A Format for Multiplier Optimization

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Abstract

Robustness problems are considered, where structural information about uncertainty is given in terms of an infinite class of integral quadratic constraints. We introduce a flexible format for representation of such information that supports the use of numerical software for solution of linear matrix inequalities. Examples are given involving bounded time-variations and nonlinearities.

1. Introduction

It has recently been demonstrated that a large number of know criteria for robustness analysis of control systems can be unified based on the concept integral quadratic constraint (IQC, see next section for definition), [6],[7],[9]. The idea is that every piece of structural information about time-varying parameters, nonlinearities and uncertain elements, that is characterized by an IQC, can be exploited by a multiplier in the robustness condition. Based on this idea, robustness can be verified using convex optimization over a set of such multipliers. This set is generally infinite dimensional.

For example, suppose that the operator Δ in the feedback configuration of Figure 1, satisfies the IQC's defined by the matrix functions $\Pi_1, \dots, \Pi_n$. In [9], the stability condition for the uncertain loop is written as existence of non-negative numbers $x_1, \dots, x_n$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \sum x_i \Pi_i(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0$$

for all $\omega \in R \cup \{\infty\}$. This infinite number of inequalities can be transformed using the Kalman-Yakubovich-Popov Lemma into a single linear matrix inequality in the variables $x_1, \dots, x_n$ and a symmetric matrix variable P of a size determined by the order of the state space representations of $\Pi_1, \dots, \Pi_n$ and G.

This approach to robustness analysis covers a large majority of stability criteria based on passivity and small gain arguments, that have been stated over the last thirty years. Indeed, the combination of a unifying formulation of the stability theorem, with recent progress on computations involving linear matrix inequalities (LMI's), opens up a new perspective for analysis of uncertain systems. Old robustness criteria, with respect to particular types of nonlinearities and time-variations, had their main limitation in the computability of multipliers. The use of LMI-code for computation of the multipliers can be useful and there is a large potential for new applications.

However, although the robustness problem has a straightforward reformulation in terms of convex LMI optimization, each particular type of uncertainty structure has its own features and it is a cumbersome task to do the reformulation well. It is therefore desirable to do as much of the work as possible once and for all, and represent the IQC's in a form that allows recycling in other problems where the same type of uncertainty structure appears. The aim of this paper is to define such a format and to demonstrate its applicability in a few examples, including stability and performance analysis for uncertain time-variations with derivative constraints, memoryless odd nonlinearities and searches for simultaneous Lyapunov functions.

In most cases the class of IQC's that describe the structure of Δ is infinite and it is not practical to describe them by some finite number of extreme points $\Pi_1 \dots, \Pi_n$. Instead we suggest in this paper the following format for description of a class of IQC's.

$$\Pi(j\omega) = \Psi(j\omega)^* M(\lambda)\Psi(j\omega)$$

where M is a fixed affine function of $\lambda \in R^n$ and the range of λ is determined by the additional constraints

$$\Phi_k(j\omega)^* M(\lambda)\Phi_k(j\omega) \leq 0, \quad k = 1, \dots, K.$$

for all $\omega \in R$. This description again allows transformation of the stability condition by the K-Y-P Lemma into a single linear matrix inequality.

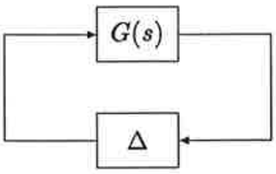

Figure 1. Perturbation in feedback form

The suggested description of IQC classes also allows convenient addition and diagonal augmentation of classes of IQC's. This is convenient in the study of robust stability or robust performance of systems composed of several uncertainties with different nature.

There are other recent LMI formulations of robustness problems. In for example, [1] and [5] it is shown that an application of the passivity theorem with appropriate multipliers gives robustness tests such as the computation of the real/mixed structured singular value. The search for suitable multipliers can then be performed using LMI optimization. Further examples of LMI formulations of robustness tests are given in [2].

Notation

Let $\mathbf{RL}_\infty$ be the set of proper (bounded at infinity) rational functions with real coefficients. The subset consisting of functions without poles in the closed right half plane is denoted $\mathbf{RH}_\infty$, and the subset of $\mathbf{RH}_\infty$ consisting of functions without poles in $\mathrm{Re}\, s \geq -\alpha$ is denoted $\mathbf{RH}_\infty(\alpha)$. The set of $m \times n$ matrices with elements in $\mathbf{RL}_\infty$ ($\mathbf{RH}_\infty$,$\mathbf{RH}_\infty(\alpha)$) will be denoted $\mathbf{RL}_\infty^{m\times n}$ ($\mathbf{RH}_\infty^{m\times n}$, $\mathbf{RH}_\infty^{m\times n}(\alpha)$.

$L_2^l[0,\infty)$ can be thought of as the space of square integrable functions $[0,\infty) \to R^l$, and the induced $\mathbf{L}_2$-norm of an operator from $\mathbf{L}_2$ into $\mathbf{L}_2$, is denoted $||\cdot||$.

Introduce the truncation operator P_T, which leaves a function unchanged on the interval $[0,T]$ and gives the value zero on $(T,\infty]$. *Causality* of an operator F means that $P_T F = P_T F P_T$. I_m denotes an $m \times m$ identity matrix.

2. IQC Based Stability Analysis

We will in this section shortly review IQC based stability analysis as presented in [9]. Consider the feedback system in Figure 1. Here G is a stable linear causal time-invariant operator with transfer function $G(s)$ in $\mathbf{RH}_\infty^{l\times m}$. Δ is a bounded causal operator from $L_2^l[0,\infty)$ to $L_2^m[0,\infty)$. We will be concerned with input/output stability in the following sense

Definition The system in (1) is input/output stable if $I - G\Delta$ has a bounded causal inverse, i.e. if there exists a $C > 0$, such that $||(I - G\Delta)^{-1}|| \leq C$.

The term IQC is defined as follows. Suppose Π is rational matrix function, which takes bounded Hermitean values on the imaginary axis. Then, Δ is said to satisfy the IQC defined by Π, if

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{u}(j\omega) \\ \widehat{v}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{u}(j\omega) \\ \widehat{v}(j\omega) \end{bmatrix} d\omega \geq 0$$

for any $\widehat{u}, \widehat{v}$ being the Fourier transforms of $u, v \in \mathbf{L}_2[0,\infty)$ with $v(t) = \Delta(u)(t)$.

It is assumed that the feedback system in Figure (1) satisfies the extended well-posedness condition, that for every $\tau \in [0,1]$ and for every $T \geq 0$, there exists a causal inverse of $I - \tau P_T G\Delta$. The main stability result of [9] is stated as

THEOREM 1
Assume that the system in Figure 1 is well posed, and that for all $\tau \in [0,1]$, $\tau\Delta$ satisfies the IQC defined by Π. If

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0 \qquad (1)$$

for all $\omega \in R \cup \{\infty\}$ then the feedback system is input/output stable.

Remark In most applications Π has a form such that $\tau\Delta$ satisfies the IQC defined by Π for all $\tau \in [0,1]$, if and only if Δ satisfies the IQC defined by Π. This is the case in all applications in this paper, and we will therefore only state the simplified assumption in what follows.

The next two obvious properties are useful when applying Theorem 1.

Property 1 Assume Δ satisfies the IQC's defined by $\Pi_1, \ldots, \Pi_n$, then Δ also satisfies the IQC defined by $\sum_{i=1}^n \alpha_i \Pi_i$, for any $\alpha_i \geq 0$, $i \in [1, \ldots, n]$

Property 2 Assume Δ has the block-diagonal structure $\Delta = \mathrm{diag}\,[\Delta_1, \ldots, \Delta_n]$, and that Δ_i satisfies the IQC defined by Π_i. Then Δ satisfies the IQC defined by $\Pi = \mathrm{daug}\,[\Pi_1, \ldots, \Pi_n]$, where the operation daug is defined as follows. If

$$\Pi_i = \begin{bmatrix} \Pi_{i1} & \Pi_{i2} \\ \Pi_{i2}{}^* & \Pi_{i3} \end{bmatrix}, \quad i = 1, 2$$

where the block structures are consistent with the size of Δ_1 and Δ_2, respectively, then

$$\mathrm{daug}\,(\Pi_1, \Pi_2) = \left[\begin{array}{cc|cc} \Pi_{11} & 0 & \Pi_{12} & 0 \\ 0 & \Pi_{21} & 0 & \Pi_{22} \\ \hline \Pi_{12}{}^* & 0 & \Pi_{13} & 0 \\ 0 & \Pi_{22}{}^* & 0 & \Pi_{23} \end{array}\right]$$

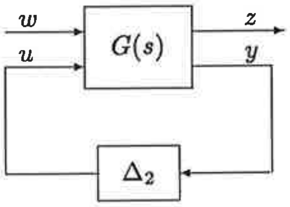

Figure 2. System setup for performance analysis.

3. IQC Based Performance Analysis

It is possible to use the ideas of the previous section for analysis of robust performance of the system in Figure 2. The transfer function has the structure

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

We define robust performance as follows.

Definition Assume that the perturbation Δ_2 satisfies the IQC defined by Π_2. Then the system in Figure 2 is said to have robust performance if

a. The system is input output stable.

b.

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{z}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \Pi_1(j\omega) \begin{bmatrix} \hat{z}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega < 0 \quad (2)$$

when $z = (G_{11} + G_{12}\Delta_2(I - G_{22}\Delta_2)^{-1}G_{21})w$.

The standard transformation from robust performance to robust stability, gives

PROPOSITION 2
Assume Δ_2 satisfies the IQC defined by Π_2. Then under some nonrestrictive conditions on Π_1 and well posedness of the loop, the system in Figure 2 has robust performance if

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \text{daug}(\Pi_1, \Pi_2)(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0 \quad (3)$$

for all $\omega \in R \cup \{\infty\}$.

Remark If we introduce the class of perturbations Δ_1 that satisfy the IQC defined by Π_1, then Theorem 2 states that the the system in Figure 2 has robust performance if the system in Figure 1 is stable according to Theorem 1, when $\Delta = \text{diag}(\Delta_1, \Delta_2)$.

Remark An example of the second condition in the definition of robust performance is when we require the induced $\mathbf{L}_2$-norm of the system to be less than one. In this case we can chose Π_1 as

$$\Pi_1 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \quad (4)$$

4. LMI Formulation of the Robustness problem

It follows from the last two sections that it is possible to analyze a wide variety of stability and performance robustness problems by means of application of Theorem 1. There are many examples of perturbation structures Δ, which satisfies a class of IQC's that can be described in terms of the following set of matrix functions, which we denote $\mathcal{P}(\Psi, \Phi, M)$.

$$\left\{ \Psi^*(j\omega)M(\lambda)\Psi(j\omega) \;\middle|\; \begin{matrix} \Phi_k^*(j\omega)M(\lambda)\Phi_k(j\omega) < 0 \\ \text{for } k = 1, \ldots, K, \lambda \in R^n \end{matrix} \right\} \quad (5)$$

Here M is an affine function of $\lambda \in R^n$, which takes values in the set of symmetric matrices in $R^{N\times N}$, and $\Psi = [\Psi^a \;\; \Psi^b] \in \mathbf{RL}_\infty^{N\times(l+m)}$, is structured consistently with the structure of Δ, i.e. Ψ^a and Ψ^b has column size l and m respectively. The inequalities involving $\Phi_k \in \mathbf{RL}_\infty^{N\times N_k}$, $k = 1, \ldots, K$ are used to restrict the range of λ. In the next section, we give several examples of perturbation structures, which can be described by IQC classes defined as above.

It is often convenient to derive the IQC class for a perturbation with complicated structure from simple IQC classes. For this means, we need to add and augment such classes. Addition and diagonal augmentation of the IQC classes defined by the sets $\mathcal{P}_1 = \mathcal{P}(\Psi_1, \Phi_1, M_1)$ and $\mathcal{P}_2 = \mathcal{P}(\Psi_2, \Phi_2, M_2)$ is performed in the following way.

$$\mathcal{P}_1 + \mathcal{P}_2 = \mathcal{P}(\begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}, \begin{bmatrix} \Phi_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \Phi_2 \end{bmatrix}, \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix})$$

and

$$\text{daug}(\mathcal{P}_1, \mathcal{P}_2) = \mathcal{P}(\Psi, \begin{bmatrix} \Phi_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \Phi_2 \end{bmatrix}, \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix})$$

where

$$\Psi = \begin{bmatrix} \Psi_1^a & 0 & \Psi_1^b & 0 \\ 0 & \Psi_2^a & 0 & \Psi_2^b \end{bmatrix},$$

Robust stability and performance analysis can now be approached in the following way. Derive a class of IQC's describing the perturbation and the performance criterion. The system is then stable or has robust performance if the following feasibility problem has a solution.

Robustness Test Find $\lambda \in R^N$, such that

$$\Phi_i^*(j\omega)M(\lambda)\Phi_i(j\omega) < 0, \quad i \in [0, \ldots, K]$$

for all $\omega \in R \cup \{\infty\}$. Here

$$\Phi_0 = \Psi \begin{bmatrix} G \\ I \end{bmatrix}$$

It is possible to obtain a LMI formulation of this robustness test as follows. Introduce representations $\Phi_i = C_i(sI - A_i)^{-1}B_i + D_i$, for $i = 0, \ldots, k$. Then by the KYP lemma as stated below, the robustness test is equivalent to the following LMI problem.

LMI Formulation of Robustness Test
Find suitable $\lambda \in R^N$ and matrices $P_i = P_i^T$, $i = 0, \ldots K$, such that

$$\begin{bmatrix} C_i^T \\ D_i^T \end{bmatrix} M(\lambda)\,[\,C_i \quad D_i\,] + \begin{bmatrix} A_i^T P_i + P_i A_i & P_i B_i \\ B_i^T P_i & 0 \end{bmatrix} < 0$$

for $i = 0, \ldots, k$.

LEMMA 3—KYP
For the system $\Phi(s) = C(sI - A)^{-1}B + D$, where $\det(j\omega - A) \neq 0$, the following inequalities are equivalent. M is a symmetric matrix.

a.

$$\Phi^*(j\omega)M\Phi(j\omega) < 0, \quad \forall \omega \in R \cup \{\infty\}$$

b. There exists a matrix $P = P^T$, such that

$$\begin{bmatrix} C^T \\ D^T \end{bmatrix} M\,[\,C \quad D\,] + \begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} < 0$$

Proof. The lemma follows in the case when (A, B) is controllable, from [12]. The formulation here does not require controllability and a proof for this case is given in [8]. QED

Remark. For Example 3 in the next section we need the lemma above for the case of non-strict inequalities. In that example we have controllability of (A, B) and equality of the non-strict versions of the statements above follows from [12].

5. Examples of IQC Classes

This section will give several examples of how important perturbation structures, can be described in terms of IQC classes defined as in (5).

Example 1 Assume Δ is defined by multiplication with a real number of magnitude less that 1, this corresponds to the real μ problem, [3]. Then Δ satisfies the IQC's defined by (we suppress some arguments for compactness of notation)

$$\Pi_X(j\omega) = \begin{bmatrix} X & 0 \\ 0 & -X \end{bmatrix}, \quad \Pi_Y(j\omega) = \begin{bmatrix} 0 & Y \\ Y^* & 0 \end{bmatrix}$$

for any $X, Y \in \mathbf{RL}_\infty^{m\times m}$, where $X(j\omega) = X^*(j\omega) \geq 0$, and $Y(j\omega) = -Y^*(j\omega)$, for all $\omega \in R$. Introduce "basis multipliers" $\tilde{R}, \tilde{S} \in \mathbf{RH}_\infty^{N\times m}$, and let $X = \tilde{R}^* U \tilde{R}$, $Y = V\tilde{S} - \tilde{S}^* V^T$, where $U \in R^{N\times N}$, with $U = U^T \geq 0$, and $V \in R^{m\times N}$. If we let λ correspond to the elements in U and V, respectively, then we can define a class of IQC's corresponding to Π_X by

$$\mathcal{P}(\begin{bmatrix} \tilde{R} & 0 \\ 0 & \tilde{R} \end{bmatrix}, \begin{bmatrix} 0 \\ I \end{bmatrix}, \begin{bmatrix} U & 0 \\ 0 & -U \end{bmatrix})$$

and a class of IQC's corresponding to Π_Y by

$$\mathcal{P}(\begin{bmatrix} I & 0 \\ 0 & I \\ 0 & \tilde{S} \\ \tilde{S} & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & V & 0 \\ 0 & 0 & 0 & -V \\ V^T & 0 & 0 & 0 \\ 0 & -V^T & 0 & 0 \end{bmatrix})$$

Note that no Φ constraints are needed for the last class of IQC's.

Example 2 Let $\delta \in \mathbf{L}_\infty$ with $|\delta|_\infty \leq 1$. Then $\Delta = \delta(t)I$, is defined by multiplication in the time domain with a time-varying scalar function, and Δ satisfies the IQC defined by

$$\Pi = \begin{bmatrix} X & Y \\ Y^T & -X \end{bmatrix} \tag{6}$$

for any $X, Y \in R^{m\times m}$ satisfying $X = X^T \geq 0$ and $Y = -Y^T$. If we let λ represent the elements of X and Y and if we let $M(\lambda)$ have the value given by (6) for a given X and Y, then we can represent a class of IQC's describing Δ, by $\mathcal{P}(I, (0 \quad I)^T, M)$.

Example 3 Consider the system

$$\dot{x} = (A + B\Delta(t)C)x \tag{7}$$

where $\Delta(t)$ is a time-varying matrix variable, which takes values in a convex polytope. Simultaneous Lyapunov theory states that this system is stable if there exists $P = P^T > 0$ such that

$$(A + B\Delta_i C)^T P + P(A + B\Delta_i C) < 0 \tag{8}$$

for all corners Δ_i of the polytope. We can view the system in (7) as a system in Figure 1, with $G(s) = (sI - A)^{-1}$ and with the perturbation $B\Delta C$. It is easy to see that the stability condition in (8) is equivalent to the robustness test of finding $Q = Q^T$, $P = P^T > 0$ and $\varepsilon > 0$, such that

$$\begin{bmatrix} (j\omega I - A)^{-1} \\ I \end{bmatrix}^* \begin{bmatrix} Q + \varepsilon I & -P \\ -P & 0 \end{bmatrix} \begin{bmatrix} (j\omega I - A)^{-1} \\ I \end{bmatrix} \leq 0$$

$\forall \omega \in R$, subj to

$$\begin{pmatrix} I \\ B\Delta_i C \end{pmatrix}^T \begin{bmatrix} Q & -P \\ -P & 0 \end{bmatrix} \begin{pmatrix} I \\ B\Delta_i C \end{pmatrix} \geq 0$$

for all corners Δ_i of the polytope. This is again of the form (5), with K equal to the number of corners of the polytope.

Example 4 If $\Delta = \delta(t)I_m$, where $\delta(t)$ is a real-valued and differentiable parameter, with $0 < k_{\min} \leq \delta(t) \leq k_{\max} < \infty$, and $-2\alpha\delta(t) \leq \dot{\delta}(t) \leq 2\beta\delta(t)$, for some $\alpha, \beta \geq 0$. Then Δ satisfies the IQC defined by

$$\Pi(j\omega) = \begin{bmatrix} 0 & M^*(j\omega) \\ M(j\omega) & 0 \end{bmatrix}$$

Here $M = M_1 + M_2^*$, where $M_1 \in \mathbf{RH}_\infty^{m\times m}(\beta)$, $M_2 \in \mathbf{RH}_\infty^{m\times m}(\alpha)$, satisfies

$$M_1(j\omega - \beta) + M_1^*(j\omega - \beta) > 0$$
$$M_2(j\omega - \alpha) + M_2^*(j\omega - \alpha) > 0$$

for all $\omega \in R$. This follows from the ideas in [10]. Introduce basis multipliers $\tilde{M}_1 \in \mathbf{RH}_\infty^{N\times m}(\beta)$, $\tilde{M}_2 \in \mathbf{RH}_\infty^{N\times m}(\alpha)$ and let $M_1 = U\tilde{M}_1$ and $M_2 = V\tilde{M}_2$, where $U, V \in R^{m\times N}$. It is now straight forward to define a representation as in (5) of a class of IQC's describing Δ.

Example 5 Suppose Δ operates scalar signals according to the nonlinear map $(\Delta v)(t) = \delta(v(t))$, where δ is an odd, monotone non-deceasing function on R with $\delta(0) = 0$ and $|\delta(x)| \leq k|x|$, for some positive constant k. Then Δ satisfies the IQC defined by

$$\Pi(j\omega) = \begin{bmatrix} 0 & I + H(j\omega) \\ I + H^*(j\omega) & -2(I + \mathrm{Re}\, H(j\omega))/k \end{bmatrix}$$

for any $H \in \mathbf{RL}_\infty$ with $\mathbf{L}_1$-norm of its impulse response is no larger than one, this follows from [13]. If we restrict our attention to the subset of $\mathbf{RL}_\infty$, which have transfer functions that can be represented in the form

$$H(s) = \sum_{i=1}^{N} \frac{b_i}{s + a_i} + d$$

then we can find an representation as in (5) in the following way. Introduce a basis multiplier on the form

$$\tilde{H} = \left[1, \frac{1}{s + a_1}, \ldots, \frac{1}{s + a_N}\right]^T$$

and define variables $\lambda_0^+, \lambda_0^-, \ldots, \lambda_N^+, \lambda_N^- \geq 0$. Then,

$$H(s) = \sum_{i=1}^{N} (\lambda_i^+ - \lambda_i^-)\frac{1}{s + a_i} + \lambda_0^+ - \lambda_0^-$$

and an upper bound of the $\mathbf{L}_1$ norm of H's impulse response is given as

$$\sum_{i=1}^{N} (\lambda_i^+ + \lambda_i^-)/|a_i| + (\lambda_0^+ + \lambda_0^-) < 1$$

This is essentially of the form (5).

6. Numerical Examples

We will in this section illustrate the previous discussion with two simple numerical examples. The LMI optimization has been done with LMI-lab, [4].

Example 1 We will consider robust performance of the system

$$P(s) = \frac{s + 1}{s^3 + (3 + \delta_1)s^2 + (4 + \delta_2)s + (3 + \delta_2)}$$

where δ_1 and δ_2 are real-valued constant parameters with magnitude less that one. We first consider the problem of finding a a stability margin for the system, i.e. we want to find an as large as possible value of γ, such that the system is stable for the perturbation $\gamma\Delta$. We represent the system as in Figure 2, with $\Delta = \mathrm{diag}(\delta_1, \delta_2 I_2)$ and with $G(s) = C(sI - A)^{-1}B$, where $C = I_3$ and

$$A = \begin{pmatrix} -3 & -4 & -3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We let the basis multipliers be $\tilde{S}_1 = (s + 1)^{-1}$, $\tilde{S}_2 = (sI - 1)^{-1}I_2$

$$\tilde{R}_1 = \begin{bmatrix} 1 \\ (s + 0.1)^{-1} \end{bmatrix}, \quad \tilde{R}_2 = \begin{bmatrix} I_2 \\ (s + 0.1)^{-1}I_2 \end{bmatrix}$$

define IQC classes, which describes δ_1 and δ_2, as in Example 1 of the last section. Addition and diagonal augmentation gives a class of IQC's that describe the perturbation Δ.

Application of the robustness test in Section 4 to the system γG, gives the maximal value $\gamma^0 = 2.162$. This is very close to the optimal value.

Next we consider the robust performance problem of finding an upper bound on the induced $\mathbf{L}_2$-norm of the system. We represent the system as in Figure 2, with

$$G(s) = \begin{pmatrix} \gamma\tilde{C} \\ C \end{pmatrix} (sI - A)^{-1} \begin{pmatrix} \tilde{B} & B \end{pmatrix}$$

where A, B and C are as above, and $\tilde{B} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$, $\tilde{C} = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix}$. We represent the performance criteria as in (4) and the perturbation Δ as above. The robustness test in section 4 is satisfied with the maximal value $\gamma^0 = 1.28$, which gives $1/\gamma^0$ as an upper bound of the induced $\mathbf{L}_2$-norm. The problem in this example can be solved in a number of simple ways. However, it is remarkable that a very simple choice of multipliers turns out to be close to optimal.

Example 2 We will here consider a robust performance problem as in figure 2, when

$$G = \begin{bmatrix} \gamma P & \gamma P \\ P & P \end{bmatrix}$$

where

$$P(s) = -\frac{(s^2+4s+11)(s^2+200s+20)}{(s^2+2s+10)(s^2+s+16)}$$

and when $\Delta = \delta(t)I$ where $\delta(t)$ is a real positive and time-varying parameter with $-\delta(t) \le \dot{\delta}(t) \le 6\delta(t)$. We will try to find an as large as possible value of γ. This gives $1/\gamma$ as an upper bound on the maximal induced $\mathbf{L}_2$-norm of the system. Let us describe the performance criteria as in (4), and the perturbation Δ by the class of IQC's in Example 4 of the last section, defined by the basis multipliers

$$\tilde{M}_1 = \left[1, \frac{1}{s+4}\right]^T$$
$$\tilde{M}_2 = \left[1, \frac{1}{s^2+4s+16}, \frac{s}{s^2+4s+16}\right]^T$$

Numerical computations in LMI-lab with these basis multipliers gives the minimal upper bound of the induced $\mathbf{L}_2$-norm to be 345 and the optimal parameter values are

$$U = (\,1.58 \quad 8.71\,)\,10^{-4}$$
$$V = (\,1.58 \quad 8.71 \quad 850\,)\,10^{-4}$$

In [10] robust stability was studied for this system system.

7. Conclusions

We have suggested a format for representation of classes of IQC's and demonstrated its applicability on a few examples. The format is designed to give conditions in terms of linear matrix inequalities and it also allows for convenient addition and diagonal augmentation of classes of IQC's. The last point is essential in the development of software for analysis of problems with combinations of several uncertainty structures.

The approach is based on finite-dimensional restrictions of convex optimization problems, that are a priori infinite-dimensional. The choice of good finite-dimensional restrictions, remains to a large extent an open problem. Also to estimate the conservatism of the finite-dimensional analysis remains to be a challenge. A approach using duality theory is reported in [11].

8. References

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