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Price Based Linear Quadratic Control Under Transportation Delay ⋆

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Abstract: We study a simple transportation problem on a string graph. The objective is to regulate the node levels of some decaying quantity to optimize dynamical performance. This can be achieved by controlling the flows, which are subject to delay, between neighbouring nodes. The problem is considered from two perspectives. In the first (the social perspective), all nodes cooperate to find the flows that maximize the aggregated utility of the entire transportation network. In the second (the user perspective), the nodes instead try to maximimize their own utility. Our main contribution is to give an implementation of the feedback law that gives the social optimum, that only depends on the local states and a set of prices defined by a distributed update rule. These prices align the social and user optimum in a budget neutral way, and give all nodes no worse cost than if they were on their own.

Keywords: Distributed control and estimation, Control of networks, Convex optimization, Distributed optimisation for large-scale systems, Systems with time-delays

1. INTRODUCTION

In this work we study the optimal transportation of a decaying quantity. The dynamics studied could for example describe a transportation network, as illustrated in Section 2. The objective is to control the node levels by regulating the transportation between the nodes to optimize performance. The challenge is to do this in a manner that scales well with network size, whilst accounting for dynamical effects such as transportation delays.

To capture the essence of the problem, we consider a string network with N nodes in discrete time. The nodes are numbered as in Fig. 1, where the most downstream node has index one, the second most downstream has index two, and so on. Furthermore, we index the links according to the node which they enter. We let the transportation delay be one time unit on every link, and define the dynamics to be

$$z_i[t+1] = \alpha\left(z_i[t] + u_i[t-1]\right) - u_{i-1}[t], \tag{1}$$

for $1 \le i \le N$. The variable $z_i[t]$ is the level of the quantity in node i (at time step t), and the control input $u_{i-1}[t]$ is the amount leaving node i (if written in state-space form, a choice of the system system state would be $\{z_i[t],\ u_i[t-1]\}$). At the boundaries we have $u_0[t] = 0$, $u_N[t] = 0$. The constant $0 < \alpha \le 1$ is the decay rate.

We assume that node i values its level $z_i[t]$ according to the quadratic function, $U_i(z_i[t]) = b_i z_i[t] - 1/2 q_i z_i[t]^2$, where $q_i > 0$, $b_i > 0$, and $b_{i+1} = \alpha b_i$. The last assumption arises naturally when considering transportation about an equilibrium, and will be motivated fully in the next section. We study the problem from two perspectives. First we consider the social optimum

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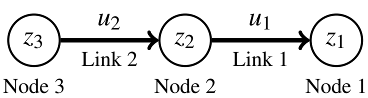

Fig. 1. Illustration of the studied problem when $N = 3$. There are three nodes whose levels z are to be controlled. Each link has a corresponding input u that is the flow entering the link and arriving one time unit later. We have also indicated the indexing conventions for the nodes and links.

problem, where the objective is to maximize the sum of local utility functions for the nodes:

$$\begin{aligned} \underset{z,u}{\text{maximize}} \quad & J(z) = \sum_{i=1}^{N}\sum_{t=1}^{T}\left(b_i z_i[t] - \frac{1}{2} q_i z_i[t]^2\right) \\ \text{subject to} \quad & \text{Dynamics in (1)} \\ & z_i[0] \text{ given.} \end{aligned} \tag{2}$$

In the above, $z[t] \in \mathbb{R}^N$ is defined for $1 \le t \le T$ and $u[t] \in \mathbb{R}^{N-1}$ is defined for $0 \le t \le T-1$. This problem is an instance of a Linear Quadratic control problem with a linear term in the optimization criterion. The absence of an input penalty allows for a highly structured solution that is efficient to calculate. This was demonstrated for the infinite horizon case in Heyden et al. (2018) (with $b_i = 0$) by finding the solution to an algebraic Riccati equation.

Our main contribution is to provide a distributed solution to eq. (2), using a pricing mechanism to implement the feedback law. The prices are used to adjust the utility of the individual nodes, so that each has an optimal level when considering its own utility. Our scheme is budget neutral, is simple to implement even in very large networks, and leaves no node worse off than if it were on its own. These results are previewed at the end of this section, and presented in Section 3.

Our results add to the growing body of work on distributed control. Early effort in this regard include team game problems, where a set of agents work towards a common goal, but with different information, see for example Radner (1962). Important work along these lines includes Rotkowitz and Lall (2006), where sufficient conditions for finding a distributed controller using convex optimization were given. More recent work includes System Level Synthesis, see for example Anderson et al. (2019), that allows for scalable synthesis and implementation of distributed controllers using a novel controller architecture. In contrast, the structure in the controllers in this paper is inherited from the plant. This is similar in nature to the work on spatially invariant systems from Bamieh et al. (2002), where the optimal control law was shown to be localized in space. The structure of the controllers derived in this work also have strong similarities to those from Shah and Parrilo (2013), where the optimal poset-causal controller is found.

The objectives of this paper are also well aligned with the theme of solving optimization problems using prices based on Lagrange multipliers. For pioneering work on the use of prices for coordination, see Cohen (1978), which was later used to control water supply networks in Carpentier and Cohen (1993). The use of Lagrange multipliers for coordination is well studied, for example as shadow prices in the work on Internet congestion control from Kelly et al. (1998), Low et al. (2002), and in the distributed MPC schemes from Giselsson et al. (2013). Lagrange multipliers have also been suggested for controlling power grids Jafarian et al. (2016), Jokić et al. (2009). Normally these problems are either static, or of high complexity. Requiring either solving for all the prices and states at the same time, or solving a Riccati equation. In our specific problem, the prices are not the Lagrange multipliers, but rather a linear combination of current node levels and goods in transit. These prices are much simpler to compute than the Lagrange multipliers are.

Preview of Results

To give an incentive for the individual nodes to follow the social optimum, we will introduce prices and study the problem from a node perspective. Each node i will be presented with a price vector $p_i[t]$ that will affect the nodes utility proportional to their levels, so that its total utility $V_i(z_i, p_i)$ is given by

$$V_i(z_i,p_i) = p_i[0]z_i[0] + \sum_{t=1}^{T}\left(b_i z_i[t] - \frac{1}{2}q_i z_i^2[t] - p_i[t]z_i[t]\right). \quad (3)$$

Typically increasing $z_i[t]$ will lead to a trade off between the increased utility from the $b_i z_i[t] - 1/2q_i z_i^2[t]$ term, and the decreased utility from the cost $p_i[t]z_i[t]$. The utility function in (3) will be further discussed in Section 2. Each node will consider the following problem

$$\begin{aligned} \underset{z_i}{\text{maximize}} \quad & V_i(z_i,p_i) \\ \text{subject to} \quad & p_i \text{ given.} \end{aligned} \quad (4)$$

We find the solution to social optimum problem by studying the Lagrangian of the problem. The main contribution lies in deriving a set of prices from the Lagrange multipliers that allows for a distributed implementation of the optimal feedback law and aligns social and user optimum. However, in contrast to 'typical' Lagrangian approaches, the prices are given by a simple, temporally decoupled, expression

$$p_i[t+1] = \begin{cases} b_i - \gamma_i \sum_{j=1}^{i} z_j[t] + u_j[t-1] & 0 \le t \le T-i \\ 0 & t > T-i. \end{cases} \quad (5)$$

In the above, γ is a constant that can be computed ahead of time by the following iteration:

$$\gamma_1 = q_1, \quad \gamma_i = \frac{\alpha^2\gamma_{i-1}q_i}{\alpha^2\gamma_{i-1}+q_i},\ i \ge 2. \quad (6)$$

With p as in (5), the optimal inputs are given by

$$u_{i-1}[t] = \alpha(z_i[t] + u_i[t-1]) - \frac{1}{q_i}\left(p_i[t+1] - b_i\right). \quad (7)$$

The combined structure of (5) and (7) allows for a simple implementation of the optimal u using only local communication. The expression for the optimal prices in (5) indicates that the price should increase the more a node values its level from the term b_i, and decrease when more goods are available.

2. MOTIVATION FOR THE PROBLEM

We will consider a simple model of a generic transportation network for a decaying quantity. This could for example be a district heating network, or an inventory control system for decaying goods. In this section we will show that when controlling such a system around an equilibrium, the dynamics in (1) arise. This could, for example, be of interest if the operating conditions changes and the system needs to be shifted from the old to the new equilibrium point.

Each node i in the network has a constant production (or consumption) w_i. Furthermore, the quantity can be transported along the links of the system. Finally, we make the simplifying assumption that the decay has a homogeneous rate $1-\alpha$ throughout the system. We can write the dynamics for the level ζ_i in each node i as

$$\zeta_i[t+1] = \alpha\left(\zeta_i[t] + v_i[t-1]\right) + w_i - v_{i-1}[t].$$

In the above v_{i-1} is the quantity leaving node i and v_i is the quantity arriving to node i. The quantity leaving the node goes immediately into transportation and will take one time unit to arrive. For the physical interpretation the flows $v[t]$ must be positive. This will generally be the case if there is producer at the top of the network.

Let the flows $v[t] = \bar{v}$ be constant. Then each node will have an equilibrium level $\bar{\zeta}_i$ where the inflow equals the outflow,

$$\bar{\zeta}_i = \frac{1}{1-\alpha}(w_i + \alpha\bar{v}_i - \bar{v}_{i-1}).$$

We assume that each node values its level according to a quadratic function $U_i(\zeta_i)$. Then the optimal equilibrium is the solution to

$$\begin{aligned} \underset{\bar{v}}{\text{maximize}} \quad & \sum_i U_i(\bar{\zeta}_i) \\ \text{subject to} \quad & \bar{\zeta}_i = \frac{1}{1-\alpha}(w_i + \alpha\bar{v}_i - \bar{v}_{i-1}). \end{aligned} \quad (8)$$

Now we study the system around this equilibrium. We introduce a new level vector $z \in \mathbb{R}^N$ describing the levels relative to the optimal levels, and a new input vector $u \in \mathbb{R}^{N-1}$ that describe the flows relative to the optimal flows,

$$u_i[t] = v_i[t] - \bar{v}_i,$$
$$z_i[t] = \zeta_i[t] - \bar{\zeta}_i.$$

Then the dynamics for z are given by (1). The utility relative to the optimum can be written as

$$U_i(\bar{\zeta}_i + z_i[t]) - U_i(\bar{\zeta}_i) = \sum_{t=1}^{T} \left(b_i z_i[t] - \frac{1}{2} q_i z_i^2[t] \right),$$

where $q_i < 0$. Note that $u_i[t]$ can take negative values.

Remark 1. Since $\bar{\zeta}_i$ solves (8), we must have that

$$b_{i+1} = \alpha b_i. \tag{9}$$

To see this, observe that if it were not the case, then the utility could be improved by making a small perturbation ε to $\bar{v}_i$ which would increase $\bar{\zeta}_i$ by $\alpha/(1-\alpha)\varepsilon$ and decrease $\bar{\zeta}_{i+1}$ by $1/(1-\alpha)\varepsilon$.

The optimal control around the equilibrium can be found by solving (2). The lack of penalty on the flows can be motivated by that the cost of changing the transportation is small. For example, if the transportation is done via trucks, then there is typically a very low, or no additional cost, if a truck transports more goods. However, there is still a loss in moving the quantity in that it is not being utilized while in transportation.

How can the user problem in (3) be motivated? It is natural that the users in the transportation network pay, or are paid, for changes in equilibrium levels. If the new equilibrium level of a node is lower, then that node would expect to be paid to actively send away some of its quantity, since this reduction will reduce its own utility. Similarly, if a node is to receive a higher level, that node would be expected to pay for it. This is captured by the term $p_i[0]z_i[0]$. Since the new equilibrium cannot be reached immediately, the nodes should also pay for the time periods where they have a higher level, and be compensated while it is too low. This is captured by the terms $p_i[t]z_i[t]$. We note that close to the equilibrium, $p_i[t] > 0$, $t \geq 1$. We shall later see that that is the case for $t = 0$ as well.

3. RESULTS

We start by giving the solution to (2), in Theorem 1 below. This result shows that the i-th entry of the optimal control input can computed based only on local measurements of the quantity z and the goods in transit u, and a local price p_i. Next we show in Proposition 2 how these prices can be used to align the user problem in (4) to the social optimum. The prices have additional appealing properties. Firstly, the node utilities are higher than if they had zero flows and no payments, and secondly, the sum of all payments equal zero. The proofs of the results presented here will be given in Section 5.

Theorem 1. Define γ_i as in (6), and $p[t]$ by

$$p_i[t+1] = \begin{cases} b_i - \gamma_i \sum_{j=1}^{i} z_j[t] + u_j[t-1], & 0 \leq t \leq T-i \\ 0 & t > T-i. \end{cases} \tag{10}$$

Then the optimal u for (2) is given by

$$u[t] = \alpha \,[\mathbf{0}\;\; I] \begin{bmatrix} z_1[t] + u_1[t-1] \\ \vdots \\ z_{N-1}[t] + u_{N-1}[t-1] \\ z_N[t] \end{bmatrix} - \begin{bmatrix} 0 & \frac{1}{q_2} & & 0 \\ & & \ddots & \\ 0 & & & \frac{1}{q_N} \end{bmatrix} (p[t+1] - b). \tag{11}$$

With $p[t] = [p_1[t], \ldots, p_N[t]]^T$ and $b = [b_1, \ldots, b_N]^T$, $\mathbf{0}$ a column vector of length $N-1$ and I an identity matrix of dimension $N-1$.

If we write out the expressions for each input we get (7). From the theorem we see that there exists a simple method for calculating the optimal feedback law, using only local states and local prices. Furthermore, $p_i[t+1]$ can be calculated recursively through the graph,

$$p_i[t+1] = \gamma_i \left(-z_i[t] - u_i[t-1] + \frac{1}{\gamma_{i-1}} p_{i-1}[t+1] \right) + (1 - \frac{1}{\alpha \gamma_{i-1}}) b_i,$$

requiring only local communication. This expression is also interesting in that each node only needs to share a combination of its level and utility function. This gives some privacy compared to sharing both the level and the utility function.

Equation (7) has a very natural interpretation from the user optimal perspective, as it is the solution to

$$\begin{aligned} \underset{u_{i-1}[t-1]}{\text{minimize}} \quad & b_i z_i[t] - \frac{1}{2} q_i z_i^2[t] - p_i[t] z_i[t] \\ \text{subject to} \quad & z_i[t] = \alpha(z_i[t-1] + u_i[t-2]) - u_{i-1}[t], \end{aligned}$$

which corresponds to the node optimizing its utility for the next time point.

Remark 2. At first sight it may seem like (11) is non causal as the input at time t depends on prices at time $t+1$. However, from (10) we can see that prices at time $t+1$ depends on state at time t, and the expression is indeed causal. As the prices are associated with the states when the input has taken affect, it is natural that the prices are one time-index ahead of the inputs.

Remark 3. It might be surprising that some of the prices are zero and thus the corresponding nodes will have optimal levels, $z_i = -b_2/q_2$, as t gets closer to T. This is due to the boundary effects of the system, where the level of a node can be increased without decreasing the value of others. This is achieved by by exploiting that the goods sent at time $T-1$ will not reach its destination within the optimization horizon .

Proposition 2. In addition to the definitions in Theorem 1, let $m = \min(T-i, N)$, and

$$p_i[0] = \sum_{t=1}^{T-(i-1)} \left(\alpha^{t-(t_0+\tau)} b_i \right) - \alpha \left(\sum_{j=i}^{m} \gamma_j \sum_{k=1}^{j} z_k[0] + u_k[-1] + \sum_{j=N+1}^{T} \gamma_N \alpha^{2(j-N)} \sum_{k=1}^{N} z_k[0] + u_k[-1] \right)$$

Then:

(i) The optimal z for (2) and (4) are equal.
(ii) The node utility satisfies

$$V(z_i, p_i) \geq \sum_{t=1}^{T} \left(b_i \alpha^t z_i[0] - q_i (\alpha^t z_i[0])^2 \right).$$

(iii) The sum of payments are equal to zero,

$$\sum_{i=1}^{N} \left(p_i[0] z_i[0] - \sum_{t=1}^{T} p_i[t] z_i[t] \right) = 0.$$

The proposition shows not only that is possible to align the user and social optimum, but also that each node is never worse off than if they had no in- or outflow. This is what would happen if the node was not a part of the transportation network. This is an important property, as otherwise the nodes would be reluctant to be part of the network. Furthermore the payment scheme is budget neutral (the payments sum to zero). This is significant as if the scheme had a budget deficit, it would be very hard to find someone to supply additional money to drive the system, while receiving nothing in return.

4. ANALYSIS OF THE LAGRANGIAN

In this section we perform the necessary analysis of the Lagrangian for (2) needed to prove Theorem 1 and Proposition 2. An important part is to construct an alternative user utility based on the Lagrange multipliers, and showing that it is equal to the original one in (4).

4.1 Lagrangian

The Lagrangian of (2) is given by

$$\begin{aligned} L(z,u,\lambda) = J(z) + \sum_{t=0}^{T-1} \Bigg[& \\ & \lambda_1[t+1]\Big\{\alpha\big(z_1[t]+u_1[t-1]\big) - z_1[t+1]\Big\} \\ & + \sum_{i=1}^{N-1} \lambda_i[t+1]\Big\{\big(\alpha(z_i[t]+u_i[t-1]) - u_{i-1}[t]\big) - z_i[t+1]\Big\} \\ & + \lambda_N[t+1]\Big\{\big(\alpha z_N[t] - u_{N-1}[t]\big) - z_N[t+1]\Big\}\Bigg]. \end{aligned} \tag{12}$$

The Lagrange dual variable has dimensions $\lambda[t] \in \mathbb{R}^N$ for $1 \le t \le T$. The dual variables $\lambda_i[t]$ has a natural economic interpretation as the marginal change in social utility when $z_i[t]$ changes.

4.2 Alternative User Optimal Problem

Based on the Lagrangian we define an alternative user utility function, and show that it is equal to the original in (3). In this formulation the node utility will include a cost based on the level change,

$$\hat{V}_i(z_i,\lambda_i) = \sum_{t=1}^{T} b_i z_i[t] - \frac{1}{2} q_i z_i^2[t] - \lambda_i[t] \underbrace{\left(z_i[t] - \alpha z_i[t-1]\right)}_{\text{change in level}}. \tag{13}$$

Note that all the terms in $\hat{V}_i$ are in the Lagrangian L. By letting

$$p_i[t] = \begin{cases} \alpha\lambda_i[1] & t = 0 \\ \lambda_i[t] - \alpha\lambda_i[t+1] & 1 \le t \le T-1 \\ \lambda_i[T] & t = T, \end{cases} \tag{14}$$

the node utility can be rewritten as

$$\begin{aligned} &\hat{V}_i(z_i,\lambda_i) \\ &= \sum_{t=1}^{T-1} b_i z_i[t] - \frac{1}{2} q_i z_i^2[t] - (\lambda_i[t] - \alpha\lambda_i[t+1]) z_i[t] \\ &\quad + \alpha\lambda_i[1] z_i[0] - \lambda_i[T] z_i[T] \\ &= \sum_{t=1}^{T} \Big(b_i z_i[t] - \frac{1}{2} q_i z_i^2[t] - p_i[t] z_i[t] \Big) \\ &\quad + p_i[0] z_i[0] = V_i(z_i, p_i). \end{aligned}$$

Thus the two different user optimal problems are equal, and we can analyze either one of them. We will use the Lagrangian version for analysis, while the p version will be used for implementation.

4.3 Optimality Conditions

The optimization problem in (2) is concave as it is the maximization of a concave cost function under affine constraints. Thus necessary and sufficient optimality conditions are given by the KKT conditions (see Boyd and Vandenberghe (2004))

$$\nabla_\lambda L = 0, \quad \nabla_u L = 0, \quad \nabla_z L = 0.$$

$\nabla_\lambda L = 0$ is equal to the dynamics constraint being satisfied.

For a standard LQ problem with a penalty on the input, $\nabla_u L = 0$ gives u as a function of λ. See Cannon et al. (2008) for a slightly more general MPC case. Here we instead get the following

$$\frac{\partial L}{\partial u_i[t]} = -\lambda_{i+1}[t+1] + \alpha\lambda_i[t+2] = 0, \quad 0 \le t \le T-1 \tag{15a}$$

$$\frac{\partial L}{\partial u_i[T-1]} = -\lambda_{i+1}[T] = 0. \tag{15b}$$

Note that it is due to the lack of penalty on u that $\nabla_u L$ is independent of u.

Next we study $\nabla_z L$. Normally this allows us to solve for λ given z, going backwards in time. Calculating the gradients gives

$$\frac{\partial L}{\partial z_i[t]} = b_i - q_i z_i[t] + \alpha\lambda_i[t+1] - \lambda_i[t] \quad 1 \le t \le T-1 \tag{16a}$$

$$\frac{\partial L}{\partial z_i[T]} = b_i - q_i z_i[T] - \lambda_i[T]. \tag{16b}$$

Combining the two optimality conditions, we get the following lemma.

Lemma 3. The optimal inventory level z satisfies

$$z_i[t] = \frac{\alpha q_{i-1}}{q_i} z_i[t+1] \tag{17}$$

for $i \ge 2$ and $t \le T-1$.

Proof. Using (16a) and (15a) gives for $t \le T-2$

$$\begin{aligned} z_i[t] &= \frac{\alpha\lambda_i[t+1] - \lambda_i[t] + b_i}{q_i} \\ &= \frac{\alpha q_{i-1}}{q_i} \frac{\alpha\lambda_{i-1}[t+2] - \lambda_{i-1}[t+1] + b_{i-1}}{q_{i-1}} + \frac{b_i - \alpha b_{i-1}}{q_i} \\ &= \frac{\alpha q_{i-1}}{q_i} z_{i-1}[t+1] \end{aligned}$$

where we have used that $\alpha b_{i-1} = b_i$. The case for $t = T-1$ follows similarly.

5. PROOF OF THEOREM 1 AND PROPOSITION 2

We are now ready to prove Theorem 1 and Proposition 2.

Proof of Theorem 1: For every input u_{i-1}, $i \le N$, we have from the dynamics that

$$\begin{aligned} z_i[t+1] = \alpha(z_i[t] + u_i[t]) - u_{i-1}[t] \Rightarrow \\ u_{i-1}[t] = \alpha(z_i[t] + u_i[t]) - z_i[t+1]. \end{aligned}$$

Using (16a–b) we get for $t \le T-2$, that the optimal u_{i-1} must satisfy

$$u_{i-1}[t] = \alpha(z_i[t] + u_i[t]) + \frac{\alpha\lambda_i[t+2] - \lambda_i[t+1] + b_i}{q_i}$$

and for $t = T-1$,

$$u_{i-1}[T-1] = z_{i+1}[T-1] + u_i[T-1] + \frac{-\lambda_i[T] + b_i}{q_i}.$$

Thus with the relation between p and λ as defined in (14) we have that the optimal u is given by (11). The expressions for p in (10) follows from Proposition 8 and Lemma 7(see the appendix).

Proof of Proposition 2: As the nodes choices of levels has no effect on the prices, the optimal level from the nodes perspective must satisfy

$$0 = \frac{\partial \hat{V}_i}{\partial z_i[t]} = \frac{\partial L}{\partial z_i[t]}.$$

This must also hold for the social optimum, thus proving (i).

Furthermore we see that choosing the social optimum inventory levels are better than choosing $z_i[t] = \alpha^t z_i[0]$, as it is not a minimizer of V_i. Thus proving (ii).

The sum of all the payments are

$$-\sum_{i=1}^{N}\sum_{t=1}^{T}\lambda_i[t]\Big(z_i[t]-\alpha z_i[t-1]\Big). \tag{18}$$

Using that

$$z_i[t]-\alpha z_i[t-1] = -u_{i-1}[t-1]+\alpha u_i[t-2],$$

The sum in (18) can be rewritten as

$$\sum_{i=1}^{N-1}\left(\sum_{t=0}^{T-2}\Big(-\lambda_{i+1}[t+1]+\alpha\lambda_i[t+2]\Big)u_i[t]-\lambda_{i+1}[T]u_i[T-1]\right).$$

This is equal to zero, since $\lambda_{i+1}[t+1] = \alpha\lambda_i[t+2]$ and $\lambda_i[T] = 0$ for $i \geq 2$. Thus proving (iii).

6. CONCLUSIONS

We have considered an optimal control problem for a simple transportation network from the social and user perspective. By solving the social problem using a Lagrange multiplier approach, we gave an implementation of the feedback law in terms of local prices and local states that allows for a distributed implementation. Furthermore, these prices aligned the two problem in a budget neutral way so that the nodes are never worse off than if they had been on their own.

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APPENDIX

In the appendix we will derive the optimal Lagrange multipliers $\lambda[t_0+\tau]$ in terms of $z[t_0-1]$ and $u[t_0-2]$. We will show that each λ can be found as a sum of the the corresponding node levels in Lemma 4. These node levels can in turn be found by studying a time shifted aggregate level as shown in Lemma 5. This shifted aggregate can then be written in terms of a non shifted aggregate at t_0-1 in Lemma 6.

Lemma 4. The optimal Lagrange multipliers are given by

$$\lambda_i[t_0] = \sum_{t=t_0}^{T}\alpha^{t-t_0}\big(b_i - q_i z_i[t]\big).$$

Proof. We have from (16) that $\lambda_i[T] = b_i - q_i z_i[T]$ and $\lambda_i[t] = b_i - q_i z_i[t] + \alpha\lambda_i[t+1]$. From that the lemma follows trivially.

Next we show how each node level can be written in terms of a time shifted level vector.

Lemma 5. The optimal inventory levels satisfy

$$z_i[t_0+k] = \begin{cases} \dfrac{\gamma_{i+k}}{\alpha^k q_i}\displaystyle\sum_{j=1}^{i+k}\frac{z_j[t_0+k+(i-j)]}{\alpha^{k+i-j}} & i+k \leq N \\ \dfrac{\gamma_N}{\alpha^{N-i} q_i}\displaystyle\sum_{j=1}^{N}\frac{z_j[t_0+k+(i-j)]}{\alpha^{N-j}} & i+k > N. \end{cases} \tag{19}$$

Proof. We start by showing the lemma for $k=0$. Using (17) gives

$$z_2[t] = \frac{\alpha q_1}{q_2} z_1[t+1] \Rightarrow$$

$$(1+\frac{\alpha^2 q_1}{q_2})z_2[t] = \frac{\alpha^2 q_1}{q_2}\left(\frac{z_1[t+1]}{\alpha}+z_2[t]\right) \Rightarrow$$

$$z_2[t] = \frac{\alpha^2\gamma_1}{q_2+\alpha^2\gamma_1}\left(\frac{z_1[t+1]}{\alpha}+z_2[t]\right).$$

Now assume that (19) holds for $i-1$ and $k=0$. Then using (17) again gives

$$z_i[t] = \frac{\alpha q_{i-1}}{q_i} z_{(i-1)}[t+1]$$
$$= \frac{\alpha q_{i-1}}{q_i} \frac{\gamma_{i-2}}{q_{i-1}+\gamma_{i-2}} \left(\sum_{j=1}^{i-1} \frac{z_j[t+((i-1)-j)]}{\alpha^{(i-1)-j}} \right)$$

Which gives that

$$\left(1+\frac{\alpha^2 \gamma_{i-1}}{q_i}\right) z_i = \frac{\alpha^2 \gamma_{i-1}}{q_i} \left(\sum_{j=1}^{i} \frac{z_j[t+(i-j)]}{\alpha^{i-j}} \right)$$

From which it follows that the lemma holds for $k=0$. Now assume that the lemma holds for $k-1$. Then if $i+k \le N$

$$z_i[t_0+k] = \frac{q_{i+1}}{\alpha q_i} z_{i+1}[t_0+k-1] =$$
$$\frac{q_{i+1}}{\alpha q_i} \frac{\gamma_{(i+1)+(k-1)}}{\alpha^{k-1} q_{i+1}} \sum_{j=1}^{i+1+k-1} \frac{z_j[t_0+(k-1)+((i+1)-j)]}{\alpha^{(k-1)+(i+1)-j}}$$
$$= \frac{\gamma_{i+k}}{\alpha^k q_i} \sum_{j=1}^{i+k} \frac{z_j[t_0+k+(i-j)]}{\alpha^{k+i-j}}$$

For $i+k > N$ define $\hat{k}$ and $\hat{t}_0$ so that

$$\begin{aligned} i+\hat{k} &= N \\ \hat{t}_0+\hat{k} &= t_0+k. \end{aligned} \tag{20}$$

Then using that $z_i[t_0+k] = z_i[\hat{t}_0+\hat{k}]$ gives the second part.

Finally, we will show that the time shifted level vector can be written in terms of $z[t_0-1]$.

Lemma 6. The optimal z for (2) satisfies for $i+k \le N$:

$$\sum_{j=1}^{i+k} \frac{z_j[t_0+k+(i-j)]}{\alpha^{k+i-j}} = \alpha \sum_{j=1}^{i+k} \left(z_j[t_0-1]+u_j[t_0-2]\right)$$

and for $i+k > N$:

$$\sum_{j=1}^{N} \frac{z_j[t_0+k+(i-j)]}{\alpha^{N-j}} = \alpha^{k+i-N+1} \sum_{j=1}^{N} \Big(z_j[t_0-1]+u_j[t_0-2]\Big)$$

Proof. We start with the first equality. Using that

$$z_j[t+n] = \alpha^{n+1}(z_j[t-1]+u_j[t-2])$$
$$- \sum_{\tau=0}^{n-1} \alpha^{(n-1)-\tau} u_{j-1}[t+\tau] + \sum_{\tau=0}^{n-2} \alpha^{n-1-\tau} u_j[t+\tau],$$

we have for $i+k \le N$:

$$\sum_{j=1}^{i} \frac{z_j[t_0+k+(i-j)]}{\alpha^{k+i-j}} = \alpha \sum_{j=1}^{n} \Big(z_i[t_0-1]+u_i[t_0-2]\Big)$$
$$+ \alpha^{-1-\tau} \Big(\sum_{j=1}^{i} \sum_{\tau=0}^{k+(i-j)-2} u_j[t_0+\tau] - \sum_{j=1}^{i} \sum_{\tau=0}^{k+(i-j)-1} u_{j-1}[t_0+\tau] \Big)$$

Since $u_0 = 0$ and $k+(i-j)-2 < 0$ for $j > i-2$ the last row equals to zero:

$$\sum_{j=1}^{i-2} \sum_{\tau=0}^{k+(i-j)-2} u_j[t+\tau] - \sum_{j=2}^{i-1} \sum_{\tau=0}^{k+(i-j)-1} u_{j-1}[t+\tau] = 0$$

For the second equality we use (20) again,

$$\sum_{j=1}^{N} \frac{z_j[t_0+k+(i-j)]}{\alpha^{N-j}} = \sum_{j=1}^{N} \frac{z_j[\hat{t}_0+\hat{k}+(i-j)]}{\alpha^{\hat{k}+i-j}}$$
$$= \alpha \sum_{j=1}^{N} \Big(z_j[\hat{t}_0-1]+u_j[\hat{t}_0-2]\Big)$$
$$= \alpha^{k-(N-i)+1} \sum_{j=1}^{N} \Big(z_j[t_0-1]+u_j[t_0-2]\Big)$$

Where we have used that $\hat{t}_0 - t_0 = k - \hat{k} = k-(N-i)$ and that the system is closed.

We also need the following lemma, which shows that there exist a boundary effect in the optimal controller that makes some of the states locally optimal.

Lemma 7. The optimal inventory levels satisfy

$$z_i[t] = \frac{b_i}{q_i} \quad \forall t \ge T-(i-2),\ i \ge 2.$$

Proof. We start by showing the lemma for $i=2$. As $u_1[T-1]$ only affects $z_2[T]$, the optimal value corresponds to maximizing the local utility, so that $z_2[T] = b_2/q_2$. Thus

$$u_1[T-1] = -\frac{b_2}{q_2} + \alpha(z_2[T-1]+u_2[T-2])$$

and $z_2[T] = b_2/q_2$, independent of all other $u_i[t]$.

Now assume that the lemma holds for all $i \le n$. Then $u_i[t]$ only needs to consider z_{i+1} for all $t \ge T-i$. Thus the optimal $u_n[t]$ satisfies

$$u_n[t] = -\frac{b_{n+1}}{q_{n+1}} + \alpha(z_{n+1}[t]+u_{n+1}[t-1])$$

$\forall t \ge T-n$ and

$$z_{n+1}[t] = \frac{b_{i+1}}{q_{i+1}} \quad \forall t \ge T-(n-1).$$

Thus the Lemma holds for all n.

We are now ready to state the following proposition, which gives expressions for the optimal λ.

Proposition 8. Let $m = \min(T-t_0-(i-1), N)$ and

$$\Xi_i(t_0,\tau) = \alpha^{1-\tau} \Bigg(\sum_{j=i+\tau}^{m} \gamma_j \sum_{k=1}^{j} z_k[t_0-1]+u_k[t_0-2]+$$
$$\sum_{j=N+1}^{T-t_0+1} \gamma_N \alpha^{2(j-N)} \sum_{k=1}^{N} z_k[t_0-1]+u_k[t_0-2] \Bigg)$$

Then the optimal λ's are given by

$$\lambda_i[t_0+\tau] = \sum_{t=t_0+\tau}^{T-(i-1)} \alpha^{t-(t_0+\tau)} b_i - \Xi_i(t_0,\tau)$$

Proof. From Lemma 4 and 7 we have that

$$\lambda_i[t_0+\tau] = \alpha^{-\tau} \sum_{k=\tau}^{T-t_0-(i-1)} \alpha^k \Big(b_i - q_i z_i[t_0+k]\Big)$$

Combining Lemma 5 and 6 gives that

$$\alpha^k q_i z_i[t_0+k] =$$
$$\begin{cases} \alpha\gamma_{i+k} \sum_{j=1}^{i+k} z_j[t_0-1]+u_j[t_0-2] & i+k \le N \\ \alpha\gamma_N \alpha^{2(k+i-N)} \sum_{j=1}^{N} z_j[t_0-1]+u_j[t_0-2] & i+k > N \end{cases}$$

Which gives that

$$
\begin{aligned}
\alpha^{-\tau} \sum_{k=\tau}^{T-t_0-(i-1)} \alpha^k q_i z_i[t_0+k] = \\
\alpha^{1-\tau}\Big(\sum_{j=i+\tau}^{m} \gamma_j \sum_{k=1}^{j} z_k[t_0-1] + u_k[t_0-2] \\
+ \sum_{j=N+1}^{T-t_0+1} \gamma_N \alpha^{2(j-N)} \sum_{k=1}^{N} z_k[t_0-1] + u_k[t_0-2] \Big)
\end{aligned}
$$