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Svensson Seth, Douglas

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On Steady Water Waves and Flows with Vorticity in Three Dimensions

Douglas Svensson Seth

DOCTORAL THESIS
Thesis advisor: Docent Erik Wahlén
Faculty opponent: Professor David Ambrose

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Abstract

In this thesis we study the steady Euler equations in three dimensions where the solution is assumed to have nonvanishing vorticity. The thesis is based on three research papers. In the first and the third we study the steady Euler equations in the context of the water wave problem, which means we are solving a free boundary problem, while in the second paper we study the equations in a fixed cylinder-like domain. In the first paper we prove the existence of small amplitude doubly periodic waves when the velocity of the water is assumed to be a Beltrami field. Divergence free Beltrami fields are special solutions to the steady Euler equations where the velocity and vorticity are parallel. In the second paper we prove the existence of solutions of the steady Euler equations in cylinder-like domains, where the fluid flows through the domain, like water through a pipe. Here the vorticity is specified by two boundary conditions on the part of the surface where the fluid flows into the domain. In the third paper we also prove the existence of small amplitude doubly periodic waves, but with a different assumption on the vorticity. This assumption is more technical in nature and comes from magnetohydrodynamics. This theory is applicable because the governing equations for the magnetic field in a magnetohydrostatic equilibrium is equivalent to the steady Euler equations.

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Douglas Svensson Seth

Lund University

Mathematics
Centre for Mathematical Sciences
Lund University
Box 118
SE-221 00 Lund
Sweden

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I would also like thank the rest of the permanent staff at the department, many of whom has been inspirational teachers to me before becoming my colleagues, as well as give a special thanks to our administrator Kerstin Rodahl, who does a great job allowing us at the department to focus on our research.

I also cannot leave out the Postdocs and the other PhD-students. They have provided great company and it is a pity that we did not have the possibility to see each other more during my final year as a PhD-student. In particular I would like to thank Dag, Bartosz, Tien, Adem, Daniele, Mats and Jonathan.

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Populärvetenskaplig sammanfattning

Leonhard Euler publicerade redan 1757 ekvationer för att matematiskt beskriva hur fluider rör sig och de bär därmed namnet Eulers ekvationer. Att fullständigt analysera konsekvenserna av dessa ekvationer är dock ännu inte ett avslutat problem inom matematiken. Detta beror dels på att ekvationerna beskriver många olika fenomen och det finns ingen övergripande teori kan hantera alla dessa samtidigt. Ett annat problem är inbyggt i själva ekvationernas struktur. Eulers ekvationer är vad som kallas icke-lineära vilket medför matematiska svårigheter.

I denna avhandling studerar vi speciellt dessa ekvationer i fall när fluiden är en vätska med vattenlikande egenskaper och nollskild vorticitet. Att beskriva vad vorticitet innebär rent konkret är inte helt enkelt, men en direktöversättning av den tyska termen för vorticitet ger ordet *virvelstyrka* vilket är talande. vorticiteten beskriver hur mycket vätskan roterar runt varje punkt, inklusive runt vilken axel och i vilken rikting vätskan roterar.

I den första artikeln utökas teorin om vattenvågor. Då måste själva vattnets rörelse uppfylla Eulers ekvationer. För vattenvågor ingår det dock också i själva problemet att hitta ytans form. Detta problem är studerat i både tvådimensionella modeller och i tre dimensioner ifall vattnet saknar vorticitet. I denna artikel visar vi existens av tredimensionella lösningar till vattenvågsproblemet med nollskild vorticitet. Vi antar att vattenflödet är ett Beltramiflöde vilket innebär att rotationsaxeln är parallell med vattenflödets riktning. Detta resultat täcker dock endast det enklaste fallet av denna relation vilket är när vattnets hastighet och vorticiteten är relaterade genom en konstant. Det betyder att förhållandet mellan vattnets hastighet och hur mycket det roterar är detsamma överallt. Lösningarna som hittas är dubbelperiodiska (se figur 1c), vilket inte kan beskrivas i tvådimensionella modeller. En viktig anledning till att studera problem som detta är att en bättre förståelse av vattenvågor ökar säkerheten för människor som befinner sig ute på havet.

I den andra artikeln studerar vi problemet att hitta lösningar till Eulers ekvationer i tre dimensioner i cylinderliknande områden. Detta är en matematisk beskrivning av vatten som rinner genom ett rör. Även här är nollskild vorticitet någonting vi lägger stor vikt vid och den tillåts i denna artikel att vara av en mer komplicerad form än i den första (och den tredje) artikeln. Den största matematiska utmaningen i artikeln är att området har skarpa kanter. I en cylinder finns kanterna precis där toppen och botten möter manteln. Att lösa differentialekvationer som Eulers ekvationer i områden med kanter är avsevärt svårare än att lösa dem i områden utan kanter, som t.ex. i ett klot. Om detta resultat går att utöka till områden med hörn (kubliknande domäner) så kan det potentiellt användas för att lösa vattenvågsproblemet med mer komplicerad vorticitet än i de andra artiklarna i denna avhandlig.

I den tredje artikeln löser vi återigen vattenvågsproblemet i tre dimensioner. Denna gång med ett annat antagande på vorticiteten. Detta antagande kommer från magnetohydrodynamiken. Det är nämligen så att ekvationerna som beskriver ett magnetfält i jämvikt inom magnetohydrodynamiken är ekvivalenta med Eulers ekvationer. Även här hittar vi dubbelperiodiska vågor över ett vattenflöde med nollskild vorticitet. Det kan dock poängteras att det definitivt är andra lösningar än de vi hittar i den första artikeln eftersom antagandena i denna artikel är oförenliga med de antaganden vi gör i den första.

List of Publications

This thesis is based on the following publications:

I **An Existence Theory for Small-Amplitude Doubly Periodic Water Waves with Vorticity**
D. Svensson Seth, E. Lokharu, E. Wahlén
Archive for Rational Mechanics and Analysis, 238(2):607–637, 2020.

II **Steady Three-Dimensional Ideal Flows with Nonvanishing Vorticity in Domains with Edges**
D. Svensson Seth
Journal of Differential Equations, 274:345–381, 2021.

III **Symmetric Doubly Periodic Gravity-Capillary Waves with Small Vorticity**
D. Svensson Seth, K. Varholm, E, Wahlén
Preprint, 2021.

Author contributions

Paper I: I mostly did work pertaining to the two-and-a-half dimensional waves, flattening, reduction to the surface, and dispersion equation (part of section 1 and sections 2 & 3). I also wrote parts of the related sections. For the remaining parts I checked the validity of the mathematics and proofread. However, all authors did contribute to almost all parts of the article.

Paper II: I did all the mathematical analysis and wrote the entire article with advice from Erik Wahlén.

Paper III: I did most of the analysis of the contraction, including regularity of compositions and fixed points, used to reduce the problem to the surface (part of section 3 and appendices A & B). Furthermore, I did the bifurcation argument to solve the surface equation and the expansion of the solutions (section 4). I also wrote large parts of the related sections.

Contents

Preface

1 Background

The history of the mathematical study of fluids in motion stretches back almost 300 years. This field of study is called fluid dynamics as opposed to fluid statics, which describes fluids at rest. Fluid statics is far older and traces its history all the way back to Archimedes of Syracuse. Fluid dynamics on the other hand can arguably be seen as starting with the publication of *Hydrodynamica* by Daniel Bernoulli [4]. It did at least lend its name to the field, which was formerly known as hydrodynamics. Subsequent progress was made through derivation of general equations governing fluid motion from more fundamental physical principles. Maybe most notably, the *Euler equations* for inviscid flow originally derived by Leonhard Euler [13] and the *Navier-Stokes equations* for viscous flow originally by Claude-Louis Navier [27] and later by George Gabriel Stokes [31]. Although these equations have been the subject of extensive study, many questions regarding their solutions remain. One reason is that they are mathematically complicated due to being non-linear. Another reason is that they serve as the equations governing a wide variety of different physical problems and there exists no mathematical theory that can be applied to all these different problems.

The equations we are mostly concerned with in this thesis are the Euler equations for an incompressible flow with constant density. If we let $\boldsymbol{v}$ denote the velocity field of the fluid, p the pressure and $\boldsymbol{g}$ external forces (in the water wave problem this is usually only gravity, i.e. $\boldsymbol{g} = -g\boldsymbol{e}_3$) then the equations are given by

$$\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \boldsymbol{g},$$
$$\nabla \cdot \boldsymbol{v} = 0.$$

We are particularly interested in the steady Euler equations, which means that the velocity field and pressure are time-independent. Hence the time derivative vanishes. In this thesis we are either working directly under the aforementioned assumption or what is known as the *travelling wave assumption*, that is, the assumption that $\boldsymbol{v} = \boldsymbol{v}(t, \boldsymbol{y}) = \boldsymbol{w}(\boldsymbol{y} - \boldsymbol{c}t)$ for some constant $\boldsymbol{c}$. Performing a *Galilean transformation* gives us the steady Euler equations. Physically this means that we view the problem in a reference frame moving with constant velocity. Mathematically we change coordinates $(t, \boldsymbol{y}) \mapsto (t, \boldsymbol{x}) = (t, \boldsymbol{y} - \boldsymbol{c}t)$. In the moving reference frame the velocity field is given by $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{w}(\boldsymbol{x}) - \boldsymbol{c}$. Since

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\boldsymbol{c} \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{w} = \partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}$$

we find that in the moving frame $\boldsymbol{u}$ satisfies the steady Euler equations

$$\begin{aligned} \boldsymbol{u} \cdot \nabla \boldsymbol{u} &= -\nabla p + \boldsymbol{g}, \\ \nabla \cdot \boldsymbol{u} &= 0. \end{aligned} \tag{1}$$

In this thesis we study the *water wave problem*. Generally speaking this means that we want to solve a *free boundary problem* involving these equations, that is, determining the shape of the domain in which equation (1) is satisfied is part of the problem. We solve equation (1) in a three-dimensional domain Ω. In the vertical direction Ω is bounded above by the free surface, described by a function η, and bounded below by some fixed bottom. In the horizontal plane we assume periodicity in two different directions. With these assumptions we get *periodic waves*. Another common assumption is that η is localized which gives *solitary waves*. The water wave problem has also been studied with a bottom that is not flat but instead given some fixed shape or at infinite depth which means Ω is unbounded below in the vertical direction. In some applications it is also impossible to make the assumption that the surface is described by a function, for example when studying overhanging or breaking waves. The time-dependent problem has also been studied extensively, see for example [9, 18, 23] and references therein.

For the problem to be well posed we also need some boundary conditions. The first is the *kinematic boundary condition*, which simply reads

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \tag{2}$$

where $\boldsymbol{n}$ denotes a normal vector to the boundary of Ω. The physical meaning of this boundary condition is that there is no flow through the boundary of Ω. The second boundary condition is the *dynamic boundary condition*, which is given by

$$p - p_a + \sigma \nabla \cdot \left(\frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}} \right) = 0. \tag{3}$$

Here p_a denotes the atmospheric pressure (or the pressure in the medium above the surface), which is assumed to be constant, while σ is the coefficient of surface tension. The physical meaning of the dynamic boundary condition is that with nonzero surface tension the pressure jump at the the surface is proportional to the mean curvature. Our methods rely on first solving equations (1) and (2) for a general but fixed η. These equations are not uniquely solvable and some additional conditions are needed to make this problem well-posed. In this thesis we use integral conditions of the form

$$\int_{\Omega_0} \boldsymbol{u} \cdot \boldsymbol{e}_i = c_i, \tag{4}$$

for either $i = 1$ or $i = 1, 2$, some symmetry assumptions, and some assumptions on the *vorticity*. The integral conditions determine the total flow in the horizontal directions through one period Ω_0 of the domain Ω. When $c_i \neq 0$ we avoid the trivial solution $(\boldsymbol{u}, p) = (0, -gx_3)$ to equations (1) and (2).

The water wave theory can be divided into three different problems: finding *gravity-capillary waves*, *gravity waves* or *capillary waves*. In this context gravity simply means that the external force in equation (1) is given by gravity, that is, $\boldsymbol{g} = -g\boldsymbol{e}_3$. On the other hand, capillary means that surface tension (cf. capillary action which is driven by surface tension) is present. Thus gravity-capillary waves means $\boldsymbol{g} = -g\boldsymbol{e}_3$ and $\sigma \neq 0$, gravity waves means $\boldsymbol{g} = -g\boldsymbol{e}_3$ and $\sigma = 0$, and capillary waves means $\boldsymbol{g} = 0$ and $\sigma \neq 0$. Under the physical conditions present at earth for water both gravity and surface tension are present. However, the effect of gravity is negligible for waves with small wavelength and the effect of surface tension is negligible for waves with large wavelength. Mathematically the presence of surface tension makes the problem of finding doubly periodic waves easier. This is due to the fact that many standard techniques for non-linear analysis hinge on the use of the implicit function theorem at some point, which in turn relies on the fact that some linearised operator has bounded inverse. For example in both [Paper I, Paper III] we reduce the problem to the surface. The linearised version of this problem, L, acts on a single Fourier mode $\eta_{\boldsymbol{k}}(\boldsymbol{x}') = e^{i\boldsymbol{k}\cdot\boldsymbol{x}'}$ through

$$L\eta_{\boldsymbol{k}}(\boldsymbol{x}') = \rho(\boldsymbol{k})\eta_{\boldsymbol{k}}(\boldsymbol{x}') = (r(\boldsymbol{k}) + g + \sigma|\boldsymbol{k}|^2)\eta_{\boldsymbol{k}}(\boldsymbol{x}'),$$

where $\boldsymbol{x}'$ denotes the horizontal variables and $r(\boldsymbol{k}) = \mathcal{O}(|\boldsymbol{k}|)$ as $|\boldsymbol{k}| \to \infty$. Due to the $\sigma|\boldsymbol{k}|^2$-term we have good control over how this operator acts on modes with sufficiently large $|\boldsymbol{k}|$. If $\sigma = 0$ then we lose this control and can (quite likely) find sequences of $\boldsymbol{k}_i$ such that $\rho(\boldsymbol{k}_i) \to 0$ as $i \to \infty$. This makes the inverse unbounded since the inverse of L clearly acts on $\eta_{\boldsymbol{k}}(\boldsymbol{x})$ by division with $\rho(\boldsymbol{k})$. This is commonly referred to as a *small divisor problem*. For this reason the results in [Paper I, Paper III] which prove the existence of gravity-capillary waves can not be replicated for gravity waves using the same techniques.

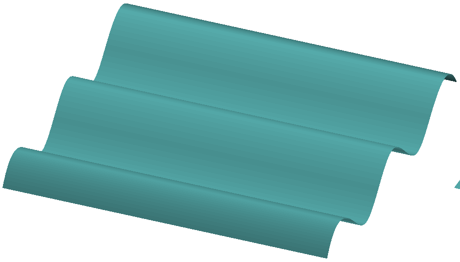
Similar problems have been extensively studied in two dimensions. Already in 1847 Stokes [32] formulated a non-linear theory for the two dimensional gravity wave problem and computed the flow up to third order with the wave amplitude as the parameter for the expansion. The first rigorous existence results were proven by Nekrasov [28] and Levi-Civita [24] independently. They both used conformal mappings known as *hodograph transforms* to reduce the problem to finding a harmonic function that satisfies a non-linear boundary condition on a fixed domain. Later the theory of local bifurcation [8], which we rely on in this thesis (see Sections 1.2 to 1.4), was also applied to the two dimensional problem for periodic waves. Another idea, proposed by Kirchgässner [21], is to treat the horizontal direction as a 'time'-variable, and to formulate an infinite-dimensional dynamical system. This approach is now known as *spatial dynamics* and allows results from dynamical systems to be applied, in particular the *center manifold theorem*. For the two dimensional problem there also exists a *global* theory, starting with Krasovskii [22]. In the context of water waves the local theory generally refers to waves of small amplitude while in global theory the amplitude of the waves is not restricted. However, to actually determine any characteristics of a global solution is a challenging problem. One notable example where this has been done is in the case of stokes waves of greatest height. Stokes conjectured that the periodic gravity waves he calculated, which forms a family of waves parametrised by the amplitude, have a wave of greatest height with sharp crests of included angle $2\pi/3$. This conjecture was confirmed by Amick, Fraenkel & Toland [3], and Plotnikov [29]. The two dimensional problem can give useful information about our three-dimensional world, but it can not account for every phenomenon. It is suitable to describe waves like the ones depicted in Figure 1a, but clearly has its shortcomings when it comes to describe waves like the ones in Figures 1b and 1c.

The three dimensional theory is not quite as old but there exist a number of results, most of which are from the last twenty years. The first existence result, though, is by Reeder & Shinbrot [30] from 1981. They prove the existence of doubly periodic gravity-capillary waves with respect to a *diamond lattice* (see Figure 1c). The waves are called doubly periodic if η is periodic with respect to two linearly independent vectors defining a lattice in the horizontal plane. A diamond lattice means that these two vectors are of the same length. This result was improved by Craig & Nicholls [6] who proved the existence of doubly periodic gravity-capillary waves with respect to a general lattice. In their method of proof they used a Hamiltonian formulation due to Zakharov [37] expressed in a more convenient form by Craig & Sulem [7], together with a variational argument. The variational problem obtained in [6] is reduced to a finite-dimensional problem using a *Lyapunov Schmidt reduction*. The approach of spatial dynamics has also been successfully applied to the three dimensional problem. Here one of the horizontal directions has to be chosen to take the role of a 'time'-variable. This approach has the benefit of not being solely focused on seeking doubly periodic solutions. If the solutions are assumed to be periodic in a transverse direction to the 'time'-direction, then the special case of periodic solutions to the dynamical system introduced gives rise to doubly periodic waves. However, these solutions can for example also be localised, which then shows the existence of waves that are solitary in one direction and periodic in another (see Figure 1b). The use of this approach for the three dimensional problem was introduced by Groves & Mielke [16] and Haragus-Courcelle & Kirchgässner [17], and generalized by e.g. Groves & Haragus [14]. These results contain the existence of doubly periodic gravity-capillary waves. There also exist results for doubly periodic gravity waves. The existence of these waves which are periodic with respect to a diamond shaped lattice was proved by Iooss & Plotnikov [20]. The result for a more general lattice was later proven by the same authors [19]. Both of these results rely on Nash-Moser type theorems, which are in essence generalized implicit function theorems that are applicable despite the small divisor problem that occurs due to the lack of surface tension. There are currently no global results for the three dimensional problem.

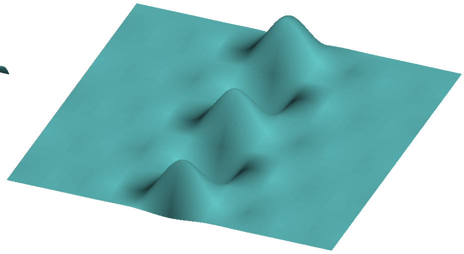
All the three dimensional results mentioned above only treat *irrotational flows*, i.e. flows with zero vorticity

$$\boldsymbol{w} := \nabla \times \boldsymbol{u} = 0.$$

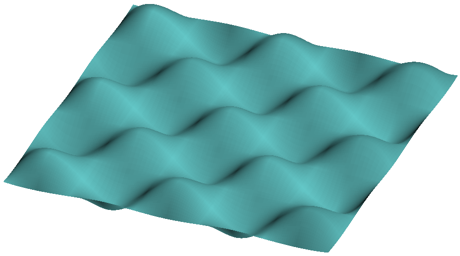
In this thesis we are in particularly interested in flows with nonzero vorticity. In the irrotational case the velocity field is given by a scalar potential. Thus the problem can be simplified to a scalar problem, which is not possible in the case of nonzero



(a) A periodic-constant wave.



(b) A periodic-solitary wave.



(c) A doubly periodic wave.

Figure 1: Three different types of waves.

vorticity. It can also be noted that in two dimensions the problem can be reduced to a scalar problem even in the case of nonzero vorticity due to what is known as a *stream function*. This can be seen as a consequence of the fact that a three dimensional divergence free field can be written as the curl of a vector potential

$$\boldsymbol{u} = \nabla \times \boldsymbol{A}.$$

Assuming that $\boldsymbol{u} = (u_1, 0, u_3)$ and independent of x_2, that is, for all intents and purposes two dimensional, leaves us with

$$-\partial_{x_3} A_2 = u_1, \qquad \partial_{x_1} A_2 = u_3.$$

Here A_2 is the stream function. Reformulation with the help of a vector potential is still a useful tool in the (truly) three dimensional case, but it does not reduce the problem to finding a scalar stream function. We also note that there are several recent results for two-dimensional steady water waves with vorticity. Constantin

& Strauss [5] proved a global result for two-dimensional gravity waves with a general vorticity distribution. The properties of these have been further studied in e.g. [1, 11, 10, 12]. Two-dimensional gravity-capillary waves have also been studied for a class of vorticity functions in e.g. [33, 35, 36]. For the three-dimensional problem with nonzero vorticity the results are yet very scarce. One of the few results is a non-existence result for constant vorticity [34]. There also exists variational formulations for waves over a three-dimensional *Beltrami flows* [15, 25]. A Beltrami flow means that the velocity field is a *Beltrami field*, that is, the velocity and vorticity are parallel. In other words there exists a function α such that

$$\boldsymbol{w} = \nabla \times \boldsymbol{u} = \alpha \boldsymbol{u}.$$

This is usually divided into the two cases of constant α, which we will refer to as a *linear* Beltrami field, and non-constant α, which we will refer to as a *nonlinear* Beltrami field. The first existence result is given in [Paper I]. There we show the existence of doubly periodic waves over a linear Beltrami flow. Another result for doubly periodic waves with non-vanishing vorticity is shown in [Paper III]. There the vorticity is assumed to have the form proposed by Lortz in [26]. The observant reader will note that [26] is not a paper on water waves, but instead treats what is called *magnetohydrodynamics*. In particular Lortz shows the existence of a magnetohydrostatic equilibrium in a toroidal region. The governing equations are

$$\begin{aligned} \boldsymbol{J} \times \boldsymbol{B} &= \nabla P, \\ \boldsymbol{J} &= \nabla \times \boldsymbol{B}, \\ \nabla \cdot \boldsymbol{B} &= 0, \end{aligned}$$

where $\boldsymbol{B}$ is the magnetic field $\boldsymbol{J}$ is the current density and P is the plasma pressure. By identifying $\boldsymbol{B} = \boldsymbol{u}$ and $P = p + gx_3 + \frac{1}{2}|\boldsymbol{u}|^2$ we can recover the steady Euler equations under the influence of gravity with vorticity $\boldsymbol{\omega} = \boldsymbol{J}$. Thus the assumption that Lortz makes for the current density can be directly applied for the vorticity. That the problem is studied in a toroidal region is also precisely what we need to obtain solutions with the periodicity we seek.

Despite the differences in the vorticity assumptions, both these existence results are proven in a similar way. First the pressure is written as a function of the velocity. Then the problem given by equations (1), (2) and (4) are solved for a given function η. This gives a solution $(\boldsymbol{u}[\eta], p[\boldsymbol{u}[\eta]])$, which we can substitute in equation (1.8c) to obtain an equation for η, which determines the free boundary. This problem is in both cases solved through a Lyapunov-Schmidt reduction together with two different extensions of the classical local bifurcation result by Crandall & Rabinowitz [8]. A summary of these results is presented below. The final paper included in this thesis [Paper II] is concerned with only the first part, that is, solving equations (1), (2) and (4) for a given function η. The result does need some improvement though if it is to be applied to the three dimensional water wave problem.

1.1 Lyapunov-Schmidt Reduction

Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{C}$ be Banach spaces and let $F \in C^1(\mathcal{X} \times \mathcal{C}, \mathcal{Y})$. Here $\mathcal{C}$ denotes a finite-dimensional parameter space. The problem we are interested in is finding $(x, c) \in \mathcal{X} \times \mathcal{C}$ that is the solution to

$$F[x, c] = 0. \tag{5}$$

Assuming that we have some trivial solution $(0, c^*)$ to (5) and that $D_1F[0, c^*] : \mathcal{X} \to \mathcal{Y}$ is an isomorphism then the implicit function theorem gives us a solution $(x(c), c)$ for every c in some neighbourhood of c^*. Here D_i denotes the Fréchet derivative with respect to the *i*:th argument.

However, if we relax the assumption to $D_1F[0, c^*] : \mathcal{X} \to \mathcal{Y}$ is a Fredholm operator, then we can no longer in general apply the implicit function theorem. Let us describe it briefly. What we can do is perform a Lyapunov-Schmidt reduction to reduce (5) to a finite-dimensional problem. Recall the definition of a Fredholm operator.

Definition 1.1. A bounded linear operator $T : \mathcal{X} \to \mathcal{Y}$ is said to be a *Fredholm operator* if:

(i) The kernel of T is finite-dimensional.

(ii) The cokernel of T is finite-dimensional.

(iii) The range of T is closed.

Morover the *index of a Fredholm operator* is defined as

$$\operatorname{ind} T = \dim \ker T - \dim \operatorname{coker} T.$$

Condition (iii) is usually included in the definition, but is actually redundant. If $D_1F[0,c^*]$ is a Fredholm operator then we can write $\mathcal{X} = \mathcal{X}_1 \oplus \tilde{\mathcal{X}}$ and $\mathcal{Y} = \mathcal{Y}_1 \oplus \tilde{\mathcal{Y}}$, where $\mathcal{X}_1 = \ker D_1F[0,c^*]$ and $\tilde{\mathcal{Y}} = \operatorname{ran} D_1F[0,c^*]$. The spaces $\tilde{\mathcal{X}}$ and $\mathcal{Y}_1$ are isomorphic to the quotient spaces $\mathcal{X}/\ker D_1F[0,c^*]$ and $\operatorname{coker} D_1F[0,c^*] = \mathcal{Y}/\operatorname{ran} D_1F[0,c^*]$ respectively. This decomposition of $\mathcal{X}$ and $\mathcal{Y}$ defines the projections Q on $\mathcal{X}$ with range $\mathcal{X}_1$ and kernel $\tilde{\mathcal{X}}$ and P on $\mathcal{Y}$ with range $\mathcal{Y}_1$ and kernel $\tilde{\mathcal{Y}}$. By applying P and $(I-P)$ to equation (5) we obtain the equivalent system of equations

$$(I-P)F[x,c] = 0, \tag{6}$$

$$PF[x,c] = 0. \tag{7}$$

Writing $x = x_1 + \tilde{x}$, where $x_1 = Qx$ and $\tilde{x} = (I-Q)x$, we can view the left hand side in equation (6) as an operator $(I-P)F : \mathcal{X}_1 \times \tilde{\mathcal{X}} \times \mathcal{C} \to \tilde{\mathcal{Y}}$. Differentiating this operator with respect to $\tilde{x}$ at $(0,0,c^*)$ gives $(I-P)D_1F[0,c^*] : \tilde{\mathcal{X}} \to \tilde{\mathcal{Y}}$, which is an isomorphism. By the implicit function theorem there exists a solution $\tilde{x}[x_1,c]$ to equation (6). Substituting this solution in equation (7) we obtain the equation

$$PF[x_1 + x_2[x_1,c],c] = 0,$$

which is equivalent to the original problem (5). This is a finite-dimensional problem because the left hand side is an operator that maps $\mathcal{X}_1 \times \mathcal{C}$ to $\mathcal{Y}_1$. This completes the Lyapunov-Schmidt reduction. To solve the finite-dimensional problem that we obtained requires some additional assumptions. An example of such assumptions are given in the local bifurcation theorem of Crandall & Rabinowitz [8].

1.2 Local Bifurcation

We keep the notation from Section 1.1 and continue studying equation (5). In this section we assume that $\mathcal{C} = \mathbb{R}$.

Theorem 1.2 (Crandall & Rabinowitz). *If $F \in C^k(\mathcal{X} \times \mathcal{C}, \mathcal{Y})$, with $k \in \mathbb{Z}$ such that $k \geq 2$, $k = \infty$ or $k = \omega$ (i.e. F is analytic), is an operator with the following properties:*

(i) $F[0, c] = 0$ *for all $c \in \mathcal{C}$ and there exists $c^* \in \mathcal{C}$ such that $D_1 F[0, c^*]$ is a Fredholm operator of index* 0.

(ii) The kernel of $D_1 F[0, c^*]$ is one dimensional, i.e. there exists some $x_1 \in \mathcal{X} \setminus \{0\}$ such that $\ker D_1 F[0, c^*] = \operatorname{span}\{x_1\}$.

(iii) $P D_1 D_2 F[0, c^*] x_1 \neq 0$.

Then there exists some ϵ such that for every $s \in (-\epsilon, \epsilon)$ equation (5) has a solution $(x[s], c[s])$. Moreover, $(x[\cdot], c[\cdot]) \in C^{k-1}((-\epsilon, \epsilon), \mathcal{X} \times \mathcal{C})$ (if k is equal to ∞ or ω then $k - 1$ is also equal to ∞ or ω respectively) and

$$x[s] = s x_1 + o(s), \qquad c[s] = c^* + \mathcal{O}(s).$$

Remark 1.3. Condition *(iii)* is known as the *transversality condition* and it is a sufficient condition to apply the implicit function theorem after performing a Lyapunov-Schmidt reduction.

Proof. By the assumptions of the theorem we can immediately perform a Lyapunov-Schmidt reduction. Writing $Qx = s x_1$ we obtain the equation

$$K(s, c) := P F[s x_1 + \tilde{x}[s, c], c] = 0.$$

Here we view K as a function $\mathbb{R}^2 \to \mathbb{R}$. Strictly speaking it maps into $\mathcal{Y}_1$, but this is a one dimensional space so we can identify it with $\mathbb{R}$. Similarly we identify $\mathcal{X}_1$ with $\mathbb{R}$ to view $\tilde{x}$ as an operator $\mathbb{R}^2 \to \tilde{\mathcal{X}}$. $\tilde{x}[0, c]$ is the unique solution to

$$F[\tilde{x}[0, c], c] = 0,$$

which means that $\tilde{x}[0, c] = 0$ because $F[0, c] = 0$. Moreover we have that $\partial_s \tilde{x}[0, c^*] = 0$ because it solves the equation

$$(I - P) D_1 F[0, c^*] \partial_s \tilde{x}[0, c^*] + (I - P) D_1 F[0, c^*](x_1) = 0,$$

where the second term is 0 because $x_1 \in \ker D_1F[0, c^*]$. Define the function

$$H(s, c) := \begin{cases} \dfrac{K(s, c)}{s} & \text{if } s \neq 0, \\ \partial_s K(0, c) & \text{if } s = 0. \end{cases}$$

We obtain

$$H(0, c) = PD_1F[0, c](x_1 + D_1\tilde{x}[0, c](x_1))$$

which means that $H(0, c^*) = 0$ and

$$\begin{aligned} \partial_c H(0, c) = PD_1D_2F[0, c](x_1 + D_1\tilde{x}[0, c](x_1), 1) \\ + PD_1F[0, c](D_1D_2\tilde{x}[0, c](x_1, 1)), \end{aligned}$$

so

$$\partial_c H(0, c^*) = PD_1D_2F[0, c^*](x_1, 1) \neq 0$$

because $D_1\tilde{x}[0, c^*](x_1) = 0$ and $PD_1F[0, c^*] = 0$. Hence we can apply the implicit function theorem to obtain a function $c[s]$ that solves $H(s, c[s]) = 0$. Tracing the argument backwards gives us the solution we seek. It is given by $c[s]$ and

$$x[s] = sx_1 + \tilde{x}[sx_1, c[s]].$$

The differentiability of the solutions follows from the implicit function theorem and that $\tilde{x}[sx_1, c[s]] = o(s)$ follows from the fact that

$$\tilde{x}[0, c^*] = D_1\tilde{x}[0, c^*] = D_2\tilde{x}[0, c^*] = 0.$$

□

1.3 Multi Parameter Bifurcation

The result in Theorem 1.2 is not sufficient for either of the papers that rely on a bifurcation argument in this thesis. Below we show a modified version used in [Paper I]. In this section we let $\mathcal{C} = \mathbb{R}^n$ (the version used in [Paper I] is in fact the special case when $n = 2$). Moreover, let $\mathcal{X}_i = \operatorname{span}\{x_i\}$ and $\mathcal{Y}_i = \operatorname{span}\{y_i\}$,

$i = 1, \ldots, n$ be one dimensional subspaces of $\mathcal{X}$ and $\mathcal{Y}$ respectively. This means that we can write

$$\mathcal{X} = \left(\bigoplus_{i=1}^{n} \mathcal{X}_i \right) \oplus \tilde{\mathcal{X}},$$
$$\mathcal{Y} = \left(\bigoplus_{i=1}^{n} \mathcal{Y}_i \right) \oplus \tilde{\mathcal{Y}},$$

where $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{Y}}$ are closed subspaces. This decomposition allows us to define the projections Q_i and P_i, $i = 1, \ldots, n$, which are projections onto $\mathcal{X}_i$ and $\mathcal{Y}_i$ along

$$\hat{\mathcal{X}}_i \oplus \tilde{\mathcal{X}} := \left(\bigoplus_{\substack{j=1 \\ j \neq i}}^{n} \mathcal{X}_j \right) \oplus \tilde{\mathcal{X}}$$

and

$$\hat{\mathcal{Y}}_i \oplus \tilde{\mathcal{Y}} := \left(\bigoplus_{\substack{j=1 \\ j \neq i}}^{n} \mathcal{Y}_j \right) \oplus \tilde{\mathcal{Y}}$$

respectively. Moreover, in this section we let $P = \sum_{i=1}^{n} P_i$ and $Q = \sum_{i=1}^{n} Q_i$.

Theorem 1.4. *If $F \in C^k(\mathcal{X} \times \mathcal{C}, \mathcal{Y})$ (with k as in Theorem 1.2) is an operator with the following properties:*

(i) $F[0, \mathbf{c}] = 0$ for all $\mathbf{c} \in \mathcal{C}$ and there exists $\mathbf{c}^* \in \mathcal{C}$ such that $D_1 F[0, \mathbf{c}^*] : \mathcal{X} \to \mathcal{Y}$ is a Fredholm operator of index 0

(ii) The kernel of $D_1 F[0, \mathbf{c}^*]$ is n-dimensional and given by $\bigoplus_{i=1}^{n} \mathcal{X}_i$.

(iii) If $P_i D_1 D_{j+1} F[0, \mathbf{c}^*](x_i, c_j - c_j^*) = \nu_{ij}(c_j - c_j^*) y_i$, then the matrix $\boldsymbol{\nu}$ given by

$$(\boldsymbol{\nu})_{i,j} = \nu_{ij}$$

is invertible.

(iv) There exist closed subspaces $\tilde{\mathcal{X}}_i$ *of* $\tilde{\mathcal{X}}$ *and* $\tilde{\mathcal{Y}}_i$ *of* $\tilde{\mathcal{Y}}$ *for each* $i = 1, \ldots, n$ *such that*

$$F(\hat{\mathcal{X}}_i \oplus \tilde{\mathcal{X}}_i, \boldsymbol{c}) \subseteq \hat{\mathcal{Y}}_i \oplus \tilde{\mathcal{Y}}_i,$$

and

$$(I - P_i) D_1 F[0, \boldsymbol{c}^*]|_{\hat{\mathcal{X}}_i \oplus \tilde{\mathcal{X}}_i} : \hat{\mathcal{X}}_i \oplus \tilde{\mathcal{X}}_i \to \hat{\mathcal{Y}}_i \oplus \tilde{\mathcal{Y}}_i$$

is a Fredholm operator of index 0 *with kernel* $\hat{\mathcal{X}}_i$.

Then there exists an ϵ *such that for every* $\boldsymbol{s} = (s_1, \ldots, s_n) \in B_\epsilon(0) = \{\boldsymbol{s} \in \mathbb{R}^n : |\boldsymbol{s}| < \epsilon\}$ *equation* (5) *has a solution* $(x[\boldsymbol{s}], \boldsymbol{c}[\boldsymbol{s}])$. *Moreover,* $(x[\cdot], \boldsymbol{c}[\cdot]) \in C^{k-1}(B_\epsilon(0), \mathcal{X} \times \mathcal{C})$ *and*

$$x[\boldsymbol{s}] = \sum_{i=1}^{n} s_i x_i + o(|\boldsymbol{s}|), \qquad \boldsymbol{c}[\boldsymbol{s}] = \boldsymbol{c}^* + \mathcal{O}(|\boldsymbol{s}|).$$

Remark 1.5. The conditions *(i)*,*(ii)*, and *(iii)* are clear analogues to the standard local bifurcation theorem by Crandall & Rabinowitz. Condition *(iv)* has no analogue because it is superfluous if $n = 1$. In the case when $n \geq 2$ it separates the domain and codomain of the operators into in a way that makes the proof of the theorem quite straightforward.

Proof. We begin by performing a Lyapunov Schmidt reduction. Writing $x = \sum_{i=1}^{n} s_i x_i + \tilde{x}$ where $s_i x_i = Q_i x$ and $\tilde{x} = (I - Q)x$ we obtain the equations

$$(I - P) F \left[\sum_{j=1}^{n} s_j x_j + \tilde{x}, \boldsymbol{c} \right] = 0,$$
$$P_i F \left[\sum_{j=1}^{n} s_j x_j + \tilde{x}, \boldsymbol{c} \right] = 0 \qquad\qquad i = 1, \ldots, n.$$

By assumption *(i)* we can apply the implicit function theorem to obtain $\tilde{x}[\boldsymbol{s}, \boldsymbol{c}]$ that solves the first equation. We note that $\tilde{x}[0, \boldsymbol{c}] = 0$ and $\partial_{s_i} \tilde{x}[0, \boldsymbol{c}^*] = 0$, $i =$

$1, \ldots, n$. Moreover, by assumption *(iv)* we can consider the restricted operator $F : \hat{\mathcal{X}}_i \oplus \tilde{\mathcal{X}}_i \to \hat{\mathcal{Y}}_i \oplus \tilde{\mathcal{Y}}_i$ and perform a Lyapunov-Schmidt reduction. It follows that $\tilde{x}[\boldsymbol{s}, \boldsymbol{c}]|_{s_i=0} \in \hat{\mathcal{X}}_i \oplus \tilde{\mathcal{X}}_i$. Due to this fact and assumption *(iv)* we get that

$$P_i F \left[\sum_{j=1}^{n} s_j x_j + \tilde{x}[\boldsymbol{s}, \boldsymbol{c}], \boldsymbol{c} \right] \Bigg|_{s_i=0} = 0$$

This means that we can write

$$P_i F \left[\sum_{j=1}^{n} s_j x_j + \tilde{x}[\boldsymbol{s}, \boldsymbol{c}], \boldsymbol{c} \right] = s_i H_i(\boldsymbol{s}, \boldsymbol{c})$$

and solve

$$H_i(\boldsymbol{s}, \boldsymbol{c}) = 0, \qquad i = 1, \ldots, n. \tag{8}$$

instead. The functions H_i are differentiable and

$$\begin{aligned} H_i(0, \boldsymbol{c}^*) &= P_i D_1 F[0, \boldsymbol{c}^*](x_i + \partial_{s_i} \tilde{x}[0, \boldsymbol{c}^*]) = 0, \\ \partial_{c_j} H_i(0, \boldsymbol{c}^*)(c_j - c_j^*) &= P_i D_1 F[0, \boldsymbol{c}^*](\partial_{s_i} \partial_{c_j} \tilde{x}[0, \boldsymbol{c}^*](c_j - c_j^*)) \\ &\quad + P_i D_1 D_{j+1} F[0, \boldsymbol{c}^*](x_i + \partial_{t_i} \tilde{x}(0, \boldsymbol{c}^*), c_j - c_j^*) \\ &= P_i D_1 D_{j+1} F[0, \boldsymbol{c}^*](x_i, c_j - c_j^*) \\ &= \nu_{ij}(c_j - c_j^*) y_i. \end{aligned}$$

Since $\boldsymbol{\nu}$ is invertible this means we can apply the implicit function theorem to obtain a differentiable function $\boldsymbol{c}(\boldsymbol{t})$ defined in $B_\epsilon(0)$ that solves equation (8).

We end by noting that we do not obtain uniqueness due to the fact that if $s_i = 0$ then $P_i F(x, \boldsymbol{c}) = 0$ is not equivalent to $H_i(\boldsymbol{s}, \boldsymbol{c}) = 0$. $\square$

1.4 Bifurcation with a Small Perturbation

In this section we return to $\mathcal{C} = \mathbb{R}$ to consider the problem that appears in [Paper III]. There we have a problem like the one in equation (5) but the operator F is not differentiable, at least not in the usual sense. This naturally leads to some difficulties because we cannot apply the implicit function theorem as in the two previous examples. Not even the Lyapunov-Schmidt reduction can be performed in the same manner. The actual result used in [Paper III] is very technical in nature. We will not repeat all the details here but show a result with weaker assumptions (and conclusions) that could have been used instead.

Theorem 1.6. *Let $G \in C(\mathcal{X} \times \mathcal{C}, \mathcal{Y})$ be the mapping*

$$G = F + \alpha R, \tag{9}$$

where $F \in C^2(\mathcal{X} \times \mathcal{C}, \mathcal{Y})$ satisfies the same assumptions as in Theorem 1.2 and $R \in \mathrm{Lip}(\mathcal{X} \times \mathcal{C}, \mathcal{Y})$ satisfies $R[0, c] = 0$ for all $c \in \mathcal{C}$. Then there exists an ϵ such that for every $s \in (-\epsilon, 0) \cup (0, \epsilon)$ there exists a nonempty open interval $\mathcal{I}(s)$ around 0 such that for every $\alpha \in \mathcal{I}(s)$ there exists $(x[s], c[s])$, which is a nontrivial solution to

$$G[x, c] = 0.$$

Proof. We begin in the same manner as if performing a Lyapunov-Schmidt reduction and write the problem as the two equations

$$(I - P)(F[sx_1 + \tilde{x}, c] + \alpha R[sx_1 + \tilde{x}, c]) = 0 \tag{10}$$

$$P(F[sx_1 + \tilde{x}, c] + \alpha R[sx_1 + \tilde{x}, c]) = 0 \tag{11}$$

Consider first equation (10). Since $L := D_1 F[0, c^*]$ is invertible as an operator $\tilde{\mathcal{X}} \to \tilde{\mathcal{Y}}$ we can instead consider

$$L^{-1}(I - P)(F[sx_1 + \tilde{x}, c] + \alpha R[sx_1 + \tilde{x}, c]) = 0.$$

Since $D_1 L^{-1}(I - P)F[0, 0](\tilde{x}) = \tilde{x}$ we get that $\tilde{x} \mapsto \tilde{x} - L^{-1}(I - P)F[sx_1 + \tilde{x}, c]$ is a contraction for (s, c) close to $(0, c^*)$. Moreover, adding $L^{-1}(I - P)\alpha R[sx_1 + \tilde{x}, c]$ with a sufficiently small α does not change that fact. Hence we get that $\tilde{x} \mapsto \tilde{x} - L^{-1}(I - P)(F[\tilde{x}, s, c] + \alpha R[\tilde{x}, s, c])$ is a contraction (for small $(s, c - c^*, \alpha)$). By Banach's fixed point theorem this operator has a unique fixed point $\tilde{x}[s, c, \alpha]$ satisfying

$$\tilde{x}[s, c, \alpha] = \tilde{x}[s, c, \alpha] - L^{-1}(I - P)(F[\tilde{x}[s, c], s, c] + \alpha R[\tilde{x}[s, c], s, c]),$$

which is equivalent to (10). It follows from the Lipschitz continuity of F and R that $\tilde{x}$ is Lipschitz continuous. Indeed, if $C[\tilde{x}, s, c, \alpha] = \tilde{x} - L^{-1}Q[F[\tilde{x}, s, c] + \alpha R[\tilde{x}, s, c]]$, then

$$\begin{aligned}
&\|\tilde{x}[s_1, c_1, \alpha_1] - \tilde{x}[s_2, c_2, \alpha_2]\|_{\mathcal{X}} \\
&\quad = \|C[\tilde{x}[s_1, c_1, \alpha_1], s_1, c_1, \alpha_1] - C[\tilde{x}[s_2, c_2, \alpha_2], s_2, c_2, \alpha_2]\|_{\mathcal{X}} \\
&\quad \leq \|C[\tilde{x}[s_1, c_1, \alpha_1], s_1, c_1, \alpha_1] - C[\tilde{x}[s_2, c_2, \alpha_2], s_1, c_1, \alpha_1]\|_{\mathcal{X}} \\
&\qquad + \|C[\tilde{x}[s_2, c_2, \alpha_2], s_1, c_1, \alpha_1] - C[\tilde{x}[s_2, c_2, \alpha_2], s_2, c_2, \alpha_2]\|_{\mathcal{X}} \\
&\quad \leq b\|\tilde{x}[s_1, c_1, \alpha_1] - \tilde{x}[s_2, c_2, \alpha_2]\|_{\mathcal{X}} \\
&\qquad + C_L(|s_1 - s_2| + |c_1 - c_2| + |\alpha_1 - \alpha_2|),
\end{aligned}$$

where $0 < b < 1$ and C_L is the Lipschitz constant of C. Rearranging proves the claim. Before we proceed, note that $\tilde{x}[s, c, 0]$ is the same $\tilde{x}$ as in the Crandall & Rabinowitz theorem. Define the function

$$K(s, c, \alpha) \coloneqq P(F[sx_1 + \tilde{x}[s, c, \alpha], c] + \alpha R[sx_1 + \tilde{x}[s, c, \alpha], c]),$$

which means that equation (11) can be written as

$$K(s, c, \alpha) = 0.$$

For a small fixed s and $\alpha = 0$ the right hand side is a monotone function with respect to c in some neighbourhood of c^*. This follows from the fact that if we divide $K(s, c, 0)$ by s we obtain $H(s, c)$ from the proof of Theorem 1.2 and

$$\partial_c H(0, c^*) \neq 0.$$

Thus the sign of $\partial_c H(s, c)$ is is constant in some rectangle $(-\epsilon, \epsilon) \times (c_1, c_2)$ around $(0, c^*)$. This means that $\partial_c K(s, c, 0) = s\partial_c H(s, c)$ has the same sign s for all $s \in (-\epsilon, 0)$ and the opposite sign for all $s \in (0, \epsilon)$. Moreover, if ϵ is small enough there exists a $\tilde{c}[s] \in (c_1, c_2)$ for every $s \in (-\epsilon, 0) \cup (0, \epsilon)$ such that $K(s, \tilde{c}[s], 0) = 0$ (the solution in Theorem 1.2). Hence $K(s, c_1, 0)$ and $K(s, c_2, 0)$ must have opposite signs. For all sufficiently small α the same is true for $K(s, c_1, \alpha)$ and $K(s, c_2, \alpha)$ because $K(s, c, \alpha)$ is continuous with respect to α. Because $K(s, c, \alpha)$ also is continuous with respect to c there must exist a $c[s] \in (c_1, c_2)$ such that $K(s, c[s], \alpha) = 0$ by the intermediate value theorem. □

2 Summary of the Research Papers

2.1 Paper I

In the first paper in this thesis we prove the existence of doubly periodic waves over a Beltrami flow. Assuming that $\boldsymbol{u}$ is a linear Beltrami field is the simplest assumption that gives nonzero vorticity, aside from constant vorticity which has been proven to yield no solutions [34]. Since this is the first three dimensional existence result for the water wave problem with nonzero vorticity, making the simplest possible assumption is reasonable. Another good reason for choosing a Beltrami flow is that a divergence free Beltrami field always solves equation (1), with

$$p = -\frac{1}{2}|\boldsymbol{u}|^2 - gx_3 + C. \tag{12}$$

This can be seen through the vector calculus identity

$$\frac{1}{2}\nabla|\boldsymbol{u}|^2 = (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \boldsymbol{u}\times(\nabla\times\boldsymbol{u}), \tag{13}$$

where the last term is always 0 for a Beltrami field. This allows us to replace equation (1) with

$$\begin{aligned} \nabla\times\boldsymbol{u} &= \alpha\boldsymbol{u} \\ \nabla\cdot\boldsymbol{u} &= 0 \end{aligned} \tag{14}$$

in Ω, simplifying the problem because we have replaced a non-linear equation with a linear one. Since we want to find doubly periodic waves we study these equations under the assumption that the solutions are periodic with respect to some lattice

$$\Lambda = \{\boldsymbol{\lambda}\in\mathbb{R}^2 : \boldsymbol{\lambda} = m\boldsymbol{\lambda}_1 + n\boldsymbol{\lambda}_2,\ m,n\in\mathbb{Z}\}. \tag{15}$$

We also define the dual lattice

$$\Lambda' = \{\boldsymbol{k}\in\mathbb{R}^2 : \boldsymbol{k} = m\boldsymbol{k}_1 + n\boldsymbol{k}_2,\ m,n\in\mathbb{Z}\},$$

where $\boldsymbol{\lambda}_i\cdot\boldsymbol{k}_j = 2\pi\delta_{ij}$. By solving equations (2), (4) and (14) we obtain a velocity field $\boldsymbol{u}[\eta, c_1, c_2]$. Through (12) we can substitute this in (1.8c) to obtain the equation

$$F[\eta, c_1, c_2] := -\frac{1}{2}|\boldsymbol{u}[\eta, c_1, c_2]|^2 + g\eta + \sigma\nabla\cdot\left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right) - Q(c_1, c_2) = 0.$$

on the boundary $x_3 = \eta$. This is an equation of the same form as equation (5) with η taking the role of x. The solution is obtained by showing that F satisfies the assumptions of Theorem 1.4. For the appropriate choice of constant $Q(c_1, c_2)$ we obtain $F[0, c_1, c_2] = 0$ for all $(c_1, c_2) \in \mathbb{R}$. Checking the rest of the conditions mainly boils down to studying the *dispersion equation*

$$\rho(\boldsymbol{c}, \boldsymbol{k}) := g + \sigma|\boldsymbol{k}|^2 - \frac{(\boldsymbol{c} \cdot \boldsymbol{k})^2}{|\boldsymbol{k}|^2}\kappa(|\boldsymbol{k}|) + \alpha\frac{(\boldsymbol{c} \cdot \boldsymbol{k})(\boldsymbol{c} \cdot \boldsymbol{k}^\perp)}{|\boldsymbol{k}|^2} = 0,$$

where $\boldsymbol{k} \in \Lambda'$ and $\boldsymbol{k}^\perp \cdot \boldsymbol{k} = 0$ (our convention is $\boldsymbol{k}$ rotated 90° counterclockwise), because $D_1F[0, c_1, c_2]e^{i\boldsymbol{k}\cdot\boldsymbol{x}'} := L(e^{i\boldsymbol{k}\cdot\boldsymbol{x}'}) = \rho(\boldsymbol{c}, \boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}'}$ for such $\boldsymbol{k}$. This means that $\ker L$ is the span of all $e^{i\boldsymbol{k}\cdot\boldsymbol{x}'}$ such that $\boldsymbol{k}$ solves the dispersion equation and $\operatorname{ran} L$ is the span of all $e^{i\boldsymbol{k}\cdot\boldsymbol{x}'}$ such that $\boldsymbol{k}$ does not solve the dispersion equation (here we mean the span in the appropriate function space). Through a geometric argument we show that we can find $\boldsymbol{c} = \boldsymbol{c}^*$ such that $\pm\boldsymbol{k}_1$ and $\pm\boldsymbol{k}_2$ are the only solutions to the dispersion equation, which means we get a four dimensional kernel and cokernel. By additional symmetry assumptions we reduce the kernel to two dimensions and get $\ker L = \operatorname{span}\{\cos(\boldsymbol{k}_1 \cdot \boldsymbol{x}'), \cos(\boldsymbol{k}_2 \cdot \boldsymbol{x}')\}$. This means that assumptions *(i)* and *(ii)* of Theorem 1.4 are satisfied. The transversality condition (assumption *(iii)* of Theorem 1.4) is equivalent to $\nabla_{\boldsymbol{c}}\rho(\boldsymbol{c}^*, \boldsymbol{k}_1)$ and $\nabla_{\boldsymbol{c}}\rho(\boldsymbol{c}^*, \boldsymbol{k}_1)$ being linearly independent. This turns out to always be true for the choice of $\boldsymbol{c}^*$ we make. The final condition follows from the fact that F maps functions independent of $\boldsymbol{k}_j \cdot \boldsymbol{x}'$ to functions independent of $\boldsymbol{k}_j \cdot \boldsymbol{x}'$. Thus we can choose $\tilde{\mathcal{X}}_i$ and $\tilde{\mathcal{Y}}_i$ as the subspaces that are independent of $\boldsymbol{k}_j \cdot \boldsymbol{x}'$, $j \neq i$.

2.2 Paper II

The results in Sections 1.2 to 1.4 together with the results from [Paper I, Paper III] show that as long as we can solve equations (1), (2) and (4) for a general η it is likely that we can solve the full water wave problem given by equations (1), (2), (4) and (1.8c). One approach to the problem given by equations (1), (2) and (4) for doubly periodic waves is to study it in a reduced domain given by one period

$$\Omega_0 = \{\boldsymbol{x} = (\boldsymbol{x}', x_3) \in \mathbb{R} : \boldsymbol{x}' = a\boldsymbol{\lambda}_1 + b\boldsymbol{\lambda}_2,\ -d < x_3 < \eta,\ a, b \in (0, 1)\}.$$

In a bounded domain like this, though, it is reasonable to drop the assumption of periodicity and the integral condition given by equation (4), and extend equation (2) to the entire boundary by

$$\boldsymbol{u} \cdot \boldsymbol{n} = \phi,$$

where $\phi = 0$ at $x_3 = -d$ and $x_3 = \eta$. This problem has been solved by Alber in smooth domains [2]. However, the results are not immediately applicable to the problem in Ω_0 because the domain has both edges and corners. Corners are more problematic than edges so instead of attempting to solve the problem in the domain Ω_0 we restricted ourselves to a domain with edges. The results are given in [Paper II] where we solve the problem in domains that are based on a cylinder. Consider the domain

$$D = \{\boldsymbol{x} = (\boldsymbol{x}', x_3) \in \mathbb{R} : 0 < x_3 < L,\ |\boldsymbol{x}'| < \eta(x_3)\},$$

where $\eta > 0$ and $\partial_{x_3}\eta|_{x_3=0,L} = 0$ (more general domains are considered in [Paper II] but this is sufficient for the discussion below). Note that in the case $\eta = const$ this is a cylinder. By assuming $\phi < 0$ at the piece of the boundary at $x_3 = 0$, referred to as the *inflow set*, and $\phi > 0$ at the piece of the boundary at $x_3 = L$, referred to as the *outflow set*, we restrict ourselves to finding solutions which describes a flow through D.

We find the solution by perturbing an irrotational solution $\boldsymbol{u}_0$. In the paper the perturbation is given as the fixed point of a contraction. To describe the contraction we consider an iterative scheme starting with $\boldsymbol{u}_0$. $\boldsymbol{u}_{n+1}$ is obtained from $\boldsymbol{u}_n$ by solving a transport equation for the vorticity $\boldsymbol{w}_n$ along the stream lines of $\boldsymbol{u}_n$. The vorticity is determined by additional boundary conditions given on the inflow set. Then $\boldsymbol{u}_{n+1} = \boldsymbol{u}_0 + \boldsymbol{v}_n$, where $\boldsymbol{v}_n$ is the solution to

$$\nabla \times \boldsymbol{v}_n = \boldsymbol{w}_n,$$
$$\nabla \cdot \boldsymbol{v}_n = 0,$$

in D and

$$\boldsymbol{v} \cdot \boldsymbol{n} = 0,$$

on ∂D. We show that the mapping $\boldsymbol{u}_n \mapsto \boldsymbol{u}_{n+1}$ is a contraction. Hence it converges to a fixed point $\boldsymbol{u}$. We also show that this fixed point indeed gives the desired solution. Aside from the fact that the problem has to be solved in the domain Ω_0 instead of D to be applicable to the water wave problem two other issues also remain:

(i) The edges impose some restrictions on the boundary condition for the vorticity on the inflow set. These are not optimal in [Paper II].

(ii) The solutions are not necessarily periodic and there is no known condition that can be imposed to make sure that we obtain periodic solutions.

To shed some more light on these issues consider the boundary conditions that determine $\boldsymbol{w}$ on the inflow set

$$\boldsymbol{w} \cdot n = h,$$
$$\boldsymbol{w}_T = \frac{h}{\phi}\boldsymbol{u}_T - \frac{1}{\phi}\boldsymbol{n} \times \nabla_T g,$$

where g and h are given functions and the subscript T denotes the part tangential to the inflow set. With no regard for the additional conditions that h and g have to satisfy, we see that setting $h = \alpha\phi$ and $g = 0$ gives the same relations as for a Beltrami field. It can be proven independently that we have Beltrami field solutions that are periodic, i.e. $\boldsymbol{u}|_{x=0} = \boldsymbol{u}|_{x=L}$. Similarly, using the method from [26, Paper III], we can find periodic solutions where the vorticity satisfies these boundary conditions with $h = 0$ and appropriately chosen ϕ and g. Due to uniqueness this shows that certainly some of the solutions in [Paper II] are periodic, but it is unknown if these are all the periodic solutions or if there exist more. The other thing we find is that we can obtain solutions that break the restrictions put on g and h. This clearly shows that these conditions are non-optimal and possibly even redundant in the periodic case. Another reason to believe they might be redundant is that in the periodic case the domain can be viewed as an infinite domain with smooth boundary.

2.3 Paper III

For the the third paper we use the ansatz from [26] to solve equations (1), (2) and (4) with nonzero vorticity. The assumption is that the vorticity takes the form

$$\boldsymbol{w} = \nabla H \times \nabla \tau$$

for some potentials H and τ. These potentials satisfy

$$\boldsymbol{u} \cdot \nabla H = 0, \qquad \boldsymbol{u} \cdot \nabla \tau = 1, \tag{16}$$

which means that we get $\boldsymbol{u} \times \boldsymbol{w} = \nabla H$. Using equation (13) we see that a divergence free field with vorticity of this form solves equation (1) if

$$p = H - \frac{1}{2}|\boldsymbol{u}|^2 - gx_3 + C.$$

If we also let $\tau|_{x_1=0} = 0$, then τ is uniquely determined by $\boldsymbol{u}$ if $u_1 > a > 0$ in our domain. The function τ gives the time it takes for a particle to travel along the streamlines from the surface $x_1 = 0$ to the point it is evaluated at. From τ we can define $q(\boldsymbol{x}) := \tau(\boldsymbol{x} + \boldsymbol{\lambda}_1) - \tau(\boldsymbol{x})$, where $\boldsymbol{\lambda}_1$ is the generator of some lattice like the one in equation (15). However in this paper we assume that we have a rectangular lattice so $\boldsymbol{\lambda_i} = \lambda_i \boldsymbol{e}_i$. The function q clearly satisfies $\boldsymbol{u} \cdot \nabla q = 0$. Hence we can make the assumption $H = \beta h(q)$ for some function h. This means we can replace equation (1) with

$$\begin{aligned} \nabla \times \boldsymbol{u} &= \beta h'(q) \nabla q \times \nabla \tau, \\ \nabla \cdot \boldsymbol{u} &= 0. \end{aligned}$$

Through a fixed point argument we find a solution to this equation and equations (2) and (4). The problem can be rewritten as $u = \beta T[u]$ and for sufficiently small β we can apply Banach's fixed point theorem. This gives us a solution $\boldsymbol{u}[\eta, c]$. However, due to the nature of equation (16) the solution is not differentiable with respect to η and c. This complicates the bifurcation argument that is used to solve equation (1.8c). More or less immediately, though, by substituting the solution $u[\eta, c]$ in equation (1.8c) we obtain an equation that satisfies the assumptions of Theorem 1.6. For $\beta = 0$ this is the problem for irrotational flows which can be treated using Theorem 1.2, so F satisfies the needed assumptions. It is also not difficult to check that the perturbation R that appears when β is nonzero satisfies the needed assumptions. This means that there definitely are some solutions to this problem, but the conclusion of Theorem 1.6 is very weak. To strengthen the results we consider the properties of $\boldsymbol{u}$. Let

$$X_s \subset X_t,\ Y_s \subset Y_t, \qquad t < s$$

Be scales of Banach spaces (based on Hölder spaces in this paper). While $\boldsymbol{u} : Y_s \times \mathbb{R} \mapsto X_s$, is not differentiable in the regular sense, there exists a linear operator $Du : Y_s \times \mathbb{R} \to \mathcal{L}(Y_s \times \mathbb{R}, X_s)$ such that

$$\lim_{\|(\eta,c)\|_{Y_r \times \mathbb{R}} \to 0} \frac{\|\boldsymbol{u}[\eta_0 + \eta, c^* + c] - \boldsymbol{u}[\eta_0, c^*] - Du[\eta_0, c^*](\eta, c)\|_{X_t}}{\|(\eta, c)\|_{Y_r \times \mathbb{R}}} = 0$$

for $(\eta, c) \in Y_s \times \mathbb{R}$ and some $r \leq s$, $t < s$. Through some technical calculations using derivatives of this form we can prove a bifurcation result more akin to Theorem 1.2 which gives a curve of solutions. This curve is also differentiable in a sense similar to $\boldsymbol{u}$, which allows us to expand the solution in terms of the parameter of the curve. The main reason for doing this is to make sure that the expression $\beta h'(q)\nabla q \times \nabla \tau \neq 0$, which proves we indeed have solutions with nonzero vorticity.

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