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A Damping Ratio Bound for Networks of Masses and Springs

Richard Pates

***Abstract*— The damping ratio is a key performance measure in systems that can be modelled as networks of masses and springs. We derive a lower bound on this quantity that applies to such networks when the masses are subject to viscous damping. The result allows the size of the damping ratio to be understood as a function of the system parameters. We use this to derive a decentralised criterion which, if satisfied, guarantees that all the modes of a swing equation power system model are sufficiently well damped, independently of its operating point and size.**

I. Introduction

Networks of masses and springs are ubiquitous throughout physics and engineering, and are used to describe the dynamical behaviour of a vast range of physical phenomena. A key performance measure for such networks is the damping ratio. This dimensionless constant describes how quickly any given oscillatory mode of the network dies out relative to its frequency of oscillation. It often plays a central role in applications, for example in the control of inter-area oscillations in electrical power systems [1].

In this paper we study networks of masses and springs with dynamics described by the differential equations

$$m_i\ddot{q}_i + c_i\dot{q}_i + k_{ii}q_i + \sum_{j\neq i}^{n} k_{ij}\left(q_i - q_j\right) = 0. \tag{1}$$

Each variable $q_i\left(t\right)$ describes the position (in generalised coordinates) of a point with mass $m_i \geq 0$. The points are interconnected by springs with stiffness constants $k_{ij} = k_{ji} \geq 0$, and are additionally subject to viscous damping with damping coefficients $c_i \geq 0$.

The damping ratios of the modes of such systems are commonly defined[1] to equal $-\cos\angle s$, where s solves the Generalised Eigenvalue Problem (GEVP)

$$\begin{bmatrix} -C & -K \\ I & 0 \end{bmatrix} v = s \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} v. \tag{2}$$

In the above M, C and K are the so called mass, damping and stiffness matrices. For eq. (1), M and C are the diagonal matrices that satisfy $M_{ii} = m_i$ and $C_{ii} = c_i$, and K the symmetric matrix given by

$$K_{ij} = \begin{cases} \sum_{k=1}^{n} k_{ik} & \text{if } i = j, \\ -k_{ij} & \text{otherwise.} \end{cases}$$

One can always compute the damping ratios for any given M, C and K by solving eq. (2), and the algorithms for solving this problem have a well established mathematical theory [4]. However in applications one is often not interested in the specific values of the damping ratios of the modes, rather that they are large enough for all possible operating points and network configurations. For example, in the power system context, the requirement is typically for damping ratios of at least 0.03-0.05 [5]. Since, as we shall see in Section II-B, in this case the system parameters depend on the operating point, methods based on direct computation are significantly less useful, since there are far too many scenarios to evaluate numerically.

Accordingly, we look for methods that can provide insight into how the system parameters affect the damping ratio. To illustrate the idea, consider the scalar instance of eq. (2). In this case the network has two modes, both with damping ratios equal to

$$\min\left\{1, \frac{c_1}{2\sqrt{m_1 k_{11}}}\right\}. \tag{3}$$

The appealing feature of this formula is the intuitive way in which the damping ratio can be understood in terms of the system parameters. Suppose for example that the value of k_{11} depends on the operating point, but is known to satisfy $0 \leq k_{11} \leq k_{\max}$. Then a damping ratio of at least 0.05 would be guaranteed for every operating point provided

$$c_1 \geq \frac{\sqrt{m_1 k_{\max}}}{10}.$$

This formula shows precisely how much viscous damping is required to meet a damping ratio performance requirement.

Our main contribution is to show that an analogous expression can be used to lower bound the damping ratios of eq. (2) even when coupling exists. This allows us to obtain damping ratio bounds valid for large, coupled networks, on the basis of locally defined parametrised inequalities. In Section II-B we will show how to use this to verify that a swing equation power system model is sufficiently well damped for all its operating points, independently of its size, and without global knowledge of its interconnection topology.

Our approach is based on lower bounding the damping ratios of the solutions to eq. (2) using a GEVP of half the size. This constitutes the main theoretical contribution of the paper, and is presented as Theorem 1 in Section II-A. This reduced order GEVP is particularly amenable to analysis

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[1]This is in line with the usual control theoretic notion of the damping ratio of a pole, see e.g. [2]. For an extensive discussion of damping, see [3].

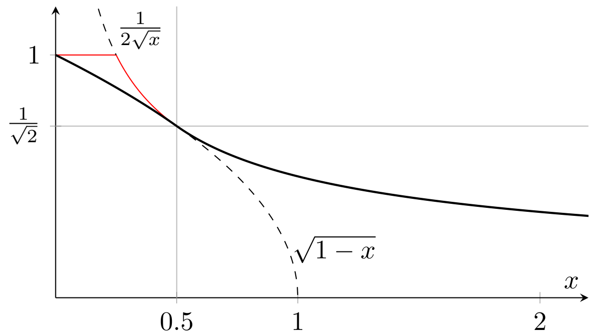

Fig. 1. Plot of $\zeta(x)$. For comparison the damping curve that would be followed if eq. (3) held with $x = m_i k_{ii}/c_i^2$ is shown in red.

with many well established eigenvalue bounds, which is how we arrive at the aforementioned bound for power system models in Section II-B.

II. Results

A. Damping Ratio Bounds Based on a Reduced Order GEVP

In this section we show that damping ratio of every solution to eq. (2) can be lower bounded based on the solution of a closely related GEVP of half the size. To state our result, we need to define two functions. The first is the extension of the spectral radius to the generalised eigenvalue problem, as given by

$$\rho(A, E) := \sup_{\lambda \in \mathbb{C}, v \in \mathbb{C}^n} |\lambda|, \text{ subject to } Av = \lambda E v.$$

Observe that this quantity is equal to the absolute value of the largest generalised eigenvalue of the pair (A, E).

The second is the function

$$\zeta(x) := \begin{cases} \sqrt{1-x} & \text{if } 0 \leq x \leq 1/2, \\ \frac{1}{2\sqrt{x}} & \text{otherwise,} \end{cases}$$

which is sketched in Figure 1.

The following theorem, which is the main theoretical contribution of the paper, shows that the modes of the mass-spring-damper network have damping ratios greater than or equal to $\zeta\left(\rho\left(MK, C^2\right)\right)$.

Theorem 1: If $M, C, K \in \mathbb{R}^{n\times n}$ are positive semi-definite, where in addition M and C are diagonal, then for every $s \notin \{0, \infty\}$ that satisfies the GEVP

$$\begin{bmatrix} -C & -K \\ I & 0 \end{bmatrix} v = s \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} v,$$

$-\cos\angle s \geq \zeta\left(\rho\left(MK, C^2\right)\right)$.

The proof of the theorem is given at the end of the section. However we first give a simple numerical example to illustrate the meaning of the theorem statement, and a remark commenting on the interpretation of the solutions to the GEVP with $s \in \{0, \infty\}$.

Example 1: Let

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \; K = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Solving eq. (2) gives the generalised eigenvalues

$$\{0, -0.625 \pm 0.927j, \infty\}.$$

The damping ratio is not defined for $\{0, \infty\}$ (c.f. the remark below), but evaulating the damping ratio for the remaining mode gives 0.559. Solving the GEVP associated with the generalised spectral radius shows that

$$\rho\left(MK, C^2\right) = 1.$$

Therefore the lower bound from Theorem 1 is $\zeta(1) = 0.5$. $\diamondsuit$

Remark 1: If for any i, $m_i = 0$, then the GEVP in eq. (2) will have solutions at infinity. In a similar fashion, the order of the differential equation in eq. (1) will reduce, since the $\ddot{q}_i$ term will vanish. In fact, these two phenomena are in direct correspondence, and every solution at infinity to the GEVP corresponds to a reduction in order of the ODE. Therefore it is of no consequence that the eigenvalues at infinity are not covered by Theorem 1. Similarly, eigenvalues on the real axis correspond to non-oscillatory modes, and therefore the omission of $s = 0$ is of no consequence. Note that the only reason these points are not covered by the theorem statement is because their argument is not defined.

Proof: A complex number s is a solution of eq. (2) if and only if there exists a non-zero $x \in \mathbb{C}^n$ such that

$$\left(Ms^2 + Cs + K\right)x = 0. \tag{4}$$

We will prove the result by showing that given any $s \notin \{0, \infty\}$ such that

$$0 \leq \angle s \leq \frac{\pi}{2} + \theta, \tag{5}$$

for eq. (4) to hold it is necessary that

$$\theta \leq \arcsin\left(\zeta\left(\rho\left(MK, C^2\right)\right)\right).$$

Since the damping ratio is defined to be $-\cos(\angle s)$ and the solutions to eq. (2) must come in conjugate pairs, this guarantees that every solution to eq. (2) has damping ratio greater than or equal to $\zeta\left(\rho\left(MK, C^2\right)\right)$.

The proof is based on the separating hyperplane theorem, and so we begin by manipulating eq. (4) into a condition on two subsets of the complex plane. Assume for now that M, C, K are positive definite, observe that

$$Ms^2 + Cs = \left(MC^{-2}\right)^{-1}\left(\left(MC^{-1}s\right)^2 + MC^{-1}s\right),$$

and define $S = MC^{-1}s$. Therefore eq. (4) is equivalent to

$$\left(\left(M^{-1}C^2\right)\left(S^2 + S\right) + K\right)x = 0.$$

Letting $\sqrt{\cdot}$ denote the positive definite square root of a positive definite matrix, rearranging the above shows that

$$\sqrt{M^{-1}}C\left(\left(S^2 + S + C^{-1}\sqrt{M}K\sqrt{M}C^{-1}\right)\right)C\sqrt{M^{-1}}x$$

equals zero. Multiplying on the left by $\left(S^2+S\right)^{-1}\sqrt{M}C^{-1}$ and rearranging gives

$$C\sqrt{M^{-1}}x = -\left(S^2+S\right)^{-1}C^{-1}\sqrt{M}K\sqrt{M}C^{-1}C\sqrt{M^{-1}}x.$$

If we then let $C\sqrt{M^{-1}}x = \left(C^{-1}\sqrt{M}K\sqrt{M}C^{-1}\right)^{-1}y$, we get that

$$\left(C^{-1}\sqrt{M}K\sqrt{M}C^{-1}\right)^{-1}y = \left(S^2+S\right)^{-1}y$$

Multiplying on the left by y^* and dividing by y^*y gives

$$\frac{y^*\left(S^2+S\right)^{-1}y}{y^*y}=-\frac{y^*\left(C^{-1}\sqrt{M}K\sqrt{M}C^{-1}\right)^{-1}y}{y^*y}. \quad (6)$$

It follows from the definition of the generalised spectral radius that for any $y\in\mathbb{C}^n$,

$$-\frac{y^*\left(C^{-1}\sqrt{M}K\sqrt{M}C^{-1}\right)^{-1}y}{y^*y}\in-\left[\frac{1}{\rho\left(KM,C^2\right)},\infty\right]=:\mathcal{I}.$$

Also observe that for any $y\in\mathbb{C}^n$,

$$\frac{y^*\left(S^2+S\right)^{-1}y}{y^*y}\in\text{Co}\left(\frac{1}{S_{ii}^2+S_{ii}}:i\in\{1,\dots n\}\right),$$

where Co denotes the convex hull of a set of points. Since $\angle S_{ii}=\angle S_{jj}$, a simple argument shows that

$$\text{Co}\left(\frac{1}{S_{ii}^2+S_{ii}}:i\in\{1,\dots n\},0\leq\angle S_{ii}\leq\frac{\pi}{2}+\theta\right)=\mathcal{S}_\theta,$$

where

$$\mathcal{S}_\theta=\text{Co}\left(\frac{1}{s^2+s}:\angle s\in\left\{0,\frac{\pi}{2}+\theta\right\}\right).$$

Therefore the two terms in eq. (6) lie in the convex sets $\mathcal{I}$ and $\mathcal{S}_\theta$. Consequently an s that satisfies eq. (5) can be a solution to eq. (2) only if these two sets intersect. We proceed to find the smallest θ such that this can happen by constructing a separating hyperplane between them. The idea behind this construction is illustrated in Figure 2. In this figure the blue line shows the separating hyperplane, the cyan curves the contours

$$\left\{\frac{1}{s^2+s}:s=re^{j\phi},r\geq0\right\}$$

for different values of $0\leq\phi\leq\frac{\pi}{2}+\theta$, the red curve the boundary of $\mathcal{S}_\theta$, and the green line $\mathcal{I}$. Consider now the function

$$f\left(s,k,\psi\right)=\text{Re}\left(\left(1+j\cot\psi\right)\left(\frac{1}{s^2+s}+k\right)\right).$$

This function has the effect of applying a shift of length k and rotation of ψ to the set $\mathcal{S}_\theta$. Therefore $\mathcal{S}_\theta$ 'lies to the right' of the hyperplane in Figure 2 if and only if

$$\left(\forall s:\angle s\in\left\{0,\frac{\pi}{2}+\theta\right\}\right),\ f\left(s,k,\psi\right)\geq0. \quad (7)$$

We will now show how to construct k and ψ as a function of θ in order to satisfy the above. The details are algebraically messy, so we include only the essential steps. We will first simplify the requirement in eq. (7). A little algebraic perseverance shows that

$$f\left(re^{j\phi},k,\psi\right)=k+\frac{\sin\left(\psi+\phi\right)+r\sin\left(\psi+2\phi\right)}{r\sin\psi\left(r^2+2r\cos\phi+1\right)}.$$

From this expression we see that provided $k\geq0$ and $-\frac{\pi}{2}\leq\psi\leq\frac{\pi}{2}$, $f\left(s,k,\psi\right)\geq0$ for all s such that $\angle s=0$. Therefore to verify eq. (7) we need only consider the case that $\angle s=\frac{\pi}{2}+\theta$. It can then be shown that

$$f\left(re^{j\left(\frac{\pi}{2}+\theta\right)},4\sin^2\theta,\theta\right)=\frac{\left(2r\sin\theta-1\right)^2\left(r\sin\theta+\cos2\theta\right)}{r\sin\theta\left(r^2-2r\sin\theta+1\right)},$$

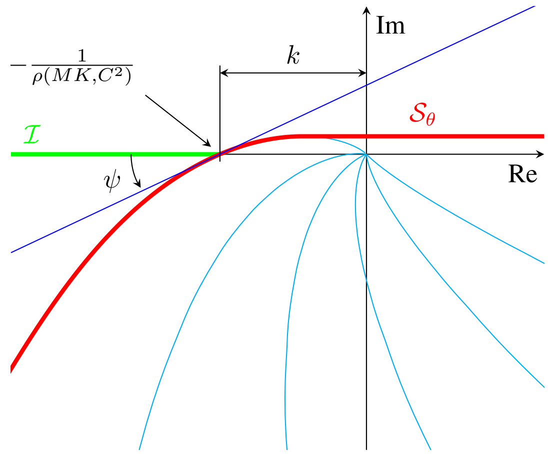

Fig. 2. Sketch of the separating hyperplane argument used in the proof of Theorem 1.

which is non-negative for all $r\geq0$ if and only if $\frac{\pi}{4}\geq\theta\geq0$. It can also be shown that

$$f\left(re^{j\left(\frac{\pi}{2}+\theta\right)},\sec^2\theta,\frac{\pi}{2}-\theta\right)=\frac{\left(r-\sin\theta\right)^2}{\cos^2\theta\left(r^2-2r\sin\theta+1\right)},$$

which is always non-negative. It then follows that if

$$\frac{1}{\rho\left(MK,C^2\right)}\geq\begin{cases}4\sin^2\theta & \text{if } 0\leq\theta\leq\frac{\pi}{4}\\ \sec^2\theta & \text{otherwise,}\end{cases}$$

then there exists a separating hyperplane between $\mathcal{I}$ and $\mathcal{S}_\theta$. It is a simple matter to verify that this is equivalent to checking that

$$\theta\leq\arcsin\left(\zeta\left(\rho\left(MK,C^2\right)\right)\right),$$

which completes the proof for the positive definite case.

To extend the above argument to the semi-definite case, let $M_\epsilon=M+\epsilon I$, with analogous expressions for C_ϵ and K_ϵ, and consider the perturbed generalised eigenvalue problem

$$\begin{bmatrix}-C_\epsilon & -K_\epsilon\\ I & 0\end{bmatrix}v=s\begin{bmatrix}M_\epsilon & 0\\ 0 & I\end{bmatrix}v. \quad (8)$$

The proof for the positive definite case therefore shows that the damping ratio of every solution to the above is greater than or equal to $\zeta\left(\rho\left(M_\epsilon K_\epsilon,{C_\epsilon}^2\right)\right)$. Since the location of the eigenvalues of eq. (8) with a finite limit vary continuously in ϵ [6], the damping ratios of the solutions to eq. (2) are greater than or equal to

$$\lim_{\epsilon\to0}\zeta\left(\rho\left(M_\epsilon K_\epsilon,{C_\epsilon}^2\right)\right),$$

from which the result immediately follows. ■

B. Decentralised Damping Ratio Guarantees for Power Systems

Consider the mechanical network in Figure 3. This model obeys the differential equations

$$m_i\ddot{\theta}_i+c_i\dot{\theta}+\sum_{j\neq i}^{n}\tilde{k}_{ij}\sin\left(\theta_i-\theta_j\right)=f_i. \quad (9)$$

In the above $\theta_i\left(t\right)$ denotes the angular position of the i-th mass on the circle, $m_i\geq0$ its mass, $c_i>0$ a viscous damping term, and f_i an external force applied to it

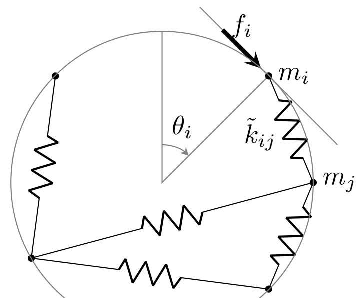

Fig. 3. Mechanical analogue of the swing equation model.

tangentially to the circle. Each $\tilde{k}_{ij} \geq 0$ denotes the spring constant of the spring connecting the *i*-th and *j*-th masses.

This is the mechanical analogue of the 'swing equation' power system model [7]. The masses are analogous to the system buses, and if $m_i > 0$, then there is a synchronous generator connected at the *i*-th bus. The springs correspond to the transmission lines, and the damping coefficients either frequency dependent loads or other sources of power dissipation. The externally applied forces correspond to equilibrium levels of power generation and consumption.

When operating around an equilibrium point $(\dot{\bar{\theta}}, \bar{\theta})$, the linearisation of eq. (9) is given by

$$m_i \ddot{q}_i + c_i \dot{q} + \sum_{j \neq i}^{n} \tilde{k}_{ij} \cos\left(\bar{\theta}_i - \bar{\theta}_j\right)(q_i - q_j) = 0.$$

In the above the variables q_i and $\dot{q}_i$ denote the deviations in θ_i and $\dot{\theta}_i$ from their equilibrium values. The difficulty in applying direct methods to compute the damping ratios of the modes of the power system model is now apparent. Since the operating point has the effect of scaling the spring constants by $\cos\left(\bar{\theta}_i - \bar{\theta}_j\right)$, if we were to evaluate the damping ratio directly, we would have to redo this calculation in response to every change in operating point. This is an issue because the operating points of power systems are changing all the time based on the precise levels of production and consumption, and are far too numerous to enumerate. And whilst we could attempt to perform this calculation to find the current damping ratio in real time, this does little to help us predict, and therefore design to prevent, scenarios in which unacceptable levels of damping arise.

This can be directly addressed with Theorem 1. Under the following very mild assumption, the linearised power system model is of precisely the form in eq. (2) with $k_{ii} = 0$ and $k_{ij} = \tilde{k}_{ij} \cos\left(\bar{\theta}_i - \bar{\theta}_j\right)$.

Assumption 1: At equilibrium, the angle difference $\left|\bar{\theta}_i - \bar{\theta}_j\right|$ across each spring is less than $\pi/2$.

This assumption is required to make $k_{ij} \geq 0$. However it is essentially without loss of generality, since thermal and voltage drop limitations for transmission lines preclude load angles anywhere near $\pi/2$ [1]. Theorem 1 then tells us that the damping ratio is lower bounded by $\zeta\left(\rho\left(MK, C^2\right)\right)$. While this in itself doesn't help address the issue at hand, we observe that since $c_i > 0$, $\rho\left(MK, C^2\right)$ is equal to the spectral radius[2] of the matrix $C^{-2}MK$. There exist many good upper bounds for this quantity [8]. Even a relatively crude argument using Gershgorin discs shows that

$$\begin{aligned} \rho\left(MK, C^2\right) &\leq \max_i \frac{m_i}{c_i^2} \sum_{j \neq i}^{n} 2\tilde{k}_{ij} \cos\left(\bar{\theta}_i - \bar{\theta}_j\right), \\ &\leq \max_i \frac{m_i}{c_i^2} \sum_{j \neq i}^{n} 2\tilde{k}_{ij}. \end{aligned}$$

Since the function $\zeta(x)$ is monotonically decreasing (c.f. Figure 1), we arrive at the following lower bound, which holds for every mode of the power system model:

$$-\cos\angle s \geq \min_i \zeta\left(\frac{m_i \kappa_i}{c_i^2}\right), \text{ where } \kappa_i = \sum_{j \neq i}^{n} 2\tilde{k}_{ij}.$$

This gives us a locally defined parametrised inequality, that lower bounds the damping ratio, and is valid for every operating point satisfying Assumption 1. Furthermore, it can be seen from the definition of $\zeta(x)$ that if $\max_i m_i \kappa_i / c_i^2 \geq 1/2$, which is very likely in applications with low levels of viscous damping such as power systems, then

$$\zeta\left(\frac{m_i \kappa_i}{c_i^2}\right) = \frac{c_i}{2\sqrt{m_i \kappa_i}}.$$

Therefore the network bound is in perfect agreement with the bound for scalar systems in eq. (3). This means that if for each i,

$$c_i \geq \frac{\sqrt{m_i \kappa_i}}{10},$$

then a damping ratio of at least 0.05 would be guaranteed for every mode in the network, and every operating point satisfying Assumption 1.

III. Conclusions

It has been shown that the damping ratio of networks of masses and springs can be lower bounded as a function of $\rho\left(MK, C^2\right)$. This was used to derive a decentralised criterion which, if satisfied, guarantees that all the modes of a swing equation power system model are sufficiently well damped, independently of its operating point and size.

References

[1] P. Kundur, *Power System Stability and Control*. McGraw-Hill Professional, 1994.

[2] K. J. Aström and R. M. Murray, *Feedback systems: an introduction for scientists and engineers*. Princeton university press, 2010.

[3] P. Lancaster, *Lambda-Matrices and Vibrating Systems*. Pergamon, 2014.

[4] I. Gohberg, P. Lancaster, and L. Rodman, *Matrix Polynomials*. Society for Industrial and Applied Mathematics, 1982.

[5] G. Rogers, *Power System Oscillations*. Springer, 1999.

[6] G. W. Stewart, "Gershgorin theory for the generalized eigenvalue problem $Ax = \lambda Bx$," *Mathematics of Computation*, vol. 29, no. 130, pp. 600–606, 1975.

[7] F. Dörfler, M. Chertkov, and F. Bullo, "Synchronization in complex oscillator networks and smart grids," *Proceedings of the National Academy of Sciences*, vol. 110, no. 6, pp. 2005–2010, 2013.

[8] R. Horn and C. Johnson, *Topics in Matrix Analysis*. Cambridge University Press, 1991.

[2] This is because if a matrix E is invertible, then the generalised eigenvalues of (A, E) equal the eigenvalues of $E^{-1}A$.