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Abstract

A new optimal stochastic control problem that minimizes the probability of a signal to cross a high level is solved. This type of problems has previously been solved by minimum variance control, which is known to have a badly behaved control signal. To overcome this, weighting on the control signal—LQG—has been proposed, but no good criteria on how to choose the weighting has been known. The solution to the new problem can be thought of as finding such optimal weightings.

1. Introduction

There are a lot of control problems where the goal not only is to keep the signal that is controlled near a certain reference value, but also to keep it below a dangerous level—dangerous in the sense that if the signal crosses the level, the controller has failed. The distance between the critical level and the reference value is normally large, since otherwise the failure rate will be intolerably high. However there may be other control-objectives that make it undesirable to choose the distance too large. An example of problems of this kind can be found in [4], where the power of an ore crusher should be kept as high as possible but not exceed a certain level. Other examples can be found in sensor-based robotics and force control, [7].

This type of problems has previously been solved by minimum variance control, [2] p. 169 and [3] pp. 159–209—the intuitively best controller. It is well-known that minimum variance control causes large variations in the

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control signal. This problem has been solved by introducing weighting on the control signal—LQG-design. However there has been no good criteria on how to choose the weighting. The controller designed below can be interpreted as chosing optimal weighting-matrices in a LQG-problem, and it is obtained by minimizing a criteria that better captures the control-objectives in the problems described above than the minimum variance criteria does.

In Section 2 the control problem is formulated. It is an optimal stochastic control problem. In Section 3 the problem presented in Section 2 is solved for the stationary case. In Section 4 the optimal controller found in Section 3 is compared with the minimum variance controller on a second order process. It is seen that the optimal controller gives a variance that is close to that of the minimum variance controller, causes a lower upcrossing intensity of high levels, and has a control signal that is better behaved. Finally in Section 5 the results are summarized.

2. Control Problem

The problems mentioned above are captured in the following simple problem formulation:

$$\min_{u(t)} \mathrm{P}\left\{\sup_{[0,T]} z(t) > z_0\right\}, \tag{1}$$

where $z(t)$ is a signal that should be kept well below z_0, and where $u(t)$ is a control signal that in some way influences $z(t)$. The minimization is of course constrained to a stable closed loop. To be able to solve the problem it will be assumed that $z(t)$ is a stationary Gaussian process with mean $m_z = \mathrm{E}\{z(t)\}$ that should be equal to a predescribed reference value, and covariance $r_z(\tau) = \mathrm{E}\{(z(t+\tau) - m_z(t+\tau))(z(t) - m_z(t))\}$. It must be required that $z(t)$ is continuous with probability one for (1) to have any meaning. A sufficient condition for this can be found in [5], p. 170. The assumptions above hold if the plant to be controlled is linear, the disturbances acting on it are stationary Gaussian processes, the controller is linear, and the closed loop is stable and has a covariance function that is continuous. Thus these are the constraints under which the minimization above will be done.

3. Regulator Design

The problem stated in Section 1 will be solved approximately for large values of $z_0 - m_z$. This is the interesting case for the type of problems discussed in the introduction.

In the first subsection some results from the theory on extremes in random processes are given, and then in the second subsection the problem is rephrased to minimization over a set of solutions to LQG-problems parameterized by a scalar. The equations for solving the LQG-problems are given in the last subsection.

Some Results from the Theory on Extremes in Random Processes

To simplify the problem in Section 2 an upper bound for the probability in (1) will be given.

THEOREM 1
If $r_z(\tau)$ has a finite second derivative for $\tau = 0$, then the probability in (1) can be bounded as

$$\mathrm{P}\left\{\sup_{[0,T]} z(t) > z_0\right\} \leq \mathrm{P}\left\{z(0) > z_0\right\} + T\mu, \tag{2}$$

where

$$\mu = \frac{1}{2\pi}\frac{\sigma_{\dot{z}}}{\sigma_z}\exp\left(-\frac{(z_0 - m_z)^2}{2\sigma_z^2}\right), \tag{3}$$

and where $\sigma_z^2 = r_z(0)$ and $\sigma_{\dot{z}}^2 = -r_z''(0)$.

Proof: The proof can be found in Theorem 7.3.2 (Rices's Formula) and Theorem 8.2.1 in [9], but a rough outline of it will be given below. Let N be the number of upcrossings of z_0 by z in $[0, T]$. Then it follows by Rice's Formula that $\mathrm{E}\{N\} = T\mu$. Further

$$\begin{aligned}\mathrm{P}\left\{\sup_{[0,T]} z(t) > z_0\right\} =& \mathrm{P}\left\{z(0) > z_0 \cup N \geq 1\right\} \\ \leq& \mathrm{P}\left\{z(0) > z_0\right\} + \mathrm{P}\left\{N \geq 1\right\} \\ \leq& \mathrm{P}\left\{z(0) > z_0\right\} + \mathrm{E}\left\{N\right\}\end{aligned}$$

□

If $z_0 - m_z$ is large, then the first term on the right hand side in (2) is neglectable. Under this assumption it can also be shown that the number of upcrossings of z_0 by z is approximately a Poisson process with intensity μ, see Theorem 9.2.1 and Corollary 9.1.3 in [9].

Solution

It is obvious from the previous subsection, that the problem in (1) can be approximately solved for large values of $z_0 - m_z$ by minimizing μ in (3). Let z be defined by

$$\begin{cases} dx = Ax\,dt + B_1du + B_2dv \\ dy = C_1x\,dt + Dde \\ z = C_2x \end{cases}, \tag{4}$$

where v and e are zero-mean Wiener-processes with $\mathrm{E}dvdv^T = R_1dt$, $\mathrm{E}dede^T = R_2dt$ and $\mathrm{E}dvde^T = 0$. The signal y is the measurement, which the control signal is constrained to be a function of.

It will be seen that the minimization of μ in (3) can be done by first minimizing

$$J = \mathrm{E}\{z^2 + \rho^2\dot{z}^2\} \tag{5}$$

over u for $\rho \geq 0$, and then minimizing μ over ρ. In the following lemma J is rewritten to fit into the usual LQG-problem formulation.

LEMMA 1
If $C_2B_2 = 0$, then J in (5) can be written

$$J = \mathrm{E}\{x^TQ_1x + 2x^TQ_{12}u + u^TQ_2u\}, \tag{6}$$

where

$$\begin{aligned} Q_1 &= C_2^T C_2 + \rho^2 A^T C_2^T C_2 A \\ Q_{12} &= \rho^2 A^T C_2^T C_2 B_1 \\ Q_2 &= \rho^2 B_1^T C_2^T C_2 B_1 \end{aligned} \quad . \tag{7}$$

Proof: The result follows immediately by using the definition of z in (4). □

The following lemma gives conditions under which there exist a unique solution to the LQG-problem of minimizing J in (6).

LEMMA 2

Suppose that (A, B_1, C_1) is SIMO, (A, B_1, C_2) is SISO, (A, B_1) is stabilizable, (C_1, A) is detectable, and $C_2B_2 = 0$. Let Q_1 be factorized into

$$Q_1 = C\bar{Q}_1C^T,$$

where C and $\bar{Q}_1$ have full rank. Further let $B_2R_1B_2^T$ be factorized into

$$B_2R_1B_2^T = B\bar{R}_1B^T,$$

where B and $\bar{R}_1$ have full rank. If $\rho C_2B_1 \neq 0$ or $C(sI - A)^{-1}B_1$ is full rank, and if DR_2D^T is positive definite or $C_1(sI - A)^{-1}B$ is full rank and $\bar{R}_1$ has rank greater than or equal to the number of measurements y, then there exist a unique feed-back control that minimizes J in (5).

Proof: If $\rho C_2B_1 \neq 0$, then $Q_2 > 0$ by (7) in Lemma 1. Lemma 1 also implies that $Q_1 - Q_{12}Q_2^{-1}Q_{12}^T = C_2^TC_2$, which is nonnegative definite, and that Q_1 is nonnegative definite. Thus it follows by [1], pp. 56–57, that the deterministic optimal control problem has a unique solution. If $\rho C_2B_1 = 0$, then $Q_{12} = Q_2 = 0$ by (7) in Lemma 1, but then it follows by [10] that the deterministic optimal control problem has a unique solution, since $C(sI - A)^{-1}B_1$ is full rank. If DR_2D^T is positive definite or $C_1(sI - A)^{-1}B$ is full rank and $\bar{R}_1$ has rank greater than or equal to the number of measurements y, then it follows by [10] that the optimal filtration problem has a unique solution. Since both the optimal deterministic control problem and the optimal filtration problem has a unique solution, there exist a unique feedback control that minimizes J in (5). □

The next lemma gives a parameterization of all jointly minimal variances σ_z^2 and $\sigma_{\dot{z}}^2$.

LEMMA 3

Suppose that there exist a unique feed-back control that minimizes J in (5) for $m_z = 0$ and $\rho \geq 0$. Then all pairs of minimal values, i.e. all pareto optimal values, of σ_z and $\sigma_{\dot{z}}$ are obtained and parameterized by ρ by minimizing J.

Proof: Let

$$\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = \begin{pmatrix} (C_2(sI - A)^{-1}B_2 \quad 0) & C_2(sI - A)^{-1}B_1 \\ (C_1(sI - A)^{-1}B_2 \quad D) & C_1(sI - A)^{-1}B_1 \end{pmatrix},$$

and let $P_{22} = N_rD_r^{-1} = D_l^{-1}N_l$ be right and left coprime factorizations of P_{22} with

$$\begin{pmatrix} V_r & -U_r \\ -N_l & D_l \end{pmatrix} \begin{pmatrix} D_r & U_l \\ N_r & V_l \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Then by Theorem 1 p. 38 in [6] all stabilizing controllers $U = HY$ of (4), where U and Y are Laplace transforms of $\dot{u}$ and $\dot{y}$, can be written $H = H_1 H_2^{-1}$, where

$$\begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} U_l & -D_r \\ V_l & -N_r \end{pmatrix} \begin{pmatrix} I \\ Q \end{pmatrix}$$

with Q being a stable transfer-function matrix. Thus the minimization over u constrained to a stable closed loop system can be rephrased to a minimization over Q, where Q belongs to the linear space of stable transfer-function matrices. By Theorem 1 p. 43 in [6]

$$Z = (P_{11} + P_{12} D_r U_r P_{21} - P_{12} D_r Q D_l P_{21}) \begin{pmatrix} V \\ E \end{pmatrix},$$

where Z, V and E are Laplace transforms of z, $\dot{v}$ and $\dot{e}$. It is seen that the transfer-function matrices from V and E to Z are affine in Q, and since the variances of z and $\dot{z}$ are convex in the transfer-function matrices, it follows that the variances are convex in Q. Then, since there exist a unique feedback control that minimizes J in (5), it follows by Theorem 2.1 in [8] that all pareto optimal values of σ_z and $\sigma_{\dot{z}}$ are obtained and parameterized by ρ by minimizing J in (5). □

It will now be shown how the minimization of μ in (3) can be rephrased to finding optimal weightings in a LQG-problem.

THEOREM 2
If $\sigma_z < z_0 - m_z$, and the conditions in Lemma 2 are fulfilled, then the minimization of μ in (3) can be performed in two steps. First J in (6) is minimized over u for $m_z = 0$ and $\rho \geq 0$. Then $\mu = \mu(\sigma_z(\rho), \sigma_{\dot{z}}(\rho))$ is minimized over ρ, where $\sigma_z(\rho)$ and $\sigma_{\dot{z}}(\rho)$ are the values of σ_z and $\sigma_{\dot{z}}$ obtained when J is minimized.

Proof: Since μ is an encreasing function of σ_z and $\sigma_{\dot{z}}$ for $\sigma_z < z_0 - m_z$, the solution is found among all pairs of minimal values of σ_z and $\sigma_{\dot{z}}$. By lemmas 1–3 all such pairs are found and parameterized by ρ by minimizing J in (6). □

LQG-equations

For short reference the equations for deriving the LQG-solution when Q_2 and R_2 are invertible are given below. The transfer function from measurement to control is

$$H(s) = -L(sI - A + B_1 L + K C_1)^{-1} K, \tag{8}$$

where L and K are given by

$$\begin{aligned} L &= Q_2^{-1}(Q_{12}^T + B_1^T S) \\ K &= P C_1^T R_2^{-1} \end{aligned}, \tag{9}$$

and where S and P are the solutions to the Riccati-equations, [1] p.56–58, and p. 168,

$$\begin{aligned} (A - B_1 Q_2^{-1} Q_{12}^T)^T S + S(A - B_1 Q_2^{-1} Q_{12}^T) \\ -S B_1 Q_2^{-1} B_1^T S + Q_1 - Q_{12} Q_2^{-1} Q_{12}^T = 0. \\ AP + PA^T + B_2 R_1 B_2^T - P C_1^T R_2^{-1} C_1 P = 0 \end{aligned} \tag{10}$$

To calculate σ_z and $\sigma_{\dot{z}}$ the following Lyapunov-equation for the closed loop system has to be solved, [3] p. 66 and pp. 290–291,

$$A_c X + X A_c^T + R_c = 0, \tag{11}$$

where

$$\begin{aligned} A_c &= \begin{pmatrix} A - B_1 L & B_1 L \\ 0 & A - K C_1 \end{pmatrix} \\ R_c &= \begin{pmatrix} B_2 R_1 B_2^T & B_2 R_1 B_2^T \\ B_2 R_1 B_2^T & B_2 R_1 B_2^T + K D R_2 D^T K^T \end{pmatrix} \end{aligned}.$$

Then σ_z and $\sigma_{\dot{z}}$ are given by

$$\begin{aligned} \sigma_z^2 &= (\, C_2 \quad 0 \,) X (\, C_2 \quad 0 \,)^T \\ \sigma_{\dot{z}}^2 &= C_2 (\, A - B_1 L \quad B_1 L \,) X (\, A - B_1 L \quad B_1 L \,)^T C_2^T \end{aligned}. \tag{12}$$

4. Evaluation

To evaluate the performance of the optimal controller obtained by minimizing (3) a second order process will be investigated. The set of LQG-solutions is calculated analytically, and then $\mu(\rho)$ is calculated numerically and plotted. The optimal controller is compared with the minimum variance controller—the intuitively best controller. It is seen that the new controller gives a lower upcrossing intensity of high levels, and that it has a control signal that is better behaved.

In the first subsection the process is defined, in the second subsection the LQG-controllers are computed, and in the last subsection the optimal controller is compared with the minimum variance controller.

Process

Let the process be given by

$$\begin{cases} dx = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x\,dt + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} du + \begin{pmatrix} 0 \\ 1 \end{pmatrix} dv \\ dy = (\, 1 \quad 0 \,) x\,dt + de \\ \;\; z = (\, 1 \quad 0 \,) x \end{cases},$$

$R_1 = \sigma_1^2 > 0$, and $R_2 = \sigma_2^2 > 0$. As long as $b_2 \neq 0$ there will by Lemma 2 exist a unique solution to the LQG-problems.

LQG-Controllers

The solutions to the Riccati-equations in (10) are

$$\begin{aligned} S &= \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix} \\ P &= \begin{pmatrix} \sigma_2\sqrt{2\sigma_1\sigma_2} & \sigma_1\sigma_2 \\ \sigma_1\sigma_2 & \sigma_1\sqrt{2\sigma_1\sigma_2} \end{pmatrix} \end{aligned}.$$

By using (9) it is found that

$$L = \begin{pmatrix} \frac{1}{\rho b_1} & \frac{1}{b_1} \end{pmatrix}$$

and that

$$K = \begin{pmatrix} \sqrt{2\frac{\sigma_1}{\sigma_2}} \\ \frac{\sigma_1}{\sigma_2} \end{pmatrix}.$$

Some more tedious calculations will give the controller $H(s)$ in (8) to be

$$H(s) = -\frac{(\sqrt{2\sigma_1\sigma_2} + \rho\sigma_1)s + \sigma_1}{b_1\rho\sigma_2 s^2 + (b_1\sigma_2 + b_1\rho\sqrt{2\sigma_1\sigma_2} + b_2\rho\sigma_2)s + b_2(\rho\sqrt{2\sigma_1\sigma_2})}.$$

It is interesting to note that if $b_1 \neq 0$, then the controller is proper for all values of ρ. For $\rho > 0$ the controller is strictly proper. When $b_1 = 0$ and $b_2 \neq 0$, the controller is proper only for $\rho > 0$. It is also seen how an integrator can be forced into the controller by having a Wiener process as load-disturbance, i.e. $b_1 \neq 0$ and $b_2 = 0$.

Optimal Controller and Minimum Variance Controller

The intensity μ and the variances of z and $\dot{z}$ has been calculated numerically for values of ρ in the range of 10^{-6} to 10^3, $m_z = 0$, $z_0 = 5$ and $b_1 = b_2 = \sigma_1 = \sigma_2 = 1$. The result is shown in Figure 1. The intensity has a minima for $\rho = 0.1$, which is $\mu = 1.1334 \cdot 10^{-4}$. Note that the variance of $\dot{z}$ decreases rapidly, while the variance of z only increases slowly for small values of ρ, $\sigma_z(10^{-6}) = 1.1892$ and $\sigma_z(10^{-1}) = 1.2305$. As ρ goes to zero—minimum variance control— the intensity goes to infinity. This is easily seen also theoretically by noting that with minimum variance control $r_z(\tau)$ is not differentiable for $\tau = 0$. The optimal controller is given by:

$$H(s) = -\frac{15.1421s + 10.0000}{s^2 + 12.4142s + 11.4142}$$

and the minimum variance controller is given by:

$$H(s) = -\frac{1.4142s + 1.0000}{s + 1.0000}.$$

Bode-diagrams of the controllers are shown in Figure 2. It is interesting to note that the difference between the minimum variance controller and the optimal controller is that the optimal controller has much lower gain for high frequences due to the optimal controller being strictly proper while the minimum variance controller being only proper.

The Amplitude-diagrams for the closed loop transfer functions from V and E to Z are shown in Figure 3. The main difference between the two control strategies are that the optimal controller causes a lower high-frequency gain from E to Z than the minimum variance controller.

Plots of z as functions of time for the two control strategies are shown in Figure 4. It is seen that the main difference is that the variance of $\dot{z}$ is larger with the minimum variance controller than with the optimal controller.

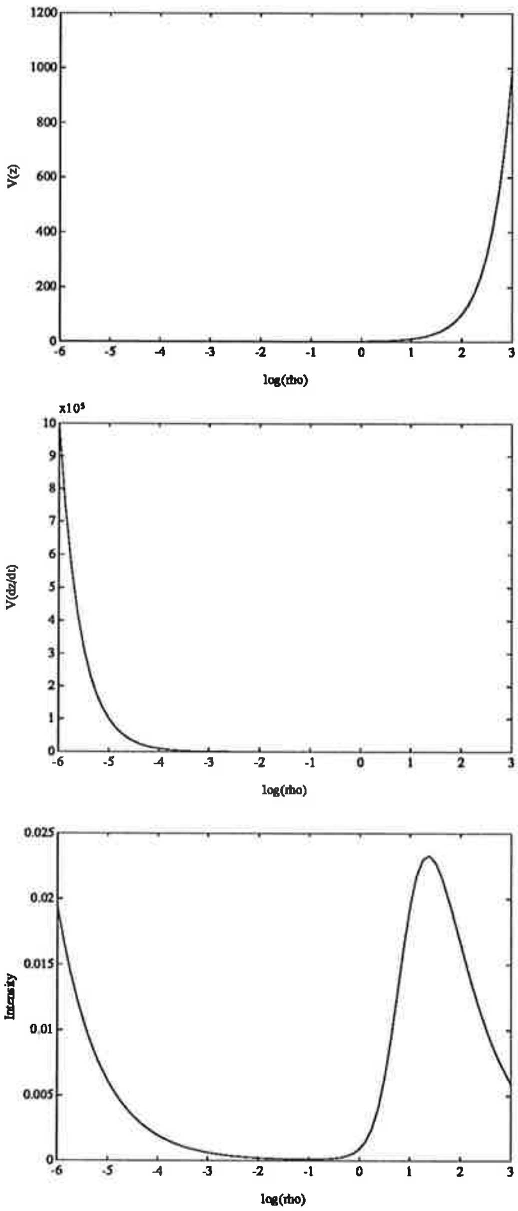

Figure 1. The variance of z—top, the variance of $\dot{z}$—middle, and the crossing intensity μ—bottom as functions of $\log(\rho)$.

5. Conclusions

A new interesting optimal stochastic control problem has been posed. It has been solved in the stationary case. The solution can be thought of as finding optimal weighting-matrices in a LQG-problem. The new controller has been compared with the minimum variance controller—the intuitively best controller—for a second order process. It has been seen that the new controller causes a lower upcrossing intensity of high levels, gives a variance that is close to that of the minimum variance controller, and that it has a relative degree that is larger. Thus the control signal will be better behaved for the new controller.

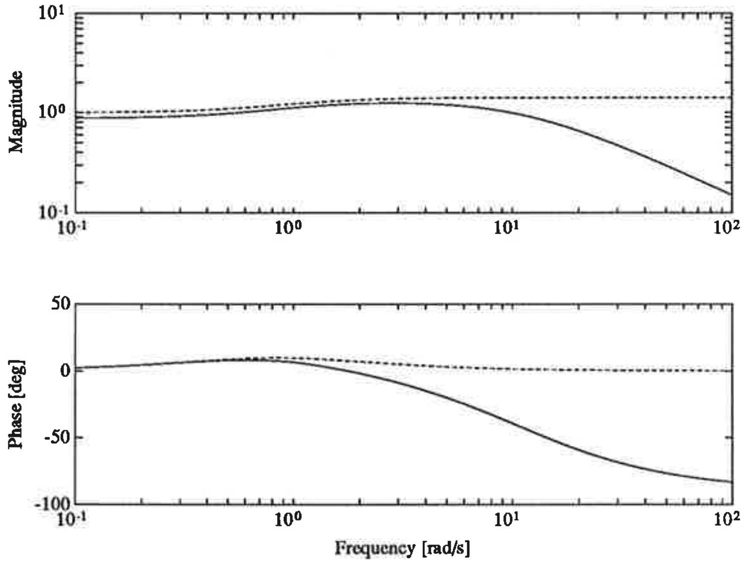

Figure 2. Bode-diagrams showing the transfer functions of the minimum variance controller—dotted line, and the optimal controller—solid line.

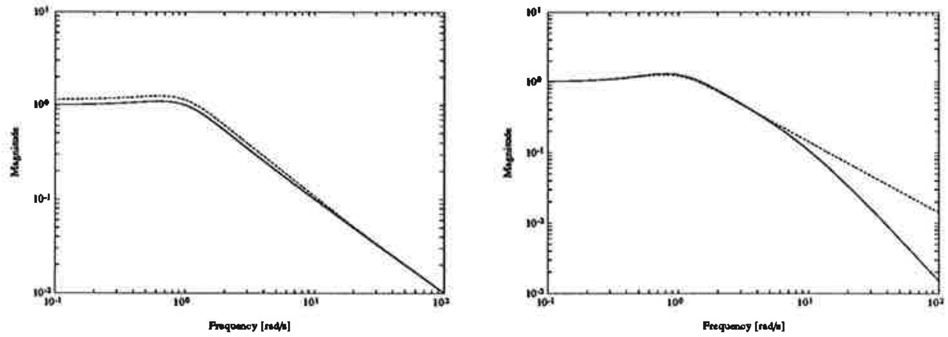

Figure 3. Bode-diagrams showing the transfer functions of the closed loop from $\dot{v}$ to z—left, and $\dot{e}$ to z—right. The solid lines are the optimal controller and the dotted lines are the minimum variance controller.

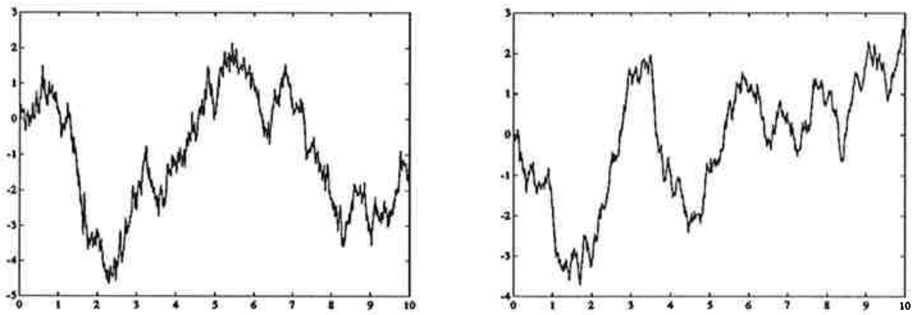

Figure 4. The signal $z(t)$ as function of time for the minimum variance controller—left, and the optimal controller—right.

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