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Abstract

A simple adaptive system consisting of an adjustable feedforward gain is analyzed with respect to bifurcations limit cycles and stability.

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1. Introduction

Adaptive systems can often exhibit interesting nonlinear dynamical behaviour. There are many accounts of this in the litterature. For example, in [Cyr et al., 1983] a Hopf bifurcation was investigated in an adaptive system with unmodeled dynamics. Chaos in adaptive systems has been studied since the mid 1980's. [Mareels and Bitmead, 1986] is one of the first reports on chaos in a discrete time adaptive system, while the continuous time case is considered in for example [Rubio et al., 1985] and [Salam and Bai, 1986].

In this report we will consider a simple continuous time adaptive system subject to a nonlinear perturbation. We will show that our system undergoes a supercritical Hopf bifurcation and both theoretical and simulation evidence of this will be given. Furthermore, the stability region of the fixed point will be determined in the case when the fixed point is stable. We will also give an example that the system has complex dynamical behaviour when sinusoidal and load disturbances are injected at the input.

We want to point out that this report is mainly concerned with the study of nonlinear phenomena in an adaptive controller rather than sound adaptive control engineering. The adaptive controller that has been studied is interesting from a nonlinear dynamical systems perspective but it is by no means the best possible solution to the control problem under consideration.

2. The Adaptive System

The system under consideration in this report is an adaptation of a feedforward gain as in [Åström and Wittenmark, 1989]. The process is thought to be described by

$$\dot{y} = -\lambda y + u$$

where

$$u = \theta u_c$$

and the desired model is

$$\dot{y}_m = -\lambda y_m + b u_c$$

The purpose of the adaptive controller is to tune the parameter θ so that the output from the process and from the model becomes equal. The rule used for changing θ is the so called MIT-rule [Åström and Wittenmark, 1989]

$$\dot{\theta} = -\gamma e \frac{\partial e}{\partial \theta}$$

In our case we have

$$e = y - y_m = \frac{1}{p + \lambda}(\theta - b) u_c$$

where p is the differential operator. From this we get

$$\frac{\partial e}{\partial \theta} = \frac{1}{p + \lambda} u_c = y/\theta$$

Assuming that θ is constant and hence can be included in γ, our adaptation law will be

$$\dot{\theta} = -\gamma (y - y_m) y$$

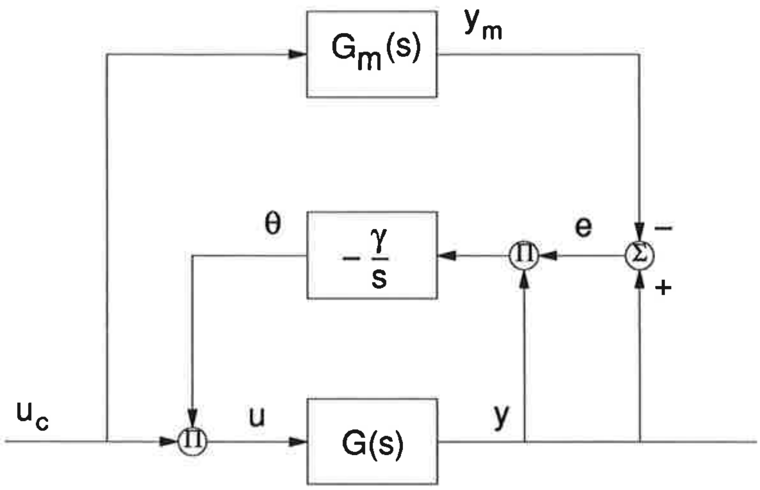

Figure 1. The adaptive system under consideration.

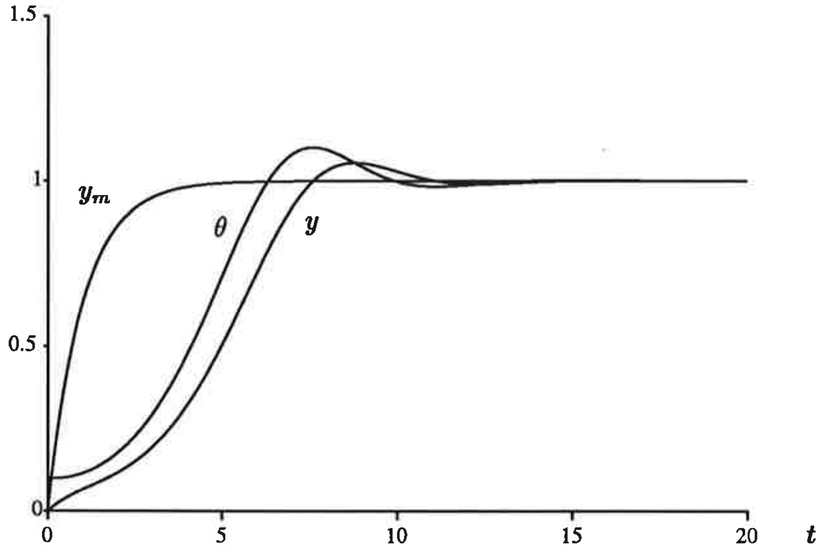

Figure 2. Step response for the adaptive system, $\lambda = 1$, $b = 1$ and $\gamma = 1$.

which, due to the time-variability of θ, does not agree truely with the MIT-rule.

Remark: Note that the adaption law in [Åström and Wittenmark, 1989] is $\dot{\theta} = -\gamma(y - y_m)y_m$, which from an engineering point of view is better than our adaptation law since it gives a system that is more robust to perturbations of the dynamics. However, our goal was to investigate nonlinear phenomena that appear when the system is perturbed and it turned out that our update law gave a system that was more interesting in this respect.

A block diagram of this system is drawn in Figure 1. The response of the system to a step input can be seen in Figure 2. The initial values are $\theta(0) = 0.1$, $y(0) = 0$ and $y_m(0) = 0$.

To reduce the number of parameters in the differential equations describing the system we introduce a new dependent variable and rescale the input and adaptation gain as follows

$$t' = \lambda t$$
$$\theta' = \frac{\theta}{b}$$
$$u_c' = \frac{bu_c}{\lambda}$$
$$\gamma' = \frac{\gamma}{b\lambda}$$

The system can then be described by

$$\begin{aligned}
\dot{y} &= -y + \theta' u_c' \\
\dot{y}_m &= -y_m + u_c' \\
\dot{\theta}' &= -\gamma'(y - y_m)y
\end{aligned}$$

where of course the derivatives now are with respect to t'. In the following we will however drop the superscript $'$ for ease of notation.

The purpose of this report is now to study how the adaptive system behaves when the differential equation for the process includes a nonlinear term as follows

$$\dot{y} = -y + ay^3 + u$$

The analysis will be on how the adaptive system behaves for different values of the parameter a while the adaptation gain γ equals 1.

3. Fixed Points of the Adaptive System

In this section the fixed points of the adaptive system and their stability type will be determined. The adaptive system derived in the last section was after a rescaling of time and input signal equal to

$$\begin{aligned}
\dot{y} &= -y - ay^3 + \theta u_c \\
\dot{y}_m &= -y_m + u_c \\
\dot{\theta} &= -\gamma(y - y_m)y
\end{aligned} \tag{1}$$

The case when $\gamma = 1$ and when u_c is a unit step will be considered. The fixed points of the system (1) are given as the solutions to the following system of nonlinear equations

$$\begin{aligned}
-y - ay^3 + \theta &= 0 \\
-y_m + 1 &= 0 \\
-(y - y_m)y &= 0.
\end{aligned}$$

The solutions are

$$\begin{cases} y_{0_1} = 0 \\ y_{m0_1} = 1 \\ \theta_{0_1} = 0 \end{cases} \tag{2}$$

and

$$\begin{cases} y_{0_2} = 1 \\ y_{m0_2} = 1 \\ \theta_{0_2} = 1 + a. \end{cases} \tag{3}$$

We will now investigate how the stability type of the fixed points depend on the parameter a. It is the eigenvalues of the Jacobian matrix of the system (1) evaluated at the fixed point that determine the stability type of the fixed point. The Jacobian matrix of (1) is

$$J = \begin{pmatrix} -1 - 3ay_0^2 & 0 & 1 \\ 0 & -1 & 0 \\ -2y_0 + y_{m0} & y_0 & 0 \end{pmatrix}$$

Which for the fixed point in (2) has the eigenvalues

$$\lambda_{1_1} = -1 \quad \lambda_{1_2} = \frac{1}{2}(-1 - \sqrt{5}) \quad \lambda_{1_3} = \frac{1}{2}(-1 + \sqrt{5}) \tag{4}$$

where λ_{1_1} and λ_{1_2} are negative and λ_{1_3} is positive, hence this fixed point is a saddle point.
For the fixed point in (3) the Jacobian has the eigenvalues

$$\begin{aligned} \lambda_{2_1} &= -1 \\ \lambda_{2_2} &= -\frac{1}{2}(3a+1) + \frac{\sqrt{3}}{2}\sqrt{3a^2 + 2a - 1} \\ \lambda_{2_3} &= -\frac{1}{2}(3a+1) - \frac{\sqrt{3}}{2}\sqrt{3a^2 + 2a - 1} \end{aligned} \tag{5}$$

The first eigenvalue, which corresponds to the desired model y_m, is always negative. The other two eigenvalues depend on the parameter a. Figure 3 shows the locus of λ_{2_2} and λ_{2_3} as a function of a.
If the system is started with the initial value of the model, $y_m(0) = 1$ then y_m will stay constant and equal to one. It can now be seen from Figure 3 that the phase portrait of y and θ will show a stable focus for $\frac{1}{3} < a < -\frac{1}{3}$, and an unstable focus when $-\frac{1}{3} < a < -1$.
Figure 4 to Figure 7 show phase portraits of the system obtained by making simulations in Simnon with several different initial values of y and θ and with $y_m(0) = 1$. The phase portraits are consistent with our conclusions from the locus of the eigenvalues in Figure 3. We also notice that for the case when a is slightly less than $-\frac{1}{3}$ the trajectories around the unstable focus converge to a stable limit cycle about the fixed point $(y_0, \theta_0) = (1, 1 + a)$. This suggests that the adaptive system undergoes a supercritical Hopf bifurcation at $a = -\frac{1}{3}$. In the next section we will give theoretical evidence of this by using center manifold theory and the theory of the Hopf bifurcation.

Remark: Note that the stability region of the fixed point $(y_0, y_{m0}, \theta_0) = (1, 1, 1 + a)$ is very restricted for values of a slightly larger than $-\frac{1}{3}$. This means that the adaptive system is of no practical interest for such values of a. We will discuss the stability region of this fixed point in Section 6.

4. A Theoretical Study of the Supercritical Hopf Bifurcation

In the last section we saw that the system in (1) undergoes a supercritical Hopf bifurcation when the desired model is started with $y_m(0) = 1$, i.e. when y_m is

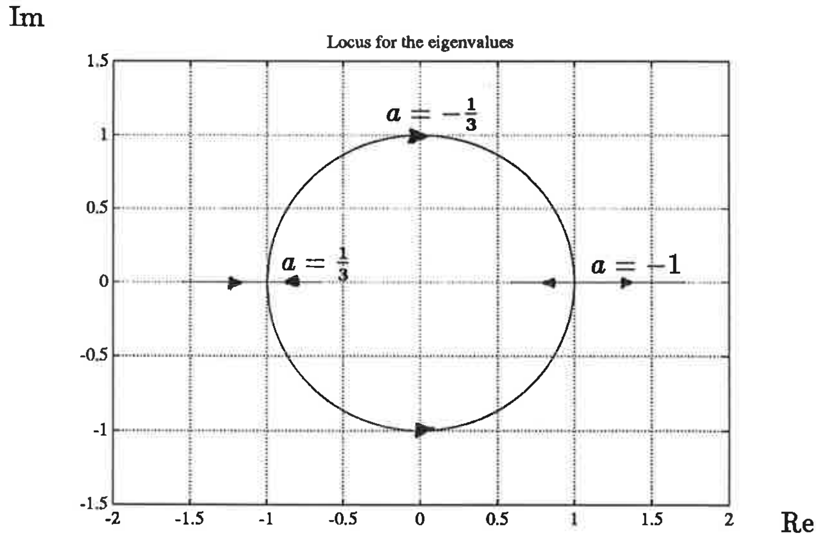

Figure 3. Locus of the eigenvalues λ_{2_2} and λ_{2_3} as function of a. The parameter a varies from $a = 0.4$ to $a = -1.1$ and the arrows indicate the direction of the locus when a is decreasing.

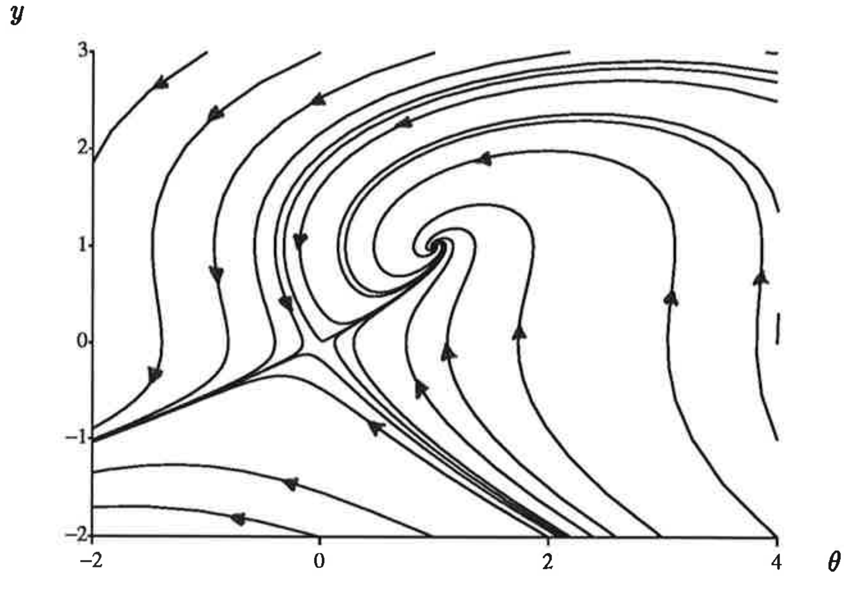

Figure 4. Phase portrait of the system in (1) when $y_m(t) \equiv 1$. This phase portrait is for the case $a = 0$. In this case there is a saddle point in $(0, 0)$ and a stable focus in $(1, 1)$.

constant and equal to one, which means that the second equation of (1) does not come into play and therefore the system is 2-dimensional in this case. We will now give theoretical evidence that the system undergoes a supercritical Hopf bifurcation even when the full 3-dimensional system is considered.

In order to determine the character of a Hopf bifurcation of a nonlinear system of ordinary differential equations with two complex conjugate eigenvalues and the other eigenvalues having negative real part we need to compute the center manifold of the vector field.

The center manifold of a nonhyperbolic fixed point (a fixed point for

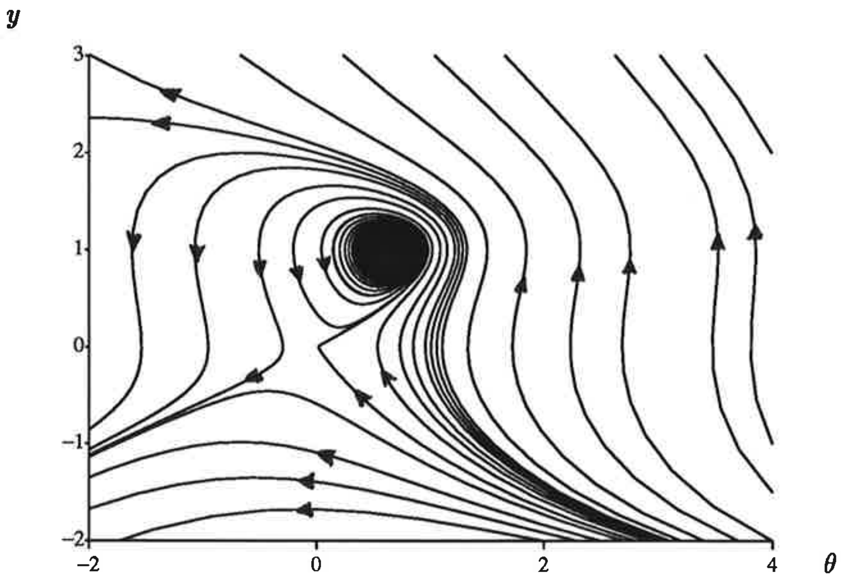

Figure 5. Phase portrait of the system in (1)when $y_m(t) \equiv 1$. This phase portrait is for the case $a = -0.33$. In this case there is a saddle point in $(0, 0)$ and a stable focus in $(0.67, 1)$.

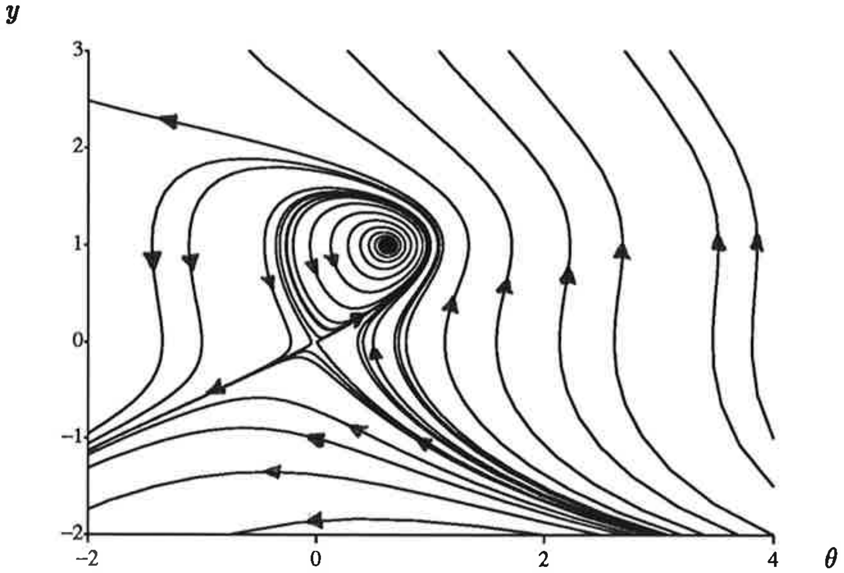

Figure 6. Phase portrait of the system in (1) when $y_m(t) \equiv 1$. This phase portrait is for the case $a = -0.37$. In this case there is a saddle point in $(0, 0)$ and an unstable focus in $(0.63, 1)$. Note the limit cycle around the unstable fixed point.

which the corresponding Jacobian has eigenvalues with zero real part) is an invariant manifold that passes through the fixed point where it is tangent to those eigenvectors of the Jacobian that corresponds to eigenvalues with zero real part. We can study the asymptotic behaviour of the system in a neighborhood of the nonhyperbolic fixed point by restricting the system to the center manifold. Center manifold theory is therefore useful for studying stability and existence of bifurcations for systems with nonhyperbolic fixed points.

The center manifold is in our case a 3-dimensional invariant manifold such that the restriction of the vector field to it, has two imaginary eigenvalues and one

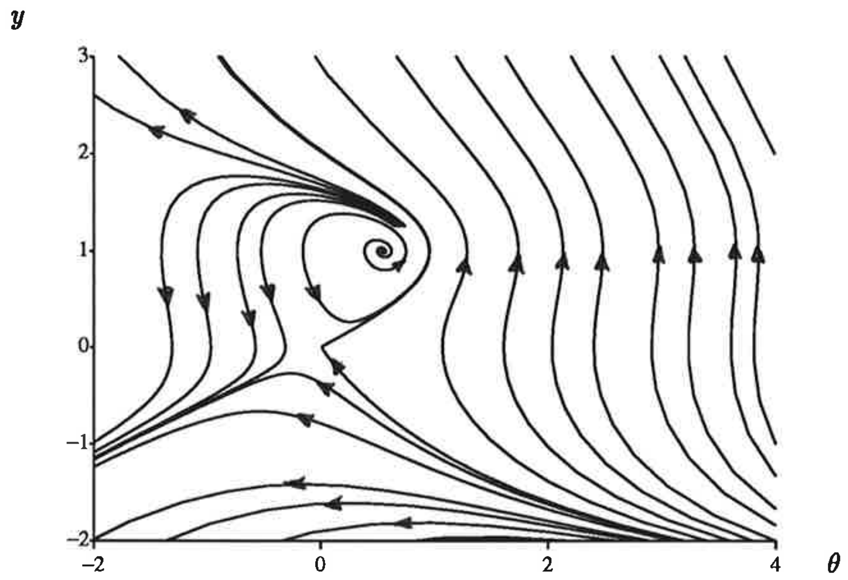

Figure 7. Phase portrait of the system in (1)when $y_m(t) \equiv 1$. This phase portrait is for the case $a = -0.45$. In this case there is a saddle point in $(0,0)$ and an unstable focus in $(0.55, 1)$. There is no limit cycle about the unstable fixed point.

zero eigenvalue. One may think that it would be enough with a 2-dimensional center manifold corresponding to the complex conjugate imaginary eigenvalues of the vector field, but we must, however, consider the parameter, in our case a, whose variation causes the change in character of the vector field about the fixed point, as a dependent variable. By doing so, we assure that the center manifold exists in a sufficiently small neighborhood about the parameter's nominal value. We collect the results in Chapter 2.1 of [Wiggins, 1990] in the following theorem

THEOREM 1
Consider a vector field of the form

$$\begin{aligned} \dot{x} &= Ax + f(x, y, \mu) \\ \dot{\mu} &= 0, \qquad\qquad (x, y, \mu) \in \mathbb{R}^c \times \mathbb{R}^s \times \mathbb{R}^p \qquad (6) \\ \dot{y} &= By + g(x, y, \mu) \end{aligned}$$

where

$$f(0,0,0) = 0, \qquad Df(0,0,0) = 0,$$
$$g(0,0,0) = 0, \qquad Dg(0,0,0) = 0,$$

and where both f and g are C^r functions with $r \geq 2$. Further, the matrix A has eigenvalues with zero real parts and the matrix B has eigenvalues with negative real parts. Note that this vector field has a fixed point at the origin. There exists a local center manifold for this vector field, which is given as

$$W^c_{loc} = \{(x, y, \mu) \in \mathbb{R}^c \times \mathbb{R}^s \times \mathbb{R}^p \mid y = h(x, \mu), \quad |x| < \delta, \quad |\mu| < \hat{\delta}, \quad h(0,0) = 0, \quad Dh(0,0) = 0\}$$

The restriction of (6) to the center manifold is for x sufficiently small given by

$$\begin{aligned} \dot{x} &= Ax + f(x, h(x, \mu), \mu), \\ \dot{\mu} &= 0, \qquad\qquad (x, \mu) \in R^c \times R^p. \end{aligned}$$

If the vector field in (6) is C^r then the center manifold W^c_{loc} is also C^r. The function h satisfies the following partial differential equation

$$N(h(x,\mu)) \stackrel{def}{=} D_x h(x,\mu)[Ax + f(x, h(x,\mu),\mu)] - Bh(x,\mu) - g(x, h(x,\mu),\mu) = 0$$

which in most cases is extremely difficult to solve. We can, however, find an approximate solution by solving for a polynomial ϕ in x and μ, which satisfies $\phi(0) = D\phi(0) = 0$ and $N(\phi(x,\mu)) = O(|(x,\mu)|^q)$ for some $q > 1$, then

$$|h(x,\mu) - \phi(x,\mu)| = O(|(x,\mu)|^q) \qquad \text{as} \quad x \to 0.$$

□

Remark: Note that we regard the parameter μ in (6) as a dependent variable in order to assure the existence of the center manifold in a neighborhood of $\mu = 0$. This is necessary when studying the dependence to changes of the parameter a for the vector field restricted to the center manifold.

We will now apply the center manifold theory to our system in (1). In order to do this we must transform (1) to the form in (6). For this, we need to translate the system so that the bifurcation point becomes $(x, y, \mu) = (0, 0, 0)$. This is done by taking $a = -(\frac{1}{3} + \mu)$, and then making the following change of variables, which translates the fixed point to the origin and changes the coordinate basis. The new variables are x_1,x_2 and y.

$$\begin{pmatrix} y - y_{0_2} \\ y_m - y_{m0_2} \\ \theta - \theta_{0_2} \end{pmatrix} = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & -2 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix}$$

The resulting system is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} + \begin{pmatrix} f_1(x,y,\mu) \\ f_2(x,y,\mu) \\ 0 \end{pmatrix} \tag{7}$$
$$\dot{\mu} = 0$$

where

$$\begin{aligned} f_1(x,y,\mu) &= \frac{1}{3}(y + x_1)(x_1(2y + 6\mu y - 3) + 9\mu(1 - x_1) + (x_1^2 + y^2)(1 + 3\mu) \\ &\qquad - 3y(1 + 3\mu)) \\ f_2 &= x_1^2 - y^2 \end{aligned}$$

Which is on the form (6).

It will soon be shown that in order to determine the character of the Hopf bifurcation, we need derivatives up to order three evaluated at the origin (i.e. the fixed point) of the restriction of the vector field to the center manifold. It is therefore enough to consider an approximation ϕ of order two, i.e. assume that ϕ has the form

$$\phi(x,\mu) = a_1 x_1^2 + a_2 x_1 x_2 + a_3 x_2^2 + a_4 \mu x_1 + a_5 \mu x_2 + a_6 \mu^2$$

By putting the second order coefficients in the expression for $N(\phi(x,\mu))$ to zero we get an equation system with the solution $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 0$, i.e. $\phi(x, \mu) = 0$. Hence, the restriction of the vector field in (4) to the center manifold is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 3\mu x_1(1 - x_1) + x_1^2((\mu + \frac{1}{3})x_1 - 1) \\ x_1^2 \end{pmatrix} \qquad (8)$$
$$\dot{\mu} = 0$$

We are now in a position to study the character of the Hopf bifurcation. The following summary of the results in Chapter 3.1B in [Wiggins, 1990] gives us the necessary tools.

THEOREM 2
Consider a vectorfield of the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \mathrm{Re}(\lambda(\mu)) & -\mathrm{Im}(\lambda(\mu)) \\ \mathrm{Im}(\lambda(\mu)) & \mathrm{Re}(\lambda(\mu)) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f^1(x, y, \mu) \\ f^2(x, y, \mu) \end{pmatrix} \qquad (9)$$

$$(x, y, \mu) \in \mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{R}^1$$

where f^1 and f^2 are nonlinear at the origin and where $\lambda(\mu) = \alpha(\mu) + i\omega(\mu)$ and $\bar{\lambda}(\mu)$ are the eigenvalues of the vectorfield linearized about the fixed point at the origin. It is assumed that $\alpha(0) = 0$ and that $\omega(0) \neq 0$, i.e. the vector field has a complex conjugate pair of eigenvalues on the imaginary axis when $\mu = 0$.

By an analytic coordinate change, (9) can be transformed to the normal form

$$\dot{x} = \alpha(\mu)x - \omega(\mu)y + (a(\mu)x - b(\mu)y)(x^2 + y^2) + O(|x|^5, |y|^5),$$
$$\dot{y} = \omega(\mu)x + \alpha(\mu)y + (b(\mu)x + a(\mu)y)(x^2 + y^2) + O(|x|^5, |y|^5).$$

The character of the Hopf bifurcation of the vector field in (9) is determined by the the following coefficents

$$d = \alpha'(0) = \frac{d}{d\mu} Re\lambda(\mu)) \,|_{\mu=0}$$
$$a_0 = a(0) = \frac{1}{16}[f^1_{xxx} + f^1_{xyy} + f^2_{xxy} + f^2_{yyy}] + \frac{1}{16\omega}[f^1_{xy}(f^1 xx + f^1_{yy}) - f^2_{xy}(f^2_{xx} + f^2_{yy}) - f^1_{xx}f^2_{xx} + f^1_{yy}f^2_{yy}] \,|_{x=y=\mu=0} \qquad (10)$$

The coefficient a_0 determines the stability of the periodic orbit. We have that the periodic orbit is asymptotically stable for $a_0 < 0$ (i.e. a supercritical Hopf bifurcation) and unstable for $a_0 > 0$ (subcritical Hopf bifurcation). The parameter d determines the direction the eigenvalues cross the imaginary axis when μ increases. We have the following characterization of the Hopf bifurcation.

a_0	d	$\mu < 0$	$\mu = 0$	$\mu > 0$
$a_0 > 0$	$d > 0$	unstable periodic orbit	unstable focus	unstable focus
$a_0 < 0$	$d > 0$	stable focus	stable focus	stable periodic orbit
$a_0 > 0$	$d < 0$	unstable focus	unstable focus	unstable periodic orbit
$a_0 < 0$	$d < 0$	stable periodic orbit	stable focus	stable focus

□

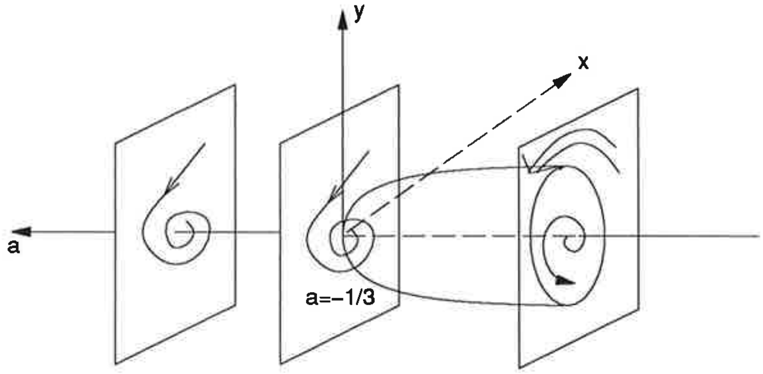

Figure 8. The character of the supercritical Hopf bifurcation

The system restricted to the center manifold must be transformed to the form in (9). This is done by considering μ as a constant parameter, and then make a linear change of basis which depends on μ and split the system into a linear part and a nonlinear part. The linear transformation

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{3}{2}\mu & -\frac{1}{2}\sqrt{4-9\mu^2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

applied to (8) results in the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \frac{3}{2}\mu & -\frac{1}{2}\sqrt{4-9\mu^2} \\ \frac{1}{2}\sqrt{4-9\mu^2} & \frac{3}{2}\mu \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f^1(x,y,\mu) \\ f^2(x,y,\mu) \end{pmatrix} \tag{11}$$

where

$$f^1(x,y,\mu) = -x^2 - 3\mu x^2 + \frac{1}{3}x^3 + \mu x^3$$
$$f^2(x,y,\mu) = x^2(2 - 3\mu - 9\mu^2 + \mu x + 3\mu^2 x)/\sqrt{4-9\mu^2}$$

which is on the desired form. From (10) in Theorem 2 we get

$$\begin{aligned} d &= 3/2 \\ a_0 &= -\frac{1}{8} \end{aligned} \tag{12}$$

which according to Theorem 2 means that we have a supercritical Hopf bifurcation. Considering that $a = -(\frac{1}{3} + \mu)$ and that $a_0 < 0$ and $d > 0$, from the table in Theorem 2 we draw the conclusion that the vector field behaves as in Figure 4 about the fixed point in (3). This is consistent with the phase portrait simulations in Section 3. Note that in Figure 7 when $a = -0.45$ the periodic orbit has broken up, which is consistent with the fact that the Hopf bifurcation is a local phenomenon.
All the calculations in this chapter have been done with maple. The maple code is included in the appendix. This code is a modification of a maple program for the discrete time case found in [Brundell, 1990].

5. Periodic Solutions

In this section we will study the possibility of periodic solutions to the studied adaptive system. As before the system is described by

$$\begin{aligned}\dot{y} &= -y + \theta u_c \\ \dot{y}_m &= -y_m + b u_c \\ \dot{\theta} &= -\gamma(y - y_m)y\end{aligned}$$

and as in Section 3 we consider the case when $y_m(t) \equiv 1$, $u_c = 1$ and $\gamma = 1$ so that we get the two dimensional system

$$\begin{aligned}\dot{y} &= -y - ay^3 + \theta \\ \dot{\theta} &= -(y-1)y\end{aligned}$$

with $\mathbb{R}^2$ as its phase space. First we will apply Bendixson's criterion which is summarized in the following theorem

THEOREM 3
Let a vector field in $\mathbb{R}^2$ be given by

$$\begin{aligned}\dot{x} &= f(x,y) \\ \dot{y} &= g(x,y)\end{aligned}$$

where f and g are at least C^1, and let $D \subset \mathbb{R}^2$ be a simply connected region where $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$ is not zero and does not change sign.
Then the system has no closed orbit lying in D.

□

In the case of our second order adaptive system this means that we have to study

$$\frac{\partial f}{\partial y} + \frac{\partial g}{\partial \theta} = -1 - 3ay^2$$

From this equation we see that if $a \geq 0$ we cannot have any closed orbit at all, and that for $a < 0$ we cannot have any closed orbit lying entirely in any of the three regions described by $y < -\sqrt{\frac{1}{3|a|}}$, $y > \sqrt{\frac{1}{3|a|}}$ and $-\sqrt{\frac{1}{3|a|}} < y < \sqrt{\frac{1}{3|a|}}$.

To further characterize any closed orbits we can use index theory [Wiggins, 1990] to see that any closed orbit must enclose the fixed point at $(y,\theta) = (1, 1+a)$ but cannot enclose the saddle point at the origin. In Figure 9 we show one possible closed orbit. The closed orbit cannot lie entirely in either of the three regions. The results in this section agrees with the phase portraits in Section 3 where a stable limit cycle occurs for $a = -0.37$.

6. Stability Region of the Stable Fixed Point

As pointed out in earlier sections, there are two fixed points of the adaptive system under study. The fixed point at the origin is hyperbolic for all values of the parameter a and hence unstable. The other fixed point is however at least locally stable for $a > -\frac{1}{3}$. In this section the stability region of this fixed point will be studied.

The first question one might ask is if there are values of a for which the system is globally stable. This would be good from the control problem point of view. That this is not the case can easily be shown. As in Section 5 we

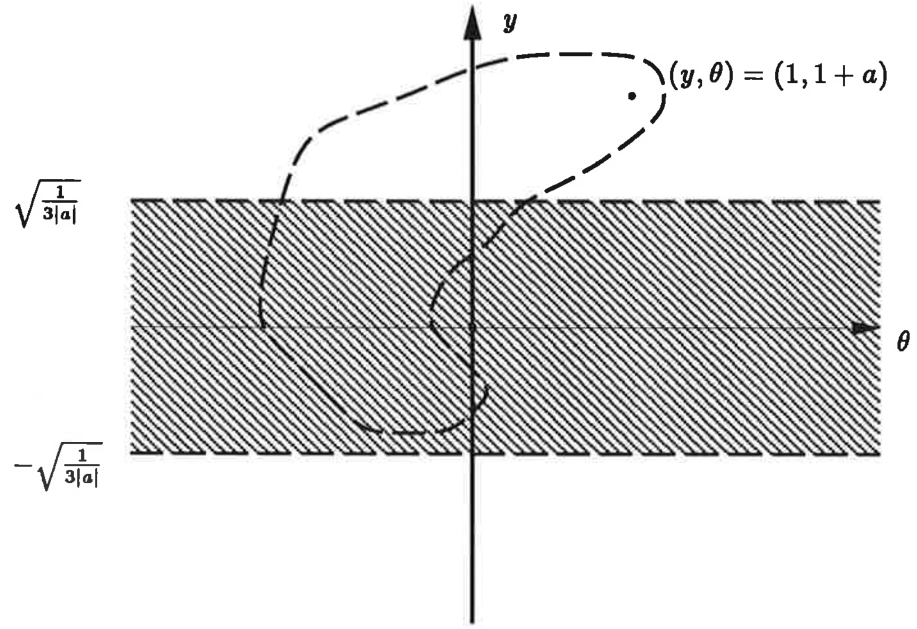

Figure 9. No closed orbit can exist entirely inside the shaded region. Furthermore, any closed orbit must enclose the fixed point at $(1, 1+a)$ but, on the other hand, cannot enclose the saddle point at the origin. The dashed orbit is a possible closed orbit.

consider the case when $y_m(t) \equiv 1$, $u_c = 1$ and $\gamma = 1$ so that we get the two dimensional system

$$\begin{aligned} \dot{y} &= -y - ay^3 + \theta \\ \dot{\theta} &= -(y-1)y \end{aligned} \tag{13}$$

Now we have to consider two cases depending on the sign of a.

$a < 0$

In this case we note that $\exists y_0 < 0$ and $\theta_0 < 0$ such that $\dot{y}|_{y,\theta} < M < 0$ and $\dot{\theta}|_{y,\theta} < N < 0$ for all $y < y_0$ and $\theta < \theta_0$ which means that both y and θ tend to $-\infty$.

$a \geq 0$

In this case we consider the solution to (13) with initial value in the area in the third quadrant described by $y < -\epsilon$ and $\theta < y - ay^3$ where $\epsilon > 0$. In this area $\dot{\theta}$ is negative and bounded away from zero so that if the solution stays in this region then at least θ will tend to $-\infty$. That both the boundaries are rejective can be shown by studying the sign of $S\dot{S}$ where S is the equation for the boundary. For the boundary $y = -\epsilon$ we have $S_1 = y+\epsilon = 0$ and $S_1\dot{S}_1 = (y+\epsilon)(-y-ay^3+\theta)$. The first factor is negative inside the region and so is the second one. Thus $y+\epsilon$ is a rejective surface when approaching it from $y < -\epsilon$. The other boundary is $S_2 = \theta - y - ay^3 = 0$ for which $S_2\dot{S}_2 = (\theta - y - ay^3)(-y^2 + y - (1+3ay^2)(-y - ay^3 + \theta))$. Now the first factor is negative when $\theta < y - ay^3$ and in the second factor the term $-y^2+y$ is negative and dominates when we are close to the boundary so that $S_2\dot{S}_2 > 0$ and the surface is rejective when approaching it from y such that $\theta - y - ay^3 < 0$. Therefore we cannot leave the region once in it and $\dot{\theta}$ is negative and bounded away from zero so that the system is unstable.

Hence the system cannot be globally stable irrespective of the parameter a.

Next the stability region of the fixed point at $y = 1, \theta = 1 + a$ will be determined. We refer to the paper [Hsiao-Dong et al., 1988] where the stability region of a stable fixed point is completely characterized under three assumptions. In our case we will show that the stability boundary is the stable manifold of the hyperbolic fixed point at the origin. From [Hsiao-Dong et al., 1988] we have the following theorem

THEOREM 4

Let the smooth vector field describing the nonlinear system satisfy the following assumptions

A1 All the equilibrium points on the stability boundary are hyperbolic.

A2 The stable and unstable manifolds of equilibrium points on the stability boundary satisfy a transversality condition.

A3 Every trajectory on the stability boundary approaches one of the equilibrium points as $t \to \infty$

Then

$$\partial A = \bigcup_i W^s(x_i)$$

i.e. the stability boundary, ∂A, of the fixed point is the union of the stable manifolds, W^s, of the fixed points, x_i, on the stability boundary.

□

Remark: By two manifolds satisfying the transversality condition, it means that at every point of intersection the tangent vectors of the two manifolds span the whole space, in this case $\mathbb{R}^2$. This means that in our two dimensional case we cannot admit the saddle point to have any homoclinic orbit , i.e. one part of the unstable manifold cannot coincide with one part of the stable manifold.

In our case we only have one fixed point except for the stable one and that is the one at the origin. Theorem 4 and the phase portraits in Section 3 suggests that the boundary of the stability region of the fixed point at $(y, \theta) = (1, 1 + a)$ is the stable manifold of the saddle point. To verify this we have to check that the assumptions in Theorem 4 are satisfied. A1 is clearly satisfied since the fixed point at the origin is a saddle point for all values of a. A2 is a little bit harder to check. In Figure 10 we see how this assumption fail to be valid when we have a homoclinic orbit. If we could guarantee that there is no homoclinic orbit then A2 would be fulfilled. Since we have no easy way of doing this we will trust the simulations and conclude that A2 is fulfilled when the simulation of the stable manifold of the saddle point in $(0, 0)$ is not homoclinic. Note that A2 is a generic condition for ODE's, i.e. is true for almost all systems, and therefore we can expect it to be fulfilled.

A3 is also hard to prove and furthermore it is not generic. In showing that A3 must be fulfilled we will give a rather heuristic reasoning where some of the details have been left out. We will start with the case $a = 0$.

Figure 11 shows a phase portrait of the vector field where the surfaces $S_1 = \{(\theta, y) \in \mathbb{R}^2 \mid y = 0\}$, $S_2 = \{(\theta, y) \in \mathbb{R}^2 \mid y = 1\}$ and $S_3 = \{(\theta, y) \in \mathbb{R}^2 \mid$

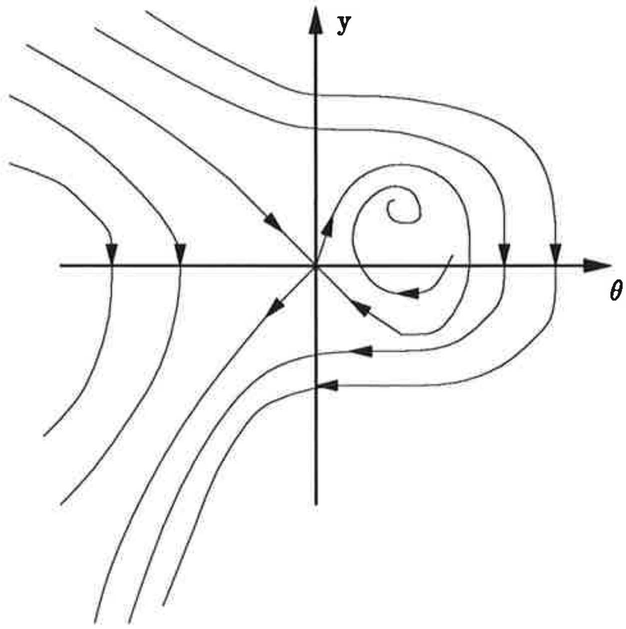

Figure 10. A possible phase portrait that do not fulfill A2.

$y = \theta\}$ divides the phase plane into six regions R_1 to R_6. It is seen from the direction of the vector field that a trajectory in R_1 must either go into R_2 or into R_6 unless it goes to the saddle point. Similarly, trajectories in R_2, R_3 and R_4 go either into R_3, R_4 and R_5 respectively or to the stable fixed point in $(1,1)$. Trajectories in R_5 can go to R_6 or to R_2 or to one of the fixed points. Finally, we know that all trajectories in R_6 go to infinity (This was shown on p. 12).

It is clear that a trajectory on the stability boundary cannot go into region R_6. This follows since in R_6 all trajectories in a neighborhood of ∂A would also go to infinity, which contradicts that ∂A is the stability boundary. We can therefore conclude that trajectories on the stability boundary must either spiral around the stable fixed point in $(1,1)$ or go to the saddle point in $(0,0)$. Here we should note that the stability boundary cannot go to the stable fixed point since there is a neighborhood about the stable fixed point which is not part of ∂A. We will now show that the first alternative is impossible. This will be done by showing that one part of the unstable manifold of the saddle point at the origin indeed is a trajectory that goes to the stable fixed point, and because of this there cannot be any trajectories spiraling around the stable fixed point. In order to show this we will use Lyapunov theory.

A Lyapunov function for the vector field in (13) can be found by using a Lyapunov function for the linearization of (13) about the fixed point in $(1,1)$. The linearization is

$$\begin{pmatrix} \dot{y-1} \\ \dot{\theta-1} \end{pmatrix} = A_{lin} \begin{pmatrix} y-1 \\ \theta-1 \end{pmatrix} \qquad \text{where} \qquad A_{lin} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

A Lyapunov function is now given by

$$V(y,\theta) = \begin{pmatrix} y-1 & \theta-1 \end{pmatrix} P \begin{pmatrix} y-1 \\ \theta-1 \end{pmatrix}$$

where the positive definite symmetric matrix P is given as the solution to the

Lyapunov equation

$$A^T P + PA = -Q$$

where Q is a symmetric and positive definite matrix. We used

$$Q = \begin{pmatrix} 1.78 & -0.525 \\ -0.525 & 1.3 \end{pmatrix} \quad \text{which gives} \quad P = \begin{pmatrix} 1.54 & -0.65 \\ -0.65 & 1.665 \end{pmatrix}.$$

Figure 12 shows the level curve $\dot{V} = 0$ together with two level curves for V. The figure indicates that the stability region of the stable fixed point given by the Lyapunov function goes almost all the way to the saddle point. To show that we actually enter this stability region we will use a Lyapunov type instability argument, see p119 [Slotine and Li, 1991], to show how the unstable manifold of the saddle point leaves the origin and enters the stability region determined by the Lyapunov function above. First however we conclude that the nonlinear terms in the differential equations are $-ay^3$ and $-y^2$. Around the origin (let us say $|\theta| < 0.01$ and $|y| < 0.01$) these are much smaller than the linear terms and in this region the solution to the differential equations along one part of the unstable manifold cannot deviate significantly from the direction of the eigenvector corresponding to the unstable eigenvalue of the linear system. Now we construct a function W defined by

$$W(y, \theta) = -16(\theta - y)^2 + y + \theta$$

This function is zero at the origin and along a parabola in the first quadrant. A few level curves of W and the contour lines where $\dot{W} = 0$ are shown in Fig. 13. Since $\dot{W} > 0$ in the area where our solution has entered the first quadrant it must go towards increasing values of W but by doing so it must pass the dotted line in Fig. 13 which is the borderline of an area inside which the Lyapunov function defined by P above implies that the stable fixed point is asymptotically stable. This means that the trajectory along the part of the unstable manifold in the first quadrant ends at the stable fixed point which we wanted to show.

We have thus shown that all the assumptions in Theorem 4 are fulfilled and therefore we know that the stabilty boundary is the stable manifold of the saddle point. The same way of reasoning work in the case when $a = 0.3$. Figure 14 shows that we, as when $a = 0$, can divide the phase plane into six regions R_1 to R_6 with exactly the same properties as for $a = 0$. We need only to find a Lyapunov function that gives stability in a region from the stable fixed point to a neigborhood of the saddle point and then the same arguments as in the case when $a = 0$ will show that A3 holds. We used the Lyapunov function

$$V(y, \theta) = \begin{pmatrix} y - 1 & \theta - 1.3 \end{pmatrix} P \begin{pmatrix} y - 1 \\ \theta - 1.3 \end{pmatrix}$$

where

$$P = \begin{pmatrix} 2.6316 & -0.5 \\ -0.5 & 2.5816 \end{pmatrix}$$

which almost includes the origin, see Fig. 15, and as in the case when $a = 0$ we use the function W and the same arguments, and from Fig. 16 we see that A3 must hold.

A similar argument would show that A3 also holds when $a = -0.3$. We will not show this here.

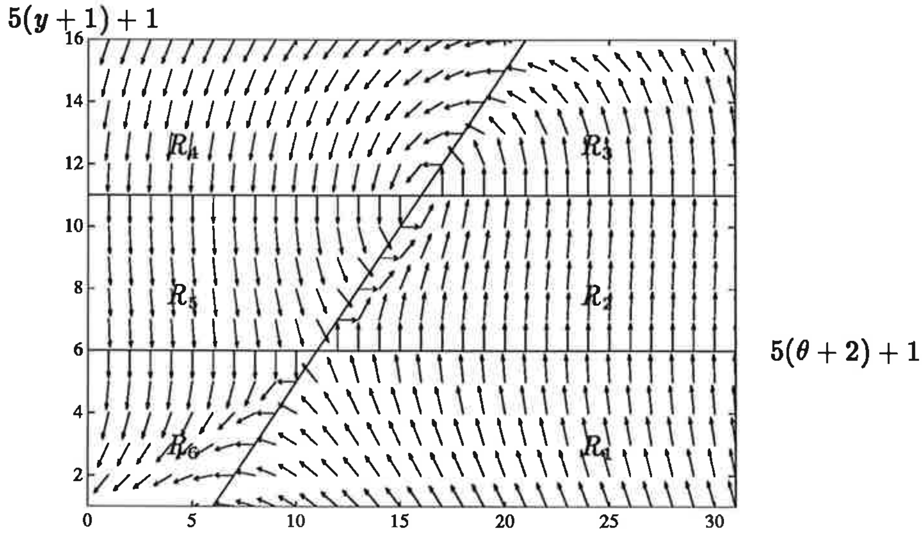

Figure 11. The normalized phase portrait when $a = 0$

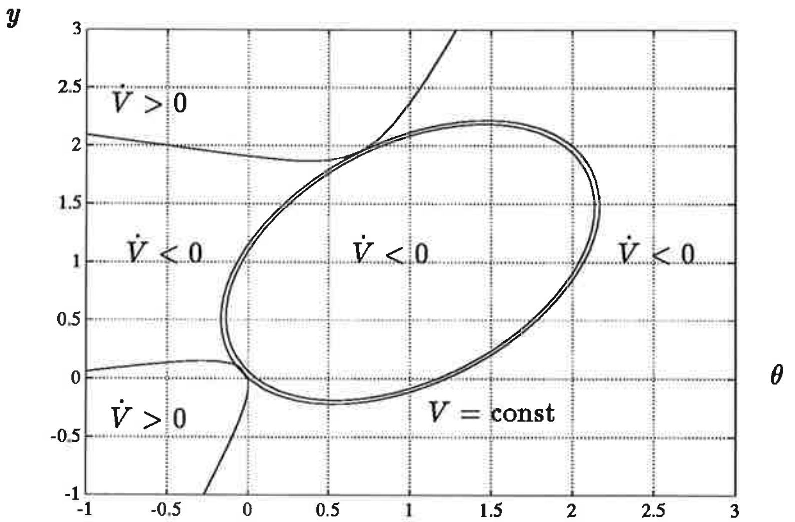

Figure 12. The level curve $\dot{V} = 0$ and two level curves for V. V is the Lyapunov function used when $a = 0$

The stable manifold of the saddle point was found by simulation in Simnon. The starting value was chosen as a small step in the direction of the eigenvector corresponding to the stable eigenvalue of the Jacobian at the origin and then the simulation was done in reversed time. The unstable manifold was also simulated for completeness. The result is in Figure 17 where the stability region for the three cases $a = -0.3$, $a = 0$ and $a = 0.3$ are shown. Note that in all three cases the unstable manifold goes to the stable fixed point which is consistent with our conclusion that the stability area of the stable fixed point is such that we get a trajectory between the fixed points.

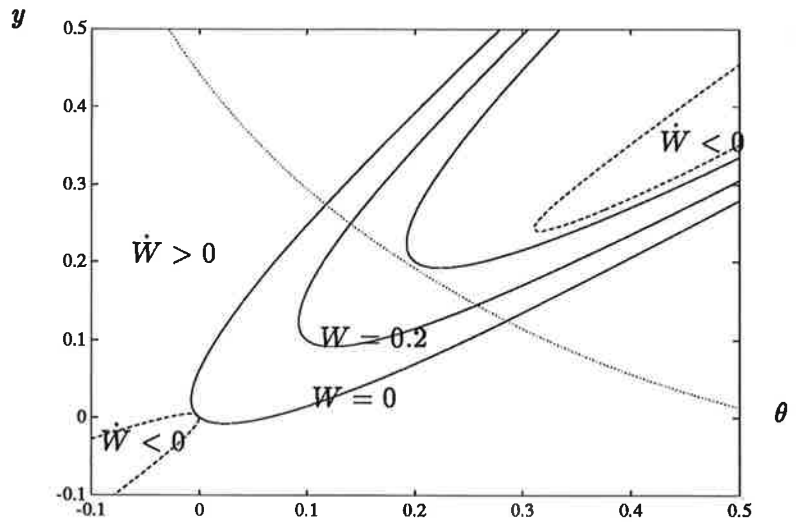

Figure 13. The level curve $\dot{W} = 0$ (dashed) and three level curves for W (solid). W is the function used when $a = 0$. Shown is also the borderline inside which the stable fixed point is asymptotically stable.

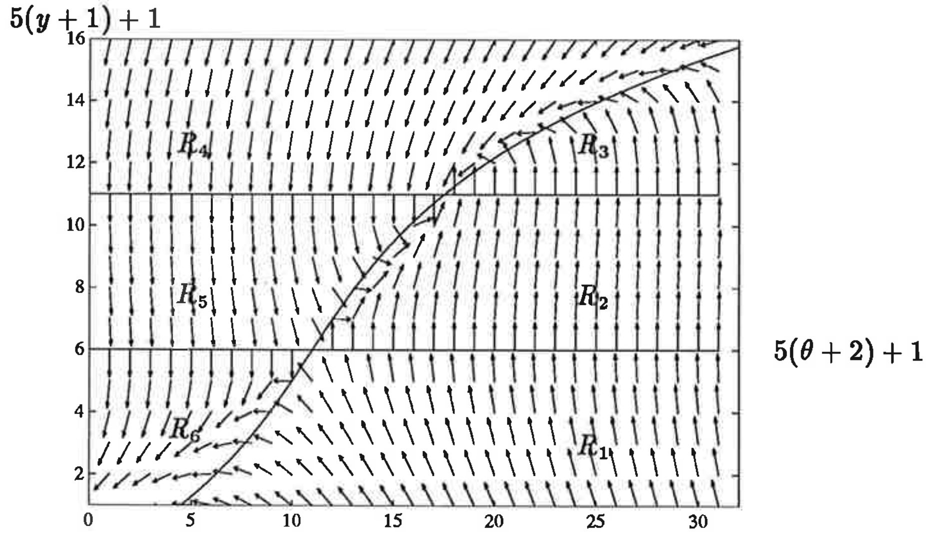

Figure 14. The normalized phase portrait when $a = 0.3$

7. Chaotic behavior

Our system will, if slightly modified, show something that might be described as chaotic behavior. The modifications we will do is to introduce a sinusoidal disturbance and a load disturbance in the process dynamics. The modified system looks as follows

$$\begin{aligned} \dot{y} &= -y - ay^3 + \theta + A\sin(\omega t) + d \\ \dot{\theta} &= -(y-1)y \end{aligned} \tag{14}$$

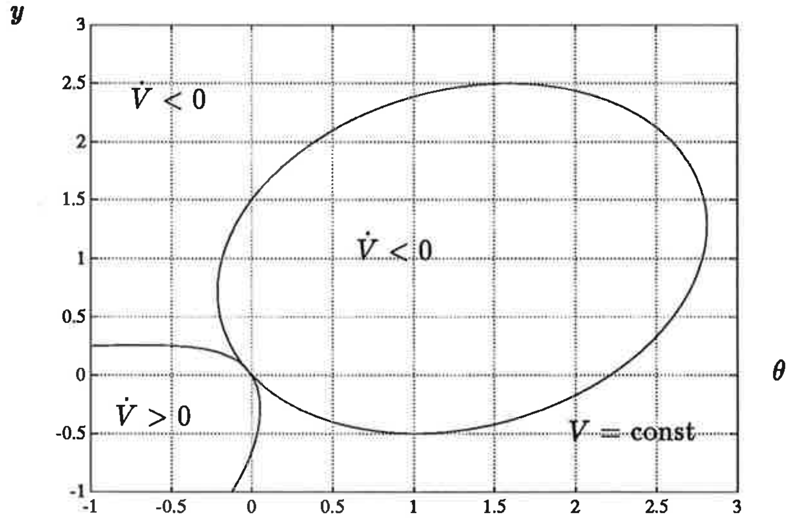

Figure 15. The level curve $\dot{V} = 0$ and a level curve for V. V is the Lyapunov function used when $a = 0.3$

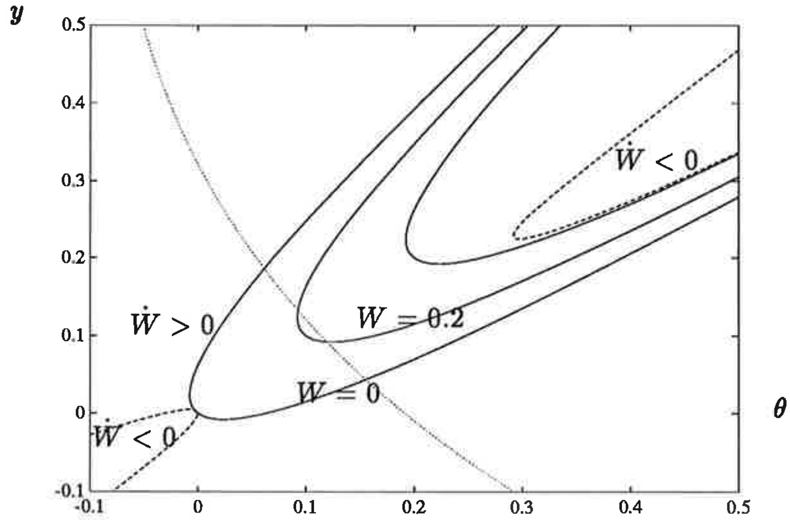

Figure 16. The level curve $\dot{W} = 0$ (dashed) and three level curves for W (solid). W is the function used when $a = 0.3$. Shown is also the borderline inside which the stable fixed point is asymptotically stable.

with $a = -0.35$, $A = 0.1$, $\omega = 2$ and $d = 0.4$. The system order is hence three which is the smallest dimension for which chaos can occur. In Figure 18 and Figure 19 we can see two trajectories of the system when initialized with $(y(0), \theta(0)) = (0.6194, 0.5)$ and $(y(0), \theta(0)) = (0.6198, 0.5)$ respectively. As can be seen the two solutions are very different even though the initial values are very close. Note that sensitivity to initial conditions is characteristic for chaotic motion. The choice of $a = -0.35$ means that without the disturbances, the system has an unstable fixed point at $(y, \theta) = (1, 0.25)$ but that there is a

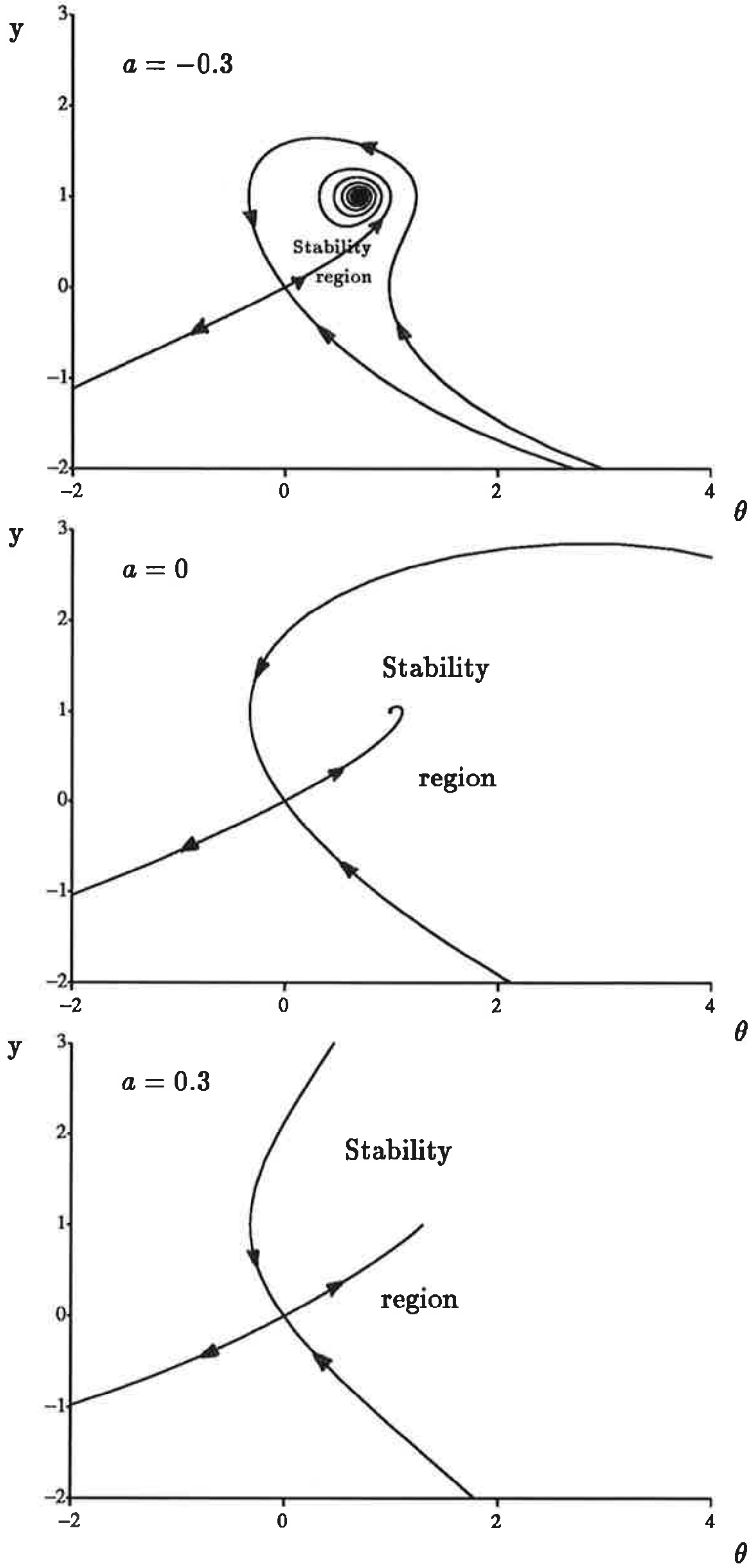

Figure 17. Stability region for $a = -0.3$, $a = 0$ and $a = 0.3$

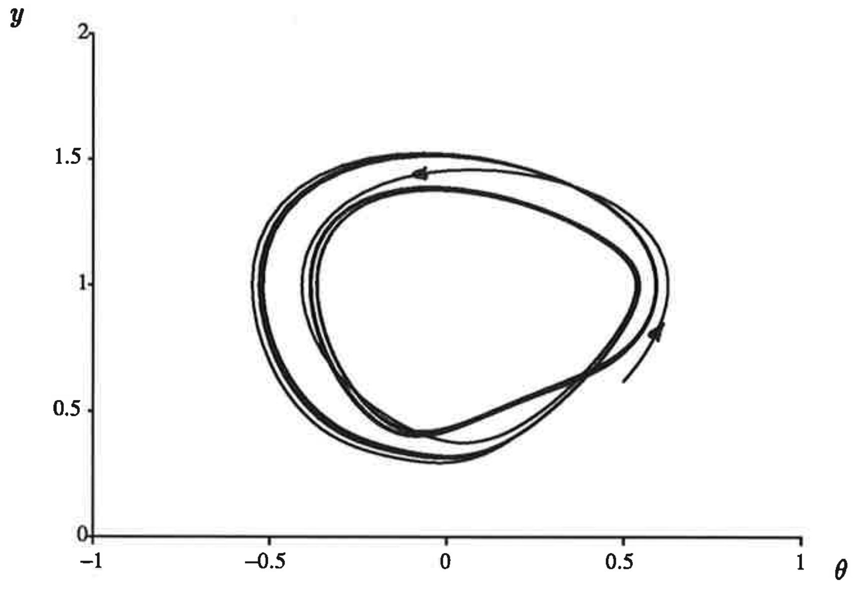

Figure 18. Solution to (14) with initial values $(y(0), \theta(0)) = (0.6194, 0.5)$.

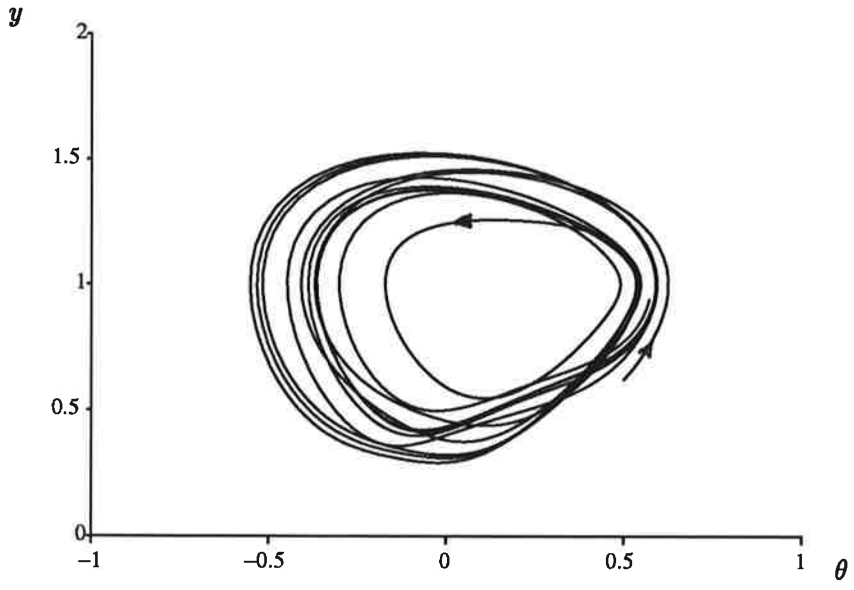

Figure 19. Solution to (14) with initial values $(y(0), \theta(0)) = (0.6198, 0.5)$.

stable limit cycle surrounding the fixed point. In Figure 18 it looks as if we are close to having a periodic limit cycle after the initial transients have died out but this quasi periodicity is highly sensitive to perturbations.

8. References

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9. Appendix: Maple code

The maple code that was used to determine the character of the Hopf bifurcation in Chapter 3 is given here.

```
#The system under consideration
#(Note that gamma=1, uc=1 and a=-1/3-mu)
f1:=-x+(1/3+mu)*x^3+z;
f2:=-y+1;
f3:=-x*(x-y);
#
#This program determines the character of the Hopf
#bifurcation of a 3-dimensional vector field, which
#linearization about its fixed point has two imaginary
#eigenvalues.
#The program consists of four steps. The first step
#calculates the fixed point and transforms the system
#so that the eigenvalues are at the origin. This step
#is not general since vector fields can have several
#fixed points not all corresponding to imaginary
#eigenvalues. The vectorfield studied here has two fixed
#points and the code is written such that the correct
#fixed point is choosen. It easy to modify this step
#for other vector fields.
#However, the second to the fourth steps are completely
#general. The second step computes the center manifold
#of the vectorfield and the third step transforms the
#center manifold to a form suitable for step 4. Finally,
#in step 4, the two parameters that characterizes the
#Hopf bifurcation are determined.
#The code is a modification of a similar code for the
#discrete time case in [Brundell].
#
#step 1: Calculate the fixed point, and transform system
#so that the fixed point for the transformed system is
#at the origin
#
with(linalg);
solset:=solve({f1=0,f2=0,f3=0},{x,y,z});
#
#choose the fixed point corresponding to imaginary
#eigenvalues.
#This part does not work for a general vectorfield.
#
x0:=subs(solset[2],x);
y0:=subs(solset[2],y);
z0:=subs(solset[2],z);
newcordset:={x=q1+x0,y=q2+y0,z=q3+z0};
f1:=subs(newcordset,f1);
f2:=subs(newcordset,f2);
f3:=subs(newcordset,f3);
#
#Taylor expand to third order around the fixed point
#Third order is enough since we need derivatives of
#the vector field up to order three in order to determine
#the character of the Hopf bifurcation.
#
readlib(mtaylor);
f1:=mtaylor(f1,[q1,q2,q3],4);
f2:=mtaylor(f2,[q1,q2,q3],4);
f3:=mtaylor(f3,[q1,q2,q3],4);
#
```

```
#Step 2:
#Compute the center manifold (step A to step B)
#For a reference see [Wiggins] chapter 2.1
#
#step A: Transform the system to the form
#
#           [x1]     [A11 A12 0 0][x1]     [f1(x1,x2,y,mu)]
#      d/dt[x2]  =  [A21 A22 0 0][x2] +  [f2(x1,x2,y,mu)]
#           [mu]     [ 0   0  0 0][mu]     [      0       ]
#           [y ]     [ 0   0  0 B][y ]     [g(x1,x2,y,mu) ]
#
#where the matrix A has two eigenvalues at the imaginary
#axis, and B has a stable eigenvalue. The third equation
#is known beforehand and will therefore not appear in the
#computations bellow.
#
J:=jacobian([f1,f2,f3],[q1,q2,q3]);
J0:=subs(q1=0,q2=0,q3=0,mu=0,evalm(J));
lambda:=eigenvals(J0);
l1:=lambda[1];
l2:=lambda[2];
if evalc(Im(l1))=0 then
   lambda[1]:=lambda[3];
   lambda[3]:=l1;
   lambda[2]:=l2;
elif evalc(Im(l2))=0 then
   lambda[2]:=lambda[3];
   lambda[3]:=l2;
   lambda[1]:=l1;
fi;
Id:=array([[1,0,0],[0,1,0],[0,0,1]]);
J1:=evalm(lambda[1]*Id-J0);
r1:=array([J1[1,1],J1[1,2],J1[1,3]]);
r2:=array([J1[2,1],J1[2,2],J1[2,3]]);
r3:=array([J1[3,1],J1[3,2],J1[3,3]]);
e1:=crossprod(r1,r3);
if e1[1]=0 and e1[2]=0 and e1[3]=0 then
   e1:=crossprod(r1,r2);
fi;
v1:=array([evalc(Re(e1[1])),evalc(Re(e1[2])),evalc(Re(e1[3]))]);
v2:=array([evalc(Im(e1[1])),evalc(Im(e1[2])),evalc(Im(e1[3]))]);
J2:=evalm(lambda[3]*Id-J0);
r1:=array([J2[1,1],J2[1,2],J2[1,3]]);
r2:=array([J2[2,1],J2[2,2],J2[2,3]]);
r3:=array([J2[3,1],J2[3,2],J2[3,3]]);
v3:=crossprod(r1,r3);
if v3[1]=0 and v3[2]=0 and v3[3]=0 then
   v3:=crossprod(r1,r2);
fi;
T:=array([[v1[1],-v2[1],v3[1]],[v1[2],-v2[2],v3[2]],
             [v1[3],-v2[3],v3[3]]]);
newvars:={q1=T[1,1]*x1+T[1,2]*x2+T[1,3]*y,q2=T[2,1]*x1+
    T[2,2]*x2+T[2,3]*y,q3=T[3,1]*x1+T[3,2]*x2+T[3,3]*y};
f1sl:=subs(newvars,f1);
f2sl:=subs(newvars,f2);
f3sl:=subs(newvars,f3);
F:=array([[f1sl],[f2sl],[f3sl]]);
Fhat:=evalm(inverse(T)&*F);
Atot:=subs(x1=0,x2=0,y=0,mu=0,jacobian([Fhat[1,1],Fhat[2,1],
             Fhat[3,1]],[x1,x2,y]));
F:=evalm(Fhat-Atot&*array([[x1],[x2],[y]]));
#
```

```
#Step B
#Determine h(x,mu) such that N(h(x,mu))=O(|x|^2) where
#N(h(x,mu))=Dxh(x,mu)[Ax+f(x,h(x,mu),mu)]-Bh(x,mu)
#            -g(x,h(x,mu),mu)
#Take h(x,mu)=a*x1^2+b*x1*x2+c*x2^2+d*mu^2+e*x1*mu+f*x2*mu
#This gives a third order approximation of vector field
#restricted to the center manifold, which is
#dx/dt=Ax+f(x,h(x,mu),mu)+O(|(x,mu)|^4)
#dmu/dt=0
#This is sufficient since third order derivatives evaluated
#in the origin are sufficient in order to determine the
#character of the Hopf bifurcation.
#
h:=a1*x1^2+a2*x1*x2+a3*x2^2+a4*mu^2+a5*mu*x1+a6*mu*x2;
Dxh:=grad(h,[x1,x2]);
A:=array([[Atot[1,1],Atot[1,2]],[Atot[2,1],Atot[2,2]]]);
B:=Atot[3,3];
f:=subs(y=h,vector(2,[F[1,1],F[2,1]]));
g:=subs(y=h,vector(1,F[3,1]));
Nsl1:=evalm(A&*array([[x1],[x2]])+f);
Nsl2:=Dxh[1]*Nsl1[1,1]+Dxh[2]*Nsl1[2,1];
N:=Nsl2-B*h-g;
fixp:={x1=0,x2=0,y=0,mu=0};
c1:=subs(fixp,diff(N,x1,x1));
c2:=subs(fixp,diff(N,x1,x2));
c3:=subs(fixp,diff(N,x2,x2));
c4:=subs(fixp,diff(N,mu,mu));
c5:=subs(fixp,diff(N,mu,x1));
c6:=subs(fixp,diff(N,mu,x2));
solset:=solve({c1=0,c2=0,c3=0,c4,c5,c6},{a1,a2,a3,a4,a5,a6});
h1:=subs(solset,h);
f1:=subs(y=h1,F[1,1]);
f2:=subs(y=h1,F[2,1]);
#
#Step3
#In this step the center manifold obtained in step 2 will be
#transformed to the form
#
#         [x] [Re(lambda(mu) -Im(lambda(mu)][x] [f1(x,y,mu)]
#    d/dt[ ]=[                             ][ ]+[          ]
#         [y] [Im(lambda(mu)  Re(lambda(mu)][y] [f2(x,y,mu)]
#
#
Ftot:=evalm(array([[f1],[f2]])+A&*array([[x1],[x2]]));
f1:=Ftot[1,1];
f2:=Ftot[2,1];
J:=jacobian([f1,f2],[x1,x2]);
J0:=subs(x1=0,x2=0,evalm(J));
lambda:=eigenvals(J0);
l1:=lambda[1];
if evalc(Im(l1))=0 then
  l2:=lambda[2];
  realpart:=(l1+l2)/2;
  impart:=sqrt(-1*(l1-l2)^2)/2;
  l1:=realpart+I*impart;
fi;
id:=array([[1,0],[0,1]]);
J1:=evalm(l1*id-J0);
evect:=array([1,evalc(-J1[1,1]/J1[1,2])]);
#
#evect:=v1-Iv2, where v1 and v2 will be columns in the
#transformation matrix T
```

```
#
v1:=array([evalc(Re(evect[1])),evalc(Re(evect[2]))]);
v2:=array([evalc(Im(-evect[1])),evalc(Im(-evect[2]))]);
T:=array([[v1[1],v2[1]],[v1[2],v2[2]]]);
fsl1:=subs(x1=T[1,1]*x+T[1,2]*y,x2=T[2,1]*x+T[2,2]*y,f1);
fsl2:=subs(x1=T[1,1]*x+T[1,2]*y,x2=T[2,1]*x+T[2,2]*y,f2);
fn:=array([[fsl1],[fsl2]]);
fhat:=evalm(inverse(T)&*fn);
fhat1:=fhat[1,1];
fhat2:=fhat[2,1];
A:=subs(x=0,y=0,jacobian([fhat1,fhat2],[x,y]));
F:=evalm(array([[fhat1],[fhat2]])-A&*array([[x],[y]]));
f1:=F[1,1];
f2:=F[2,1];
#
#Step4
#The coefficients that determines the character of the
#Hopf bifurcation are given as
#d=d/dmu(Re(lambda(mu))
#and
#a=[f1xxx+f1xyy+f2xxy+f2yyy]/16+[f1xy(f1xx+f1yy)-
#         f2xy(f2xx+f2yy)-f1xxf2xx+f1yyf2yy]/16omega
#
# where the derivatives are evaluated att (xt,yt,mu)=(0,0,0)
#
#For a reference see [Wiggins] chapter 3.1B
#
omega:=subs(mu=0,A[2,1]);
fixset:={x=0,y=0,mu=0};
f1xxx:=subs(fixset,diff(f1,x$3));
f1xyy:=subs(fixset,diff(f1,x,y,y));
f1xy:=subs(fixset,diff(f1,x,y));
f1xx:=subs(fixset,diff(f1,x$2));
f1yy:=subs(fixset,diff(f1,y$2));
f2yyy:=subs(fixset,diff(f2,y$3));
f2xxy:=subs(fixset,diff(f2,x,x,y));
f2xy:=subs(fixset,diff(f2,x,y));
f2xx:=subs(fixset,diff(f2,x$2));
f2yy:=subs(fixset,diff(f2,y$2));
a:=(f1xxx+f1xyy+f2xxy+f2yyy)/16+(f1xy*(f1xx+f1yy)
    -f2xy*(f2xx+f2yy)-f1xx*f2xx+f1yy*f2yy)/(16*omega);
Relamda:=A[1,1];
d:=subs(mu=0,diff(Relamda,mu));
#
#References
#----------
#Brundell, J.E.(1990) "stability and bifurcation of an
#     adaptive controller" Technical Report TFMA-5013,
#     Dept of Mathematics, Lund Institute of Technology,
#     Lund , Sweden.
#
#Wiggins, S.(1990) "Introduction to applied nonlinear
#      dynamical systems and chaos", Springer-Verlag.
#
```