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Abstract

In [Jönsson and Olsson, 1993] stability theorems for feedback systems were derived using Lipschitz and Dahlquist functionals. With stability it was meant that the signals in the feedback system were bounded and depend Lipschitz continuously on the input signals. However, the stability theorems were derived in a Banach space, which imply that boundedness in many cases is a very restricted notion. To aliviate this, we will in this report introduce the extended space of a Banach space. We will then define the Lipschitz and the Dahlquist functionals for operators defined in the extended space. It will be shown that the stability theorems in [Jönsson and Olsson, 1993] also hold in extended space.

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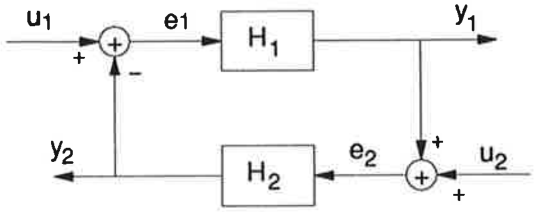

Figure 1. Feedback system under consideration

1. Introduction

In [Jönsson and Olsson, 1993] stability theorems for the feedback system in Figure 1 were derived using Lipschitz and Dahlquist constants of the operators H_1 and H_2, when H_1 and H_2 are defined on a Banach space. It was, however, noted that there are drawbacks with studying stability in a Banach space. For example, several signals that often appear in practice are not bounded in the norm of the Banach space, which deny them from consideration. It is possible to get around this problem by introducing an extended space of the Banach space. We will in this report define the Lipschitz and the Dahlquist functionals of operators defined on an extended space and then show that the stability theorems of [Jönsson and Olsson, 1993] also hold in this case.

2. Extended spaces, existence of a solution and causality

Classical stability theory results such as the small gain theorem and the passivity theory were derived in extended spaces, see [Desoer and Vidyasagar, 1975], [Zames, 1966] and [Willems, 1971]. There are several reasons for introducing extended spaces when studying stability. For example, the conditions for existence and uniqueness of a solution are very weak in extended spaces. It is also possible to consider system responses to unbounded signals and to consider unstable operators in an extended space setting. Furthermore, it is natural to study causality in extended spaces.

This rather long section will start with a discussion about extended spaces. Next, the notion of causality will be discussed, and finally some theorems giving sufficient conditions for existence of a unique solution in extended space to the feedback system in Figure 1 will be derived.

Good references to the material in this section are [Desoer and Vidyasagar, 1975] and [Willems, 1971].

Extended spaces

In [Jönsson and Olsson, 1993] all stability theorems were derived in Banach spaces X, specified to be signal spaces of the following type.

DEFINITION 1
The space X is a linear space of functions x of the type

$$x : \mathbf{T} \mapsto V$$

where $\mathbf{T}$ is a half infinite subset of the real numbers $\mathbb{R}$, i.e. $\mathbf{T}=[T_0, \infty) \subset \mathbb{R}$ or the integers Z, i.e. $\mathbf{T}=[T_0, \infty) \subset Z$. It is assumed that X is a Banach space with norm bounded elements, i.e. $x \in X \Rightarrow \|x\| < \infty$. □

Examples of signal spaces X are the $L_p^n[T_0, \infty)$ and the $l_p^n[T_0, \infty)$ spaces, where $p \in [1, \infty]$. This is a very restrictive class of signals, which does not include several in practice frequently appearing signals. For example, sinusoidal signals are not in $L_p[0, \infty)$, when $p < \infty$. One way to get around this problem is to use modified norms. In the example with sinusoidals we could, for example, use the norm $\|x\| = \sqrt[1/p]{\int_0^\infty |e^{-\lambda t}(x(t))|^p dt}$, with $\lambda > 0$. The drawback with this approach is that we must know what signals to expect in order to choose λ. For example if e^t is an expected signal then we must take $\lambda > 1$. A much more flexible approach is to use extended spaces.

The idea behind extended spaces is to consider the signals only for a finite interval of time $[0, T]$, so that the norm of the truncated signal is bounded.

Before we define the extended space X_e of the Banach space X, we need to define the truncation operator P_T.

Definition 2
Let $t \in \mathbf{T}$ then for every function $x : \mathbf{T} \to V$, P_T is the linear projection defined as

$$P_T(x)(t) = \begin{cases} x(t), & t \le T \\ 0, & \text{otherwise} \end{cases}$$

where $\mathbf{T}$ and V are defined as in Definition 1. □

Remark. Note that $P_T^2 = P_T$, i.e. P_T is a linear projection operator.

The extended space X_e of X is defined as the set of signals whose truncation at any finite time are in X. We have

Definition 3

$$X_e = \{x : \mathbf{T} \to V | P_T x \in X, \forall T \in \mathbf{T}, T < \infty\}$$

□

Remark 1. The extended space X_e is not a normed space. However, for truncated signals in X_e we use the norm in X, i.e. $\|P_T x\|$ is well defined for every $x \in X_e$ and for all $T \in \mathbf{T}$.

Remark 2. Since the Banach space X is assumed to contain only bounded signals, i.e. elements with finite norm, we see that X_e is an extension of X, which contains also unbounded signals with infinite escape time.

Remark 3. The definition above implies that the space X must contain discontinuous signals.

We will from now on make the following further assumptions on X

1. $\forall x \in X$, we have $\lim_{T\to\infty} \|P_T x\| = \|x\|$, where $\|\cdot\|$ is the norm of the Banach space X.
2. The space $P_T X_e = \{P_T x | x \in X_e\}$ is a Banach space for all fixed $T \in \mathbf{T}$.
3. The space $(P_{T_1} - P_{T_2})X_e = \{P_{T_1} x - P_{T_2} x | x \in X_e\}$ is a Banach space for all fixed $T_2, T_1 \in \mathbf{T}$ with $T_2 > T_1$.
4. If $x \in X_e$ then x is also in X if and only if $\lim_{T\to\infty} \|P_T x\| < \infty$.

Remark 1. Examples of spaces with these properties are the $L_P^n[0,\infty)$ and the $l_P^n[0,\infty)$ spaces.

We will next introduce two classes of nonlinear time invariant operators, $\mathbf{H}$ and $\mathbf{H}_e$, that operate on the Banach space X and the extended space X_e respectively.

DEFINITION 4
An operator $H \in \mathbf{H}$ maps X into itself and has the property $H(0) = 0$. □

DEFINITION 5
An operator $H \in \mathbf{H}_e$ maps X_e into itself and has the property $H(0) = 0$. □

Remark 1. The condition $H(0) = 0$ is not a restriction since an operator can always be redefined so that $H(0) = 0$.

Remark 2. The definition of extended spaces and the class of operators $\mathbf{H}_e$ allows us to consider unbounded operators.

Remark 3. It is possible to draw conclusions also in X from an stability analysis in X_e as will be shown in Section 4.

Remark 4. It is assumed in this report that all operators are defined on all of their domain space, i.e. $\text{Dom}(H) = X$ or $\text{Dom}(H) = X_e$.

Causality

In this subsection we will discuss the concept of causality.

DEFINITION 6
$H \in \mathbf{H}_e$ $(\mathbf{H})$ is causal if and only if $P_T H P_T = P_T H, \quad \forall T \in \mathbf{T}$ □

There is also an alternative definition of causality

DEFINITION 7
$H \in \mathbf{H}_e$ $(\mathbf{H})$ is causal if and only if $\forall T \in \mathbf{T}$ and $\forall u, v \in X$ (X_e) $P_T u = P_T v \Rightarrow P_T H u = P_T H v$ □

Remark. These definitions imply that the output of a causal operator does not depend on future inputs.

We will now show that the two definitions of causality are equivalent

THEOREM 1
Definition 6 and Definition 7 are equivalent.

Proof: Definition 6 $\Rightarrow$ 7: Take arbitrary $u, v \in X_e$ (X) and an arbitrary $T \in \mathbf{T}$ then if $P_T u = P_T v$, we have $P_T H u = P_T H P_T u = P_T H P_T v = P_T H v$. Definition 7 $\Rightarrow$ 6: Let $u = P_T v$ for an arbitrary $v \in X_e$ (X), then $P_T u = P_T v$, so by assumption we have $P_T H u = P_T H v$ and since $u = P_T v$, the theorem follows. □

From now on we will use the following classes of causal operators

DEFINITION 8
The subset of causal operators in $\mathbf{H}_e$ $(\mathbf{H})$ is denoted $\mathbf{H}_e^+$ $(\mathbf{H}^+)$. □

The following theorem will be used frequently in this report

THEOREM 2
The composition H_1H_2 of two operators $H_1, H_2 \in \mathbf{H}_e^+$ ($\mathbf{H}^+$) is also in $\mathbf{H}_e^+$ ($\mathbf{H}^+$).

Proof:

$$P_T H_1 H_2 = P_T H_1 P_T H_2 = P_T H_1 P_T H_2 P_T = P_T H_1 H_2 P_T$$

where we used Definition 6 in all equalities. The theorem follows since $P_T H_1 H_2 = P_T H_1 H_2 P_T$ for any $T \in \mathbf{T}$ by Definition 6 implies that $H_1 H_2$ is causal. □

It is obvious that if an operator H is in both $\mathbf{H}$ and in $\mathbf{H}_e$ then H is in $\mathbf{H}^+$ if and only if it is in $\mathbf{H}_e^+$. This fact will be used later.

Existence and unicity of solutions in extended space

We will in this subsection give theorems that state conditions under which there exist a unique solution in the extended space X_e to the feedback system in Figure 1. The solution e_1, e_2, y_1 and y_2 will depend causally on the input signals u_1 and u_2. Such a solution will be called causal. We start with a theorem from [Desoer and Vidyasagar] from which we will derive some corollaries that give conditions that are easy to check in practice. The material in this subsection is only concerned with signal spaces with continuous time, such as the $L_p^n[0, \infty)$-spaces.

The equations of the feedback system in Figure 1 are

$$\begin{aligned} e_1 &= u_1 - y_2 \\ e_2 &= u_2 + y_1 \\ y_1 &= H_1 e_1 \\ y_2 &= H_2 e_2 \end{aligned} \tag{1}$$

We assume that $H_1, H_2 \in \mathbf{H}_e^+$. It should be noted that these equations only make sense if there exists a solution to the feedback system for the input signals $u_1, u_2 \in X_e$. The theorem, which give conditions for existence of a unique and causal solution to the feedback system in Figure 1 for any $u_1, u_2 \in X_e$, will use the following equivalent form of the feedback equations in (1).

$$\begin{aligned} e &= u + He \\ y &= He \end{aligned}$$

where $e = (e_1, e_2)$, $u = (u_1, u_2)$, and $y = (-y_2, y_1)$, and where the operator $H : X_e \times X_e \mapsto X_e \times X_e$ is defined as $He = (-H_2 e_2, H_1 e_1)$. We make the assumption that the operators H_1 and H_2 are models of systems with stable non-controllable and non-observable states. We also assume zero initial conditions or otherwise that we can let the initial conditions response be part of the input signal vector u.

We have the following theorem from [Desoer and Vidyasagar, 1975]

THEOREM 3
Define the projection operator

$$\tilde{P}_{t,\Delta} = P_{t+\Delta} - P_t, \quad t, t+\Delta \in \mathbf{T}$$

Then the feedback system in Figure 1 has a unique causal solution in extended space, if $H_1, H_2 \in \mathbf{H}_e^+$, and if for all compact intervals $I \subset \mathbf{T}$, there are numbers $\gamma(I) < 1$ and $\Delta(I) > 0$ such that $\forall t \in I$ and $\forall e, e' \in X_e$ subject to $P_t e = P_t e'$,

$$\|\tilde{P}_{t,\Delta(I)}(He - He')\| \leq \gamma(I)\|\tilde{P}_{t,\Delta(I)}(e - e')\|.$$

Remark. The space $\tilde{P}_{t,\Delta} X_e$ is a Banach space according to our assumptions above.

The proof is adopted from [Desoer and Vidyasagar, 1975].

Proof: Assume that we have computed $e \in X_e \times X_e$ up to time $t > 0$. We will show how the solution can be extended to all of $\mathbf{T}$. Let I be a compact interval including t such that $I \cap (t, \infty)$ is nonempty. If Δ corresponds to I, then we can define $\tilde{e}_\Delta = \tilde{P}_{t,\Delta} e = P_{t+\Delta} e - P_t e$ and $\tilde{u}_\Delta$ similarly. Let $\tilde{P}_{t,\Delta}$ operate on the system equation $e = u - He$, then we get.

$$\begin{aligned} \tilde{e}_\Delta &= \tilde{u}_\Delta - (P_{t+\Delta} He - P_t He) \\ &= \tilde{u}_\Delta - (P_{t+\Delta} H P_{t+\Delta} e - P_t H P_t e) \\ &= \tilde{u}_\Delta - (P_{t+\Delta} H(P_t e + \tilde{e}_\Delta) - P_t H P_t (P_t e + \tilde{e}_\Delta)) \\ &= \tilde{u}_\Delta - \tilde{P}_{t,\Delta} H(P_t e + \tilde{e}_\Delta) \end{aligned} \tag{2}$$

where the second equality follows from the causality of H, the third equality follows from the definition of $\tilde{e}_\Delta$ and the properties of the projection operator P_t, and the last equality follows from the causality of H and the definition of $P_{t,\Delta}$. Equation (2) is on the form $\tilde{e}_\Delta = f(\tilde{e}_\Delta)$. This follows since $\tilde{u}_\Delta$ is given and $P_t e$ is by assumption known. We will next show that $f(\cdot)$ is a contraction, which imply that we can find the unique extension of the solution e up to time $t + \Delta$, by using the iteration $\tilde{e}_\Delta^{n+1} = f(\tilde{e}_\Delta^n)$, $n = 1, 2, 3.....$ We have for all $\tilde{e}_\Delta, \tilde{e}'_\Delta \in \tilde{P}_{t,\Delta} X_e \times \tilde{P}_{t,\Delta} X_e$,

$$\begin{aligned} \|f(\tilde{e}_\Delta) - f(\tilde{e}'_\Delta)\| &= \|\tilde{P}_{t,\Delta}(H(P_t e + \tilde{e}_\Delta) - H(P_t e + \tilde{e}'_\Delta))\| \\ &\leq \gamma(I)\|\tilde{P}_{t,\Delta}(\tilde{e}_\Delta - \tilde{e}'_\Delta)\| \\ &\leq \gamma(I)\|\tilde{e}_\Delta - \tilde{e}'_\Delta\| \end{aligned}$$

so $f(\cdot)$ is a contraction. Note that we used that e was assumed to be known up to time t, i.e. $P_t e$ is known. We can now find the solution to the feedback system by dividing $\mathbf{T}$ into a countable number of compact intervals and then piece by piece construct the solution by using iterations as above. Causality follows since the fixed point iterations involve only casual operators. □

This theorem is not very nice to use in practice since we need to compute the gains $\gamma(I)$ for the composite operator H. In the following corollary conditions for existence and unicity of a solution in extended space will be stated in terms of the product of the gains of the two operators H_1 and H_2.

COROLLARY 1

If $H_1, H_2 \in \mathbf{H}_e^+$, and if for all compact intervals $I \in \mathbf{T}$, there are numbers $\Delta(I)$ and gains $\gamma_1(I)$, $\gamma_2(I)$ defined such that $\forall t \in I$ and $\forall e, e' \in X_e$ subject to $P_t e = P_t e'$,

$$\|\tilde{P}_{t,\Delta(I)}(H_1 e - H_1 e')\| \leq \gamma_1(I)\|\tilde{P}_{t,\Delta(I)}(e - e')\|.$$

and

$$\|\tilde{P}_{t,\Delta(I)}(H_2 e - H_2 e')\| \leq \gamma_2(I)\|\tilde{P}_{t,\Delta(I)}(e - e')\|.$$

respectively, then there exists a unique causal solution in X_e to the feedback system in Figure 1 if $\gamma_1(I)\gamma_2(I) < 1$, for all compact intervals $I \in \mathbf{T}$.

Proof: Inspiration for this proof is taken from [Willems, 1971]. We can define the norm on $\tilde{P}_{t,\Delta(I)}X_e \times \tilde{P}_{t,\Delta(I)}X_e$ as

$$\|\tilde{P}_{t,\Delta(I)}e\| = \|(\tilde{P}_{t,\Delta(I)}e_1, \tilde{P}_{t,\Delta(I)}e_2)\| \stackrel{def}{=} \beta_1\|P_{t,\Delta(I)}e_1\| + \beta_2\|P_{t,\Delta(I)}e_2\|$$

where $\beta_1 = \beta_1(I)$ and $\beta_2 = \beta_2(I)$, i.e. they depend on the interval I. Since $\gamma_1(I)\gamma_2(I) < 1$ it is possible to choose $\beta_1, \beta_2 > 0$ and $\gamma \in (0,1)$ such that $\gamma^2\gamma_1(I)^{-1} > \gamma\beta_2\beta_1^{-1} > \gamma_2(I)$. If $\gamma_1 > 1$ then choose $\gamma^2\gamma_2(I)^{-1} > \gamma\beta_1\beta_2^{-1} > \gamma_1(I)$. We get

$$\begin{aligned}\|\tilde{P}_{t,\Delta(I)}(He - He')\| &= \beta_1\|\tilde{P}_{t,\Delta(I)}(H_2e_2 - H_2e_2')\| + \beta_2\|\tilde{P}_{t,\Delta(I)}(H_1e_1 - H_1e_1')\| \\ &< \beta_1\gamma_2(I)\|\tilde{P}_{t,\Delta(I)}(e_2 - e_2')\| + \beta_2\gamma_1(I)\|\tilde{P}_{t,\Delta(I)}(e_1 - e_1')\| \\ &< \gamma\beta_1\|\tilde{P}_{t,\Delta(I)}(e_1 - e_1')\| + \gamma\beta_2\|\tilde{P}_{t,\Delta(I)}(e_2 - e_2')\| \\ &< \gamma\|\tilde{P}_{t,\Delta(I)}(e - e')\|\end{aligned}$$

and hence the proof follows from Theorem 3. Note that β_1, β_2 and γ depend on I. □

The following two corollaries are easy to apply in practice

COROLLARY 2
If $H_1, H_2 \in \mathbf{H}^+$, and if either of H_1 or H_2 delays all inputs then there exist a unique and causal solution in extended space to the feedback system in Figure 1.

Proof: Assume H_1 delays all inputs T_d time units, where $T_d > 0$. If we choose $\Delta(I)$ such that $0 < \Delta(I) < T_d$ for all I, then $\forall t \in I$ and all $e, e' \in X_e$ subject to $P_T e = P_T e'$,

$$\|\tilde{P}_{t,\Delta(I)}(H_1e - H_1e')\| = 0$$

This means that $\gamma_1 = 0$. Hence, the condition $\gamma_1\gamma_2 < 1$ is satisfied and the corollary follows from Corollary 1. The case when H_2 delays all inputs is similar. □

For the next result we will consider feedback systems where the operators H_1 and H_2 are state space systems on the form

$$\begin{aligned}\dot{x}_i &= f_i(x_i) + g_i(x_i)u_i \\ y_i &= h_i(x_i) + D_i(u_i), \qquad i = 1,2\end{aligned} \tag{3}$$

where $f_i : \mathbb{R}^{n_i} \mapsto \mathbb{R}^{n_i}$, $g_i : \mathbb{R}^m \mapsto \mathbb{R}^{n_i}$, $h_i : \mathbb{R}^{n_i} \mapsto \mathbb{R}^m$ and $D_i : \mathbb{R}^m \mapsto \mathbb{R}^m$ are Lipschitz continuous and such that there exists a solution to (3) in extended space. We will also assume that $f_i(0) = 0, h_i(0) = 0$ and $D_i(0) = 0$. Then it follows that the systems above are causal operators, which map zero into zero, i.e they are in $\mathbf{H}_e^+$. We have the following corollary that gives sufficient conditions for existence of a unique causal solution in extended space to the feedback system in Figure 1.

COROLLARY 3
If H_1 and H_2 are state space operators as in (3), then there exists a unique causal solution in extended space if $L[D_1]L[D_2] < 1$, where the Lipschitz constant $L[D]$ is defined as

$$L[D] = \sup_{x \neq x'} \frac{\|D(x) - D(x')\|}{\|x - x'\|}, \quad x, x' \in \mathbb{R}^n$$

□

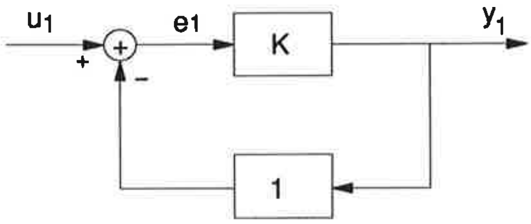

Figure 2. The feedback system of Example 1.

For the proof of the corollary we need the following lemma.

LEMMA 1
If H is a state space system as defined above, then the gain of H, $\gamma(I)$, defined as in Corollary 1 is bounded above by $\gamma(I) \leq L[D] + \mathcal{O}(\Delta(I))$ on each interval I with corresponding $\Delta(I)$.

Proof: With the notations in Corollary 1 and with $P_t u = P_t u'$, we get

$$\begin{aligned}
\|\tilde{P}_{t,\Delta}(Hu - Hu')\| &= \|\tilde{P}_{t,\Delta}[(h(x(u)) + D(u)) - (h(x(u')) + D(u'))]\| \\
&\leq \|\tilde{P}_{t,\Delta}(h(x(u)) - h(x(u')))\| + \|\tilde{P}_{t,\Delta}(D(u)) - D(u'))\| \\
&= \|(h(\tilde{P}_{t,\Delta}x(u)) - h(\tilde{P}_{t,\Delta}x(u')))\| + \|(D(\tilde{P}_{t,\Delta}u)) - D(\tilde{P}_{t,\Delta}u'))\| \\
&\leq L[h]\|\tilde{P}_{t,\Delta}(x(u)) - x(u'))\| + L[D]\|\tilde{P}_{t,\Delta}(u - u')\|
\end{aligned}$$

where the second equality follows since h and D are memoryless nonlinear functions which map zero into zero. Since we assume zero initial conditions it follows that $P_t u = P_t u' \Rightarrow P_t x(u) = P_t x(u')$. Therefore we have $\|\tilde{P}_{t,\Delta}(x(u) - x(u'))\| = \mathcal{O}(\Delta)\|\tilde{P}_{t,\Delta}(u - u')\|$, when Δ is small. Hence we get

$$\|\tilde{P}_{t,\Delta}(Hu - Hu')\| \leq (L[D] + L[h]\mathcal{O}(\Delta))\|\tilde{P}_{t,\Delta}(u - u')\|.$$

□

Proof: [of Corollary 3] We have for each interval I with corresponding $\Delta(I)$

$$\gamma_1(I)\gamma_2(I) \leq (L[D_1] + \mathcal{O}(\Delta(I)))((L[D_2] + \mathcal{O}(\Delta(I))) = L[D_1]L[D_2] + \mathcal{O}(\Delta(I))$$

Since $L[D_1]L[D_2] < 1$ there must be a $\delta > 0$ such that $L[D_1]L[D_2] \leq 1 - \delta$. Hence, by taking $\Delta(I)$ small enough, we can get $\gamma_1(I)\gamma_2(I) \leq L[D_1]L[D_2] + \mathcal{O}(\Delta(I)) < 1$ and the Corollary follows from Corollary 1. □

Corollary 3 essentially says that for a feedback interconnection of state space systems satisfying the assumptions above, we can guarantee existence and uniqueness of a causal solution in extended space when the direct feedthrough in the system is less that one. We will now give an example that discuss what may happen when the direct feedthrough is larger that one. This example is adopted from [Willems, 1971] and it also appeared in [Jönsson and Olsson, 1993].

EXAMPLE 1
Consider the feedback system in Figure 2. If a mathematical point of view is taken when analyzing the response of the system, then $y(t) = \frac{K}{1+K}u_1(t)$ when $K \neq -1$. However, in practice we will always have some slight delay in the system. This follows since the transmission speed of the signals in the system

is finite. A simple calculation shows that the response to a unit pulse at $t = 1$ (i.e. the response to $u_1(t) = \theta(t) - \theta(t-1)$ at $t = 1$) when a delay e^{-sT_ϵ} is inserted in the feedback system is

$$y(1) = K\sum_{n=0}^{n_\epsilon}(-K)^n, \quad \text{where} \quad n_\epsilon = \lfloor \frac{1}{T_\epsilon} \rfloor$$

where $\lfloor \cdot \rfloor$ denotes the truncation operator, which gives the integer part of its argument. It follows that when $|K| > 1$, $y(1) \to \infty$ as $T_\epsilon \to 0$, i.e. the output may be come extremely large when the transmission time is small. The conclusion is that the system in Figure 2 makes no sense from a practical point of view when $|K| > 1$. It is also easy to see that the case $K = -1$ gives a system response that tend to infinity as $T_\epsilon \to 0$ and the case when $K = 1$ gives a system response, which during the pulse oscillates with a frequency that tend to infinity as $T_\epsilon \to 0$. Our conclusion is that from a practical point of view, a solution to the feedback system in Figure 2 exists only if $|K| < 1$. □

We will from now on use the following definition of well posedness of the feedback system in Figure 1

DEFINITION 9
The feedback system in Figure 1 is called well posed if there exists a unique and causal solution to it in extended space. □

Remark. Well posedness has nothing to do with stability since it only assure that there exists a unique solution to the feedback system that depends causally on the inputs u_1 and u_2. A well posed system may very well give output signals in X_e that are unbounded in the norm of the Banach space X, even though the input signals are in X. Such systems are not stable.

A good model of a physical feedback system should be well posed, since otherwise either the model or the physical system makes no practical sense. So why bother about well posedness? One reason why we should be aware of well posedness of a mathematical model of a feedback system is that there are reduction methods and design methods that may affect the direct feedthrough of the system. One example is model reduction based on balanced state space realizations of a system, see for example [Johansson, 1993].

3. Lipschitz and Dahlquist functionals in extended space

The Lipschitz and the Dahlquist functionals presented in [Söderlind, 1984], [Söderlind, 1986] and [Söderlind, 1992] will be defined for operators in extended space in this section.

Lipschitz functionals

The upper and lower Lipschitz functionals of an operator $H \in \mathbf{H}$ are defined as, see [Söderlind, 1992]

$$L[H] = \sup_{u \neq v} \frac{\|H(u) - H(v)\|}{\|u - v\|}, \qquad l[H] = \inf_{u \neq v} \frac{\|H(u) - H(v)\|}{\|u - v\|} \tag{4}$$

where $u, v \in X$.

The definition of these functionals for operators defined on an extended space is not obvious since an extended space is not a normed space. Inspired by [Willems, 1971], we say that an operator $H \in \mathbf{H}_e^+$ is Lipschitz continuous in an extended space X_e with Lipschitz constant $L_e[H]$ if

$$L_e[H] = \sup_{T \in \mathbf{T}} L_T[H] < \infty$$

where

$$L_T[H] = \sup_{P_T u \neq P_T v} \frac{\|P_T(Hu - Hv)\|}{\|P_T(u - v)\|}, \quad \text{with} \quad u, v \in X_e$$

It follows from the assumptions on the space X in Section 2 that $L_T[H]$ is a non-decreasing function of T, and therefore $L_e[H] = \lim_{T \to \infty} L_T[H]$. It should be intuitively clear that $L_e[H] = L[H]$, i.e. H is Lipschitz continuous in X_e if it is Lipschitz continuous in X with the same Lipschitz constant. We actually have the theorem [Willems, 1971].

THEOREM 4
If $H \in \mathbf{H}_e^+$ is Lipschitz continuous, then H is also a Lipschitz continuous operator in $\mathbf{H}^+$ and the Lipschitz constants of H on X_e and X are equal. Conversely, if $H \in \mathbf{H}^+$ is Lipschitz continuous on X, then H is also Lipschitz continuous in $\mathbf{H}_e^+$ and the Lipschitz constants of H on X and X_e are equal.

Proof: See [Willems, 1971]. □

Note also that every Lipschitz continuous operator H in $\mathbf{H}_e$ is causal, i.e. is in $\mathbf{H}_e^+$. This follows since if H is Lipschitz continuous, then $\forall T \in \mathbf{T}$ and all $u, u' \in X_e$ we have

$$\|P_T(Hu - Hu')\| \leq L[H]\|P_T(u - u')\|$$

Therefore $P_T u = P_T u' \Rightarrow P_T H u = P_T H u'$, and causality follows from Definition 7. It is of course also the case that a Lipschitz continuous operator $H \in \mathbf{H}$ is causal.

Theorem 4 essentially says that Lipschitz continuity in $\mathbf{H}_e^+$ is equivalent to Lipschitz continuity in $\mathbf{H}^+$. We will therefore from now on only use the notation $L[\cdot]$ regardless of if we are considering Lipschitz continuous operators in $\mathbf{H}^+$ or in $\mathbf{H}_e^+$. It should also be noted that we can compute $L[\cdot]$ for a Lipschitz continuous operator in $\mathbf{H}_e^+$ by considering only signals in X. This means that the formulas, derived in [Jönsson and Olsson, 1993], for computing the Lipschitz functionals of certain linear time invariant operators and static diagonal nonlinearities also hold in extended space.

We will not use a lower Lipschitz functional in extended space but the truncated lower Lipschitz functional defined as

$$l_T[H] = \inf_{p_T u \neq p_T v} \frac{\|P_T(Hu - Hv)\|}{\|P_T(u - v)\|}, \quad \text{with} \quad u, v \in X_e$$

will be used in the proof of the next lemma and the main theorem of this section. The results in [Söderlind, 1992] relating the lower and the upper Lipschitz functionals to each other are also valid for the truncated Lipschitz functionals. If $H, H_1, H_2 \in \mathbf{H}_e^+$ then

1. $L_T[H] \geq 0$

2. $L_T[\alpha H] = |\alpha| L_T[H]$
3. $L_T[H_1] - L_T[H_2] \le L_T[H_1 + H_2] \le L_T[H_1] + L_T[H_2]$
4. $l_T[H_2] L_T[H_1] \le L_T[H_2 H_1] \le L_T[H_2] L_T[H_1]$
5. $0 \le l_T[H] \le L_T[H]$
6. $l_T[\alpha H] = |\alpha| l_T[H]$
7. $l_T[H_1] - L_T[H_2] \le l_T[H_1 + H_2] \le l_T[H_1] + L_T[H_2]$
8. $l_T[H_2] l_T[H_1] \le l_T[H_2 H_1] \le L_T[H_2] l_T[H_1]$

The first four results involving only $L_T[\cdot]$ are of course also valid for $L[\cdot]$ of operators in the extended space X_e.

The following Lemma is useful when considering stability of feedback systems.

LEMMA 2
If $H \in \mathbf{H}_e^+$ with $L[H] < 1$, then $(I+H)^{-1} \in \mathbf{H}_e^+$ with $L[(I+H)^{-1}] \le \frac{1}{1-L[H]}$.

Proof: The proof follows if we can show that for any $T \in \mathbf{T}$, and for any $y \in X_e$, the equation $P_T(I+H)x = P_T y$ has a unique solution $x \in X_e$ that depend causally and Lipschitz continuously on y. The equation can be rewritten as $P_T x = P_T y - P_T H P_T x \overset{def}{=} f(y,x)$, where $f(y,x)$ easily can be shown to be a contraction in $P_T X_e$ for any $y \in X_e$ and any $T \in \mathbf{T}$. This follows since for arbitrary $x, x' \in P_T X_e$ we have $\|f(y,x) - f(y,x')\| = \|P_T H P_T x - P_T H P_T x'\| \le L_T(H)\|P_T x - P_T x'\| \le L(H)\|P_T x - P_T x'\|$, and from the fact that $L(H) < 1$. Hence, it follows from Banach's fixed point theorem that $P_T x = P_T y - P_T H P_T x$ has a unique solution in $P_T X_e$, for any $y \in X_e$ and any $T \in \mathbf{T}$. This solution is also in X_e. Causality follows since the solution can be obtained by a fixed point iteration involving only causal operators.

It remains to show the Lipschitz bound on $(I+H)^{-1}$. For any $y, y' \in X_e$, with corresponding $x = (I+H)^{-1} y \in X_e$ and $x' = (I+H)^{-1} y' \in X_e$, we have

$$\begin{aligned} \|P_T(y - y')\| &= \|P_T((I+H)x - (I+H)x')\| \ge l_T[I+H]\|P_T(x - x')\| \\ &\ge (1 - L_T[H])\|P_T(x - x')\| \ge (1 - L[H])\|P_T(x - x')\| \end{aligned}$$

Hence

$$L[(I+H)^{-1}] = \lim_{T\to\infty} \sup_{y \ne y'} \frac{\|P_T x - P_t x'\|}{\|P_T y - P_T y'\|} \le \frac{1}{1 - L[H]}$$

□

Dahlquist functionals

The upper and lower Dahlquist functionals of a Lipschitz continuous operator $H \in \mathbf{H}$ are defined as, see [Söderlind, 1992]

$$M[H] = \lim_{\varepsilon \to 0+} \frac{L[I + \varepsilon H] - 1}{\varepsilon}; \qquad m[H] = \lim_{\varepsilon \to 0-} \frac{L[I + \varepsilon H] - 1}{\varepsilon} \tag{5}$$

As remarked in [Söderlind, 1992], it is easy to show that these functionals are well defined. If we take $0 < \varepsilon_1 < \varepsilon_2$, then

$$\begin{aligned} \frac{1}{\varepsilon_1}(L[I + \varepsilon_1 H] - 1) &= \frac{1}{\varepsilon_1}(L[\frac{\varepsilon_1}{\varepsilon_2}(I + \varepsilon_2 H) + (1 - \frac{\varepsilon_1}{\varepsilon_2})I] - 1) \\ &\le \frac{1}{\varepsilon_2}(L[I + \varepsilon_2 H] - 1) \end{aligned}$$

were we used the properties of $L[\cdot]$ and that $L[I] = 1$. So $\frac{L[I+\varepsilon H]-1}{\varepsilon}$ is monotone non-increasing, as $\varepsilon \searrow 0+$, and bounded below by $-L[H]$. Hence, it follows that the limit exists, and $M[H]$ is well defined. It is now obvious that $m[H] = -M[-H]$ is also well defined.

We take the following definitions for the Dahlquist functionals of a Lipschitz continuous operator $H \in \mathbf{H}_e^+$.

$$M_e[H] = \sup_{T\in\mathbf{T}} M_T[H], \quad \text{where} \quad M_T[H] = \lim_{\varepsilon\to 0+} \frac{L_T[I+\varepsilon H]-1}{\varepsilon}$$

and

$$m_e[H] = \inf_{T\in\mathbf{T}} m_T[H], \quad \text{where} \quad m_T[H] = \lim_{\varepsilon\to 0-} \frac{L_T[I+\varepsilon H]-1}{\varepsilon}$$

respectively.

It is easy to see that $M_e[\cdot]$ and $m_e[\cdot]$ are well defined. For example, $M_T[H]$ is a monotone non-decreasing function of T with upper bound $M[H]$. Hence, it follows that $M_e[H] = \lim_{T\to\infty} M_T[H] = M[H]$. Similarly, $m_e[H] = \lim_{T\to\infty} m_T[H] = m[H]$. This means that the Dahlquist functionals of a Lipschitz continuous operator in $\mathbf{H}_e^+$ can be computed by considering only signals in X. This means that the formulas derived in [Jönsson and Olsson, 1993], for computing the Dahlquist functionals of certain linear time invariant operators and static diagonal nonlinearities also hold in extended space. We can therefore from now on skip the index on the Dahlquist functionals of operators in $\mathbf{H}_e^+$ and simply denote them $M[\cdot]$ and $m[\cdot]$ respectively. The truncated Dahlquist functionals are related to the lower and upper Lipschitz functionals as, see [Söderlind, 1992].

1. $-l_T[H] \leq M_T[H] \leq L_T[H]$
2. $M_T[H + zI] = M_T[H] + \text{Re}z$
3. $M_T[\alpha H] = \alpha M_T[H], \quad \alpha \geq 0$
4. $m_T[H_1] + M_T[H_2] \leq M_T[H_1 + H_2] \leq M_T[H_1] + M_T[H_2]$
5. $-L_T[H] \leq m_T[H] \leq l_T[H]$
6. $m_T[H + zI] = m_T[H] + \text{Re}z$
7. $m_T[\alpha H] = \alpha m_T[H], \quad \alpha \geq 0$
8. $m_T[H_1] + m_T[H_2] \leq m_T[H_1 + H_2] \leq M_T[H_1] + m_T[H_2]$

Note that the results above not involving $l_T[\cdot]$ also hold for the functionals $L[\cdot]$,$M[\cdot]$ and $m[\cdot]$ of operators in $\mathbf{H}_e^+$.

The stability theorems of the next section are based on the following result

THEOREM 5
If $H \in \mathbf{H}_e^+$ is Lipschitz continuous, and if $m[H] > -1$ then $(I + H)^{-1} \in \mathbf{H}_e^+$ and $L[(I + H)^{-1}] \leq \frac{1}{1+m[H]}$.

Proof: The truncated lower Lipschitz functional satisfy

$$l_T[I + H] \geq m_T[H] + I \geq m[H] + I$$

where the inequalities follows from properties of the truncated functionals stated above and the fact that $m_T[\cdot]$ is monotonically decreasing to $m[\cdot]$ as

$T \to \infty$. Since $m[H] > -1$ we have $l_T[H] > 0, \quad \forall T \in \mathbf{T}$, and hence $\forall x, x' \in X_e$ and $\forall T \in \mathbf{T}$ we have

$$||P_T((I+H)x - (I+H)x')|| \geq (1 + m[h])||P_T(x - x')||$$

This implies that $I + H$ is injective on X_e. However, we have not proved that the inverse is defined on all of X_e. Before we do this we need to prove the bound on the Lipschitz functional of the inverse $(I + H)^{-1}$. From the above result we easily derive

$$\begin{aligned} L_T[(I+H)^{-1}] &= \sup_{y \neq y'} \frac{||P_T((I+H)^{-1}y - (I+H)^{-1}y')||}{||P_T(y - y')||} \\ &= \left(\inf_{P_T x \neq P_T x'} \frac{||(P_T((I+H)x - (I+H)x')||}{||P_T(x - x')||} \right)^{-1} \\ &= \frac{1}{l_T[I+H]} \leq \frac{1}{1 + m[H]}. \end{aligned}$$

where in the first equality $y, y' \in (I + H)X_e$. Since this result holds for all $T \in \mathbf{T}$, we have

$$L[(I+H)^{-1}] \leq \frac{1}{1 + m[H]}$$

It remains to prove that the inverse of $I + H$ is causal and defined on all of X_e. Introduce the operator $F(\alpha) = I + \alpha H$, for $\alpha \in [0, 1]$. It is clearly true that $F(0) = I$ is causally invertible with the inverse defined on all of X_e. We want to show that $F(1) = I + H$ has the same property. Assume that $F(\alpha)$ is invertible, then the following identity holds $F(\alpha') = (I + (\alpha' - \alpha)H(I + \alpha H)^{-1})F(\alpha)$. If $|\alpha' - \alpha| < \frac{1}{\beta}$, where $\beta = \max(\frac{L[H]}{1+m[H]}, L[H]) < \infty$, then $L[(\alpha' - \alpha)H(I + \alpha H)^{-1}] < 1, \forall \alpha \in [0, 1]$, and it follows from Lemma 2 that the first factor on the right hand side of the identity above is causally invertible on all of X_e. Divide $[0, 1]$ into N intervals $[\alpha_i, \alpha_{i+1}]$ each of length smaller than $1/\beta$, i.e. $\lceil \beta \rceil < N < \infty$, where $\lceil \cdot \rceil$ denote the smallest integer larger than the argument. Then causal invertibility of $F(\alpha_i)$ on all of X_e implies causal invertibility of $F(\alpha_{i+1})$ on all of X_e, and since $F(0)$ is invertible on all of X_e the theorem follows from an induction type argument. □

Remark. Note that it is by no means obvious that an inverse is causal. We know from the proof above that $m[H] > -1$ implies that $l_T[I + H] > 0$, i.e. that the system is injective and therefore invertible on $\text{Im}(I + H)$. However $l_T(I + H) > 0$ is not enough to ensure causality of the inverse. A simple counter example is when $H = e^{-sT_d} - 1$, where e^{-sT_d} is the linear operator that delay a signal T_d time units, where $T_d > 0$. We have $l_T(I + H) = l_T(e^{-sT}) > 0$, when $T > T_D$. The inverse is $(I + H)^{-1} = e^{sT}$, which is noncausal. Note for example, that when we use the space $L_{2e}[0, \infty)$, we get $m_2[e^{-sT_d}] = -1$, which does not imply causal invertibility.

We are now in a position to derive stability theorems in extended space. This is the topic of the next section.

4. Stability Theorems using Dahlquist functionals

We will in this section show that the stability theorems of [Jönsson and Olsson, 1993] also hold in extended space. The following theorem correspond to Theorem 12 in [Jönsson and Olsson, 1993].

Theorem 6
If the feedback system in Figure 1 is well posed with $H_2 = G_2 \in \mathbf{H}_e^+$ linear and with $H_1 \in \mathbf{H}_e^+$, then a sufficient condition for the solution $e_1, e_2, y_1, y_2 \in X_e$ to depend Lipschitz continuously and causally on $u_1, u_2 \in X_e$ is that H_1 and G_2 are Lipschitz continuous with $m[G_2H_1] > -1$. Further, if $u_1, u_2 \in X$ then $e_1, e_2, y_1, y_2 \in X$.

Proof: From the feedback equations (1) we have for any $T \in \mathbf{T}$

$$P_Te_1 = P_Tu_1 - P_TG_2(u_2 + H_1e_1) \tag{6}$$

which is well defined by the well posedness assumption. Adding $P_TG_2H_1e_1$ on both sides of this equation gives

$$\begin{aligned} P_T(I + G_2H_1)P_Te_1 &= P_Tu_1 + P_TG_2H_1P_Te_1 - P_TG_2(P_Tu_2 + P_TH_1P_Te_1) \\ &= P_Tu_1 - P_TG_2P_Tu_2 \end{aligned}$$

From Theorem 5 we know that $m(G_2H_1) > -1$ is a sufficient condition for the operator on the left hand side to be causally invertible in extended space. Hence, we get

$$P_Te_1 = P_T(I + G_2H_1)^{-1}(P_Tu_1 - P_TG_2P_Tu_2)$$

It is now easy to obtain the bound

$$\|P_T(e_1 - e_1)'\| \le \frac{1}{1 + m(G_2H_1)}(\|P_T(u_1 - u_1')\| + L(G_2)\|P_T(u_2 - u_2')\| \tag{7}$$

where e_1 and e_1' are the solutions to (6) when the inputs are $u_1, u_2 \in X_e$ and $u_1', u_2' \in X_e$ respectively. Since $P_Ty_1 = P_TH_1P_Te_1$, $P_Te_2 = P_Tu_2 + P_Ty_1$ and $P_Ty_2 = P_TG_2P_Te_2$, and since G_2 and H_1 are Lipschitz continuous it follows that all the signals in the feedback system depend Lipschitz continuously and causally on u_1 and u_2. Further if we take $u_1' = u_2' = 0$ and $u_1, u_2 \in X$ then we can let $T \to \infty$ on the right hand side of (7). We get

$$\|P_Te_1\| \le \frac{1}{1 + m(G_2H_1)}(\|u_1\| + L(G_2)\|u_2\|, \quad \forall T \in \mathbf{T}$$

This means that $e_1 \in X$ and it is easy to see that e_2, y_1 and y_2 are also in X. □

The following two theorems are also easily proven

Theorem 7
If the feedback system in Figure 1 is well posed with $H_1 = G_1$ and with $H_2 \in \mathbf{H}_e^+$, then a sufficient condition for the solution $e_1, e_2, y_1, y_2 \in X_e$ to depend Lipschitz continuously and causally on $u_1, u_2 \in X_e$ is that G_1 and H_2 are Lipschitz continuous with $m[G_1H_2] > -1$. Further, if $u_1, u_2 \in X$ then $e_1, e_2, y_1, y_2 \in X$.

Theorem 8
If $u_2 \equiv 0$ and if the feedback system in Figure 1 is well posed with $H_1, H_2 \in \mathbf{H}_e^+$ then a sufficient condition for a solution $e_1, e_2, y_1, y_2 \in X_e$ to depend Lipschitz continuously and causally on $u_1, u_2 \in X_e$ is that H_1 and H_2 are Lipschitz continuous with $m[H_2H_1] > -1$. Further, if $u_1, u_2 \in X$ then $e_1, e_2, y_1, y_2 \in X$.

Remark 1. The causality is actually guaranteed by the well posedness assumption.

The well posedness assumption in the theorems above is essential to sort out pathological cases such as the following.

EXAMPLE 2
In Example 1 we have a feedback system were both operators $H_1 = K$ and $H_2 = 1$ are linear and Lipschitz continuous. Further, we have that $m[H_2 H_1] = m[K] = K$. Therefore the conditions for Lipschitz continuity of Theorem 6 are fulfilled when $K > -1$. However, if $K > 1$ then the analysis in Example 1 shows that the system is unstable when a practical point of view is taken. The system should be regarded as not being well posed when $K > 1$.

□

5. Conclusions

We have in this report discussed extended spaces, causality of operators and well posedness of feedback systems. Further we have shown that the stability theorems involving Dahlquist functionals in [Jönsson and Olsson, 1993] also hold in extended space. This allows us to consider stability of a feedback system when the input-signals are not in a Banach space.

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