Bandwidth constraints for passive superluminal propagation through metamaterials

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Abstract

Superluminal transmission of electromagnetic waves is usually observed in a narrow bandwidth range and the velocity outside this range is subluminal. In this paper, it is shown that the transmission coefficient for superluminal propagation through a periodic metamaterial structure satisfies a sum rule. The sum rule and its corresponding physical bound relate frequency regions with a phase velocity above an arbitrary threshold with the thickness of the slab. The theoretical results are illustrated with numerical examples.

1. Introduction

Superluminal propagation of electromagnetic waves refers to propagation faster than the speed of light in free space [1–6]. It is common to distinguish between group and phase velocity. The phase velocity of electromagnetic waves in waveguides and periodic structures can exceed the speed of light in free space. Material that supports superluminal propagation are dispersive [4–6] and causality pose some restrictions on the frequency dependence [7–9]. These restriction are not severe for active material models [8] but very powerful for passive models. The bounds in [9] show that the deviation of the refractive index from the desired superluminal value is proportional to the bandwidth [9]. The material models are very useful but not sufficient to fully understand inhomogeneous metamaterial slabs. It is also known that the use of scattering data to determine equivalent material parameters [10] is increasingly problematic as the wavelength approaches the periodicity.

In this paper, superluminal propagation through arbitrary reciprocal periodic metamaterial slabs is analyzed by comparison with the transmission through an idealized non-dispersive superluminal slab. The use of transmission coefficients offers a tool to analyze inhomogeneous structures without defining an effective refractive index. It is shown that the bandwidth is inversely proportional to $1 - n_{\mathrm{m}}$, where $n_{\mathrm{m}} < 1$ is the refractive index of the desired ideal homogeneous superluminal slab.

The bounds are based on integral identities for Herglotz functions and follow the general approach introduced in [11]. Similar approaches have previously been successfully applied to derive bounds on lossless matching networks [12], radar absorbers [13], high impedance surfaces [14], scattering and absorption of electromagnetic waves [15–17], antennas [18], extra ordinary transmission [19], and temporal dispersion of metamaterials [9].

This paper is organized as follows. Superluminal transmission through periodic slabs are analyzed in Sec. 2. Numerical examples of a homogeneous Drude dispersive slab and a periodic wire medium are presented in Sec. 3. Two appendices follow the conclusions in Sec. 4. First, sum rules for Herglotz functions and the Herglotz pulse function are reviewed in App. A. Transmission coefficients are analyzed in App. B.

2. Superluminal transmission

An ideal superluminal homogeneous slab has refractive index $n_{\mathrm{m}} < 1$, see Fig. 1. There are refractive indices with $n(\omega_0) \approx n_{\mathrm{m}}$ for a range of (angular) frequencies around ω_0. It is known that these refractive indices are dispersive (frequency dependent) and it is shown that dispersion restricts passive material models in many ways [8, 9]. In [9], it is shown that the temporal dispersion of the refractive index is restricted by

$$\max_{\omega_1\leq\omega\leq\omega_2} |\omega(n(\omega) - n_\infty)| \geq \nu(n_\infty - n_{\mathrm{m}})(\omega_2 - \omega_1), \quad (1)$$

where $n_\infty = n(\infty)$ is the high-frequency limit [9] and the parameter ν quantifies the losses in the material and increases continuously from the lossy to the lossless case [9] with the extreme values

$$\nu = \begin{cases} 1/2 & \text{lossy} \\ 1 & \text{lossless.} \end{cases} \quad (2)$$

The bound (1) shows that the deviation of the refractive index $n(\omega)$ from the desired superluminal refractive index n_{m} is proportional to the fractional bandwidth, B, and $n_\infty - n_{\mathrm{m}}$. The high-frequency limit n_∞ is in many cases considered to be unity [7].

Consider the synthesis of a periodic metamaterial designed for superluminal transmission. The bound (1) is applicable if the material can be considered homogeneous and the high-frequency limit n_∞ is known. However, periodic metamaterials cannot in general be modeled as homogeneous materials and it is obvious that the material description is problematic for high frequencies where the wavelength is of the order of the material structure. It is also known that computations of material parameters [10], such as the permittivity and permeability, from scattering data can produce parameter values that are not consistent with

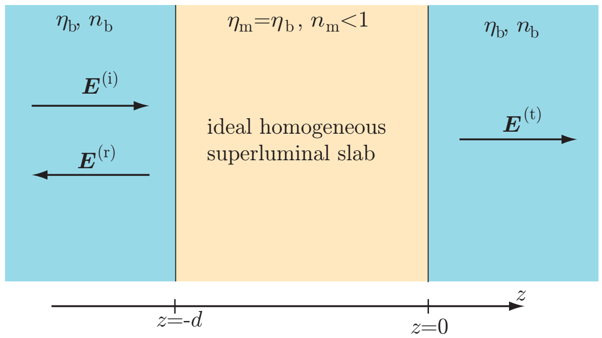

Figure 1: Transmission through a superluminal homogeneous slab with impedance $\eta_m = \eta_b$ and refractive index $n_m < 1$ in $z \in [-d, 0]$ immersed between two half spaces with impedance η_b and refractive index n_b.

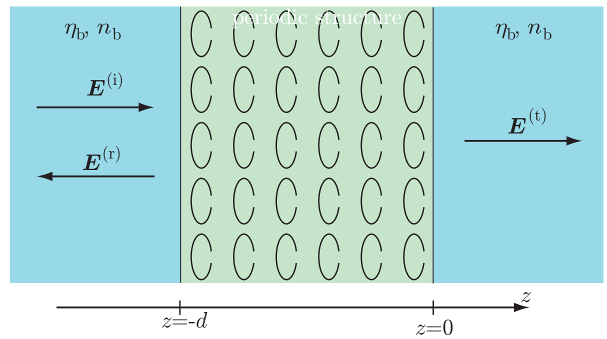

Figure 2: Transmission through a periodic structure in $z \in [-d, 0]$ immersed between two half spaces with impedance η_b and refractive index n_b.

passive material models. This implies that it is essential to investigate the properties of metamaterial slabs without referring to effective material parameters.

Here, we compare the transmission through arbitrary periodic metamaterial slabs with the transmission coefficient of the desired homogeneous superluminal slab, see Fig. 1. The transmitted field is $\boldsymbol{E}^{(t)} = T_m \boldsymbol{E}^{(i)}$, where the transmission coefficient for the slab in Fig. 1 is

$$T_m(k) = \frac{(1-r_0^2)e^{i(n_m-1)kd}}{1-r_0^2 e^{2in_m kd}} = e^{i(n_m-1)kd} \qquad (3)$$

as the impedance mismatch, $r_0 = (\eta_b - \eta_m)/(\eta_b + \eta_m) = 0$, where $k = \omega/c_0$ is the free space wavenumber and c_0 the speed of light in free space. Note that the transmission coefficient has a reversed phase, $n_m - 1 < 0$, compared to ordinary materials with refractive index $n \geq 1$.

Consider the synthesis of a periodic metamaterial with desired refractive index $n(k) \approx n_m < 1$ and impedance $\eta(k) \approx \eta_m$ for a range of wavenumbers around k. The metamaterial is placed between two slabs made from lossless non-dispersive materials with refractive index $n_b \geq 1$ and impedance $\eta_b = \eta_m$. Here, we assume that these materials exist. Note that they can be synthesized from ordinary materials with $\epsilon_r \geq 1$ and $\mu_r \geq 1$ over a sufficiently large bandwidth.

The transmitted field $\boldsymbol{E}^{(t)}(k, \boldsymbol{r})$ is expanded in a Fourier series (Floquet modes) [20]. It is only the lowest order mode that propagates for low frequencies. Let T denote the co-polarized component of the lowest order mode of the transmission dyadic [17]. Use that the periodic structure is composed of ordinary materials where the wavefront velocity is limited by the speed of light c_0. This means that the transmitted field depends causally with respect to the incident field, *i.e.*, the transmitted field cannot precede the incident field. The transmission coefficient is hence a holomorphic function of k in $\operatorname{Im} k > 0$. Moreover passivity imply that $|T(k)| \leq 1$ for $\operatorname{Im} k \geq 0$. This gives the representation $T(k) = B_p(k)e^{ih(k)}$, where $h(k)$ is a Herglotz function with the high-frequency asymptotic $h(k) \sim \alpha kd$ as $k \hat{\to} \infty$ for some $\alpha \geq 0$ and $B_p(k)$ is a Blaschke product, see [11] and App. A.

The metamaterial slab is designed to mimic the ideal superluminal slab with refractive index n_m and transmission coefficient T_m. It is possible to synthesize metamaterials with an effective refractive index $n(k_0) \approx n_m$ and transmission coefficient $T(k_0) \approx T_m(k_0)$ in narrow bandwidths around k_0. To circumvent the difficulties defining effective material parameters over large bandwidth, we analyze fundamental constraints on the transmission coefficient $T(k) \approx T_m(k)$. Consider the quotient

$$T(k)/T_m(k) = B_p(k)e^{i(h(k)+kd(1-n_m))}, \qquad (4)$$

where the bandwidth, $k_1 \leq k \leq k_2$, with $T(k)/T_m(k) \approx 1$ is determined from compositions with a pulse Herglotz function see App. A. The asymptotic low- and high-frequency expansions of the argument are

$$h(k)+kd(1-n_m) \sim \begin{cases} \mathcal{O}(k^{-1}) & \text{as } k \hat{\to} 0 \\ (\alpha + 1 - n_m)kd & \text{as } k \hat{\to} \infty \end{cases} \qquad (5)$$

giving the bound (30)

$$\max_{k_1 \leq k \leq k_2} |T(k) - T_m(k)| \geq \frac{(1-n_m)Bk_0 d}{2}\nu \qquad (6)$$

for fractional bandwidths $B \ll 1$, ν is given in (2), $k_0 = (k_1 + k_2)/2$, and $B = (k_2 - k_1)/k_0$. The equivalent bound on the fractional bandwidth is

$$B \leq B_{\text{bound}} = 2\frac{\max_{k_1 \leq k \leq k_2} |T(k) - T_m(k)|}{(1-n_m)k_0 d\nu}. \qquad (7)$$

It is illustrative to compare the bound (6) with (1). Replace ω in (1) with k and use an arbitrary small impedance mismatch, $r_0 = 0$, and $|k(n-1)| \ll 1$ to get

$$\max_{k_1 \leq k \leq k_2} |T(k) - T_m(k)| \geq (1-n_m)Bk_0 d\nu \qquad (8)$$

for $B \ll 1$. The results (6) and (8) differ by a factor of two. This is explained by the fact that the transmission coefficients of homogeneous slabs have no transmission zeros, *i.e.*, there are no Blaschke product in the representation

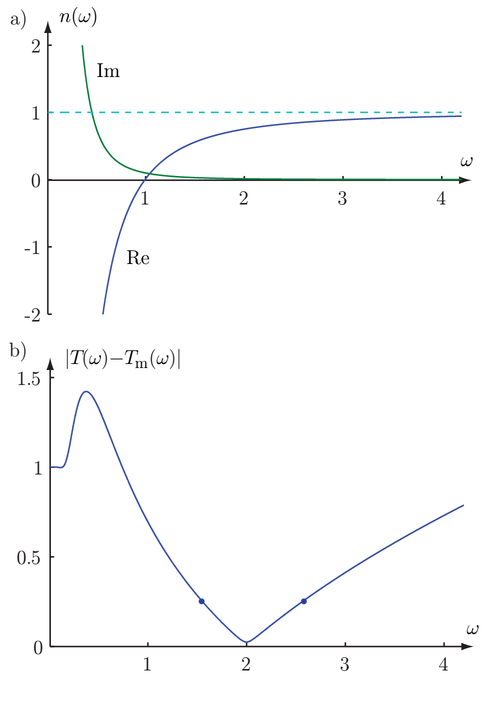

Figure 3: Drude dispersive model (9) with $\omega_\mathrm{p} = 1$ and $\omega_\mathrm{r} = 0.1$. b) difference $|T(k) - T_\mathrm{m}(k)|$ with $n_\mathrm{m} = 0.75$ giving $B \approx 0.49$ and $B/B_\mathrm{bound} \approx 0.50$ where the threshold $\max_{k\in\mathcal{B}} |T(k) - T_\mathrm{m}(k)| = 0.25$ is used.

$T(k) = \mathrm{e}^{\mathrm{i}h(k)}$. The bound is hence reduced to the analysis of $|h(k)| \leq \varDelta$ over a bandwidth $\mathcal{B}$. This is the case analyzed in [9] using the pulse Herglotz function. Note that $|h(k) + m2\pi| < \varDelta$ also adds a factor of two to the bound if $m \neq 0$.

3. Numerical examples

3.1. Drude dispersive slab

A homogeneous slab with Drude models for the permittivity and permeability is used to illustrate the bounds (6) and (8). Let

$$\epsilon(\omega) = \mu(\omega) = 1 + \frac{\omega_\mathrm{p}^2}{-\mathrm{i}\omega(-\mathrm{i}\omega + \omega_\mathrm{r})}, \tag{9}$$

where dimensionless units are used for the thickness $d/\mathrm{c}_0 = 1$, plasma frequency $\omega_\mathrm{p} = 1$ and the resonance frequency $\omega_\mathrm{r} = 0.1$, see Fig. 3a. The difference $|T(\omega) - T_\mathrm{m}(\omega)|$ is depicted in Fig. 3b for the refractive index $n_\mathrm{m} = 0.75$. It is observed that $T(\omega) \approx T_\mathrm{m}(\omega)$ and $n(\omega) \approx n_\mathrm{m}$ for $\omega = \omega_0 \approx 2$. The fractional bandwidth is $B \approx 0.49$ that is comparable to the bound (8) and a factor of two below (6), where the lossless cases are used. Note, that the bound (6) overestimates the bandwidth for the homogeneous slab since it has no transmission zeros and that it is used for $|h| \ll 1$.

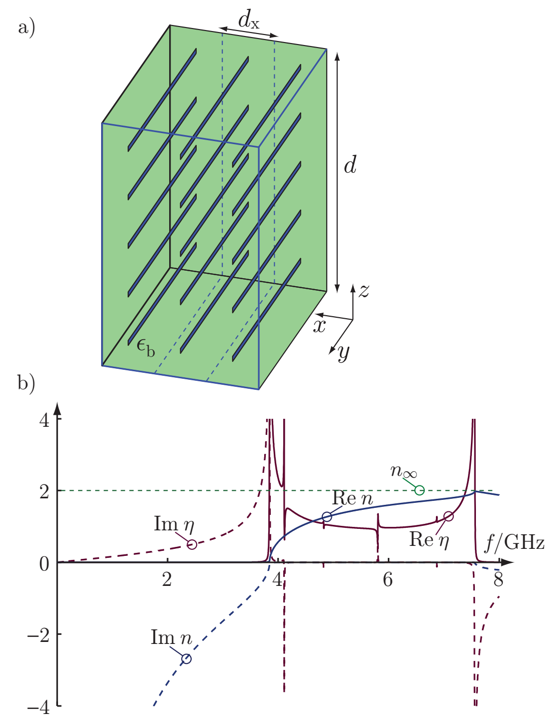

Figure 4: Transmission through a periodic wire medium. a) geometry. b) reconstructed refractive index, n, and relative impedance, η using an equivalent homogeneous slab [10].

3.2. Periodic wire medium

The transmission is also illustrated with a periodic structure composed of a wire medium, *i.e.*, infinite strips in the y-direction, see Fig. 4a. This is a classical structure that emulates a plasma [21] with a low-frequency permittivity of the form (9). The strips are aligned with the y-direction and periodic in the x-direction with the unit cell length $d_\mathrm{x} = 10\,\mathrm{mm}$. Five strips with width $1\,\mathrm{mm}$ are placed in the z-direction at distances $10\,\mathrm{mm}$. The strips are embedded in a non-magnetic homogeneous slab with permittivity $\epsilon_\mathrm{b} = 4$ and thickness $d = 50\,\mathrm{mm}$. This means that the speed of the wave front is limited by c_0/n_∞, where $n_\infty = \sqrt{\epsilon_\mathrm{b}} = 2$ is the index if refraction for high frequencies.

The transmission and reflection coefficient are computed for $f \leq 8\,\mathrm{GHz}$ using the F-solver in CST Microwave studio. The relative impedance η and refractive index n of an equivalent homogeneous slab using the inversion [10] are depicted in Fig. 4b. The low-frequency results resembles the permittivity from a non-magnetic Drude model (9). It is observed that $\eta \approx 1$ and $\mathrm{Im}\, n \approx 0$ for $4\,\mathrm{GHz} < f < 7.5\,\mathrm{GHz}$ giving a transmission $|T(\omega)| \approx 1$. The refractive index increases from 0 to $2 = n_\infty$ in this frequency range giving a phase velocity below c_0/n_∞. The difference $|T(\omega) - T_\mathrm{m}(\omega)|$ with $n_\mathrm{m} = 1.5 < n_\infty$ is depicted in Fig. 5. Here, it is observed that $T(\omega) \approx T_\mathrm{m}(\omega)$ for $\omega/(2\pi) = f_0 = 5.5\,\mathrm{GHz}$ in agreement with the reconstruction in Fig. 4b, where $n(f_0) \approx 1.5$ and $\eta(f_0) \approx 1$. The

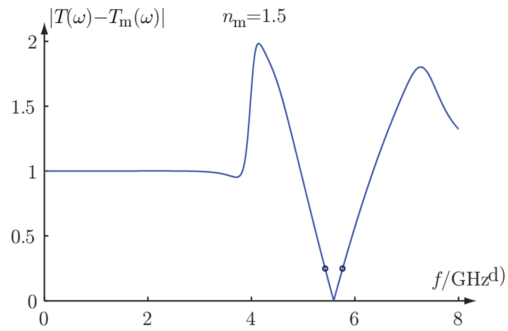

Figure 5: Transmission through a periodic wire medium. a) difference $|T(k) - T_{\mathrm{m}}(k)|$ with $n_{\mathrm{m}} = 1.5$, $B \approx 6\%$, and $B/B_{\mathrm{bound}} \approx 0.36$.

bound (7) is determined for $\max |T - T_{\mathrm{m}}| = 0.25$ giving

$$B \leq B_{\mathrm{bound}} = \frac{2\max_{k_1\leq k\leq k_2} |T - T_{\mathrm{m}}|}{(n_\infty - n_{\mathrm{m}})k_0 d} = \frac{1}{2(n_\infty - n_{\mathrm{m}})k_0 d}. \quad (10)$$

The result is $B \approx 6\%$ and $B/B_{\mathrm{bound}} \approx 0.36$. This is close to the result for the homogeneous slab (8). Also the reconstructed refractive index in Fig. 4b satisfies the Kramers-Kronig relations with a high-frequency response $n(\omega) \to n_\infty = 2$ as $\omega \hat{\to} \infty$. The results are hence restricted by the sharper bound in (8).

4. Conclusions

Bandwidth constraints on superluminal wave propagation through passive metamaterial slabs are analyzed in this paper. It is shown that the bandwidth is inversely proportional to $1 - n_{\mathrm{m}}$, where n_{m} is the refractive index of the desired superluminal slab. The derivation follows the approach in [11], where Herglotz functions are used to construct sum rules and physical bounds. The results resemble the bounds in [9] but differ by a factor of two due to the possible zeros of the transmission coefficient. Passivity is essential as it offers energy conservation used to bound the transmission coefficients by unity. It is also known that the temporal dispersion of causal but active material models is not very restrictive [8].

Acknowledgement

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A. Pulse Herglotz function

Herglotz functions, $h(z)$, are holomorphic in the upper half plane $\operatorname{Im} z > 0$ and map the upper half plane into itself, *i.e.*, $\operatorname{Im} h(z) \geq 0$, see [11, 22]. Here, we also restrict the analysis to symmetric Herglotz function $h(z) = -h^*(-z^*)$. They have at most linear growth as $z \hat{\to} \infty$ and at most a simple pole as $z \hat{\to} 0$, where $\hat{\to}$ denotes limits for $0 < \alpha \leq \arg(z) \leq \pi - \alpha$. Their asymptotic expansions are hence of the form

$$h(z) = \sum_{n=0}^{N_0} a_{2n-1} z^{2n-1} + o(z^{2N_0-1}) \quad \text{as } z \hat{\to} 0 \quad (11)$$

and

$$h(z) = \sum_{n=0}^{N_\infty} b_{1-2n} z^{1-2n} + o(z^{1-2N_\infty}) \quad \text{as } z \hat{\to} \infty \quad (12)$$

for some $N_0 \geq 0$ and $N_\infty \geq 0$, see [11]. The expansions (11) and (12) guarantees that $\operatorname{Im}(x)$ satisfy the integral identities

$$\frac{2}{\pi}\int_0^\infty \frac{\operatorname{Im}\{h(x)\}}{x^{2n}}\,\mathrm{d}x = \begin{cases} -b_{2n-1} & n < 0 \\ a_{-1} - b_{-1} & n = 0 \\ a_1 - b_1 & n = 1 \\ a_{2n-1} & n > 1, \end{cases} \quad (13)$$

where $n = 1 - N_\infty, ..., N_0$, see [11] for details. Note that a simplified notation is used in this paper where the limits in (13) are dropped, *i.e.*, the integrand in (13) is the limit $h(x + \mathrm{i}y)$ as $y \to 0$.

Composition of Herglotz functions is a powerful method to construct new Herglotz functions and integral identities (13). A pulse function is used in [9, 11] to bound the amplitude of Herglotz functions. Here, the pulse function is generalized to bound the variation of Herglotz functions from a constant. Consider the Herglotz function generated by a constant $\operatorname{Im} h(x)$ in $|x \mp x_0| < \varDelta$, *i.e.*,

$$\begin{aligned} h_{x_0,\varDelta}(z) &= \frac{1}{\pi}\int_{|\xi\pm x_0|\leq\varDelta} \frac{1}{\xi - z}\,\mathrm{d}\xi \\ &= \frac{1}{\pi}\ln\frac{z - x_0 - \varDelta}{z - x_0 + \varDelta} + \frac{1}{\pi}\ln\frac{z + x_0 - \varDelta}{z + x_0 + \varDelta} \\ &= \frac{1}{\pi}\ln\frac{(z-\varDelta)^2 - x_0^2}{(z+\varDelta)^2 - x_0^2} \sim \begin{cases} \frac{4}{\pi}\frac{\varDelta}{x_0^2-\varDelta^2} z \text{ as } z \to 0 \\ -\frac{4}{\pi}\frac{\varDelta}{z} \text{ as } z \to \infty, \end{cases} \end{aligned} \quad (14)$$

where $x_0 > \varDelta$, see Fig. 6. The generalized pulse function has the properties

$$\begin{cases} \operatorname{Im} h_{x_0,\varDelta}(z) \leq 1 \\ \operatorname{Im} h_{x_0,\varDelta}(x) = 1 \quad \text{for } |x \mp x_0| < \varDelta \\ \operatorname{Im} h_{x_0,\varDelta}(x) = 0 \quad \text{for } |x \mp x_0| > \varDelta \\ \operatorname{Im} h_{x_0,\varDelta}(z) \geq 1/2 \quad \text{for } |z \mp x_0| < \varDelta. \end{cases} \quad (15)$$

Note that the case $x_0 < \varDelta$ reduces to the pulse Herglotz function [9] with $\varDelta \to x_0 + \varDelta$. It has similar properties as in (15) but the high-frequency asymptotic is smaller, *i.e.*, $2(x_0 + \varDelta) < 4\varDelta$.

B. Passive transmission coefficients

Passive transmission coefficients can be decomposed as

$$T(k) = B_{\mathrm{p}}(k)\mathrm{e}^{\mathrm{i}h(k)}, \quad (16)$$

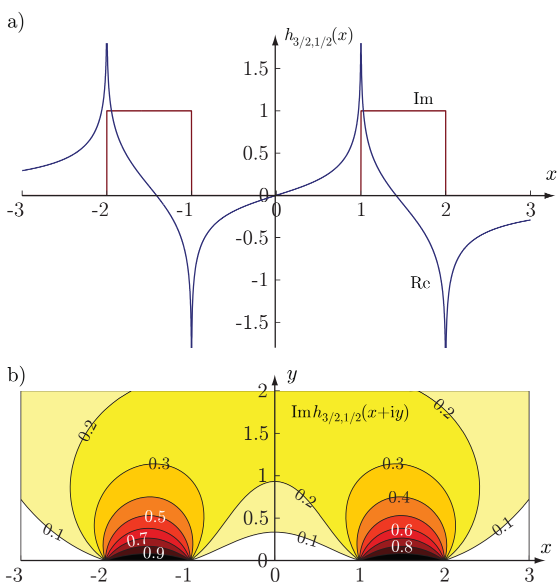

Figure 6: The generalized pulse Herglotz function $h_{3/2,1/2}$ with $x_0 = 3/2$ and $\mathit{\Delta} = 1/2$. a) real and imaginary parts of $h_{3/2,1/2}(x)$. b) contour plot of the imaginary part as function of $z = x + \mathrm{i}y$.

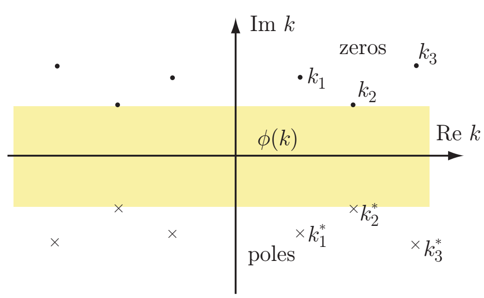

Figure 7: Zeros and poles of a Blaschke product (17) and the region of definition for the argument $\phi(k)$ (shaded region).

where $B_{\mathrm{p}}(k)$ is a Blaschke product and $h(k)$ is a Herglotz function [11]. The symmetry $T(k) = T^*(-k^*)$ implies the symmetries $B_{\mathrm{p}}(k) = B_{\mathrm{p}}^*(-k^*)$ and $h(k) = -h^*(-k^*)$. Blaschke products have the amplitude $|B_{\mathrm{p}}(k)| = 1$ for $k \in \mathbb{R}$ and can be constructed from the zeros, k_n, of $T(k)$ in $\mathrm{Im}\, k > 0$, *i.e.*, $T(k_n) = 0$. They have the representation

$$B_{\mathrm{p}}(k) = \prod_n \frac{k - k_n}{k - k_n^*}, \quad \mathrm{Im}\, k_n > 0, \tag{17}$$

where the symmetry $B_{\mathrm{p}}^*(-k^*) = B_{\mathrm{p}}(k)$ implies that if k_n is a zero than so is $-k_n^*$.

Blaschke products can also be written as functions $B_{\mathrm{p}}(k) = \mathrm{e}^{\mathrm{i}\phi(k)}$, where the argument $\phi(k) = \phi^*(k^*)$ is an odd function for $k \in \mathbb{R}$, *i.e.*, $\phi(-k) = -\phi(k)$. The argument is determined from

$$\phi(k) = -\mathrm{i} \int_0^k \frac{B_{\mathrm{p}}'(\kappa)}{B_{\mathrm{p}}(\kappa)} \,\mathrm{d}\kappa \tag{18}$$

that shows that the argument is monotone for $k \in \mathbb{R}$, *i.e.*,

$$\begin{aligned}\phi'(k) = -\mathrm{i}\frac{B_{\mathrm{p}}'(k)}{B_{\mathrm{p}}(k)} &= \sum_n \frac{2\,\mathrm{Im}\, k_n}{(k - k_n^*)^2}\frac{k - k_n^*}{k - k_n} \\ &= \sum_n \frac{2\,\mathrm{Im}\, k_n}{(k - \mathrm{Re}\, k_n)^2 + (\mathrm{Im}\, k_n)^2} \geq 0. \end{aligned}\tag{19}$$

The function $\phi(k)$ is not holomorphic in the entire upper half plane, $\mathrm{Im}\, k > 0$, but it is holomorphic in a region around $k \in \mathbb{R}$. Its Taylor series expansion around a point $k_0 \in \mathbb{R}$ is

$$\phi(k) = \sum_{n=0}^{\infty} \alpha_n (k - k_0)^n, \tag{20}$$

where the convergence radius is related to the distance $\min_n |k_n - \mathrm{e}^{\mathrm{i}\phi(k_0)}|$. The inequality (19) ensures that the linear term $\alpha_1 \geq 0$. This expresses the transmission coefficient $T(k) = B_{\mathrm{p}}(k)\mathrm{e}^{\mathrm{i}h(k)}$ as

$$T(k) = \mathrm{e}^{\mathrm{i}(h(k)+\alpha_0+\alpha_1 k-\alpha_1 k_0)}\mathrm{e}^{\mathrm{i}\sum_{m=2}^{\infty}\alpha_m (k-k_0)^m} \tag{21}$$

for $\mathrm{Im}\, k > 0$ and $|k - k_0| < \mathrm{Im}\, k_n$. It has a similar expansion around $k = -k_0$.

The results in this paper rely on the analysis of transmission coefficients that have $T(k) \approx 1$ around $k = \pm k_0$ and are generated by Herglotz functions $h(k)$ having the high-frequency asymptotic $h(k) \sim b_1 k$ as $k \hat{\to} \infty$. Note that we analyze the modified transmission coefficient (4). Consider the difference $1 - T(k)$ around $k = k_0$. Use (21), to get

$$1 - T(k) = \big(1 - \mathrm{e}^{\mathrm{i}(h(k)+\alpha_1 k+\alpha_0-\alpha_1 k_0)}\big)\big(1 + \mathcal{O}(B^2)\big) \tag{22}$$

as $B \to 0$. The difference is hence governed by the variation of $\psi(k) = h(k) + \alpha_1 k + \alpha_0 - \alpha_1 k_0$ around $k = k_0$, *i.e.*, one needs to have a function

$$h(k) + \alpha_1 k \approx \begin{cases} -\alpha_0 + \alpha_1 k_0 & \text{for } k \approx k_0 \\ +\alpha_0 - \alpha_1 k_0 & \text{for } k \approx -k_0. \end{cases} \tag{23}$$

This is a Herglotz function and its minimal variation can be determined by composition of $h(k) + \alpha_1 k$ with the pulse Herglotz function $h_{-\alpha_0+\alpha_1 k_0, \mathit{\Delta}}$. This new Herglotz function has the asymptotic expansions

$$h_{-\alpha_0+\alpha_1 k_0,\mathit{\Delta}}\big(h(k) + \alpha_1 k\big) \sim \begin{cases} \mathcal{O}(1) \text{ as } k \hat{\to} 0 \\ -\frac{4\mathit{\Delta}}{\pi k (b_1 + \alpha_1)} \text{ as } k \hat{\to} \infty \end{cases} \tag{24}$$

that follows from the expansion $h(k) + \alpha_1 k \sim (b_1 + \alpha_1)k$ as $k \hat{\to} \infty$. It satisfies the $n = 0$ sum rule (13)

$$\int_0^{\infty} \mathrm{Im}\{h_{-\alpha_0+\alpha_1 k_0,\mathit{\Delta}}\big(h(k) + \alpha_1 k\big)\}\,\mathrm{d}k = \frac{2\mathit{\Delta}}{b_1 + \alpha_1}. \tag{25}$$

The integral is bounded from below by its minimum value over a bandwidth $\mathcal{B} = [k_1, k_2]$ times the bandwidth, *i.e.*,

$$\begin{aligned} &Bk_0 \min_{k\in\mathcal{B}} \mathrm{Im}\{h_{-\alpha_0+\alpha_1 k_0,\mathit{\Delta}}\big(h(k) + \alpha_1 k\big)\} \\ &\qquad \leq \frac{2\mathit{\Delta}}{b_1 + \alpha_1} \leq \frac{2\mathit{\Delta}}{b_1}, \end{aligned}\tag{26}$$

where $B = (k_2 - k_1)/k_0$ is the fractional bandwidth and $k_0 = (k_1 + k_2)/2$ is the center wavenumber. Now, consider a bandwidth such that

$$\max_{k\in\mathcal{B}} |h(k) + \alpha_1 k + \alpha_0 - \alpha_1 k_0| = \varDelta. \tag{27}$$

Use (26) and the properties (15) to get

$$\max_{k\in\mathcal{B}} |h(k) + \alpha_1 k + \alpha_0 - \alpha_1 k_0| \geq \frac{b_1 B k_0}{2}\nu, \tag{28}$$

where ν is given in (2) and $\operatorname{Im} h(k) = 0$ in the lossless case.

For the superluminal case, we consider $|1 - T(k)|$. Set $\psi = \psi_\mathrm{r} + \mathrm{i}\psi_\mathrm{i} = h(k) + \alpha_1 k + \alpha_0 - \alpha_1 k_0$ and use

$$|1 - \mathrm{e}^{\mathrm{i}\psi}| \approx |\psi| + \mathcal{O}(|\psi|^2) \quad \text{as } |\psi| \to 0 \tag{29}$$

as $|\psi| \to 0$. This gives the final estimates

$$\begin{aligned}
&\max_{k\in\mathcal{B}} |1 - T(k)| \\
&\quad= \max_{k\in\mathcal{B}} |1 - \mathrm{e}^{\mathrm{i}(h(k)+\alpha_1 k+\alpha_0-\alpha_1 k_0)}|\big(1 + \mathcal{O}(B^2)\big) \\
&\quad\geq |h(k) + \alpha_1 k + \alpha_0 - \alpha_1 k_0|\big(1 + \mathcal{O}(B^2)\big) \\
&\quad\geq \frac{b_1 B k_0}{2}\nu + \mathcal{O}(B^3) \quad \text{as } B \to 0.
\end{aligned} \tag{30}$$

References

[1] R. Tomaschitz, "Refraction angles and transmission rates of polarized superluminal radiation," *Physica B: Condensed Matter*, vol. 404, no. 8-11, pp. 1383–1393, 2009.

[2] D. Sokolovski and R. Mayato, "Superluminal transmission via entanglement, superoscillations, and quasi-dirac distributions," *Physical Review A*, vol. 81, no. 2, p. 022105, 2010.

[3] S. Ojha, K. Thapa, and S. Singh, "Superluminal propagation in plasma photonic band gap materials," *Optik-International Journal for Light and Electron Optics*, vol. 119, no. 2, pp. 81–85, 2008.

[4] L. J. Wang, A. Kuzmich, and A. Dogariu, "Gain-assisted superluminal light propagation," *Nature*, vol. 406, pp. 277–279, Jul. 2000.

[5] D. Mugnai, A. Ranfagni, and R. Ruggeri, "Observation of superluminal behaviors in wave propagation," *Phys. Rev. Lett.*, vol. 84, no. 21, pp. 4830–4833, 2000.

[6] R. W. Ziolkowski, "Superluminal transmission of information through an electromagnetic metamaterial," *Phys. Rev. E*, vol. 63, no. 4, p. 46604, 2001.

[7] L. D. Landau, E. M. Lifshitz, and L. P. Pitaevskiĭ, *Electrodynamics of Continuous Media*, 2nd ed. Oxford: Pergamon, 1984.

[8] B. Nistad and J. Skaar, "Causality and electromagnetic properties of active media," *Phys. Rev. E*, vol. 78, no. 3, 2008.

[9] M. Gustafsson and D. Sjöberg, "Sum rules and physical bounds on passive metamaterials," *New Journal of Physics*, vol. 12, p. 043046, 2010.

[10] A. M. Nicolson and G. F. Ross, "Measurement of the intrinsic properties of materials by time-domain techniques," *IEEE Trans. Instrumentation and Measurement*, vol. 19, pp. 377–382, 1970.

[11] A. Bernland, A. Luger, and M. Gustafsson, "Sum rules and constraints on passive systems," *J. Phys. A: Math. Theor.*, vol. 44, no. 14, p. 145205, 2011.

[12] R. M. Fano, "Theoretical limitations on the broadband matching of arbitrary impedances," *Journal of the Franklin Institute*, vol. 249, no. 1,2, pp. 57–83 and 139–154, 1950.

[13] K. N. Rozanov, "Ultimate thickness to bandwidth ratio of radar absorbers," *IEEE Trans. Antennas Propagat.*, vol. 48, no. 8, pp. 1230–1234, Aug. 2000.

[14] M. Gustafsson and D. Sjöberg, "Physical bounds and sum rules for high-impedance surfaces," *IEEE Trans. Antennas Propagat.*, vol. 59, no. 6, pp. 2196–2204, 2011.

[15] C. Sohl, M. Gustafsson, and G. Kristensson, "Physical limitations on broadband scattering by heterogeneous obstacles," *J. Phys. A: Math. Theor.*, vol. 40, pp. 11 165–11 182, 2007.

[16] M. Gustafsson, "Time-domain approach to the forward scattering sum rule," *Proc. R. Soc. A*, vol. 466, pp. 3579–3592, 2010.

[17] M. Gustafsson, I. Vakili, S. E. B. Keskin, D. Sjöberg, and C. Larsson, "Optical theorem and forward scattering sum rule for periodic structures," *IEEE Trans. Antennas Propagat.*, vol. 60, no. 8, pp. 3818–3826, 2012.

[18] M. Gustafsson, C. Sohl, and G. Kristensson, "Physical limitations on antennas of arbitrary shape," *Proc. R. Soc. A*, vol. 463, pp. 2589–2607, 2007.

[19] M. Gustafsson, "Sum rule for the transmission cross section of apertures in thin opaque screens," *Opt. Lett.*, vol. 34, no. 13, pp. 2003–2005, 2009.

[20] J. van Bladel, *Electromagnetic Fields*, 2nd ed. Piscataway, NJ: IEEE Press, 2007.

[21] F. Capolino, Ed., *Applications of Metamaterials*. CRC Press, 2009.

[22] H. M. Nussenzveig, *Causality and dispersion relations*. London: Academic Press, 1972.