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Hansson, Anders

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LUND UNIVERSITY

PO Box 117
221 00 Lund
+46 46-222 00 00

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Anders Hansson

Department of Automatic Control
Lund Institute of Technology
August 1991

Department of Automatic Control
Lund Institute of Technology
P.O. Box 118
S-221 00 Lund Sweden

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Anders Hansson

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Abstract

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Control of Level-Crossings in Stationary Gaussian Random Sequences

Anders Hansson
Department of Automatic Control
Lund Institute of Technology
Box 118, S-221 00 Lund, SWEDEN
Phone +46 46 108792, Fax +46 46 138118
E-mail andersh@control.lth.se

Abstract. A new optimal stochastic control problem that minimizes the probability that a signal upcrosses a level is solved by rewriting it as a one-parametric optimization problem over a set of LQG-problem solutions. The solution can sometimes be thought of as finding optimal weightings in an LQG-problem.

Keywords: Level-Crossings, LQG

1. Introduction

There are a lot of control problems where the goal is not only to keep the controlled signal near a certain reference value, but also to prevent it from upcrossing a level. The distance between the level and the reference value is normally not small, since otherwise the upcrossing probability will be intolerably high. However, there may be other control-objectives that make it undesirable or impossible to choose the distance large. Examples of problems of this kind can be found for example in sensor-based robotics and force control, [7].

The controller obtained below is obtained by solving a one-parametric optimization problem over a set of LQG-problem solutions, and it can sometimes be interpreted as choosing optimal weighting-matrices in an LQG-problem.

In [5] and [6] the problem was solved in the continuous time case; here the discrete time case is treated. The continuous time case has also been accepted for publication in IEEE Transactions on Automatic Control.

In Section 2 the control problem is formulated. It is an optimal stochastic control problem. In Section 3 the problem presented in Section 2 is solved. In Section 4 the optimal controller found in Section 3 is computed for a first order process and different values of the critical level. Finally in Section 5 the results are summarized.

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2. Control Problem

Let z be a stationary Gaussian process defined by

$$\begin{cases} x(k+1) = Ax(k) + B_1u(k) + B_2v(k) \\ y(k) = C_1x(k) + De(k) \\ z(k) = C_2x(k) \end{cases} \tag{1}$$

where v and e are Gaussian white noise sequences with $\mathrm{E}vv^T = R_1$, $\mathrm{E}ee^T = R_2$ and $\mathrm{E}ve^T = R_{12} = 0$. The signal y is the measurement, which the control signal u is constrained to be a linear time-invariant feedback of. It is assumed that the mean $m_z = \mathrm{E}\{z\}$ of z is equal to a predescribed reference value. It is possible to consider more general process models than (1), but to not get swamped with technicalities they will not be discussed here.

Let $\mathcal{D}$ be the set of linear time-invariant stabilizing feedbacks of (1), and let $\mathcal{D}_z$ be the set of linear time-invariant stabilizing feedbacks of (1) for which $\sigma_z \leq z_0 - m_z$ holds, where σ_z^2 is the variance of z.

The control-problems mentioned in Section 1 are captured in the following problem formulation:

$$\min_{H\in\mathcal{D}_z} \mu \tag{2}$$

where

$$\mu = \mathrm{P}\left\{z(k) \leq z_0 \cap z(k+1) > z_0\right\} \tag{3}$$

and where z_0 is the level that should not be upcrossed. The quantity μ is called the upcrossing probability. The restriction on σ_z will exclude the trivial solution $\sigma_z = \infty$ for minimizing μ.

The following lemma gives an expression for the upcrossing probability μ in (3) in terms of a double integral.

LEMMA 1
It holds

$$\mu = \mathrm{P}\{z(0) \leq z_0 \cap z(1) > z_0\} = \int_0^\infty \frac{1}{\sigma_\beta}\phi\left(\frac{y}{\sigma_\beta}\right)\int_{2u-y}^{2u+y}\frac{1}{\sigma_\alpha}\phi\left(\frac{x}{\sigma_\alpha}\right)dxdy$$

where σ_α^2 and σ_β^2 are the variances of

$$\begin{cases} \alpha(k) = z(k+1) + z(k) \\ \beta(k) = z(k+1) - z(k) \end{cases} \tag{4}$$

and where $\phi(x) = \frac{1}{\sqrt{2\pi}}\exp(-x^2/2)$ and $u = z_0 - m_z$.

Proof: Since α and β are independent, the result follows immediately. □

3. Regulator Design

In the first subsection the problem of minimizing the upcrossing probability is rephrased to a one-parametric minimization over a set of solutions to LQG-problems. The equations for solving the LQG-problems are given in the second subsection. In the last subsection the results are summarized.

Solution

It will be seen that the minimization of μ in (3) over $\mathcal{D}_z$ can be done by first minimizing

$$J = \mathrm{E}\{(1-\rho)\alpha^2 + \rho\beta^2\} \tag{5}$$

for $\rho \in [0,1]$ and $m_z = 0$ over $\mathcal{D}$, and then minimizing μ over the solutions obtained in the first minimization, i.e. over

$$\mathcal{V}_J = \left\{(\sigma_\alpha(H), \sigma_\beta(H)) \in \mathcal{V}_z \middle| H \in \mathcal{D}_J\right\}$$

where

$$\mathcal{V}_z = \left\{(\sigma_\alpha(H), \sigma_\beta(H)) \in R^2 \middle| H \in \mathcal{D}_z\right\}$$
$$\mathcal{D}_J = \left\{H \in \mathcal{D} \middle| H = \mathrm{argmin}\, J(H,\rho),\ \rho \in [0,1]\right\}$$

and where σ_α^2 and σ_β^2 are the variances of α and β. Note that it is only assumed that $m_z = 0$ when J is minimized, not when μ is minimized.

In the following lemma J is rewritten to fit into the usual LQG-problem formulation.

LEMMA 2
The loss function J in (5) can be written

$$J = \bar{J} + \mathrm{E}\{v^T B_2^T C_2^T C_2 B_2 v\}$$

where

$$\bar{J} = \mathrm{E}\{x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u\}, \tag{6}$$

and where

$$\begin{aligned} Q_1 &= C_2^T C_2 + A^T C_2^T C_2 A + 2(1-2\rho) C_2^T C_2 A \\ Q_{12} &= (A^T + (1-2\rho)I) C_2^T C_2 B_1 \\ Q_2 &= B_1^T C_2^T C_2 B_1 \end{aligned} \tag{7}$$

Proof: The result follows immediately by using the definitions of z in (1), and of α and β in (4) and by noting that v is uncorrelated with x. □

Remark. For $\rho = 0.5$ the controller that minimizes $\bar{J}$ is the minimum variance controller, since $J = \mathrm{E}\{z(k+1)^2 + z(k)^2\}$ for $\rho = 0.5$.

The next lemma shows how all jointly minimal variances of α and β can be obtained by minimizing $\bar{J}$ in (6) for $\rho \in [0,1]$. But first a precise definition of jointly minimal will be given.

DEFINITION 1—Pareto optimality
Let $\mathcal{X}$ denote an arbitrary nonempty set. Let $f_i : \mathcal{X} \to R^+$, $i \in \underline{s}$ be s nonnegative functionals defined on $\mathcal{X}$. Then a point x^0 is said to be Pareto optimal with respect to the vector-valued criterion $f = (f_1, f_2, \cdots, f_s)$ if there does not exist $x \in \mathcal{X}$ such that $f_i(x) \leq f_i(x^0)$ for all $i \in \underline{s}$, and $f_k(x) < f_k(x^0)$ for some $k \in \underline{s}$. □

LEMMA 3
Suppose that (A, B_1) is stabilizable, and that (C_1, A) is detectable. Let σ_α^2 and σ_β^2 be the variances of α and β defined on $\mathcal{D}$. Then the set $\mathcal{D}_P$ of Pareto optimal controllers with respect to $(\sigma_\alpha^2, \sigma_\beta^2)$ is a subset of $\mathcal{D}_J$.

Proof: The proof follows from chapters 6.5.2, 7.2 and 12.2 in [3] and Theorem 1 in [8], but a rough outline will be given below. By Lemma 2 $\bar{J}$ can be considered instead of J. Further by using the so called Q-parameterization of all stabilizing controllers it is seen that the variances are convex in Q, [3]. Since Q belongs to the linear space of stable transfer-function-matrices the result follows by Theorem 1 in [8]. □

Remark 1. All controllers obtained by minimizing J for $\rho \in (0,1)$ are Pareto optimal by Lemma 17.1 in [9]. If the solutions obtained for $\rho = 0$ and $\rho = 1$ are unique, then they are also Pareto optimal by Lemma 17.2 in [9].

Remark 2. Remark 1 and Definition 1 implies that $\mathcal{V}_J$ can be parameterized by a scalar. This is not necessarily the case for $\mathcal{D}_J$.

Remark 3. Remark 1 implies that if the controllers obtained by minimizing J for $\rho \in [0,1]$ are unique, then a parameterization of $\mathcal{D}_P = \mathcal{D}_J$ by ρ is obtained, [8], p. 16.

LEMMA 4
If $\sigma_\alpha \leq 2(z_0 - m_z)$, then the upcrossing probability μ in (3) has strictly positive partial derivatives with respect to σ_α and σ_β, i.e. $\frac{\partial\mu}{\partial\sigma_\alpha} > 0$ and $\frac{\partial\mu}{\partial\sigma_\beta} > 0$.

Proof: See Appendix □

It will now be shown how the minimization of μ in (3) can be rephrased to a minimization over a set of LQG-problem-solutions.

Theorem 1
Suppose that (A, B_1) is stabilizable, and that (C_1, A) is detectable. Then

$$\min_{H \in \mathcal{D}_z} \mu(\sigma_\alpha(K), \sigma_\beta(K)) = \min_{(\sigma_\alpha, \sigma_\beta) \in \mathcal{V}_J} \mu(\sigma_\alpha, \sigma_\beta)$$

if it exists.

Proof: It is obvious that

$$\min_{H \in \mathcal{D}_z} \mu(\sigma_\alpha(K), \sigma_\beta(K)) = \min_{(\sigma_\alpha, \sigma_\beta) \in \mathcal{V}_z} \mu(\sigma_\alpha, \sigma_\beta)$$

so that it only remains to be shown that the minimization on the right hand side can be restricted to $\mathcal{V}_J$. Let

$$\mathcal{V}_P = \{(\sigma_\alpha(H), \sigma_\beta(H) \in \mathcal{V}_z | H \in \mathcal{D}_P\}$$

and let $\bar{\mathcal{V}}_z$ be the closure of $\mathcal{V}_z$. Since $\bar{\mathcal{V}}_z$ is a compact set, since the inequality $\sigma_z \leq z_0 - m_z$ is equivalent to that $\sigma_\alpha^2 + \sigma_\beta^2 \leq 4(z_0 - m_z)^2$, and since μ is differentiable and by Lemma 4 has strictly positive partial derivatives with respect to σ_α and σ_β on $\bar{\mathcal{V}}_z$, the minimum of μ on $\bar{\mathcal{V}}_z$is attained on the boundary $\partial\bar{\mathcal{V}}_z$ of $\bar{\mathcal{V}}_z$. If the minimum exist on $\mathcal{V}_z$, then it is attained on $\partial\mathcal{V}_z \cap \mathcal{V}_z$. Further by Definition 1 the minimum is not attained on $(\partial\mathcal{V}_z \cap \mathcal{V}_z) \backslash \mathcal{V}_P$, since there are points to the left ore below these in the $(\sigma_\alpha, \sigma_\beta)$-plane in $\mathcal{V}_z$ for which μ attains a smaller value. Since by Lemma 3 $\mathcal{V}_P \subseteq \mathcal{V}_J$, the result follows. □

Remark 1. Note that the problem of minimizing over $\mathcal{V}_J$ is a one-parametric optimization problem by Remark 2 of Lemma 3.

Remark 2. If for each $\rho \in [0, 1]$ the minimum of J is unique, then by Remark 3 of Lemma 3 the minimization of μ can be thought of as finding optimal weightings in an LQG-problem.

LQG-equations

For short reference the equations for deriving the solution that minimizes $\bar{J}$ in (6) in Lemma 2 when the controller H is allowed to have a direct-term are given below. The transfer function from measurement to control is

$$H(q) = -L_x(qI - A + B_1 L_x + KC_1)^{-1} K_x - L_y \tag{8}$$

where L_x, L_y and K are given by

$$\begin{aligned}
L_x &= L - L_y C_1 \\
L_y &= L K_f \\
L &= (Q_2 + B_1^T S B_1)^{-1}(B_1^T S A + Q_{12}^T) \\
K_x &= K - B_1 L_y \\
K &= A K_f \\
K_f &= P C_1^T (D R_2 D^T + C_1 P C_1^T)^{-1}
\end{aligned}$$

where S and P are the solutions to the Riccati-equations, Chapter 11.4 in [2], and [4],

$$\begin{aligned}
A^T S A - S + Q_1 - (A^T S B_1 + Q_{12}) \\
(Q_2 + B_1^T S B_1)^{-1}(Q_{12}^T + B_1^T S A) = 0 \\
A P A^T - P + B_2 R_1 B_2^T \\
-A P C_1^T (D R_2 D^T + C_1 P C_1^T)^{-1} C_1 P A^T = 0
\end{aligned} \tag{9}$$

and where Q_1, Q_2 and Q_{12} are given by (7) in Lemma 2. To calculate σ_z, σ_u, σ_α and σ_β the following Lyapunov-equation for the closed loop system should be solved, [1] p. 49,

$$A_c X A_c^T + B_c R B_c^T = X \tag{10}$$

where

$$A_c = \begin{pmatrix} A - B_1 L & B_1 L_x \\ 0 & A - K C_1 \end{pmatrix}$$

$$B_c = \begin{pmatrix} B_2 & -B_1 L_y D \\ B_2 & -K D \end{pmatrix}$$

$$R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}$$

Then σ_α, σ_β, σ_z and σ_u are given by

$$\begin{aligned}
\sigma_\alpha^2 &= (\,C_2 \quad 0\,) \\
&\quad ((A_c + I) X (A_c + I)^T + B_c R B_c^T)(\,C_2 \quad 0\,)^T \\
\sigma_\beta^2 &= (\,C_2 \quad 0\,) \\
&\quad ((A_c - I) X (A_c - I)^T + B_c R B_c^T)(\,C_2 \quad 0\,)^T \\
\sigma_z^2 &= (\,C_2 \quad 0\,) X (\,C_2 \quad 0\,)^T \\
\sigma_u^2 &= (\,-L \quad L_x\,) X (\,-L \quad L_x\,)^T + L_y D R_2 D^T L_y^T
\end{aligned} \tag{11}$$

Due to the triangularity of A_c it is possible to split up (10) into three equations, where one of the solutions is P in (9), which reduces the complexity of the problem.

Summary

It has been shown how the minimization of the upcrossing probability can be rephrased to a minimization over a set of LQG-problem solutions parameterized by a scalar, regardless of the uniqueness of the solutions to the LQG-problems. However, if the solutions to the LQG-problems are unique, then the problem of minimizing the upcrossing probability can be thought of as finding optimal weightings in an LQG-problem. Note that the Lyapunov equation (10) is linear, and thus does not add any significant complexity compared to an ordinary LQG-problem.

4. Evaluation

To evaluate the performance of the optimal controller obtained by minimizing (3) a first order process will be investigated. The set of LQG-solutions is calculated

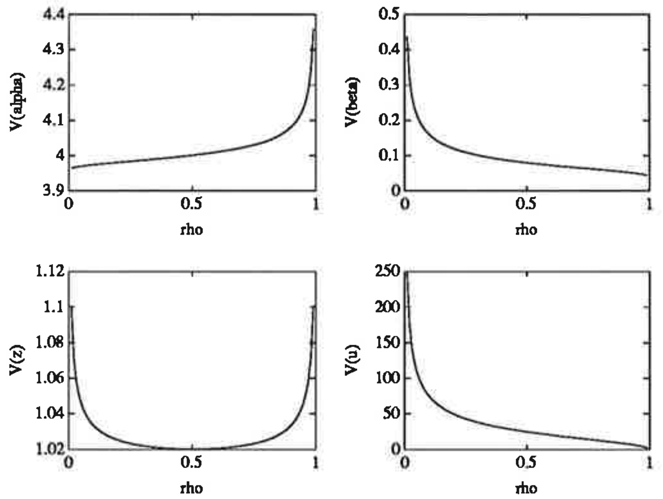

Figure 1. The variances of α—top left, β—top right, z—bottom left, and u—bottom right as functions of ρ.

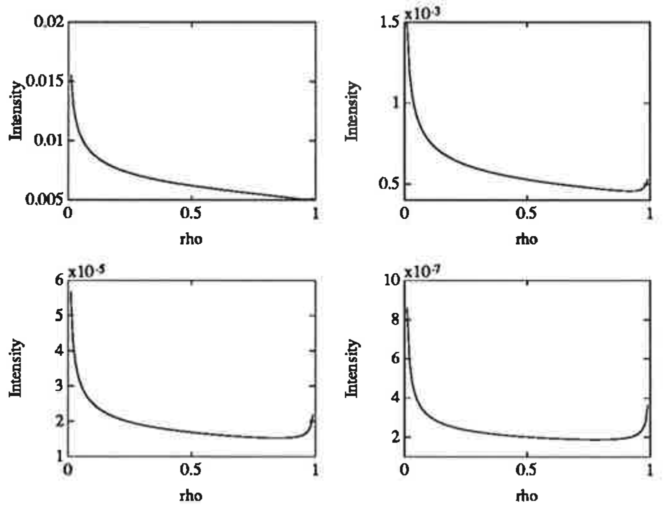

Figure 2. The probability μ as function of ρ for $z_0 = 2$—top left, $z_0 = 3$—top right, $z_0 = 4$—bottom left, and $z_0 = 5$—bottom left.

analytically, and then $\mu(\rho)$ is calculated numerically and plotted.

Let the process be given by

$$\begin{cases} x(k+1) = x(k) + 0.04u(k) + 0.2v(k) \\ y(k) = x(k) + 5e(k) \\ z(k) = x(k) \end{cases}$$

and $R_1 = 1$, $R_2 = 1$ and $R_{12} = 0$. The weighting-matrices in (7) are

$$Q_1 = 4(1-\rho)$$
$$Q_{12} = 0.08(1-\rho)$$
$$Q_2 = 0.16$$

and the solutions to the Riccati-equations in (9) are

$$S = 2\sqrt{\rho(1-\rho)}$$
$$P = \frac{0.04 + \sqrt{4.0016}}{2}$$

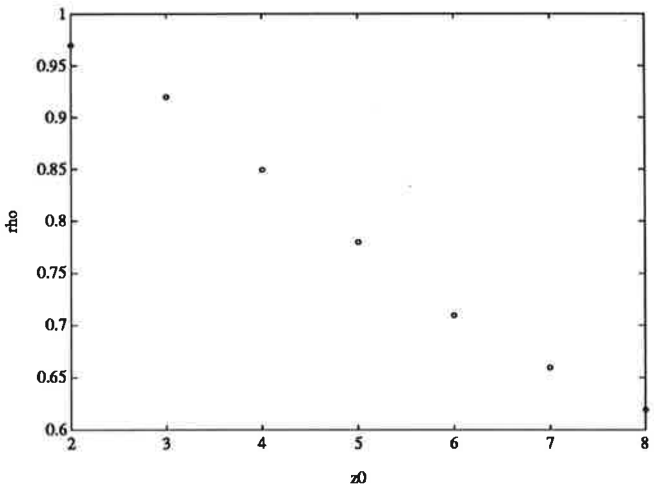

Figure 3. The optimal values of ρ as function of z_0

Some more tedious calculations will give the controller $H(q)$ in (8) to be

$$H(q) = -\frac{s_0 q}{r_0 q + r_1}$$

where

$$s_0 = 2\sqrt{\rho(1-\rho)} + 2(1-\rho))(0.04 + \sqrt{4.0016})$$
$$r_0 = 0.04(2\sqrt{\rho(1-\rho)} + 1)(50.04 + \sqrt{4.0016})$$
$$r_1 = 2(1-2\rho)$$

The probability μ and the variances of α, β, z and u have been calculated numerically for values of ρ in steps of 0.01 in the range of 0.01 to 0.99, $m_z = 0$, and $z_0 = 2$, 3, 4, and 5. The results are shown in figures 1 and 2.

In Figure 3 it is seen how the optimal value of ρ decreases as z_0 increases. This indicates by the remark of Lemma 2 that the optimal controller and the minimum variance controller are approximately the same for large values of z_0.

5. Conclusions

A new optimal stochastic control problem that minimizes the probability that a signal upcrosses a level has been solved.

The new controller is obtained as the solution to a one-parametric optimization problem over a set of LQG-problem solutions, and thus the complexity is not significantly larger than for an ordinary LQG-problem. Further it can sometimes be thought of as finding optimal weightings in an LQG-problem.

The new controller has been computeed for a first order process for different values of the critical level. It has been seen that the optimal controller approaches the minimum variance controller as the distance to the critical level from the reference value increases. However, for moderate values of the critical level, which is the interesting case for the examples in Section 1, the optimal controller outperforms the minimum variance controller, since the upcrossing probability and the variance of the control signal are significantly lower.

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7. Appendix

The proof of Lemma 4 is tedious but simple. Let $u = z_0 - m_z$. By Lemma 1 and by integrating by parts

$$\frac{\partial\mu}{\partial\sigma_\alpha} = \int_0^\infty \frac{1}{\sigma_\beta}\phi\left(\frac{y}{\sigma_\beta}\right)\int_{2u-y}^{2u+y}\left(-\frac{1}{\sigma_\alpha^2}\phi\left(\frac{x}{\sigma_\alpha}\right) - \frac{x}{\sigma_\alpha^3}\phi'\left(\frac{x}{\sigma_\alpha}\right)\right)dx dy$$
$$= \int_0^\infty \frac{1}{\sigma_\beta}\phi\left(\frac{y}{\sigma_\beta}\right)\left(-\frac{2u+y}{\sigma_\alpha^2}\phi\left(\frac{2u+y}{\sigma_\alpha}\right) + \frac{2u-y}{\sigma_\alpha^2}\phi\left(\frac{2u-y}{\sigma_\alpha}\right)\right)dy$$

Let

$$f(y) = -\frac{2u+y}{\sigma_\alpha^2}\phi\left(\frac{2u+y}{\sigma_\alpha}\right) + \frac{2u-y}{\sigma_\alpha^2}\phi\left(\frac{u-y}{\sigma_\alpha}\right)$$

Since $f(0) = 0$ and since, if $\sigma_\alpha \le 2u$ and $y > 0$,

$$\frac{df(y)}{dy} = \left(\frac{(2u+y)^2}{\sigma_\alpha^2} - 1\right)\phi\left(\frac{2u+y}{\sigma_\alpha}\right) + \left(\frac{(2u-y)^2}{\sigma_\alpha^2} - 1\right)\phi\left(\frac{2u-y}{\sigma_\alpha}\right)$$
$$> \left(\frac{8u^2+2y^2}{\sigma_\alpha^2} - 2\right)\phi\left(\frac{2u+y}{\sigma_\alpha}\right) > 0$$

it follows that $\frac{\partial\mu}{\partial\sigma_\alpha} > 0$. Further

$$\frac{\partial\mu}{\partial\sigma_\beta} = \int_0^\infty \left(-\frac{1}{\sigma_\beta^2}\phi\left(\frac{y}{\sigma_\beta}\right) - \frac{y}{\sigma_\beta^3}\phi'\left(\frac{y}{\sigma_\beta}\right)\right)\int_{2u-y}^{2u+y}\frac{1}{\sigma_\alpha}\phi\left(\frac{x}{\sigma_\alpha}\right)dx dy =$$
$$-\int_0^\infty \frac{1}{\sigma_\beta^2}\phi\left(\frac{y}{\sigma_\beta}\right)\int_{2u-y}^{2u+y}\frac{1}{\sigma_\alpha}\phi\left(\frac{x}{\sigma_\alpha}\right)dx dy$$
$$-\left[\frac{y}{\sigma_\beta^2}\phi\left(\frac{y}{\sigma_\beta}\right)\int_{2u-y}^{2u+y}\frac{1}{\sigma_\alpha}\phi\left(\frac{x}{\sigma_\alpha}\right)dx\right]_0^\infty$$
$$+\int_0^\infty \frac{1}{\sigma_\beta^2}\phi\left(\frac{y}{\sigma_\beta}\right)\left(\int_{2u-y}^{2u+y}\frac{1}{\sigma_\alpha}\phi\left(\frac{x}{\sigma_\alpha}\right)dx + y\frac{d}{dy}\int_{2u-y}^{2u+y}\frac{1}{\sigma_\alpha}\phi\left(\frac{x}{\sigma_\alpha}\right)dx\right)dy$$
$$= \int_0^\infty \frac{y}{\sigma_\beta^2}\phi\left(\frac{y}{\sigma_\beta}\right)\frac{1}{\sigma_\alpha}\left(\phi\left(\frac{2u+y}{\sigma_\alpha}\right) + \phi\left(\frac{2u-y}{\sigma_\alpha}\right)\right)dy > 0$$

□