Non-Linear Stochastic Control of Critical Processes

Hansson, Anders

1993

Document Version:
Publisher's PDF, also known as Version of record

Link to publication

Citation for published version (APA):
Hansson, A. (1993). *Non-Linear Stochastic Control of Critical Processes*. (Technical Reports TFRT-7503). Department of Automatic Control, Lund Institute of Technology (LTH).

Total number of authors:
1

LUND UNIVERSITY

PO Box 117
221 00 Lund
+46 46-222 00 00

ISSN 0280–5316
ISRN LUTFD2/TFRT--7503--SE

Non-Linear Stochastic Control of Critical Processes

Anders Hansson

Department of Automatic Control
Lund Institute of Technology
March 1993

Department of Automatic Control
Lund Institute of Technology
P.O. Box 118
S-221 00 Lund Sweden

Document name
INTERNAL REPORT

Date of issue
March 1993

Document Number
ISRN LUTFD2/TFRT--7503--SE

Author(s)
Anders Hansson

Supervisor

Sponsoring organisation
Landshövding Nils Hörjels forskningsfond and The Swedish Institute

Title and subtitle
Non-Linear Stochastic Control of Critical Processes

Abstract

In this report stochastic optimal control of critical processes are investigated. The so called running max is introduced to describe this type of problems. Both continuous time and discrete time are treated. In the continuous time case the full information problem is discussed, whereas in discrete time both full information and partial information is discussed. For some simple examples explicit solutions for the optimal controller are obtained. In general, however, the resort seems to be numerical simulations and computations.

Key words
Stochastic control, Critical processes, Optimal Control

Classification system and/or index terms (if any)

Supplementary bibliographical information

ISSN and key title
0280–5316

ISBN

Language
English

Number of pages
20

Recipient's notes

Security classification

1. Introduction

Many processes in industry are critical. They are often critical in the sense that they have a limiting level. This can be either physical or artificial. Examples of the former are such levels that cannot be exceeded without catastrophic consequences, e.g. explosion. One example on the latter is alarm levels, which if they are exceeded will initiate emergency shutdown or a change in operational conditions. Another example is quality levels, which if they are exceeded will cause unsatisfied customers. Common to the critical processes are that they enter their critical region abruptly as a signal exceeds a limiting level.

The distance between the limiting or critical level and the reference value is normally not small, since otherwise the number of exceedances of the level by the controlled signal will be intolerably high. However, there may be other control-objectives that make it undesirable or impossible to choose the distance large. An example of problems of this kind can be found in Borisson and Syding (1976), where the power of an ore crusher should be kept as high as possible but not exceed a certain level, in order that the overload protection does not cause shutdown. Another example is moisture control of a paper machine, where it is desired to keep the moisture content as high as possible without causing wet streaks, Åström (1970), pp. 188–209. Yet another example is power control of wind power plants, where the supervisory system initiates emergency shutdown, if the generated power exceeds 140% of rated power, Mattsson (1984). Other examples can be found in sensor-based robotics and force control, Hansson and Nielsen (1991), and control of non-linear plants, where the stability may be state dependent, Shinskey (1967).

Previous Work

In a deterministic framework this type of problems could be solved by minimizing

$$\max_{d} ||z||_{\infty}$$

where z is the controlled signal and d is a disturbance acting on z. Problems of this type have been studied extensively. Assuming a linear process and bounded energy on the disturbance gives the well-known H_2-controller, Vidyasagar (1986). Other types of disturbances have also been considered. In Vidyasagar (1986) and Dahleh and Pearson (1987) the disturbance has bounded supremum norm, and in Liu and Zakian (1990) it has bounded increments.

Common to the deterministic criteria is the design for worst case disturbances, which may seem somewhat too conservative. The classical way to overcome this is to consider a stochastic formulation. This has been described in Hansson (1992), where approximate solutions by means of the so called Minimum Upcrossing (MU) controller has been obtained for minimizing the criterion

$$P\left\{\max_{0\leq k\leq N} z(k) > z_0\right\}$$

where z_0 is the distance to the critical level. This criterion can also be approximately minimized by Minimum Variance (MV) control, see Åström (1970), pp. 159–209, Åström and Wittenmark (1990), p. 203, and Borisson and Syding (1976). The gain of the minimum variance controller depends critically on the sampling period. Too small a sampling period leads to large variations in the control signal, Åström and Wittenmark (1990), pp. 316–317. This problem has been solved by introducing weighting on the control signal—LQG-design.

However, there has been no good criteria for choosing the weighting. The MU controller can be interpreted as choosing an optimal weighting in an LQG-problem. In Hansson (1992) the MU controller and the MV controller are compared with respect to the criterion above. It is seen that there exist examples where the MU controller has as much as 10 % better performance. The continuous time version of this is described in Hansson (1991).

In the papers mentioned above only linear processes and controllers are discussed. The problem of non-linear processes and controllers is to some extent addressed in Heinricher and Stockbridge (1991). There continuous time full information optimal stochastic control of the running max is considered. This has applications to e.g. optimal control of wear. The ideas presented in this paper can also be applied to critical processes. This will be discussed further in Section 2.

The papers mentioned above all deal with the case of a constant reference value. The case of a varying reference value is more difficult. However, some attempts have been made to address this question. In Hansson (1993) the idea of optimally modifying reference values for critical processes utilizing the information from alarm signals is proposed. The motivation for this problem formulation is that most critical processes are equipped with supervision in the sense that an alarm is given when a signal crosses a certain alarm level, and then an operator, depending on the situation, either initiates emergency shutdown or a temporary change of the operational conditions. This temporary change could be to choose a new reference value in order to avoid that the controlled signal continues to increase and thus preventing an exceedance of a higher more dangerous level. This change of reference value is mostly done in an ad hoc fashion, and it is of interest to make the modification in a more controlled and automatic way.

Examples

In this report non-linear stochastic control of critical processes will be discussed. To get a feeling for what problem-formulations are relevant some examples will be investigated.

The first example will be a simple continuous-time linear first order process controlled with a proportional controller. Let the process be given by the stochastic differential equation

$$dx(t) = [ax(t) + bu(t)]dt + \sigma dw(t) \tag{1}$$

where x is the state of the process and where w is a standard Wiener-process. Assume that the controller has full information, i.e. that the control signal $u(t)$ is a function of $x(t)$. Now, consider a simple proportional controller $u(t) = -kx(t)$. The closed loop system is then governed by

$$dx(t) = (a - bk)x(t)dt + \sigma dw(t) \tag{2}$$

If k is chosen such that $a - bk < 0$, then the closed loop will be stable. Then, for any initial value, it is easy to show that the solution to this equation in stationarity is a Gaussian process with zero mean and covariance $P = \sigma^2/[2(bk - a)]$, Åström (1970). By letting bk go to infinity it follows that $x(t)$ can be made equal to zero in mean square. Notice that there is no problem with respect to stability in doing this. However, the variance of the control signal is $k^2\sigma^2/[2(bk - a)]$ and converges to infinity as bk goes to infinity.

In the next example it will be shown that in order to prevent the closed loop system to enter a critical region infinite gain only has to be applied at the boundary of the critical region. Consider the same process as before, but let the controller be $u(t)dt = -a/bx(t)dt + dg(x(t))$, where $g = g^+ - g^-$, and where g^+ and g^- are defined as in Karatzas (1983)

$$\left\{ \begin{array}{lcl} g^+(t) & = & \max[0, \max_{0 \le s \le t}[-x(s) + g^+(s) - x_0]] \\ g^-(t) & = & \max[0, \max_{0 \le s \le t}[x(s) + g^-(s) - x_0]] \end{array} \right. \tag{3}$$

The closed loop system will now be governed by

$$dx(t) = dg(x(t)) + \sigma dw(t) \tag{4}$$

This equation and similar ones were studied already in Åström (1961), where it was found by solving the Fokker-Planck equation, assuming the initial value $x(0)$ to be in $I = (-x_0, x_0)$, that the density function $p(t,x)$ of $x(t)$ has compact support on I and that it is given by

$$p(t,x) = \frac{1}{2x_0}\sqrt{\frac{T}{\pi t}} \sum_{n=-\infty}^{\infty} \exp\left[-\frac{T}{t}\left(\frac{x}{2x_0} + n\right)^2\right] \tag{5}$$

where $T = 2x_0^2/\sigma^2$. This density converges, as t approaches infinity, to a uniform distribution on I. Thus the probability of $x(t)$ being in the critical region $R \setminus I$ is zero for each t. In Åström (1961) no equations or explicit expressions for g were given. It was only assumed that there existed a g such that the density of $x(t)$ would have compact support on I. It was later shown that such a g indeed existed, and that it was uniquely given by the above equations. The total variation of g is given by $g^+ + g^-$, and it is bounded for all t. Had g been differentiable, which is not the case, then $u(t) = -a/bx(t) + \dot{g}(t)$ and the total variation of g would have been

$$\int_0^t |\dot{g}(s)| ds \tag{6}$$

Hence, formally, the control signal above is such that the integrated absolute value of it is bounded. This type of control problems are known as singular stochastic control problems, since the control signal is not absolutely continuous with respect to Lebesgue measure, see Karatzas (1983) for a good survey. In fact g behaves lake a Wiener-process when it is not constant. Thus since a Wiener process is a.s. nowhere differentiable, formally, it holds that $\dot{g}$ is either 0, $+\infty$ or $-\infty$. The deterministic counterpart to this type of control is known as impulse control.

From a practical point of view it seams strange that such good performance can be obtained. This is due to the fact that infinite control signals may be applied to the process without causing instability. In order to get more interesting problems different approaches can be taken. Considering discrete time problems often removes the pathological behaviour encountered in the examples above. This is due to the fact that high gain usually will cause instability. When considering continuous time problems one attractive way of ruling out infinite control signals is to limit the control signal to a certain set. Another way is to consider optimal control problems with sufficiently large weighting on the control signal, e.g. quadratic weighting. Sometimes, as in Heinricher and Stockbridge (1991), non-trivial problems can be obtained by considering non-controllable processes. Another way of obtaining well-formulated problems is to consider the case of partial information.

Outline

The remaining part of the report is organized as follows. In Section 2 continuous time problems will be investigated. The models used for the open loop system will be stochastic differential equations. The controls will be state-feedbacks obtained by solving an optimal stochastic control problem with full information of the states. It will be seen that sufficient conditions for optimal solutions similar to the ones obtained in Heinricher and Stockbridge (1991) will apply to critical processes as well. In Section 3 discrete time problems will be investigated. In this section the focus will be on linear stochastic difference equations. The criteria that will be considered are related to the discrete time version of the running max. First the full information case will be investigated. This can be solved by considering the Bellman-equation. In a special case explicit solutions will be obtained. Then the partial information case will be considered. This problem can be split up into two problems—the filtering problem and the control problem. The filtering problem will be nonlinear, and the control problem can still be solved via the Bellman-equation, but this time the argument is an n-dimensional density function, and not only an n-dimensional vector. Thus to get computationally tractable problems approximate nonlinear filtering will be discussed in order to obtain simple parametrizations of the density function. Finally, in Section 4 some conclusions will be drawn, and suggestions for future research will be given.

2. Continuous Time

In this section full information optimal stochastic control for critical processes will be treated. The processes considered will be stochastic differential equations. For an introduction to these see e.g. Oksendal (1989). A more complete treatment is given in Karatzas and Shreve (1991). In order to address critical processes the notion of the running max will be introduced as in Heinricher and Stockbridge (1991). Different relevant control objectives will be discussed by considering fairly general optimization problems. Sufficient conditions for these problems in terms of Hamilton-Jacobi-Bellman (HJB) equations will be obtained. It will be seen how it is possible to solve the HJB-equation explicitly for an example.

Model

Let the open loop system be modeled by the following stochastic differential equation:

$$dx(t) = f(x(t), u(t))dt + \sigma(x(t), u(t))dw(t), \quad x(0) = x \tag{7}$$

where w is a standard n-dimensional Wiener process, and where f and σ are n-dimensional vector functions and $n \times n$-matrix functions of the n-dimensional state $x(t)$ and the m-dimensional control $u(t)$. The assumptions that have to be imposed on f and σ for (7) to have a well-defined solution can be found in e.g. Fleming and Soner (1993).

In order to be able to address critical processes, introduce the running max of $g(x(t))$, which is defined as

$$y(t) = \max\{g(x(s)) : 0 \leq s \leq t\} \vee y, \quad y(0) = y \geq x \tag{8}$$

where g is a real-valued differentiable function of the n-dimensional state $x(t)$. By defining the set $A = \{t \in R : g(x(t)) = y(t) \cap dg(x(t)) > 0\}$ it is possible to express dy as

$$dy(t) = I_A \frac{dg(x(t))}{dx} [f(x(t), u(t))dt + \sigma(x(t), u(t))dw(t)] \tag{9}$$

In the sequel (7) will be augmented with this equation. The augmented state $(x(t)^T \ y(t))^T$ is a strong Markov process, but it is not a diffusion due to the fact that I_A is not adapted to the σ-algebra $\mathcal{F}(t) = \sigma\{w(s) : 0 \leq s \leq t\}$. Notice however, that $y(t)$ is adapted to $\mathcal{F}(t)$; it is also increasing. These facts will be used later in Section 2.3.

Control Objectives

One control objective inspired by Section 1 is obtained by considering the following criterion function

$$J_f(x, y, u(\cdot)) = E\left\{\int_0^T h(x(t), y(t), u(t))dt + \Psi(T, x(T), y(T))\right\} \tag{10}$$

where h and Ψ are real-valued functions of the state, the running max and the control. Another possible control objective is the so called discounted cost criterion

$$J_d(x, y, u(\cdot)) = E\left\{\int_0^\infty e^{-\beta t} h(x(t), y(t), u(t))dt\right\} \tag{11}$$

where $\beta > 0$. The set of controls over which the minimization of the criterion functions is to be performed will be the set of admissible controls as defined in Fleming and Soner (1993). The former criterion function will result in time-dependent control laws, whereas the latter will result in control laws independent of time. The conditions that have to be imposed on h and Ψ for the control problem to be well defined are given in Fleming and Soner (1993).

The Hamilton-Jacobi-Bellman Equation

Now sufficient conditions for optimality in terms of HJB-equations will be given. The results are variants of the result in Heinricher and Stockbridge (1991), and they follow the path of standard verification theory as presented in Fleming and Soner (1993).

First consider the problem of minimizing J_f. Introduce the following partial differential equation, called the Hamilton-Jacobi-Bellman (HJB) equation

$$V_t + \min_u \left\{V_x^T f + \frac{1}{2} tr V_{xx} \sigma \sigma^T + h\right\} = 0 \tag{12}$$

for $V = V(t, x, y)$ on $\mathcal{D}_f = \{(t, x, y) \in R^3 : g(x) \leq y, \ 0 \leq t \leq T\}$ with terminal condition $V(T, x, y) = \Psi(T, x, y)$ and boundary condition $V_y(t, x, y) = 0$ for $g(x) = y$ and $0 < t < T$. Assume that this equation has a solution V on $\mathcal{D}_f$ that fulfils all the assumptions for a classical solution as defined in Fleming and Soner (1993).

Since $y(t)$ is adapted to $\mathcal{F}(t)$ and increasing, it follows by the Itô-formula, Karatzas and Shreve (1991), Theorem 3.6, that

$$V(T, x(T), y(T)) = V(0, x, y)$$

$$
\begin{aligned}
&+ \int_0^T \Big[V_t(t,x(t),y(t)) \\
&+ V_x^T(t,x(t),y(t))f(x(t),u(t)) \\
&+ \frac{1}{2} tr V_{xx}(t,x(t),y(t))\sigma(x(t),u(t))\sigma^T(x(t),u(t)) \\
&+ I_A V_y(t,x(t),y(t)) g_x^T(x(t)) f(x(t),u(t))\Big] dt \\
&+ \int_0^T \Big[V_x^T(t,x(t),y(t)) \\
&+ I_A V_y(t,x(t),y(t)) g_x^T(x(t))\Big] \sigma(x(t),u(t)) dw(t)
\end{aligned}
$$

Noting that $V_y(t,x,y) = 0$ for $g(x) = y$ and $0 < t < T$, and that $V_x^T(t,x(t),y(t))\sigma(x(t),u(t))$ is adapted to $\mathcal{F}(t)$, it follows by taking expectations that

$$
\begin{aligned}
E\{V(T,x(T),y(T))\} = {} & V(0,x,y) \\
&+ E\Big\{\int_0^T \Big[V_t(t,x(t),y(t)) \\
&+ V_x^T(t,x(t),y(t))f(x(t),u(t)) \\
&+ \frac{1}{2} tr V_{xx}(t,x(t),y(t))\sigma(x(t),u(t))\sigma^T(x(t),u(t))\Big] dt\Big\}
\end{aligned}
$$

Now, by adding and subtracting $h(x(t),y(t),u(t))$ in the integral, using (12), and noting that $V(T,x(T),y(T)) = \Psi(T,x(T),y(T))$ it holds by (10) that

$$
V(0,x,y) \le J(x,y,u(\cdot)) \tag{13}
$$

with equality for the $u(\cdot)$ that solves (12). This shows that the optimal control can be obtained by solving the HJB-equation under the condition that this solution fulfils the assumptions that justifies the calculations above, i.e. has a classical solution in the sense of Fleming and Soner (1993). Notice that the differentiability assumption on g is not necessary, since $I_A V_y(t,x(t),y(t)) = 0$.

Similar techniques as above can be used to show that the existence of a classical solution to

$$
-\beta V + \min_u \left\{ V_x^T f + \frac{1}{2} tr V_{xx}\sigma\sigma^T + h \right\} = 0 \tag{14}
$$

for $V = V(x,y)$ on $\mathcal{D}_d = \{(x,y) \in R^2 : g(x) \le y\}$ with boundary condition $V_y(x,y) = 0$ for $g(x) = y$ and terminal condition

$$
\lim_{t\to\infty} e^{-\beta t} E\{V(x(t),y(t))\} = 0 \tag{15}
$$

is a sufficient condition for minimizing J_d. In Heinricher and Stockbridge (1991) a stopping problem is considered where the sufficient condition is the same as the one for the discounted const criterion above with $\beta = 0$ and with the additional assumption of $V(y,y) = 0$.

Example

It turns out that the time-independent HJB-equations are much easier to solve than the time-dependent. Thus a discounted const criterion problem will be considered. Let the process be linear, i.e. let

$$
dx(t) = [ax(t) + bu(t)]dt + \sigma dw(t) \tag{16}
$$

and let the criterion be given by

$$E\left\{\int_0^\infty e^{-\beta t}\frac{1}{2}\left[y^2(t)+\rho u^2(t)\right]dt\right\} \tag{17}$$

which is a type of discounted Linear Quadratic (LQ) control problem, but not in the state x as is usual, but in the running max y. Assume that $g(x)=x$. Easy calculations show that the optimal control is given by $u=-b/\rho V_x$, and that the HJB-equation for this control becomes

$$-\beta V+axV_x-\frac{b^2}{2\rho}V_x^2+\frac{1}{2}\sigma^2V_{xx}+\frac{1}{2}y^2=0 \tag{18}$$

Inspired by the solution to the to the standard discounted LQ problem the solution $V=K_1x^2+K_2xy+K_3y^2+K_4$ for K_i, $i=1,\ldots 4$ being some constants, will be investigated. Some calculations show that this is indeed a solution, if $a=0$, and it is given by

$$\begin{aligned}
K_1 &= -\frac{\rho\beta}{2b^2}\\
K_2 &= -2K_3\\
K_3 &= \frac{-\rho\beta-\sqrt{\rho^2\beta^2+4\rho b^2}}{4b^2}\\
K_4 &= -\frac{\rho\sigma^2}{2b^2}
\end{aligned}$$

The resulting control signal is given by

$$u(t)=\frac{\beta}{b}x(t)-\frac{\rho\beta+\sqrt{\rho^2\beta^2+4\rho b^2}}{2\rho b}y(t) \tag{19}$$

and it is always negative. Strangely enough bu is increasing as a function of the state. It is interesting to note that the standard discounted LQ-controller is given by, Fleming and Soner (1993)

$$u(t)=-\frac{\rho\beta+\sqrt{\rho^2\beta^2+4\rho b^2}}{2\rho b}x(t) \tag{20}$$

Hence the discounted LQ-controller for the running max just replaces the state in the standard problem with the running max and adds a new feedback from the state. If $\beta=0$, which corresponds to the average criterion

$$\lim_{T\to\infty}\frac{1}{T}J_f(x,y,u) \tag{21}$$

then the state feedback is not present and the controller is given by

$$u(t)=-\frac{sign(b)}{\sqrt{\rho}}y(t) \tag{22}$$

The assumptions that justifies the use of the HJB-equation to derive the optimal controller for the discounted cost problem has to be shown to hold for the solution obtained. The only assumption that is difficult to verify for the discounted cost problem is

$$\lim_{t\to\infty}e^{-\beta t}E\left\{V(x(t),y(t))\right\}=0 \tag{23}$$

where $(x(t),y(t))^T$ is the state obtained by applying the candidate optimal control law above. This is, however, not easy.

Summary

Optimal stochastic control problems for the running max have been treated. Sufficient conditions in terms of HJB-equations have been given. For an LQ type of problem for the running max, assuming an integrator process, the HJB-equation has been solved explicitly. For more complicated processes the solution is not known. This seems to be an inherent problem in control of the running max, see Heinricher and Stockbridge (1991), where explicit solutions also only are obtained for integrator processes.

The condition $V_y(x, y)$ for $g(x) = y$ was imposed above in order to use Itô-calculus for obtaining the sufficient condition in terms of the HJB-equation. At a first glance it seems to be interesting to instead try to obtain the backward evolution operator for the Markov process $(x(t)^T\ y(t))^T$ and the corresponding Dynkin formula to derive a sufficient condition for the problem. It, however, turns out that the condition $V_y(x, y)$ for $g(x) = y$ is a necessary condition for the backward evolution operator to exist. Thus nothing is gained by this alternative approach.

In this section only full information control has been treated. The partial information case is much more complicated. For an introductory treatment of partial information optimal stochastic control in continuous time see e.g. Wonham (1968).

3. Discrete Time

In this section both full information and partial information optimal stochastic control for critical processes will be treated. The processes considered will be linear Gaussian stochastic difference equations. For a simple introduction to optimal stochastic control in discrete time see Åström (1977). A more rigorous treatment is given in Bertsekas (1978). As in the previous section the running max will be introduced. In discrete time the running max will actually obey a difference equation. This will simplify things as compared to the continuous time case. Some control objectives relevant to critical processes will be discussed by considering different optimization problems involving the running max. In a special full information case it will actually be possible to solve the Bellman-equation related to the optimal control problem explicitly. The solution will be the full information MV-controller. In the partial information case the problem can be split into one filtering problem and one control problem. The filtering problem will be nonlinear, and the control problem can still be solved by solving the Bellman-equation. However, this time the Bellman-equation is defined on the functional-space related to the filtering problem and not on a vector space as in the full information case. Thus to obtain computationally tractable solution procedures approximations of the filtering problem will be discussed. This will result in parametrizations of the density function for the filtering problem, and thus simplify the argument of the Bellman-equation to a vector.

Model

Let the open loop system be modeled by the following linear stochastic difference equation:

$$\begin{cases} x(k+1) &= Ax(k) + Bu(k) + v(k) \\ y(k) &= C_1x(k) + e(k) \\ z(k) &= C_2x(k) \end{cases} \tag{24}$$

where v and e are sequences of independent Gaussian random variables with zero means and covariances $Ev(k)v^T(k) = \sigma_v^2 = R_1$ and $Ee(k)e^T(k) = \sigma_e^2 = R_1$. The initial value $x(0) = x_0$ of the state x is assumed to be Gaussian with mean m_0 and covariance R_0. The signal y is the measurement signal, u is the control signal, and z is the signal that is to be controlled. Define ξ to be the discrete time running running max of $g(z(k))$ by

$$\xi(k) = \max\{g(z(i)) : 0 \leq i \leq k\} \tag{25}$$

Notice that

$$\xi(k+1) = \max[\xi(k), g(z(k+1))] = \max[\xi(k), g(C_2(Ax(k) + Bu(k) + v(k)))] \tag{26}$$

with $\xi(0) = \xi_0 = g(C_2 x_0)$. Due to this difference equation for the running max it is possible to describe the behaviour of the augmented system with state $\bar{x}(k) = (x^T(k)\ \xi(k))^T$ by the nonlinear stochastic difference equation

$$\left\{ \begin{array}{rcl} x(k+1) & = & Ax(k) + Bu(k) + v(k) \\ \xi(k+1) & = & \max[\xi(k), g(C_2(Ax(k) + Bu(k) + v(k)))] \\ y(k) & = & C_1 x(k) + e(k) \end{array} \right. \tag{27}$$

This description will be used in the sequel. Depending on g different joint initial distributions for the augmented state will be obtained. For the case when $g(z) = z$ it holds that distribution is Gaussian with mean $(m_0^T\ C_2 m_0)^T$ and covariance

$$\begin{pmatrix} R_0 & R_0 C_2^T \\ C_2 R_0 & C_2 R_0 C_2^T \end{pmatrix} \tag{28}$$

It should be noted that the difference equation above defines a discrete time Markov process. It will for later purposes be convenient to introduce the transition densities for this process, assuming that they exist. The notations $p(x(k+1), \xi(k+1)|x(k)), \xi(k)), p(y(k)|x(k))$, etc. with obvious interpretations will be abused.

Control Objectives

The control objectives that will be considered can all be expressed in the general form of

$$J(p(x_0, \xi_0), u(\cdot)) = E\left\{\sum_{k=0}^{N} h(k, x(k), \xi(k), u(k))\right\} \tag{29}$$

where h is a real-valued function of time, the augmented state, and the control. The admissible controls u, which J will be minimized over, will be functions either of the augmented state—full information case—or of the sequence of measurements y available when the control is to be applied—partial information case. In this subsection the dynamic programming equation, or the Bellman-equation, which gives a solution procedure for the minimization problem above will be derived. It should be noted that it is difficult to make this derivation rigorous due to the fact that the minima computed in the sequel may not be measurable. These questions will not be addressed here. For a more complete treatment see Bertsekas (1978).

Let $\mathcal{Y}(k)$ be the sequence of information available to the controller at time k. For the full information case this is $\{x(i), \xi(i) : 0 \leq i \leq k\}$ and for the partial information case it is $\{y(i) : 0 \le i \le k\}$. Further introduce

$$V(k, p(x(k), \xi(k)|\mathcal{Y}(k))) = \min_{\{u(i):k \le i \le N\}} E\left\{\sum_{i=k}^{N} h(i, x(i), \xi(i), u(i))|\mathcal{Y}(k)\right\} \quad (30)$$

It is now obvious that

$$E\{V(0, p(x_0, \xi_0|\mathcal{Y}(0)))\} = \min_{\{u(k):0 \le k \le N\}} J(p(x_0, \xi_0), u(\cdot)) \quad (31)$$

Further by the principle of optimality it holds that

$$\begin{aligned} V(k, p(x(k), \xi(k)|\mathcal{Y}(k))) = \min_{u(k)} E\{&h(k, x(k), \xi(k), u(k)) \\ &+ V(k+1, p(x(k+1), \xi(k+1)|\mathcal{Y}(k+1)))|\mathcal{Y}(k)\} \end{aligned}$$

for $0 \le k \le N-1$ with final value

$$V(N, p(x(N), \xi(N)|\mathcal{Y}(N))) = \min_{u(N)} E\{h(N, x(N), \xi(N), u(N))|\mathcal{Y}(N)\} \quad (32)$$

This equation is called the Bellman-equation and gives a recursion for the optimal value of the cost function J. Notice that the argument is a density function. In the following sections more specialized versions will be derived in order to cope with the specific cases treated there.

The Full Information Case

In this subsection the full information case will be treated, i.e. $\mathcal{Y}(k) = \{x(i), \xi(i) : 0 \le i \le k\}$. For this case the argument of the Bellman-equation reduces from a vector-valued function to a vector. In a special case an explicit solution of this equation will be obtained.

The General Problem Since for the full information case $p(x(k), \xi(k)|\mathcal{Y}(k)) = p(x(k), \xi(k)|x(k), \xi(k))$ is a degenerate distribution, it can be represented by its mean, which is a vector. Hence the Bellman-equation can be written

$$\begin{aligned} V(k, x(k), \xi(k)) &= \min_{u(k)} E\{h(k, x(k), \xi(k), u(k)) \\ &\quad + V(k+1, x(k+1), \xi(k+1))|x(k), \xi(k)\} \\ &= \min_{u(k)} [h(k, x(k), \xi(k), u(k)) \\ &\quad + E\{V(k+1, x(k+1), \xi(k+1))|x(k), \xi(k)\}] \end{aligned}$$

for $0 \le k \le N-1$ with final value

$$V(N, x(N), \xi(N)) = \min_{u(N)} E\{h(N, x(N), \xi(N), u(N))\} \quad (33)$$

This simplifies the complexity of computing the recursion considerably. The first step in computing the recursion is to evaluate the expectation $E(x(k), \xi(k), u(k)) = E\{V(k+1, x(k+1), \xi(k+1))|x(k), \xi(k)\}$ and express it in terms of $x(k)$, $\xi(k)$ and $u(k)$. The second step is to perform the minimization with respect to $u(k)$. The expectation can be computed as

$$
\begin{aligned}
E(x(k), \xi(k), u(k)) = \int & p(x(k+1), \xi(k+1) | x(k), \xi(k)) \\
& \cdot V(k+1, x(k+1), \xi(k+1)) dx(k+1) d\xi(k+1)
\end{aligned}
$$

where the integration is to be performed over the value space of the augmented state. This will be illustrated more in detail in the special case that follows.

A Special Case Consider the case when the loss function is given by

$$
J = P\{\xi(N) > \xi^0\} \tag{34}
$$

for the critical level ξ^0. Further let $g(z) = |z|$. This is easily seen to be a special case of the problem formulation above by taking $h(k, x(k), \xi(k), u(k)) = 0$ for $0 \leq k \leq N-1$ and $h(N, x(k), \xi(k), u(k)) = I_{\{\xi(N) > \xi^0\}}$ For this case the Bellman-equation becomes

$$
\begin{aligned}
V(k, x(k), \xi(k)) = \min_{u(k)} \int & p(x(k+1), \xi(k+1) | x(k), \xi(k)) \\
& \cdot V(k+1, x(k+1), \xi(k+1)) dx(k+1) d\xi(k+1)
\end{aligned}
$$

for $0 \leq k \leq N-1$ with final value

$$
V(N, x(N), \xi(N)) = I_{\{\xi(N) > \xi^0\}} \tag{35}
$$

Assume that there exist a solution such that $V(k, x(k), \xi(k))$ is not a function of $x(k)$. Note that this assumption holds for $k = N$. Then by integrating out the state variable $x(k+1)$ the Bellman-equation reads

$$
\begin{aligned}
V(k, \xi(k)) = \min_{u(k)} \int & p(\xi(k+1) | x(k), \xi(k)) \\
& \cdot V(k+1, \xi(k+1)) d\xi(k+1)
\end{aligned}
$$

Some calculations show that the conditioned density in the equation above is

$$
p(\xi(k+1) | x(k), \xi(k)) = \frac{1}{\sigma} \varphi \left(\frac{\xi(k+1) - m(k)}{\sigma} \right) + \frac{1}{\sigma} \varphi \left(\frac{\xi(k+1) + m(k)}{\sigma} \right) \tag{36}
$$

if $\xi(k) \leq \xi(k+1)$ and that it is zero if $\xi(k) > \xi(k+1)$, where

$$
\begin{aligned}
m(k) &= C_2(Ax(k) + Bu(k)) \\
\sigma^2 &= C_2 R_1 C_2^T
\end{aligned}
$$

and where φ is the standardized normal density function. Some further calculations show that the optimal choice of $u(k)$ is given by $m(k) = 0$ if there exist a solution to this equation. It is easily seen that the resulting $V(k, \xi(k))$

is indeed independent of $x(k)$. Thus by induction the optimal control law is given by the equation above for all $0 \leq k \leq N$. The existence of a solution to $m(k) = 0$ is e.g. in the case of a single-input system equivalent to $C_2B \neq 0$, and for this case the solution $u(k)$ is given by

$$u(k) = -\frac{C_2A}{C_2B}x(k) \tag{37}$$

If, however, $C_2B = 0$, then $u(k)$ can be taken arbitrarily, and the assumption made above about $V(k, \xi(k))$ being independent of $x(k)$ does not hold. It should be noted that the resulting control law above is the same as the MV control law for the full information case when minimizing the variance of z.

Summary In this subsection the Bellman-equation for the full information case of optimal control of the running max has been derived. In a special case it was possible to solve the equation explicitly under the assumption that the controlled process had relative degree one. The resulting controller was the same as the MV controller. Nothing is known about the controller for higher relative degrees. Probably the Bellman-equation can only be solved numerically in these cases.

The Partial Information Case

In this subsection the partial information case will be treated, i.e. $\mathcal{Y}(k) = \{y(i) : 0 \leq i \leq k\}$. In this case the argument of the Bellman-equation is a density function. This density function is the conditioned density of the augmented state given the information available when the control signal is to be applied. It can be obtained by solving a non-linear filtering problem. Simulations of the filter will suggest approximations of the filtering problem in terms of parametrizations of the density function by means of its mean and covariance. This will reduce the complexity of the Bellman-equation, since the argument may be taken to be the parameters determining the density instead of the whole density. In this way a finite-dimensional argument will be obtained just as in the full information case.

The Control Problem Remember that the general Bellman-equation of Section 3.2 reads

$$\begin{aligned} V(k, p(x(k), \xi(k)|\mathcal{Y}(k))) = \min_{u(k)} E\{&h(k, x(k), \xi(k), u(k)) \\ &+ V(k+1, p(x(k+1), \xi(k+1)|\mathcal{Y}(k+1)))|\mathcal{Y}(k)\} \end{aligned}$$

for $0 \leq k \leq N-1$ with final value

$$V(N, p(x(N), \xi(N)|\mathcal{Y}(N))) = \min_{u(N)} E\{h(N, x(N), \xi(N), u(N))|\mathcal{Y}(N)\} \tag{38}$$

The wright hand side can be evaluated by computing the integrals

$$\int h(k, \bar{x}(k), u(k)) p(\bar{x}(k)|\mathcal{Y}(k)) d\bar{x}(k) \tag{39}$$

and

$$\int V(k+1, p(\bar{x}(k+1)|\mathcal{Y}(k+1))) p(y(k+1)|\mathcal{Y}(k)) dy(k+1) \tag{40}$$

respectively. Thus the only remaining question is how to express $p(\bar{x}(k+1)|\mathcal{Y}(k+1))$ and $p(y(k+1)|\mathcal{Y}(k))$ as functions of $y(k+1)$ and $u(k)$. This will be answered by the filtering equations.

The Filtering Problem Introduce the following operator

$$\mathcal{A}_{y(k+1)}p(\bar{x}(k)|\mathcal{Y}(k)) = p(y(k+1)|\bar{x}(k+1)) \int p(\bar{x}(k+1)|\bar{x}(k))p(\bar{x}(k)|\mathcal{Y}(k))d\bar{x}(k) \tag{41}$$

Then it is well known that, see e.g. Åström (1977), that the conditioned density obeys the following recursion

$$p(\bar{x}(k+1)|\mathcal{Y}(k+1)) = \frac{\mathcal{A}_{y(k+1)}p(\bar{x}(k)|\mathcal{Y}(k))}{p(y(k+1)|\mathcal{Y}(k))} \tag{42}$$

where

$$p(y(k+1)|\mathcal{Y}(k)) = \int \mathcal{A}_{y(k+1)}p(\bar{x}(k)|\mathcal{Y}(k))d\bar{x}(k+1) \tag{43}$$

Notice that the right hand sides in the two above equations both are functions of $u(k)$. This will be more apparent later on. From the special structure of the equations for the open loop system it follows that

$$p(y(k+1)|\bar{x}(k+1)) = p(y(k+1)|x(k+1)) \tag{44}$$

The filter-equation may now be written

$$p(\bar{x}(k+1)|\mathcal{Y}(k+1)) = \frac{p(y(k+1)|x(k+1))p(\bar{x}(k+1)|\mathcal{Y}(k))}{p(y(k+1)|\mathcal{Y}(k))} \tag{45}$$

where

$$p(\bar{x}(k+1)|\mathcal{Y}(k)) = \int p(\bar{x}(k+1)|\bar{x}(k))p(\bar{x}(k)|\mathcal{Y}(k))d\bar{x}(k) \tag{46}$$

Notice that the first factor in the denominator and the whole nominator is easily computed from the Kalman-filter for the state $x(k)$. Thus the only difficult task in the non-linear filter equation above is the computation of $p(\bar{x}(k+1)|\mathcal{Y}(k))$.

To further study the filtering equation assume that $x(k)$ is a scalar and that $g(z) = z$. Introduce the less sloppy notation

$$w(k, x, \xi) = p(x(k) = x, \xi(k) = \xi|\mathcal{Y}(k)) \tag{47}$$

Then some calculations give that $p(x(k+1), \xi(k+1)|\mathcal{Y}(k))$ is equal to

$$\int \frac{1}{\sigma_v}\varphi\left(\frac{x((k+1) - Ax(k) - Bu(k)}{\sigma_v}\right) w(k, x(k), \xi(k+1))dx(k) \tag{48}$$

if $x(k+1) \leq \xi(k+1)/C_2$, and that it is zero if $x(k+1) > \xi(k+1)/C_2$. From the Kalman-filter for $x(k)$ it follows that

$$p(y(k+1)|x(k+1)) = \frac{1}{\sigma_e}\varphi\left(\frac{y(k+1) - C_1x(k+1)}{\sigma_e}\right) \tag{49}$$

and that

$$p(y(k+1)|\mathcal{Y}(k)) = \frac{1}{\sigma}\varphi\left(\frac{y(k+1) - C_1(A\hat{x}(k) + Bu(k))}{\sigma}\right) \tag{50}$$

where

$$\sigma^2 = C_1 R_1 C_1^T + R_2 \tag{51}$$

and where $\hat{x}(k)$ is the Kalman-filter estimate of $x(k)$. These equations together with the Kalman-filter explicitly give the expressions for the filter. Notice that $p(x(k), \xi(k)|\mathcal{Y}(k)) = 0$ for $x(k) > \xi(k)/C2$. Thus the integration in (48) only has to be performed over the interval $[-\infty, \xi(k)/C2]$.

To get a feeling for how the filter performs a simulation has been done. The parameters used were $A = 0.5$, $B = 0.04$, $C_1 = C_2 = 1$, $R_0 = R_1 = R_2 = 1$, and $m_0 = 0$. The control signal was taken to be zero. The marginal density functions for $x(k)$ and $\xi(k)$ as well as their joint density function has been calculated numerically for $0 \leq k \leq 10$. The results are shown if figures 1–11. It is seen how the density for $x(k)$ is Gaussian as expected. Moreover the density for $\xi(k)$ also seems to be almost Gaussian. When $x(k)$ is far from $\xi(k)$ the joint density is almost Gaussian, and $x(k)$ are $\xi(k)$ are almost independent. When $x(k)$ is close to $\xi(k)$ this is not the case due to the fact that the joint density is zero for $x(k) > \xi(k)$. However, it seems to be very close to a truncated Gaussian density.

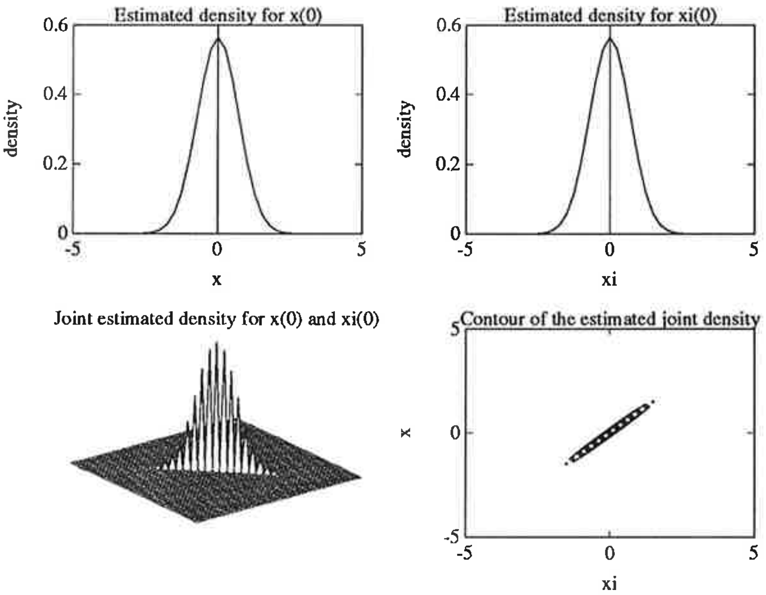

Figure 1. $k = 0$

Summary In this subsection the partial information case has been treated. It has been seen how the problem can be solved by solving the Bellman-equation. The argument of the Bellman-equation is for this case a density function obtained by solving a nonlinear filtering problem. Simulations of the filter has shown that the density function is almost a truncated Gaussian density. This suggests that it can be approximately parameterized by its mean and covariance. This will reduce the complexity of the Bellman-equation, since the argument may be taken to be the parameters determining the density instead of the whole density. In this way a finite-dimensional argument will be obtained just as in the full information case.

Summary

In this section discrete time stochastic optimal control of the running max has been discussed. Both the full information case and the partial information case

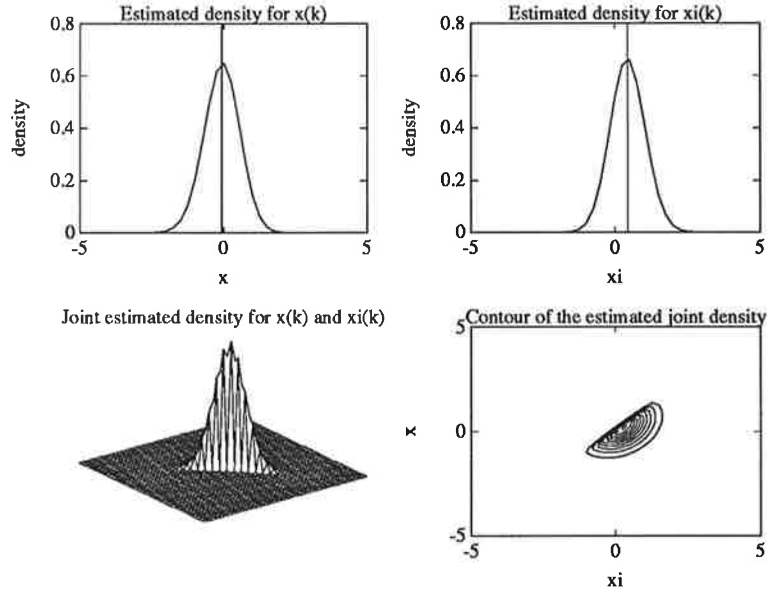

Figure 2. $k = 1$

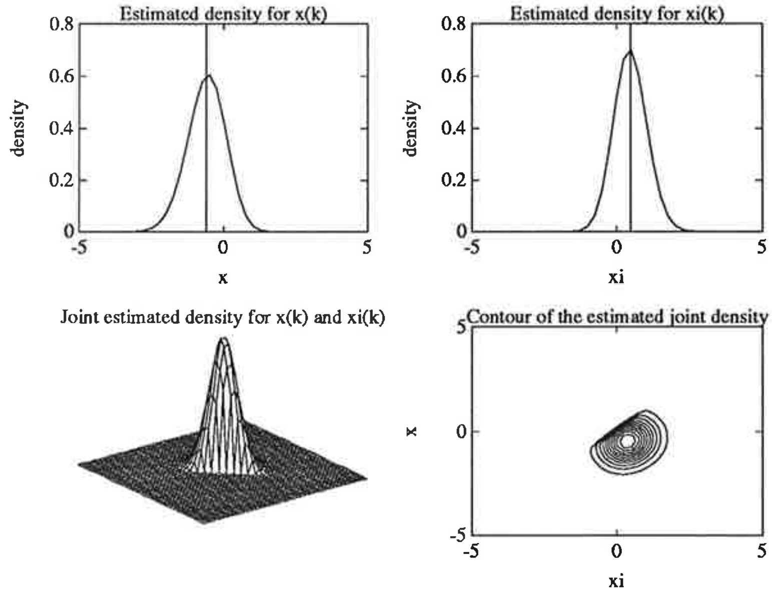

Figure 3. $k = 2$

have been treated. In a special case for the full information case an explicit solution to the Bellman equation was obtained. For the partial information case the argument of the Bellman equation is a density function and not only a vector as in the full information case. This makes the partial information case much more complicated. The nonlinear filtering problem, which determines the argument of the Bellman-equation, has been studied in a simulation. It seems to be possible to approximately parameterize the density with a truncated Gaussian density. This will reduce the complexity of the Bellman-equation considerably, since the argument of the equation then may be taken to be the mean and covariance of the density instead of the whole density. In spite of the simplification obtained by considering the parameterization of the density, it

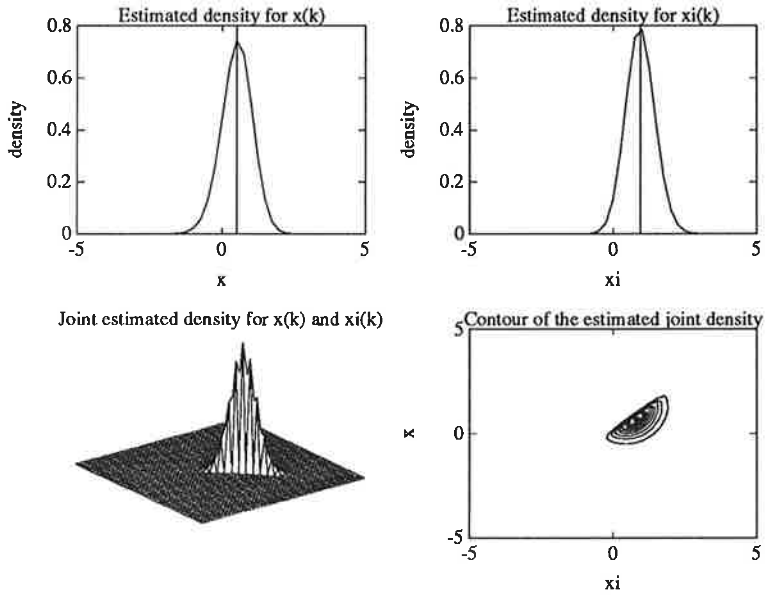

Figure 4. $k = 3$

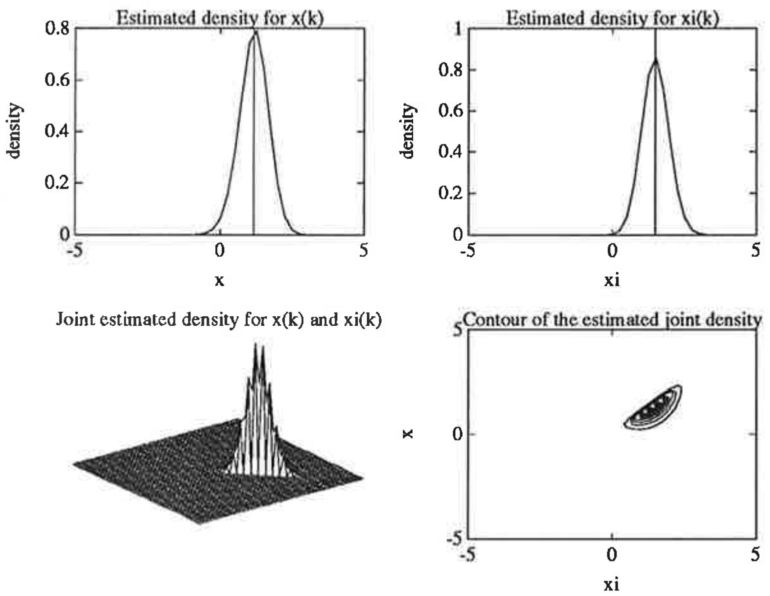

Figure 5. $k = 4$

does not seem to be possible to solve the Bellman-equation analytically. The resort seems to be numerical solutions.

4. Conclusions

In this report nonlinear stochastic optimal control of critical processes has been treated. An overview of the current status of research in critical processes, with focus on the stochastic case, has been given. Some simple examples have been investigated to get a feeling for what problems are relevant from an engineering point of view. The running max seems to be a fruitful concept in

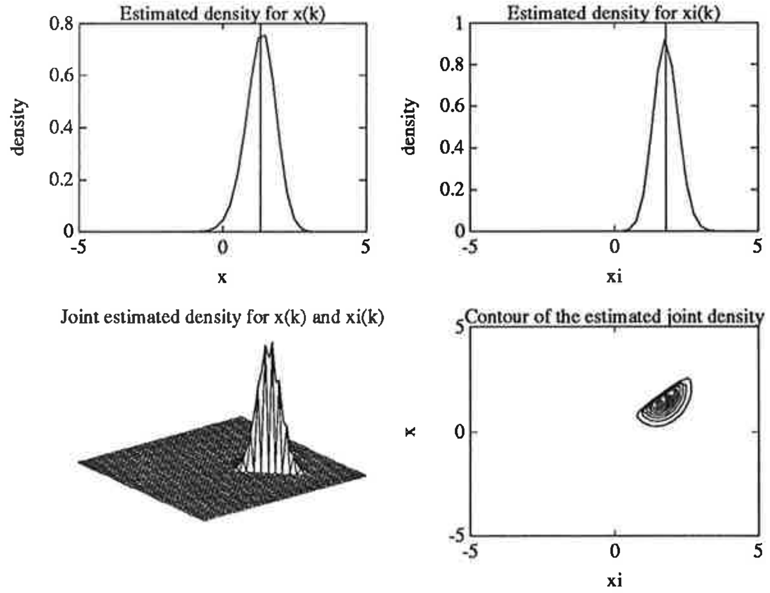

Figure 6. $k = 5$

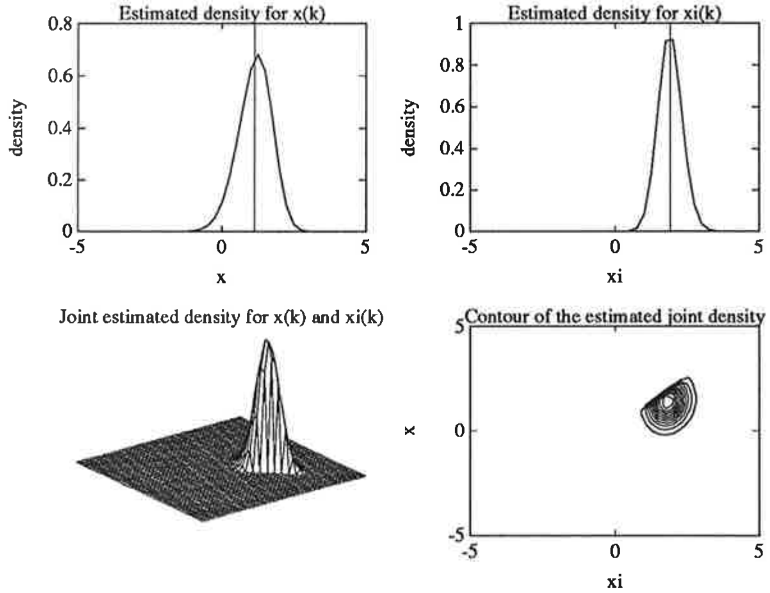

Figure 7. $k = 6$

order to deal with critical processes in a stochastic context. Both continuous time and discrete time have been addressed. Explicit solutions to some simple control problems have been obtained.

Future Research

Numerical computations and simulations seem to be the approach to enable more insight into stochastic optimal control of critical processes. In continuous time numerical solutions of the HJB-equation would be of interest. The method to use could be the one described in Kushner and Dupuis (1992). In discrete time numerical solutions of the Bellman-equation would be of interest. Here, a deeper investigation of the approximations of the nonlinear filtering problem

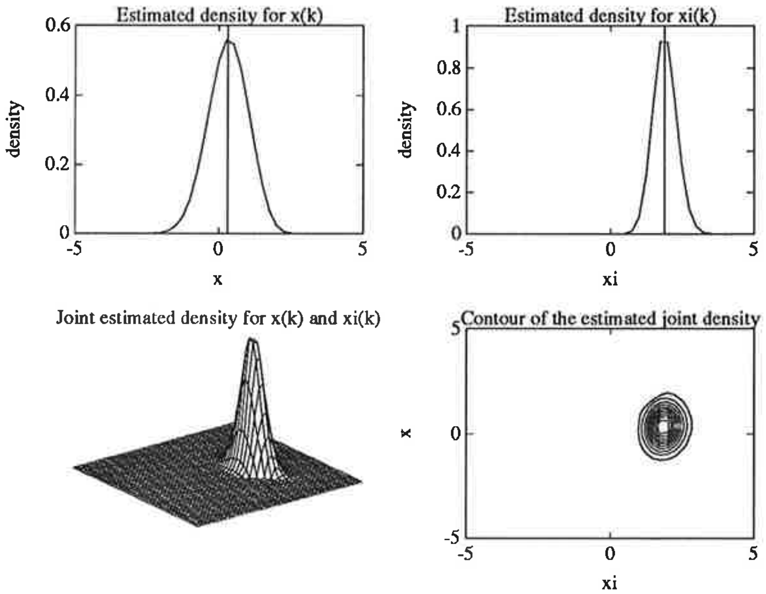

Figure 8. $k = 7$

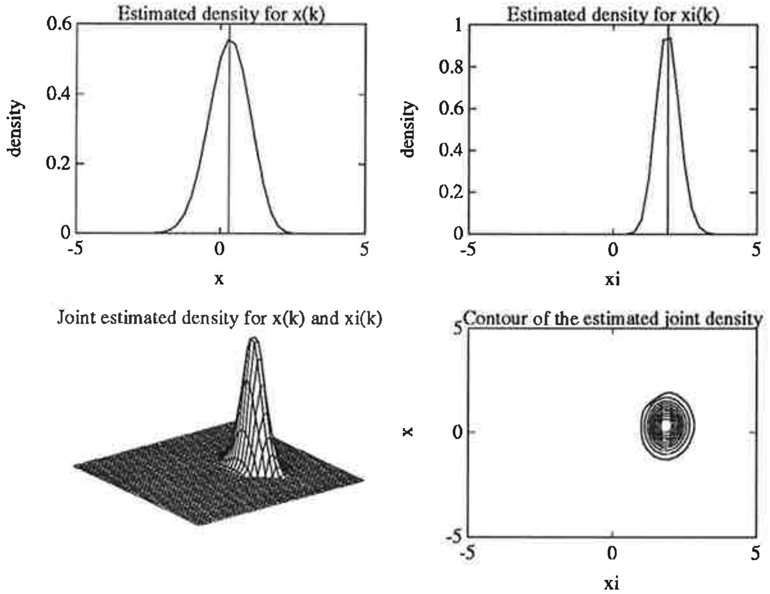

Figure 9. $k = 8$

would perhaps give further insight. It would also be interesting to compare the solution obtained by the procedure suggested in this report with the certainty equivalence solution obtained when making the very simple estimate of the running max $\hat{\xi}(k) = \max\{\hat{x}(i) : 0 \leq i \leq k\}$, where $\hat{x}(k)$ is the Kalman-filter estimate of $x(k)$. For this case no non-linear filter has to be implemented.

Acknowledgements I would like to thank Prof. Mark H. A. Davis for interesting discussions and for the friendly and stimulating atmosphere which he created during my visit to Imperial College. I would also like to thank Prof. Karl J. Åström and Prof. Björn Wittenmark for suggesting the visit and making it financially possible. For financial support I would also like to thank

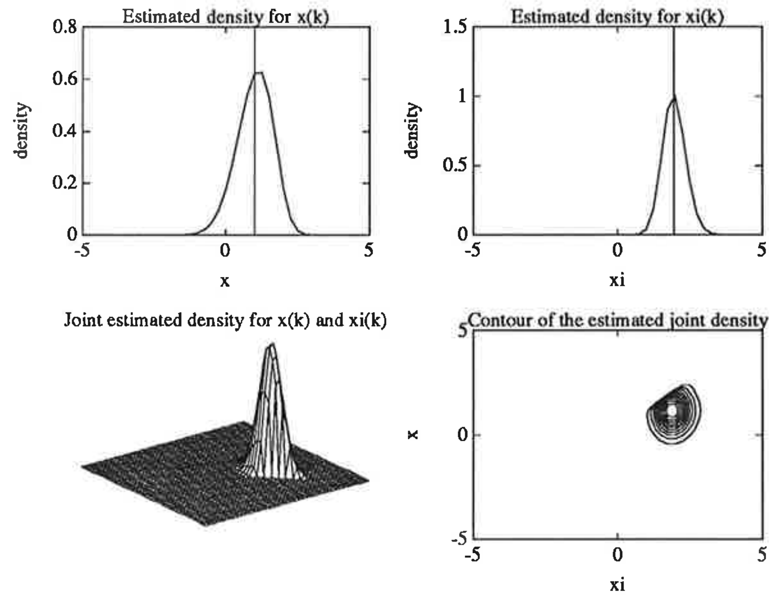

Figure 10. $k = 9$

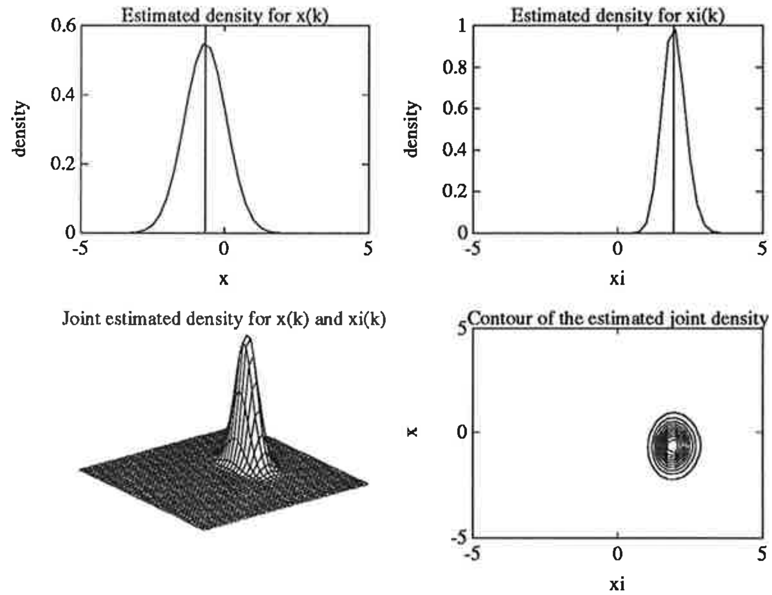

Figure 11. $k = 10$

Landshövding Nils Hörjels forskningsfond and The Swedish Institute.

5. References

ÅSTRÖM, K. J. (1961): "Analysis of a first order nonliner system with a white noise forcing function." Technical Report TN 18.057, IBM, Nordiska Laboratorier, Stockholm, Sweden.

ÅSTRÖM, K. J. (1970): *Introduction to Stochastic Control Theory*. Academic Press, Inc., San Diego.

ÅSTRÖM, K. J. (1977): "Stochastic control problems." Technical Report TFRT–3147, Department of Automatic Control, Lund Institute of Technology, Lund, Sweden.

ÅSTRÖM, K. J. and B. WITTENMARK (1990): *Computer Controlled Systems—Theory and Design*. Prentice-Hall, Englewood Cliffs, New Jersey.

BERTSEKAS, D. P. (1978): *Stochastic Optimal Control: The Discrete Time Case*. Academic Press, New York.

BORISSON, U. and R. SYDING (1976): "Self-tuning control of an ore crusher." *Automatica*, **12**, pp. 1–7.

DAHLEH, M. A. and J. B. PEARSON (1987): "l^1-optimal feedback controllers for mimo discrete-time systems." *IEEE Transactions on Automatic Control*, **32**, pp. 314–322.

FLEMING, W. H. and H. M. SONER (1993): *Controlled Markov Processes and Viscosity Solutions*. Springer-Verlag, New York.

HANSSON, A. (1991): "Control of extremes and level-crossings in stationary gaussian random processes." In *IEEE Conference on Decision and Control*.

HANSSON, A. (1992): "Minimum risk control." Technical Report TFRT–3210, Department of Automatic Control, Lund Institute of Technology, Lund, Sweden. Licentiate Thesis.

HANSSON, A. (1993): "Optimal modification of reference signals using alarm signals." In *IFAC World Congress*.

HANSSON, A. and L. NIELSEN (1991): "Control and supervision in sensor-based robotics." In *Proceedings—Robotikdagar—Robotteknik och Verkstadsteknisk Automation—Mot ökad autonomi*, pp. C7–1–10, S-581 83 Linköping, Sweden. Tekniska Högskolan i Linköping.

HEINRICHER, A. C. and R. H. STOCKBRIDGE (1991): "Optimal control of the running max." *SIAM Journal on Control and Optimization*, **29:4**, pp. 936–953.

KARATZAS, I. (1983): "A class of singular stochastic control problems." *Adv. Appl. Prob.*, **15**, pp. 225–254.

KARATZAS, I. and S. E. SHREVE (1991): *Brownian Motion and Stochastic Calculus*. Springer-Verlag, New York.

KUSHNER, H. J. and P. G. DUPUIS (1992): *Numerical Methods for Stochastic Control Problems in Continuous Time*. Springer-Verlag, New York.

LIU, G. P. and V. ZAKIAN (1990): "Sup regulators." In *IEEE Conference on Decision and Control*.

MATTSSON, S. E. (1984): "Modelling and control of large horizontal axis wind power plants." Technical Report TFRT–1026, Department of Automatic Control, Lund Institute of Technology, Lund, Sweden. Doctoral Dissertation.

OKSENDAL, B. (1989): *Stochastic Differential Equations—An Introduction with Applications*. Springer-Verlag, Berlin.

SHINSKEY, F. G. (1967): *Process-Control Systems*. McGraw-Hill, Inc., New York.

VIDYASAGAR, M. (1986): "Optimal rejection of persistent bounded disturbances." *IEEE Transactions on Automatic Control*, **31**, pp. 527–534.

WONHAM, W. M. (1968): "On the separation theorem of stochastic control." *SIAM J. Control*, **6:2**, pp. 312–326.