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Minimum Upcrossing Control of *ARMAX*-Processes

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Abstract An optimal stochastic control problem that minimizes the probability that a signal upcrosses a level is solved by rewriting it as a one-parametric optimization problem over a set of minimum variance control problem solutions. A necessary and sufficient condition for the existence of the optimal controller is given, which is easy to investigate.

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1. Introduction

In many control problems the primary goal is to keep the controlled signal near a certain reference value. Sometimes it is also of interest to consider a secondary goal of preventing the controlled signal from upcrossing a level, where the upcrossing would cause some undesirable event such as e.g. emergency shutdown or instability. The distance between the level and the reference value is normally not small, since otherwise the upcrossing intensity will be intolerably high. However, there may be other control-objectives that make it undesirable or impossible to choose the distance large. An example of problems of this kind can be found in [Borisson and Syding, 1976], where the power of an ore crusher should be kept as high as possible but not exceed a certain level, in order that the overload protection does not cause shutdown. Another example is moisture control of a paper machine, where it is desired to keep the moisture content as high as possible without causing wet streaks, [Åström, 1970, pp. 188–209]. Yet another example is power control of wind power plants, where the supervisory system initiates emergency shutdown, if the generated power exceeds 140% of rated power, [Mattsson, 1984]. Other

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examples can be found in sensor-based robotics and force control, [Hansson and Nielsen, 1991], and control of non-linear plants, where the stability may be state dependent, [Shinskey, 1967].

The proposed controller—the minimum upcrossing (MU) controller—is obtained by minimizing the mean number of upcrossings of the critical level per unit time. The problem of level crossings in the context of stochastic processes was studied already in [Rice, 1936].

In [Hansson, 1991a], [Hansson, 1991b] and [Hansson, 1992a] the problem was solved in the continuous time case; here the discrete time case is treated, which previously has been described in [Hansson, 1992b] and [Hansson, 1991c]. There the minimum upcrossing controller is compared with the minimum variance controller. It is shown that the minimum upcrossing controller performs better than the minimum variance controller. In this paper an existence proof for the optimal controller is given, when the controlled process is an $ARMAX$-process. Further, it is shown that in this case the optimal controller can be found by solving a set of minimum variance control problems parameterized by a scalar. For the more general process models treated in [Hansson, 1992b] and [Hansson, 1991c] the solution can be found by solving a set of LQG-problems parameterized by a scalar. No existence proof is known for the general case.

Only the case of a linear process controlled with a linear controller will be treated, since then, if the disturbances acting on the process are Gaussian, the closed loop system will also be Gaussian. It is very likely that a nonlinear controller will do better. However, then the non-Gaussianity of the closed loop system will make the analysis much harder.

The paper is organized as follows. In Section 2 the control problem is formulated. It is an optimal stochastic control problem. In Section 3 the problem presented in Section 2 is solved. In Section 4 some examples are given. Finally in Section 5 the results are summarized.

2. Control Problem

Let y be a stationary Gaussian process defined by

$$A(q)y(k) = B(q)u(k) + C(q)e(k) \tag{1}$$

where e is a zero mean Gaussian white noise sequences with variance σ_e^2, and where $A(q)$, $B(q)$, and $C(q)$ are polynomials in the forward shift operator q. With no loss in generality, assume that $\deg A(q) = \deg C(q) = n$. Factorize $B(q)$ as $B(q) = B^+(q)B^-(q)$ with all zeros of $B^+(q)$ inside the unit circle, and all zeros of $B^-(q)$ outside or on the unit circle. Furthermore, assume that $A(q)$ and $B^-(q)$ are relatively prime, that $B(q)$ has no zeros on the unit circle, that $\deg A(q) > \deg B(q) = n - d$, and that $C(q)$ has all its zeros inside the unit circle. The signal y is the measurement signal, and u is the control signal. Denote the variance of y with σ_y^2 and the mean of y with m_y.

Consider a linear time-invariant feedback H, linear in both the measurement signal y and the constant reference value r. Let $\mathcal{D}$ be the set of linear time-invariant stabilizing feedbacks of (1), and let $\mathcal{D}_y$ be the set of linear time-invariant stabilizing feedbacks of (1) for which $\sigma_y \leq y_0 - m_y$ holds, where y_0 is the level that should not be upcrossed.

The control-problems mentioned in Section 1 are captured in the following problem formulation:

$$\min_{H \in \mathcal{D}_y} \mu \tag{2}$$

subject to $m_y = r$, where

$$\mu = \mathrm{P}\left\{y(k) \leq y_0 \cap y(k+1) > y_0\right\} \tag{3}$$

The quantity μ is called the upcrossing probability, and it is equal to the mean number of upcrossings in the interval $[0, 1)$, see e.g. [Cramér and Leadbetter, 1967, p. 281]. The restriction on σ_y will exclude the degenerated solution $\sigma_y = \infty$ for minimizing μ. With a change of variables it may be assumed that $m_y = r = 0$.

The following lemma gives an expression for the upcrossing probability μ in (3) in terms of a double integral.

Lemma 1

It holds that

$$\begin{aligned}\mu =& \mathrm{P}\left\{y(0) \leq y_0 \cap y(1) > y_0\right\} \\ =& \int_0^\infty \phi(y) \int_{x_l}^{x_u} \phi(x) dx dy\end{aligned}$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$, $x_l = (2y_0 - \sigma_\beta y)/\sigma_\alpha$, and $x_u = (2y_0 + \sigma_\beta y)/\sigma_\alpha$, and where σ_α^2 and σ_β^2 are the variances of the independent variables

$$\begin{cases} \alpha(k) = y(k) + y(k-1) \\ \beta(k) = y(k) - y(k-1) \end{cases} \tag{4}$$

Proof: Since α and β are independent it holds that

$$\begin{aligned}\mu =& \mathrm{P}\left\{|\alpha - 2y_0| < \beta\right\} \\ =& \int\int_{|x-2y_0|<y} \frac{1}{\sigma_\alpha}\phi\left(\frac{x}{\sigma_\alpha}\right) \frac{1}{\sigma_\beta}\phi\left(\frac{y}{\sigma_\beta}\right) dx dy\end{aligned}$$

from which the result follows by a change of variables. □

Thus μ is easily calculated with some numerical routine. Further μ only depends on the variances of α and β and the critical level y_0.

3. Regulator Design

In the first subsection the problem of minimizing the upcrossing probability is rephrased to a one-parametric minimization over a set of minimum variance problem solutions. In the second subsection the equations for solving the minimum variance problems are given, and the existence of the optimal controller is discussed. Finally, in the last subsection, the results are summarized.

Solution via Pareto Optimal Controllers

It will be seen that the minimization of μ in (3) over $\mathcal{D}_y$ can be done by first minimizing

$$J = \mathrm{E}\{(1-\rho)\alpha^2 + \rho\beta^2\} \tag{5}$$

for $\rho \in [0,1]$ over $\mathcal{D}$, and then minimizing μ over the solutions obtained in the first minimization, i.e. over $\mathcal{V}_J \cap \mathcal{V}_y$, where

$$\begin{aligned}
\mathcal{V}_J &= \Big\{(\sigma_\alpha(H), \sigma_\beta(H)) \in R^2 \Big| H \in \mathcal{D}_J\Big\} \\
\mathcal{V}_y &= \Big\{(\sigma_\alpha, \sigma_\beta) \in R^2 \Big| \sigma_y \le y_0,\ \sigma_\alpha \ge 0,\ \sigma_\beta \ge 0\Big\} \\
\mathcal{D}_J &= \Big\{H \in \mathcal{D} \Big| H = \operatorname{argmin} J(H,\rho),\ \rho \in [0,1]\Big\}
\end{aligned}$$

and where σ_α^2 and σ_β^2 are the variances of α and β.

The next lemma shows how all jointly minimal variances of α and β can be obtained by minimizing J in (5) for $\rho \in [0,1]$. But first a precise definition of jointly minimal will be given.

DEFINITION 1—Pareto Optimality

Let $\mathcal{X}$ denote an arbitrary nonempty set. Let $f_i : \mathcal{X} \to R^+,\quad 1 \le i \le s$ be s nonnegative functionals defined on $\mathcal{X}$. A point x^0 is said to be Pareto optimal with respect to the vector-valued criterion $f = (f_1, f_2, \cdots, f_s)$ if there does not exist $x \in \mathcal{X}$ such that $f_i(x) \le f_i(x^0)$ for all $i,\quad 1 \le i \le s$, and $f_k(x) < f_k(x^0)$ for some $k,\quad 1 \le k \le s$. □

LEMMA 2

The set $\mathcal{D}_P$ of Pareto optimal controllers with respect to $(\sigma_\alpha^2, \sigma_\beta^2)$ is a subset of $\mathcal{D}_J$.

Proof: Let $D(q)$ be the greatest common factor of $A(q)$ and $B(q)$. Notice that $D(q)$ has all its zeros inside the unit circle. Let $R(q)$ and $S(q)$ be solutions to

$$A(q)R(q) + B(q)S(q) = D(q)$$

which exist, since the greatest common factor of $A(q)$ and $B(q)$ is a factor of $D(q)$ by construction, [Åström and Wittenmark, 1990, Theorem 10.1 p. 292].

Then by [Francis, 1986, Theorem 1, p. 38], all stabilizing controllers $H(q)$ of (1) can be written

$$H(q) = \frac{A(q)Q(q) - S(q)}{B(q)Q(q) + R(q)}$$

with $Q(q)$ being a stable transfer-function. Thus the minimization of J over $\mathcal{D}$ can be rephrased to a minimization over Q, where Q belongs to the linear space of stable transfer-functions. Simple calculations give

$$y(k) = \frac{C(q)}{D(q)}[B(q)Q(q) + R(q)]e(k)$$

It is seen that the transfer-function from e to y is affine in Q, and since the variances of α and β are convex in the transfer-function, it follows that the variances are convex in Q. The result now follows by [Khargonekar and Rotea, 1991, Theorem 1]. □

Remark 1. All controllers obtained by minimizing J for $\rho \in (0, 1)$ are Pareto optimal by Lemma 17.1 in [Leitmann, 1981]. If the solutions obtained for $\rho = 0$ and $\rho = 1$ are unique, then they are also Pareto optimal by Lemma 17.2 in [Leitmann, 1981].

Remark 2. Remark 1 and Definition 1 implies that $\mathcal{V}_J$ can be parameterized by a scalar. This is not necessarily the case for $\mathcal{D}_J$.

Remark 3. Remark 1 implies that if the controllers obtained by minimizing J for $\rho \in [0, 1]$ are unique, then a parameterization of $\mathcal{D}_P = \mathcal{D}_J$ by ρ is obtained, [Khargonekar and Rotea, 1991, p. 16].

LEMMA 3

Let

$$\mathcal{V}(r) = \left\{ (\sigma_\alpha, \sigma_\beta) \in R^2 \middle| \sigma_y \le r,\ \sigma_\alpha > 0,\ \sigma_\beta > 0 \right\}$$

where $r > 0$. Then the upcrossing probability μ in (3) has strictly positive partial derivatives with respect to both σ_α and σ_β on $\mathcal{V}(r)$, if and only if $r \le y_0$.

Proof: See Appendix. □

It will now be shown how the minimization of μ in (3) can be rephrased to a minimization over a set parameterized by a scalar.

THEOREM 1

Let

$$\mathcal{V}_\mu = \left\{ (\sigma_\alpha(H), \sigma_\beta(H)) \in \mathcal{V}_y \middle| H \in \mathcal{D}_\mu \right\}$$
$$\mathcal{D}_\mu = \left\{ H \in \mathcal{D}_y \middle| H = \operatorname{argmin} \mu(\sigma_\alpha(H), \sigma_\beta(H)) \right\}$$

Then it holds that

$$\mathcal{D}_\mu \subseteq \mathcal{D}_P \cap \mathcal{D}_y$$
$$\mathcal{V}_\mu \subseteq \mathcal{V}_J \cap \mathcal{V}_y$$

Proof: Assume that the minimum of μ on $\mathcal{D}_y$ is attained for some $H \notin \mathcal{D}_P \cap \mathcal{D}_y$. For all $H \notin \mathcal{D}_P \cap \mathcal{D}_y$ there exist by Definition 1 $\bar{H} \in \mathcal{D}_y$ such that $\sigma_i(\bar{H}) < \sigma_i(H)$ for at lest one of $i = \alpha, \beta$. Since μ is differentiable and by Lemma 3 has strictly positive partial derivatives with respect to σ_α and σ_β on $\mathcal{V}(y_0)$, it follows that $\mu(\sigma_\alpha(\bar{H}), \sigma_\beta(\bar{H})) < \mu(\sigma_\alpha(H), \sigma_\beta(H))$. This is a contradiction, and thus the minimum of μ is attained on $\mathcal{D}_P \cap \mathcal{D}_y$, if it exists on $\mathcal{D}_y$. Further $\mathcal{D}_P \subseteq \mathcal{D}_J$ by Lemma 2, which concludes the proof. □

Remark 1. Note that the minimization of μ can be done over $\mathcal{V}_J \cap \mathcal{V}_y$. This is a one-parametric optimization problem by Remark 2 of Lemma 2.

Remark 2. If for each $\rho \in [0, 1]$ the minimizing H of J is unique, then by Remark 3 of Lemma 2 the minimization of μ can be thought of as finding an optimal value of ρ.

Existence of the MU Controller

The existence of the minimum upcrossing controller will now be established. In the following lemma J is rewritten to fit into the standard minimum variance problem formulation.

LEMMA 4

The loss function J in (5) can be written

$$J = \mathrm{E}\{\bar{y}^2\}$$

where

$$\bar{y}(k) = \sqrt{1-\rho}\,\alpha(k) + \sqrt{\rho}\,\beta(k)$$

The signal $\bar{y}(k)$ fulfills

$$\bar{A}(q)\bar{y}(k) = \bar{B}(q)u(k) + \bar{C}(q)e(k)$$

with

$$\begin{aligned}
\bar{A}(q) &= qA(q)\\
\bar{B}(q) &= X(q)B(q)\\
\bar{C}(q) &= X(q)C(q)\\
X(q) &= (\sqrt{1-\rho} + \sqrt{\rho})q + \sqrt{1-\rho} - \sqrt{\rho})
\end{aligned}$$

where $X(q)$ has its zero strictly inside the unit circle for $\rho \in (0, 1)$.

Proof: The result follows immediately by using the definitions of y in (1), and of α and β in (4) and by noting that α is uncorrelated with β. □

Remark. For $\rho = 0.5$ the controller that minimizes J is the minimum variance controller, since $J = \mathrm{E}\{y(k)^2 + y(k-1)^2\}$ for $\rho = 0.5$.

The equations for solving the minimum variance problems are given in the theorem below. It is seen that the solutions are unique.

THEOREM 2

If $\rho \in (0,1)$, then the unique controller that minimizes J in (5) is given by

$$H(q) = -\frac{G(q)}{qB^+(q)F(q)}$$

where $F(q)$ and $G(q)$ are obtained as the unique solution to

$$q^{d-1}C(q)X(q)B^{-*}(q) = qA(q)F(q) + B^-(q)G(q) \tag{6}$$

with $\deg F(q) = d + \deg B^-(q) - 1$ and $\deg G(q) < \deg A(q) + 1$, and where $B^{-*}(q)$ is the reciprocal polynomial of $B^-(q)$. Further the closed loop system is governed by

$$y(k) = \frac{F(q)}{q^{d-2}X(q)B^{-*}(q)}e(k)$$

Proof: By Lemma 4 the problem of minimizing J in (5) is a minimum variance problem. Since $\bar{A}(q)$ and $\bar{B}^-(q)$ do not have any common factors, and since $\bar{C}(q)$ has all its zeros inside the unit disk for $\rho \in (0,1)$, the solution is given by

$$u(k) = -\frac{G(q)}{\bar{B}^+(q)F(q)}\bar{y}(k)$$

where $G(q)$ and $F(q)$ are the solutions to

$$q^{d-1}\bar{C}(q)\bar{B}^{-*}(q) = \bar{A}(q)F(q) + \bar{B}^-(q)G(q)$$

with $\deg F(q) = d + \deg\bar{B}^-(q) - 1$ and $\deg G(q) < \deg\bar{A}(q)$, [Åström and Wittenmark, 1990, Theorem 12.3, pp. 380–383]. The uniqueness follows from [Åström and Wittenmark, 1990, Theorem 10.1, p.292] and the closed loop behavior follows from Lemma 4. □

Remark. Notice that (6) has a unique solution for all $\rho \in [0,1]$, but that this solution is not necessarily a solution that minimizes J for $\rho = 0, 1$.

The following lemmas are needed in the investigation of the existence of the optimal controller.

LEMMA 5

The variances σ_α^2 and σ_β^2 of α and β are continuous functions of ρ on $(0,1)$.

Proof: From Lemma 1 and Theorem 2 it follows that

$$\begin{aligned} q^{d-1}X(q)B^{-*}(q)\alpha(k) &= (q+1)F(q)e(k) \\ q^{d-1}X(q)B^{-*}(q)\beta(k) &= (q-1)F(q)e(k) \end{aligned} \tag{7}$$

Since $F(q)$ is obtained as the unique solution of a linear system of equations, where the coefficients are continuous functions of ρ, it follows by Lemma 9 in the Appendix that $F(q)$ is a continuous function of ρ. Further, the variances of α and β can be obtained by solving the Yule-Walker-equations, which is equivalent to solving two linear systems of equations. The coefficients in these equations are continuous functions of ρ, since the polynomials in (7) are continuous functions of ρ. Further the solutions are unique. The result now follows by once again applying Lemma 9. $\square$

Since $X(q)$ has its zero on the unit circle for $\rho = 0, 1$, and since by (7) this zero is a pole for the closed loop system from e to α and β, this may in some cases cause infinite variances of α and β. The following lemma will clarify this.

LEMMA 6

The following statements are equivalent:

1) There exist $\rho \in [0, 1]$ such that $X(q)$ is a common factor of $F(q)$ and $G(q)$
2) For all $\rho \in [0, 1]$ it holds that $X(q)$ is a common factor of $F(q)$ and $G(q)$
3)
$$\limsup_{\rho \to 1} \sigma_\alpha^2 < \infty$$
$$\limsup_{\rho \to 0} \sigma_\beta^2 < \infty$$

Proof: That 1) and 2) are equivalent follows from the remark of Theorem 2. That 2) implies 3) follows from noting that if $X(q)$ is a factor in $F(q)$, then the transfer functions from e to α and β in (7) have no poles on the unit circle. That 3) implies 1) follows by noting that since $X(q)$ has a zero at -1 for $\rho = 0$, $X(q)$ must by (7) be a factor in $F(q)$ for $\rho = 0^+$. Further (6) implies that $X(q)$ is also a factor in $G(q)$ for $\rho = 0^+$. $\square$

Remark. Notice that σ_α and σ_β are independent of $\rho \in (0, 1)$ and finite, if $X(q)$ is a common factor of $F(q)$ and $G(q)$.

DEFINITION 2—Convexity

A curve $C \subset R^2$ is convex, if for all (x_1, y_1), (x_2, y_2) and $(x, y) \in C$ such that $x_1 \leq x \leq x_2$ and $y_2 \leq y \leq y_1$ it holds that $x \leq a$ and $y \leq b$, where a and b are solutions to
$$(x_2 - x_1)(y - y_1) = (y_2 - y_1)(a - x_1)$$
$$(x_2 - x_1)(b - y_1) = (y_2 - y_1)(x - x_1)$$
$\square$

LEMMA 7

Let
$$\mathcal{V}_J' = \left\{ (\sigma_\alpha(H), \sigma_\beta(H)) \in R^2 \middle| H \in \mathcal{D}_J' \right\}$$
$$\mathcal{D}_J' = \left\{ H \in \mathcal{D} \middle| H = \operatorname{argmin} J(H, \rho),\ \rho \in (0, 1) \right\}$$

Then the curve $\mathcal{V}_J'$ is connected and convex.

Proof: The connectedness follows from the continuity of Lemma 5. Assume that $\mathcal{V}_J'$ is not convex. Then there exist by Definition 2 (x_1, y_1), (x_2, y_2) and

$(\sigma_\alpha, \sigma_\beta) \in \mathcal{V}_J'$ such that $x_1 \leq \sigma_\alpha \leq x_2$ and $y_2 \leq \sigma_\beta \leq y_1$ for which it holds that either $\sigma_\alpha > a$ or $\sigma_\beta > b$, where a and b are the solutions to

$$(x_2 - x_1)(\sigma_\beta - y_1) = (y_2 - y_1)(a - x_1)$$
$$(x_2 - x_1)(b - y_1) = (y_2 - y_1)(\sigma_\alpha - x_1)$$

Further, for all tangents to $\mathcal{V}_J'$ in $(\sigma_\alpha, \sigma_\beta)$ the points (x_1, y_1), (x_2, y_2) can be chosen such that the tangents are parallel with the line defined by

$$\{(x, y) \in R^2 | (x_2 - x_1)(y - y_1) = (y_2 - y_1)(x - x_1)\}$$

This implies that $(\sigma_\alpha, \sigma_\beta)$ cannot be obtained as a solution to minimizing J in (5) for any $\rho \in (0, 1)$. Thus $\mathcal{V}_J'$ is not connected, which is a contradiction. □

THEOREM 3
It holds that

$$\mathcal{D}_\mu \subseteq \mathcal{D}_J \cap \mathcal{D}_y$$
$$\mathcal{V}_\mu \subseteq \mathcal{V}_J' \cap \mathcal{V}_y$$

where the left hand sides are non-empty if and only if $\mathcal{D}_y$ is non-empty.

Proof: Assume that $X(q)$ is not a common factor of $F(q)$ and $G(q)$. Then the result follows from lemmas 6 and 7 and the boundness of $\mathcal{V}_y$ by examining the proof of Theorem 1.

Assume that $X(q)$ is a common factor of $F(q)$ and $G(q)$. Then by the remark of Lemma 6 σ_α and σ_β are independent of $\rho \in (0, 1)$. It now only remains to investigate $\rho = 0$ and $\rho = 1$. If there does not exist any Pareto optimal solutions that minimizes J in (5) for these values of ρ, the result follows by Theorem 1.

Thus assume that there exist a Pareto optimal solution that minimizes J for $\rho = 0$. Denote by $(\sigma_\alpha^2(0), \sigma_\beta^2(0))$ the Pareto optimal variances of α and β obtained by minizing J for $\rho = 0$. Further let $(x^2, y^2) \in \mathcal{V}_J'$, which is unique.

Suppose that $\sigma_\alpha(0) < x$. Then, if $\sigma_\beta(0) \leq y$, it follows that (x^2, y^2) is not Pareto optimal, which is a contradiction, and if $\sigma_\beta(0) > y$, it follows that $(\sigma_\alpha^2(0), \sigma_\beta^2(0))$ not only corresponds to $\rho = 0$, but also to some $\rho \in (0, 1)$, which is in contradiction to the uniqueness of the solutions to (6). Similar arguing can be used to show that it does not hold that $\sigma_\alpha(0) > x$. Thus $\sigma_\alpha(0) = x$. Further since $(\sigma_\alpha^2(0), \sigma_\beta^2(0))$ is Pareto optimal, it follows that $\sigma_\beta(0) = y$.

Assuming that there exist a Pareto optimal solution that minimizes J for $\rho = 1$, similar arguing as above can be used to show that $\sigma_\alpha(1) = x$ and $\sigma_\beta(1) = y$, where $(\sigma_\alpha^2(1), \sigma_\beta^2(1))$ are the Pareto optimal variances of α and β obtained by minizing J for $\rho = 1$.

Thus σ_α and σ_β are independent of $\rho \in [0, 1]$, and the result follows by Theorem 1. This concludes the proof. □

Remark 1. By theorems 1 and 2 it follows that the minimization of μ can be thought of as finding an optimal value $\rho \in (0,1)$ for the minimum variance problem in Theorem 2. Notice that the solutions to these problems are unique for each value of ρ, but that this does not imply that a solution that minimizes μ is unique.

Remark 2. Notice that the non-emptiness of $\mathcal{D}_y$ can easily be checked by computing the minimum variance controller.

Summary

It has been seen how the minimization of the upcrossing probability can be rephrased to a one-parametric minimization over a set of minimum variance problem solutions. Thus the complexity is only one order of magnitude higher than for an ordinary minimum variance problem. Further a necessary and sufficient condition for the existence of the minimum upcrossing controller has been given.

4. Examples

In this section two examples will be given to illuminate the behavior of the minimum upcrossing controller.

EXAMPLE 1—First order process

Let the process-polynomials be given by

$$\begin{aligned} A(q) &= q + a \\ B(q) &= b \\ C(q) &= q + c \end{aligned}$$

with $|c| < 1$, and let the variance of e be 1. Some calculations will give the controller to be

$$H(q) = -\frac{[\rho^- + (c-a)\rho^+]q + c\rho^-}{b\rho^+ q}$$

where

$$\begin{aligned} \rho^+ &= \sqrt{1-\rho} + \sqrt{\rho} \\ \rho^- &= \sqrt{1-\rho} - \sqrt{\rho} \end{aligned}$$

Further calculations give

$$\begin{aligned} \mathrm{E}y^2 &= \frac{1}{2} + \frac{1}{4\sqrt{\rho}\sqrt{1-\rho}} \\ \mathrm{E}\alpha^2 &= \left(1 + 2\sqrt{\frac{\rho}{1-\rho}}\right) \\ \mathrm{E}\beta^2 &= \left(1 + 2\sqrt{\frac{1-\rho}{\rho}}\right) \end{aligned}$$

It is seen that the variance of y is minimized by $\rho = 0.5$ for which the controller is a proportional controller. Note that the variances of α and β approaches infinity as ρ goes to 0 and 1 respectively. Further note that the variances of y, α and β do not depend on the values of a, b, and c. □

The next example is constructed such that $X(q)$ will be a common factor of $F(q)$ and $G(q)$.

EXAMPLE 2—Second order process
Let the process-polynomials be given by

$$\begin{aligned} A(q) &= q(q+a) \\ B(q) &= q+b \\ C(q) &= q(q+c) \end{aligned}$$

with $|b| > 1$ and $|c| < 1$. Suppose that $F(q) = F'(q)X(q)$ and $G(q) = G'(q)X(q)$. Then it is seen that there exist a solution to (6) with $F'(q) = f$ and $G'(q) = gq$. Some calculations will give the following set of equations

$$\begin{cases} f = b \\ af + g = 1 + bc \\ bg = c \end{cases}$$

These equations do not have a solution for all values of a, b, and c. However, for $a = -1$, $b = 2$, and $c = 1/2$ they do. For this case the controller is

$$H(q) = -\frac{1}{8}$$

which is independent of ρ. □

Thus there are cases for which the minimum upcrossing controller and the minimum variance controller are the same.

5. Conclusions

A new optimal stochastic control problem has been investigated. The solution minimizes the probability for the controlled signal to upcross a level given a certain reference value. There are many examples of control problems for which this approach is appealing, i.e. problems for which there exist a level such that a failure in the controlled system occurs when the controlled signal upcrosses the level. One important class of such problems is processes equipped with supervision, where upcrossings of alarm levels may initiate emergency shutdown causing loss in production.

The problem of minimizing the upcrossing probability over the set of stabilizing linear time-invariant controllers has been rephrased to a minimization over minimum variance problem solutions parameterized by a scalar, and thus the complexity is only one order of magnitude larger than for an ordinary minimum variance problem. The key to the new method is the reformulation using the independent variables α and β making it possible to quantify by Lemma 1 the upcrossing probability in terms of the variances of α and β. The set of closed loop variances of α and β obtained by solving the set of minimum variance problems has been characterized. It has been seen that this set is connected and convex. This has made it possible to give a necessary and sufficient condition for the existence of the minimum upcrossing controller, which is easy to check.

Some examples have been investigated, and it has been seen that there exist degenerated cases for which the minimum variance controller and the minimum upcrossing controller are the same.

The approach taken to minimize μ can easily be generalized to problems of the type

$$\min_u \left\{(1-\rho)\mu + \rho \mathrm{E}u^2\right\}, \quad \rho \in [0,1]$$

by considering

$$\min_u \mathrm{E}\left\{\lambda_1\alpha^2 + \lambda_2\beta^2 + \lambda_3 u^2\right\}$$

subject to $\sum_{i=1}^{3} \lambda_i = 1$, and for each obtained variance x of u minimizing μ over the variances of α and β parameterized by the set

$$\left\{\lambda \in R^3 \,\middle|\, \sum_{i=1}^{3} \lambda_i = 1,\ \mathrm{E}u^2 = x\right\}$$

Thus the problem above can be reformulated to a two-dimensional optimization problem over LQG-problem-solutions. This will give all Pareto optimal controllers with respect to μ and $\mathrm{E}u^2$.

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7. Appendix

In this appendix the proof of Lemma 3 will be given together with some lemmas about continuity.

The proof of Lemma 3 goes as follows. It holds that

$$\frac{\partial \mu}{\partial \sigma_\beta} = \int_0^\infty \phi(y) \left(\frac{y}{\sigma_\alpha} \phi(x_u) + \frac{y}{\sigma_\alpha} \phi(x_l) \right) dy > 0$$

Further let $x_l = (2z_0 - \sigma_\beta y)/\sigma_\alpha$, and $x_u = (2z_0 + \sigma_\beta y)/\sigma_\alpha$. Using Lemma 1 gives

$$\frac{\partial \mu}{\partial \sigma_\alpha} = \int_0^\infty \phi(y) \left(\frac{x_l}{\sigma_\alpha} \phi(x_l) - \frac{x_u}{\sigma_\alpha} \phi(x_u) \right) dy$$

By completing the squares in the exponents and by a change of coordinates it is possible to express the integral in terms of $\Phi(x) = \int_{-\infty}^x \phi(t)dt$, and $\sigma_z^2 = (\sigma_\alpha^2 + \sigma_\beta^2)/4$

$$\begin{aligned} \frac{\partial \mu}{\partial \sigma_\alpha} = & \frac{\sigma_\alpha}{8\pi\sigma_z^2} \exp\left(-\frac{\gamma^2}{2}\right) \\ & \cdot \left[\sqrt{2\pi}\gamma \left(2\Phi(\eta) - 1\right) - 2\frac{\eta}{\gamma} \exp\left(-\frac{\eta^2}{2}\right) \right] \end{aligned}$$

where $\eta = \gamma\sqrt{\xi}$, $\xi = (\sigma_\beta/\sigma_\alpha)^2$, and $\gamma = z_0/\sigma_z > 0$. It is seen that $\frac{\partial \mu}{\partial \sigma_\alpha} > 0$ if and only if

$$2\Phi(\eta) - 1 > \sqrt{\frac{2}{\pi}} \frac{\eta}{\gamma^2} \exp\left(-\frac{\eta^2}{2}\right)$$

So if $\frac{\partial \mu}{\partial \sigma_\alpha} > 0$ on $\mathcal{V}(r)$, then the inequality above holds for all values of $\eta > 0$, since $\gamma > 0$, and since it must hold for all values of $\xi > 0$. A Taylor-expansion round $\eta = 0$ gives

$$\sqrt{\frac{2}{\pi}}\eta > \sqrt{\frac{2}{\pi}} \frac{\eta}{\gamma^2} + \mathcal{O}(\eta^2)$$

So for the inequality to hold for small values of η, it must be that $\gamma \geq 1$, which is equivalent to $r \leq z_0$. Now suppose that $r \leq z_0$, which implies $\gamma \geq 1$. Then

$$\begin{aligned} & (2\Phi(\eta) - 1)^2 \\ & \geq 1 - \exp\left(-\frac{2\eta^2}{\pi}\right) - \frac{2(\pi - 3)}{3\pi^2}\eta^4 \exp\left(-\frac{\eta^2}{2}\right) \\ & \geq 1 - \exp\left(-\frac{2\xi}{\pi}\right) - \frac{2(\pi - 3)}{3\pi^2}\xi^2 \exp\left(-\frac{\xi}{2}\right) \end{aligned}$$

where the first inequality follows from [Abromowitz and Stegun, 1968, Formula 26.2.25] and the second one from $\gamma \geq 1$. Further

$$\left(\sqrt{\frac{2}{\pi}} \frac{\eta}{\gamma^2} \exp\left(-\frac{\eta^2}{2}\right) \right)^2 \leq \frac{2}{\pi}\xi \exp(-\xi)$$

To show $\frac{\partial \mu}{\partial \sigma_a} > 0$, it is now sufficient to show $L > R$ for $\xi > 0$, where

$$
\begin{aligned}
L =& \exp\left(\frac{\xi}{2}\right) \\
R =& \frac{2}{\pi}\xi \exp\left(-\frac{\xi}{2}\right) + \exp\left(\left(\frac{1}{2} - \frac{2}{\pi}\right)\xi\right) \\
&+ \frac{2(\pi - 3)}{3\pi^2}\xi^2
\end{aligned}
$$

Some calculations give

$$
\begin{aligned}
L &\geq 1 + \frac{1}{2}\xi + \frac{1}{8}\xi^2 \\
R &\leq 1 + \frac{1}{2}\xi + \left(\frac{1}{8} - \frac{1}{3\pi}\right)\xi^2
\end{aligned}
$$

From this it follows that $L > R$ for $\xi > 0$, so $\frac{\partial \mu}{\partial \sigma_a} > 0$. □

Lemma 8

Let $a_{ij}(\xi)$, $1 \leq i \leq m$, $1 \leq j \leq n$ be real-valued functions of $\xi \in \mathcal{X} \subset R$, and let $A(\xi) = (a_{ij}(\xi))$. Define the 1-norm of $A(\xi)$ as

$$
||A(\xi)|| = \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}(\xi)|
$$

Then $A(\xi)$, i.e. $a_{ij}(\xi)$, $1 \leq i \leq m$, $1 \leq j \leq n$, are continuous functions on $\mathcal{X}$ if and only if $\forall \varepsilon\ \exists \delta$ such that for $|h| < \delta$ it holds that

$$
||A(\xi + h) - A(\xi)|| \leq \varepsilon
$$

Proof: The proof is trivial. □

Lemma 9

Let the $n \times n$-matrix $A(\xi)$ and the $n \times 1$-matrix $b(\xi)$ be continuous functions of $\xi \in \mathcal{X} \subset R$. If

$$
A(\xi)x(\xi) = b(\xi)
$$

has a unique solution for every $\xi \in \mathcal{X}$, then $x(\xi)$ is a continuous function on $\mathcal{X}$.

Proof: It is obviously sufficient to show that $A^{-1}(\xi)$ is continuous. This follows from the following chain of inequalities and Lemma 8:

$$
\begin{aligned}
&||A^{-1}(\xi + h) - A^{-1}(\xi)|| \\
&\quad \leq ||A^{-1}(\xi + h)|| \cdot ||(I - A(\xi + h))A^{-1}(\xi)|| \\
&\quad \leq ||A^{-1}(\xi + h)|| \cdot ||A^{-1}(\xi)|| \\
&\quad\ \cdot ||A(\xi + h) - A(\xi)||
\end{aligned}
$$

□