Constraints on the temporal dispersion of metamaterials

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Abstract—**The frequency dependence of the permittivity, permeability, and index of refraction restrict metamaterial applications such as cloaking and perfect lenses. Here, the principles of causality and passivity together with identities for Herglotz functions are used to construct various sum rules. The sum rules relate the frequency dependence of the material parameters with their high- and low-frequency values. The corresponding physical bounds determine minimum variations of the material parameters over a frequency interval.**

I. Introduction

It is well known that metamaterials are temporally dispersive, *i.e.*, the permittivity and permeability depend on frequency. The Kramers-Kronig relations [1], [2] are commonly used to model the dispersion as they relate the real and imaginary parts of the permittivity and permeability for causal material models. The ideal behavior of the (relative) permittivity and (relative) permeability in applications such as the perfect lens are that $\epsilon(\omega) \approx -1$ and $\mu(\omega) \approx -1$, respectively, over a range of angular frequencies, ω.

The results in [2] show that $\omega\frac{\partial\epsilon}{\partial\omega} > 4$ in frequency intervals $\omega \in \mathcal{B} = [\omega_1, \omega_2]$ where the model is lossless, *i.e.*, $\operatorname{Im}\epsilon(\omega) = 0$. This can be rewritten

$$\max_{\omega\in\mathcal{B}} |\epsilon(\omega) + 1| \geq 2B, \tag{1}$$

where B is the fractional bandwidth $B = (\omega_2 - \omega_1)/\omega_0$ and $\omega_0 = (\omega_2 + \omega_1)/2$. The requirements of lossless material models are removed in [3], where it is shown that

$$\max_{\omega\in\mathcal{B}} |\epsilon(\omega) + 1| \geq B \tag{2}$$

for all square integrable susceptibilities $\epsilon(\omega) - 1$.

In this paper, sum rules are used to derive constraints on the constitutive relations. The sum rules relate weighted integrals of the constitutive parameter over all spectrum with the instantaneous and static response of the material model. Various sum rules are presented that constrain the dispersion of the constitutive relations. The sum rules evaluate how close, *e.g.*, $\epsilon(\omega)$ can be to -1 over a frequency interval. These are the main results and they bound integrals of the type $\ln|\epsilon(\omega)+1|$ integrated over a bandwidth. They also give lower bounds on the variation, *e.g.*, that the maximum of $|\epsilon(\omega) + 1|$ over a frequency interval is greater than a number determined by the difference between the static and instantaneous permittivity, $\epsilon_{\mathrm{s}} - 1$. This is particularly important for the permeability, μ, as there is no magnetic static conductivity and the static permeabilities of many materials are close to the free space permeability, $\mu = 1$.

II. Herglotz functions and constitutive relations

The permittivity and permeability are functions of the angular frequency ω, and their frequency dependence is restricted by the Kramers-Kronig relations [1], [4]. These relations follow from the analytic properties of $\epsilon(\omega)$ in $\operatorname{Im}\omega > 0$ (using time dependence $\mathrm{e}^{-\mathrm{i}\omega t}$) together with basic assumptions on the asymptotic properties of ϵ for low and high frequencies. Here, an alternative approach is considered that is based on the additional assumption of passivity. This restricts ϵ such that $h_\epsilon = \omega\epsilon(\omega)$ is a Herglotz function [1], [5], *i.e.*, $h_\epsilon = h_\epsilon(\omega)$ is holomorphic and $\operatorname{Im} h_\epsilon(\omega) \geq 0$ in the upper halfplane $\operatorname{Im}\omega > 0$. The permeability defines a similar Herglotz function $h_\mu(\omega) = \omega\mu(\omega)$. For simplicity it is assumed that the instantaneous responses are $\epsilon(\omega) \to 1$ and $\mu(\omega) = 1$ as $\omega\hat{\to}\infty$, where symbol $\hat{\to}$ is a short hand notation for limits such that $\alpha < \arg\omega < \pi - \alpha$ for some $\alpha > 0$.

Sum rules and bounds on several electromagnetic problems are obtained by considering weighted integrals applied to symmetric, $h(\omega) = -h^*(-\omega^*)$, Herglotz functions under the assumption of the following asymptotic expansions at low frequencies: $h(\omega) = \sum_n a_{2n-1}\omega^{2n-1} + \mathrm{o}(\omega^{2N_0-1})$ as $k\hat{\to}0$, and at high frequencies, $h(\omega) = \sum_n b_{2n-1}\omega^{1-2n} + \mathrm{o}(\omega^{-2N_\infty+1})$ as $k\hat{\to}\infty$, where a star denotes the complex conjugate, see *e.g.*, [6]–[11]. These Herglotz functions satisfy the following family of integral identities:

$$\frac{2}{\pi}\int_0^\infty \frac{\operatorname{Im} h(\omega)}{\omega^{2n}}\,\mathrm{d}\omega = a_{2n-1} - b_{1-2n}, \tag{3}$$

for $1 - N_\infty \leq n \leq N_0$. Bounding the integral from below by restricting it to a finite interval produces various physical bounds [6], [7], [10]–[12].

The sum rules generated by h_ϵ and h_μ themselves are identical to the ones obtained from the Kramers-Kronig relations in, *e.g.*, [1], [2], [4]. These sum rules relate the losses to the asymptotic values. Compositions of Herglotz functions can be used to create new Herglotz function that instead relate negative values of the real-valued part and the variation around a fixed value to the corresponding asymptotes. In this paper, the cases $-1/h$, $\mathrm{i}\sqrt{-h_1h_2}$, $\mathrm{i}\ln(1 - \mathrm{i}h)$, and $\frac{1}{\pi}\ln\frac{z-\Delta}{z+\Delta}$ are investigated. Here, the square root and logarithm have their branch cuts along the negative real axis and the $-$sign and imaginary unit, i, are essential to preserve the symmetry and the Herglotz property.

III. Sum rules for metamaterials

Candidates to create sum rules that are suited for the case where ideally $\epsilon(\omega) \approx -1$ over a frequency interval are *e.g.*, based on the Herglotz function $h_{\mathrm{m1}}(\omega) = -1/(\omega(\epsilon(\omega)+1))$ that has a $n=0$ sum rule. To bound the bandwidth of ϵ when $\epsilon \approx -1$, it is necessary to consider a Herglotz function that has a large imaginary part when $|\epsilon+1|$ is small. This is achieved with compositions with the logarithm, *e.g.*, $\mathrm{i}\ln(1-\alpha\omega_0 \mathrm{i} h_{\mathrm{m1}})$, giving

$$h_3(\omega) = \mathrm{i}\ln\big(1+\mathrm{i}\frac{\alpha\omega_0}{\omega(\epsilon(\omega)+1)}\big), \tag{4}$$

where $\omega_0 > 0$ and $\alpha > 0$. This function has the appropriate properties as $\operatorname{Im} h_3 \approx \ln \alpha/|\epsilon+1|$ if $|\epsilon+1| \ll 1$ and $\omega \approx \omega_0$. It has the asymptotes $h_3(\omega) = \mathrm{o}(\omega^{-1})$ as $\omega \hat{\to} 0$ and $h_3(\omega) = -\alpha\omega_0/(2\omega) + \mathrm{o}(\omega^{-1})$ as $\omega \hat{\to} \infty$.

Apply the integral identities (3) with $n=0$ to get

$$\frac{2}{\pi}\int_0^\infty \ln\left|\frac{\omega\big(\epsilon(\omega)+1\big)+\mathrm{i}\alpha\omega_0}{\omega\big(\epsilon(\omega)+1\big)}\right| \mathrm{d}\omega = \frac{\alpha\omega_0}{2}. \tag{5}$$

This integral is bounded in several steps. As the primary interest is for $|\epsilon(\omega)+1| \ll 1$, it is practical to simplify the integrand. Consider a frequency interval $\mathcal{B} = [\omega_1, \omega_2]$, with the center angular frequency $\omega_0 = (\omega_1+\omega_2)/2$, and the fractional bandwidth $B = (\omega_2-\omega_1)/\omega_0$. Use that $\operatorname{Im}\omega(\epsilon+1) \geq 0$ implies $\ln|\omega(\epsilon+1)/(\alpha\omega_0)+\mathrm{i}| \geq 0$ and $\int_{\omega_1}^{\omega_2} \ln(\omega_0/\omega)\,\mathrm{d}\omega > 0$ to get

$$\frac{1}{\omega_0}\int_{\omega_1}^{\omega_2} \ln|\frac{\alpha}{\epsilon(\omega)+1}|\,\mathrm{d}\omega \leq \frac{\pi\alpha}{4}. \tag{6}$$

Assuming that $\epsilon(\omega)$ is point wise defined in (6) gives the bound

$$B \min_{\omega\in\mathcal{B}} \ln|\frac{\alpha}{\epsilon(\omega)+1}| \leq \frac{\pi\alpha}{4} \tag{7}$$

that can be written $\max_{\omega\in\mathcal{B}}|\epsilon(\omega)+1| \geq \alpha\mathrm{e}^{-\frac{\pi\alpha}{4B}}$. This is valid for all $\alpha > 0$ and using $\alpha\mathrm{e}^{-\alpha/\beta} \leq \beta/\mathrm{e}$ to maximize the right-hand side over α gives the final bound

$$\max_{\omega\in\mathcal{B}}|\epsilon(\omega)+1| \geq \frac{4B}{\pi\mathrm{e}}. \tag{8}$$

This bound is not as sharp as (1) and (2), but an additional sum rule that also incorporates the static value ϵ_{s} is instead constructed from the Herglotz function

$$h_{\mathrm{m2}}(\omega) = -\frac{\omega_0}{\omega}\frac{\epsilon(\omega)+1}{\epsilon(\omega)-1} = \frac{-\omega_0}{\omega}\left(\frac{\epsilon_{\mathrm{s}}+1}{\epsilon_{\mathrm{s}}-1}\right) \quad \omega\hat{\to}0 \tag{9}$$

that has the property $|h_{\mathrm{m2}}(\omega)| \approx 0$ if $\epsilon(\omega) \approx -1$. Compose h_{m2} with the logarithm as

$$h_4(\omega) = \mathrm{i}\ln\left(1+\frac{\mathrm{i}\alpha}{\omega_0 h_{\mathrm{m2}}(\omega)}\right) = \frac{\omega\alpha}{\omega_0}\left(\frac{\epsilon_{\mathrm{s}}-1}{\epsilon_{\mathrm{s}}+1}\right) + \mathrm{o}(\omega) \tag{10}$$

as $\omega\hat{\to}0$ and $h_4(\omega) = \mathrm{o}(\omega)$ as $\omega\hat{\to}\infty$. This offers a $n=1$ sum rule according to (3), *viz.*,

$$\frac{2}{\pi}\int_0^\infty \frac{1}{\omega^2}\ln\left|\frac{\omega_0 h_{\mathrm{m2}}(\omega)+\mathrm{i}\alpha}{\omega_0 h_{\mathrm{m2}}(\omega)}\right| \mathrm{d}\omega = \frac{\alpha}{\omega_0}\frac{\epsilon_{\mathrm{s}}-1}{\epsilon_{\mathrm{s}}+1}. \tag{11}$$

Simplifications in analogy with the (5) case produce a bound on the permittivity

$$\frac{1}{\omega_0}\int_{\omega_1}^{\omega_2} \ln\left|\alpha\frac{\epsilon(\omega)-1}{\epsilon(\omega)+1}\right| \mathrm{d}\omega \leq \frac{\pi\alpha}{2}\frac{\epsilon_{\mathrm{s}}-1}{\epsilon_{\mathrm{s}}+1} \tag{12}$$

that is valid for all $\alpha < 0$. A bound of the integral together with maximization over α show that the deviation around ϵ_{m} is at least

$$\max_{\omega\in\mathcal{B}}\left|\frac{\epsilon(\omega)+1}{\epsilon(\omega)-1}\right| \geq \frac{2B}{\pi\mathrm{e}}\frac{\epsilon_{\mathrm{s}}+1}{\epsilon_{\mathrm{s}}-1} \tag{13}$$

The sum rule (11) and bounds (12) and (13) show that the constraints on the variation of ϵ around -1 is proportional to the fractional bandwidth B and inversely proportional to the difference $\epsilon_{\mathrm{s}}-1$.

As an example, a model with negative index of refraction and low losses over a broad frequency range is considered [13]. It is generated by the Kramers-Kronig relations [1] using the imaginary parts

$$\operatorname{Im}\epsilon(\omega) = 0.9\frac{\omega(\omega^2-25)^2}{\omega^8+5.5}, \operatorname{Im}\mu(\omega) = 0.7\frac{\omega(\omega^2-25)^2}{\omega^8+4.2}. \tag{14}$$

The index of refraction $n = \mathrm{i}\sqrt{-\epsilon\mu}$ is depicted in 1a and it has $n(0) \approx 79$, $n_\infty = 1$, $\omega_{\mathrm{p}} \approx 6.7$, and $n(\omega_0) \approx -1$, where $\omega_0 \approx 4.9$. The normalized integrands in the sum rules (5) and (11) are depicted in Fig. 1. It is observed that the area is concentrated around ω_0, where $n(\omega) \approx -1$. A box with unit area is included in the figure to illustrate the associated physical bounds. The approximate integrands in (6) and (12) are also depicted.

The constraints (8) and (13) are not as sharp as (1) and (2) for the $\epsilon_{\mathrm{s}} \gg 1$ case. This is due to the bounds of the integrals in (6) and (12). Compositions with the Herglotz function

$$h_\Delta(z) = \frac{1}{\pi}\int_{-\Delta}^{\Delta}\frac{1}{\xi-z}\,\mathrm{d}\xi = \frac{1}{\pi}\ln\frac{z-\Delta}{z+\Delta} \tag{15}$$

improve the bounds as $\operatorname{Im}\{h_\Delta(z)\} \approx 1$ for $|z| < \Delta$ and $\operatorname{Im} z \approx 0$. Composition of $-1/h_{\mathrm{m1}}$ with h_Δ and use of the identity (3) leads to the bounds

$$\max_{\omega\in\mathcal{B}}|\epsilon(\omega)+1| \geq \frac{2B}{1+B/2}\begin{cases}1/2 & \text{lossy case}\\ 1 & \text{lossless case,}\end{cases} \tag{16}$$

that is identical to (1) and (2) for $B \ll 1$. Similarly, h_{m2} gives the constraints

$$\max_{\omega\in\mathcal{B}}\frac{|\epsilon(\omega)+1|}{|\epsilon(\omega)-1|} \geq \frac{B}{1+B/2}\frac{\epsilon_{\mathrm{s}}+1}{\epsilon_{\mathrm{s}}-1}\begin{cases}1/2 & \text{lossy case}\\ 1 & \text{lossless case}\end{cases} \tag{17}$$

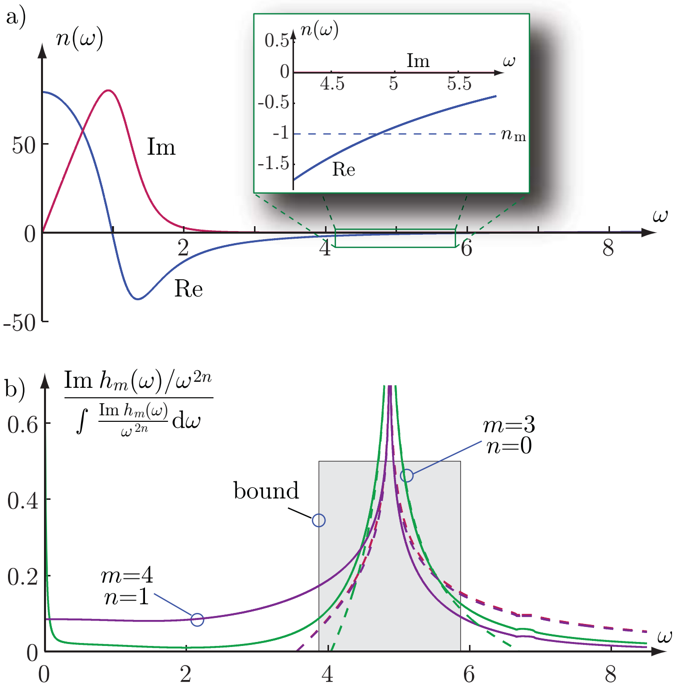

Fig. 1. Illustrations of the model (14) with $n_{\rm m} = -1$. a) the refractive index $n(\omega)$. b) normalized integrands (solid curved) of the sum rules (5) and (11). The dashed curves show the approximations (6) and (12).

IV. Conclusions

Here, sum rules are presented that constrain the dispersion of passive metamaterials. The limitations on the variation of the material parameters over a bandwidth are expressed in the differences between the low- and high-frequency permittivity.

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References

[1] F. W. King, *Hilbert Transforms, Volume 2*. Cambridge University Press, 2009.

[2] L. D. Landau, E. M. Lifshitz, and L. P. Pitaevskiĭ, *Electrodynamics of Continuous Media*, 2nd ed. Oxford: Pergamon, 1984.

[3] Ø. Lind-Johansen, K. Seip, and J. Skaar, "The perfect lens on a finite bandwidth," *J. Math. Phys.*, vol. 50, p. 012908, 2009.

[4] J. D. Jackson, *Classical Electrodynamics*, 3rd ed. New York: John Wiley & Sons, 1999.

[5] H. M. Nussenzveig, *Causality and dispersion relations*. London: Academic Press, 1972.

[6] R. M. Fano, "Theoretical limitations on the broadband matching of arbitrary impedances," *Journal of the Franklin Institute*, vol. 249, no. 1,2, pp. 57–83 and 139–154, 1950.

[7] C. Sohl, M. Gustafsson, and G. Kristensson, "Physical limitations on broadband scattering by heterogeneous obstacles," *J. Phys. A: Math. Theor.*, vol. 40, pp. 11 165–11 182, 2007.

[8] M. Gustafsson, C. Sohl, and G. Kristensson, "Physical limitations on antennas of arbitrary shape," *Proc. R. Soc. A*, vol. 463, pp. 2589–2607, 2007.

[9] C. Sohl, C. Larsson, M. Gustafsson, and G. Kristensson, "A scattering and absorption identity for metamaterials: experimental results and comparison with theory," *J. Appl. Phys.*, vol. 103, no. 5, p. 054906, 2008.

[10] M. Gustafsson, C. Sohl, and G. Kristensson, "Illustrations of new physical bounds on linearly polarized antennas," *IEEE Trans. Antennas Propagat.*, vol. 57, no. 5, pp. 1319–1327, May 2009.

[11] M. Gustafsson, "Sum rule for the transmission cross section of apertures in thin opaque screens," *Opt. Lett.*, vol. 34, no. 13, pp. 2003–2005, 2009.

[12] M. Gustafsson, C. Sohl, C. Larsson, and D. Sjöberg, "Physical bounds on the all-spectrum transmission through periodic arrays," *EPL Europhysics Letters*, vol. 87, no. 3, p. 34002 (6pp), 2009.

[13] T. G. Mackay and A. Lakhtakia, "Comment on"Criterion for Negative Refraction with Low Optical Losses from a Fundamental Principle of Causality"," *Phys. Rev. Lett.*, vol. 99, no. 18, p. 189701, 2007.