See) the following property, if $(G,\mathcal{G})$ is any measurable space and m is any finite nonnegative measure on $(\mathbb{R}^d\times G,\mathbb{B}(\mathbb{R}^d)\times\mathcal{G})$ with G-marginal $\hat{m}(A)=m(A\times\mathbb{R}^d)$ then there exists a transition kernel α from $(G,\mathcal{G})$ into $(\mathbb{R}^d,\mathbb{B}(\mathbb{R}^d))$ such that $m(B)=\int\int 1_B(g,x)\alpha(g,dx)\hat{m}(dg)$ for all $B\in\mathcal{G}\times\mathbb{B}(\mathbb{R}^d)$.

Theorem 3. *If μ is an optional $\mathcal{P}'$-σ-finite measure then there exists a predictable random measure ν, called the compensator of μ, which is unique up to a $\mathbb{P}$-null set satisfying either one of the following two equivalent conditions*
(i) $\mathbb{E}\left[\int_{\mathbb{R}^+\times\mathbb{R}^d} W(\omega,s,x,)\nu_\omega(ds,dx)\right]=\mathbb{E}\left[\int_{\mathbb{R}^+\times\mathbb{R}^d} W(\omega,s,x,)\mu_\omega(ds,dx)\right]$ *for every non-negative $\mathcal{P}'$-measurable random function W on Ω'.*
(ii) For every $\mathcal{P}'$-measurable function W on Ω' such that $\int_{[0,t]\times\mathbb{R}^d}|W(\omega,s,x,)|\mu_\omega(ds,dx)$ is an integrable process then $\int_{[0,t]\times\mathbb{R}^d}|W(\omega,s,x,)|\nu_\omega(ds,dx)$ is also integrable and $\int_{[0,t]\times\mathbb{R}^d}|W(\omega,s,x,)|\nu_\omega(ds,dx)$ is the compensator process of $\int_{[0,t]\times\mathbb{R}^d}|W(\omega,s,x,)|\mu_\omega(ds,dx)$.
Moreover there exists a (predictable) increasing and integrable process A and a transition kernel $K(\omega,t,dx)$ from $(\Omega\times\mathbb{R}^+,\mathcal{P})$ into $(\mathbb{R}^d,\mathbb{B}(\mathbb{R}^d))$ such that

$$\nu_\omega(dt,dx)=dA_t(\omega)K(\omega,t,dx)$$

Definition 6. *We say that μ is an integer valued random measure if it satisfies:*
1) $\mu_\omega(\{t\}\times\mathbb{R}^d)\le 1$ a.s. for all $t\in\mathbb{R}^+$,
2) for each $A\in\mathbb{B}(\mathbb{R}^+)\times\mathbb{B}(\mathbb{R}^d)$, $\mu_(A)\in\bar{\mathbb{N}}$
3) μ is optional and $\mathcal{P}'$-σ-finite.

Let ν denote the compensator of an integer valued random measure μ. Denote

$$a_t(\omega)=\nu(\omega,\{t\}\times\mathbb{R}^d)$$

Lemma 1. *There exists a version of the compensator ν of μ such that*

$$a_t(\omega) \le 1.$$

We say that a random set D is thin if $D = \bigcup_{n\ge1}[[T_n]]$ where $\{T_n\}_n$ are stopping times and $[[T_n]] = \{(\omega,t) : t \in \mathbb{R}^+, T(\omega) = t\}$

Proposition 1. *If μ is an integer-valued random measure, there exists a thin random set D and an $\mathbb{R}^d$-valued optional process β such that*

$$\mu_\omega(A,B) = \sum_{s\in A} 1_D(\omega,s)\delta_{(s,\beta(s))}(A,B),$$

for $A \in \mathbb{B}(\mathbb{R}^+)$ and $B \in \mathbb{B}(\mathbb{R}^d)$.

Proposition 2. *Let X be an $\mathbb{R}^d$-valued càdlàg process then*

$$\mu(A,B) = \sum_s 1_{\Delta X_s\neq 0}\delta_{(s,\Delta X_s)}(A,B)$$

with $A \in \mathbb{B}(\mathbb{R}^+)$ and $B \in \mathbb{B}(\mathbb{R}^d)$ defines an integer-valued random measure with $D = \{\Delta X_s \neq 0\}$ being the thin set and $\beta(s) = \Delta X_s$ the optional process in the previous proposition.

Let μ be an integer valued random measure, ν it's compensator and such that $|U|\cdot\nu$ is a locally integrable process then so is $|U|\cdot\nu$

Let μ be an integer valued random measure, define the integer valued measure p by

$$p(\omega,\Gamma) = \sum_{s\in\Gamma} 1_{a_s(\omega)>0}(1-\mu(\omega,\{s\}\times\mathbb{R}^d)),$$

for every $\Gamma \in \mathbb{B}(\mathbb{R}^+)$. The compensator of p is the ,measure q given by (see section 5 of chapter 3 in [7])

$$q(\omega,\Gamma) = \sum_{s\in\Gamma} 1_{a_s(\omega)>0}(1-a_s(\omega)).$$

Let U be a $\mathcal{P}'$-measurable function such that for each stopping time T

$$\int_{\mathbb{R}^d} |U(\omega,T,x)\nu(\omega,\{T\},dx)1_{T<\infty} < \infty \text{ a.s.}$$

and define for such U

$$\hat{U}(\omega,t) = \int_{\mathbb{R}^d} U(\omega,t,x)\nu(\omega,\{t\},dx).$$

Define now the process

$$G(U) = \frac{(U-\hat{U})^2}{1+|U-\hat{U}|}\cdot p + \frac{\hat{U}^2}{1+\hat{U}}\cdot q.$$

If $G(U)$ is locally integrable then the integral $U\cdot(\mu-\nu)$ is a well defined local martingale (again, see section 5 of chapter 3 in [7] for a proof).

3 Dirichlet processes

Originally studied by Föllmer in the paper REF for the purpose of developing a pathwise Ito calculus. Today there are several definitions of Dirichlet processes (which are not all equivalent). We will present the original definition proposed by Föllmer (in [5]), but first we introduce the notion of a refining seqeunce. Given some $t>0$ we say that $\{D_k\}_k$ is a refining sequence of each D_k is a partition of $[0,t]$, $D_k \subseteq D_{k+1}$ and the mesh of D_k tends to zero as $k\to\infty$.

Definition 7. *X is said to be a Dirichlet process (in Föllmer sense) if for any $t>0$, and for some refining sequence $\{D_k\}_k$ of $[0,t]$,*

$$\sup_{l\geq k}\sum_{t_i\in D_k}\mathbb{E}\left[\left(\sum_{t_i\leq s_j\leq t_{i+1},\, s_j\in D_l}\mathbb{E}\left[X_{s_{j+1}}-X_{s_j}\,|\,\mathcal{F}_{s_j}\right]\right)^2\right],$$

a.s. converges to zero as $k\to\infty$.

Föllmer also showed in [6] that this definition is equivalent to saying that X can be (uniquely) decomposed into a square integrable martingale plus an adapted continuous process starting in 0 with zero quadratic variation along $\{D_k\}_k$. We will work with a definition not equivalent Föllmers, which is much weaker.

Definition 8. *A cadlag process X is called a semimartingale if $X=Z+C$ where Z is a semimartingale and C is an adapted continuous process of zero quadratic variation along some refining sequence $\{D_k\}_{k\geq 1}$.*

By the above definition any semimartingale is a Dirichlet process, something which is obviously not true by Föllmer's definition. By transforming a semimartingale by a C^2 function we know that by Ito's formula we get yet another semimartingale. If we however transform a semimartingale by a C^1 function then in general we do not get a semimartingale but a Dirichlet process. In fact, it was shown in [9] that if we transform a Dirichlet process by a C^1 function we retain another Dirichlet process. In the same article the following representation formula was also proven

Theorem 4. *Let $X = Z + C$ where Z is a semimartingale and C has zero quadratic variation and f be a C^1-function. We have $f(X_s) = Y_s + \Gamma_s$ where Y is a semimartingale, Γ is continuous and $[\Gamma]_t = 0$ for all $t > 0$. The expression for Y is given by*

$$
\begin{aligned}
Y_t &= f(X_0) + \sum_{s \le t} \left(f(X_s) - f(X_{s-}) - \Delta X_s f(X_{s-})\right) I_{|\Delta X_s|>1} + \int_0^t f'(X_{s-}) dZ_s \\
&+ \int_0^t \int_{|x|\le 1} \left(f(X_{s-}+x) - f(X_{s-}) - x f'(X_{s-})\right)(\mu - \nu)(ds, dx) \\
&+ \sum_{s\le t} \int_{|x|\le a} \left(f(X_{s-}+x) - f(X_{s-}) - x f'(X_{s-})\right) \nu(\{s\}, dx).
\end{aligned} \tag{3.2}
$$

4 Background for papers A, B and C

Papers A and B (and to a lesser extent paper C) are directly related to the paper by [11], in this paper a number of open problems were posed. Among these problems was one referred to as "a repeated Keynesian beauty contest". Fix a parameter $p > 0$. Start with a uniform array of N elements on $[0, 1]$. At each step, compute the mean μ of the N elements, and replace by a $U[0, 1]$ random variable the element that is farthest (amongst all the N points) from $p\mu$. Thus at each step, either the minimum or maximum is replaced, depending on the current configuration. This is related to the "p-beauty contest" [27, p. 72] in which N players choose a number between 0 and 100, the winner being the player whose choice is closest to p times the average of all the N choices. The stochastic process described above is a repeated, randomized version of this game (without any learning, and with random player behaviour) in which the worst performer is replaced by a new player. In the paper "Convergence in a multidimensional randomized Keynesian beauty contest" in [2] the case $p = 1$ was studied and also generalized to the multivariate case, i.e. considering points in $\mathbb{R}^n$ and with $U[0, 1]^n$ distributed replacement points.

5 Summary of paper A

In this paper we generalize the results of those in [2]. We also redefine the process so that we allow for $K < N/2$ points to be replaced at each step. The criteria for which points that are to be replaced is that we replace those K points that minimize the sample variance of the remaining $N - K$ points (this minimization is the reason for the given name, Jante's law process). If $K = 1$ then this is actually the same as to replace the point furthest from the center of mass, so it coincides with the model in [2]. We also allow for much more general distributions.

5.1 Main outline of paper

In section one we define the model and introduce some auxiliary tools. Section two is devoted to studying the "dissipation" of the system (or to put it more succinctly, convergence of the sample variance to zero). Theorem 1 tells us that if $K \leq N - 2$ (this result is thus more general than our other results which require that $K < N/2$) regardless of the sampling distribution (on $\mathbb{R}^d$), the system can only have two long term behaviours; either the sample variance converges to zero or the system "runs of to infinity" (i.e. the absolute value of at least one of the core points goes to infinity). As a corollary to this theorem we show that when $d = 1$ and the sampling distribution is singular, the core converges to a single point.

In Section 2.1 we prove in Theorem 2 that in the real-valued case, when we remove only one point, under a rather weak assumption on the distribution of the tail, the sample variance converges almost surely to zero. It is possible to construct counterexamples to this assumption by hand, but most common continuous distributions will in fact fulfil the proposed assumption. In addition we also prove that if the core eventually stays in the tail region then the process converges to a single point almost surely; if the tail condition is valid on the whole line then the process converges to a single point (almost surely). An additional result has been added to this theorem that was not present in the published article namely, we prove that if the sampling distribution has finite first moment with the tail condition being valid through all of the support then limit of the expected value of the order statistics (up to $N - 1$) of the core all exist and coincide.

In Section three we first introduce a "local" regularity assumption for distributions on $\mathbb{R}^d$, this is a very weak assumption that we have yet to find any counterexamples to.

In Theorem 3, we prove that if $K < N/2$, either the core drifts off to infinity or it almost surely converges to a single point. In particular, if the sampling distribution has compact support then the core converges almost surely. We also introduce an even weaker assumption (the "matryoshka" condition) when $d = 1$, and show that in the real-valued case we have convergence of the core under this weaker condition. Finally we show that if one combines the "matryoshka" condition in some bounded region and assumes the tail condition used for Theorem 2 then we also have almost sure convergence of the core.

6 Summary of paper B

We study a discrete-time Markovian system of $N \geq 3$ number of points, $\{X_1^0, ..., X_N^0\}$ taking values in $[0, 1]$. Given a fixed parameter $p \in \mathbb{R}^+$ we start with our N points at time $t = 0$, compute their average $\mu(X_1^0, ..., X_N^0)$, remove the point furthest from $p\mu(X_1^0, ..., X_N^0)$ (the remaining $N-1$ points are called "the core" similarly to the Jante's law process) and then replace it with a point independent from all past points, having some fixed distribution ζ, with full support in $[0, 1]$. This procedure is then repeated indefinitely and we study convergence of the core. Study of this process (with $\zeta \in U([0, 1])$) was posed as an open problem in [11].

6.1 Main outline of paper B

The problem is divided into two different cases, $p < 1$ and $p > 1$ (The case $p = 1$ was dealt with in [2] and generalized in [8]). In Section 1 we formalize the model and introduce some notation. In section 2 we tackle the case $p < 1$ when the sampling distribution is uniform on $[0, 1]$. In Theorem 1 we prove that the core almost surely converges to zero. In part, this is based on some very tedious but elementary inequalities that have been deferred to the Appendix. In section three we study the case $p > 1$. We consider all sampling distributions with full support on $[0, 1]$ and establish that the core must converge almost surely either to zero or one, and we provide examples when both outcomes are possible, i.e. when the stationary distribution is a true Bernoulli distribution. In Section four we study the case when $N = 3$, ζ has a nondecreasing density in some neighbourhood of zero and show that in this case $\mathcal{X}'(t) \to 1$ a.s.. The second section in the Appendix contains the original proof (the published one) of

Theorem 2.

7 Summary of paper C

We study the behaviour of an interacting particle system, related to the Bak-Sneppen model and Jante's law process defined in [8]. Let $N \geq 3$ vertices be placed on a circle, such that each vertex has exactly two neighbours. To each vertex we assign a real number, called *fitness*. We pick the vertex which fitness deviates most from the average of the fitnesses of its two immediate neighbours (in case of a tie, draw uniformly among such vertices), and replace it by a random value drawn independently according to some distribution ζ. We show that when ζ is finitely supported on a uniform grid or has a continuous uniform distribution, all the fitnesses, except one, converge to the same value. The model we study in this paper is a "marriage" between Jante's law process (defined in [8]) and the Bak-Sneppen (BS) model. In the BS model, N species are located around a circle, and each of them is associated with a so-called "fitness", which is a real number. The algorithm consists in choosing the least fit individual, and then replacing it *and both of its two closest neighbours* by a new species, with a new random and independent fitness.

7.1 Main outline of paper C

In Section one we formally define the model and introduce necessary notation. In Section two we study the discrete case and show that the process is a finite state space Markov chain which almost surely gets absorbed, regardless of initial state. In Section three we deal with the case when the sampling distribution is $U[0,1]$ and show that the core converges almost surely to a single point.

8 Summary of paper Paper D

We consider limits for sequences of the type $\int Y_- df_n(X^n)$, for semimartingale integrands, where both the functions $\{f_n\}_n$ and the processes $\{X^n\}_n$ tend to some limits, f and X respectively. An important ingredient is then to study the limit of $[f_n(X^n) - f(X)]$ which is an interesting problem in its own right. We provide an important application which is jump removal. We consider processes $\{X^n\}_n$ admitting to quadratic variations and absolutely continuous functions $\{f_n\}_n$ which are dominated by some locally integrable function and study convergence in the UCP setting. We also consider the case when $\{X^n\}_n$ are Dirichlet processes and $\{f_n\}_n$ are C^1 functions whose derivatives converge uniformly on compacts. We provide important examples of how to apply this theory for sequential jump removal.

8.1 Main outline of paper D

In the first section we introduce the notion of quadratic- and covariation along a refining sequence. In the second section we go through some notation, recall some results from the theory of Dirichlet processes and prove lemma's that will be used in the main results. Of more general interest we prove that processes admitting to quadratic variations are closed under absolutely continuous transforms and that these processes are well-defined integrators for semimartingale integrands. In the third section we state our main results and prove the results concerning stability of integrators under either C^1- (for Dirichlet processes) or absolutely continuous transforms (for processes admitting to quadratic variations), in either in uniform L^p convergence setting (for Dirichlet processes) or in the UCP setting (for processes admitting to quadratic variations). In section four we provide examples for the results of the previous section in terms of so-called jump truncations. Of particular interest are processes with jumps of finite variation where we can simply remove jumps on a by-modulus basis for smaller and smaller truncation levels. We also show certain commutation properties of such jump truncation with regards to our earlier stability results. In the Appendix we give proofs of results deferred to this section.

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