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1996

ISSN 0280–5316
ISRN LUTFD2/TFRT--7554--SE

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November 1996

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Document name
INTERNAL REPORT

Date of issue
November 1996

Document Number
ISRN LUTFD2/TFRT--7554--SE

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Supervisor

Sponsoring organisation

Title and subtitle
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Key words
discrete-time algebraic Riccati equations, linear-quadratic control, numerical algorithm, Schur form, reduction scheme, well-posed, Matlab code

Classification system and/or index terms (if any)

Supplementary bibliographical information

ISSN and key title
0280–5316

ISBN

Language
English

Number of pages
32

Recipient's notes

Security classification

How to Solve Ill-Posed Semidefinite Discrete-Time Algebraic Riccati Equations

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November 4, 1996

Abstract

In this paper is discussed how to solve ill-posed semidefinite discrete-time algebraic Riccati equations in a numerically efficient and stable way. The idea is to decompose the equation into a trivial part and a reduced order well-posed equation. The solver is evaluated on some examples, and it is seen that it performs well.

1 Introduction

The interest in the Algebraic Riccati Equation (ARE) was revived by the theory of H_∞ control. There the weighting matrix in the performance index is not positive definite but usually indefinite. Not even the case when the weighting matrix is positive semidefinite, which is the case in standard H_2 control, was fully understood, and the question of how to implement good numerical solvers for these equations when they are ill-posed is still a research area. This is despite the fact that several algorithms have been proposed; either the algorithms are numerically unstable for the general case, or the complexity of the algorithms are such that they are not efficient.

The aim of the work presented here is to propose a way of circumventing the problems todays numerical solvers have for the semidefinite case. The idea is to reduce the problem to a well-posed one that can be solved in a numerically stable and efficient way by well-known solvers such as the Schur-method or the Newton-method. The proposed reduction scheme works only for Discrete-time ARE:s (DARE):s. It is numerically stable and efficient. Similar ideas can most likely be applied to the Continuous-time ARE (CARE).

The paper is organized as follows. In Section 2 background material on numerical methods for solving the DARE is given. The proposed reductions scheme is motivated. Then in Section 3 some analysis of the DARE is done, and in Section 4 the reduction scheme is derived. In Section 5 it is numerically evaluated. Finally, in Section 6 some concluding remarks together with suggestions for future research are given.

2 Background

In this section background material on the Riccati equation will be given, and especially on numerical methods for solving it. The origin of the Riccati equation goes back to Count Riccati in 1724, who studied

$$a(t)\frac{dy(t)}{dt} = b_2(t)y^2(t) + b_1(t)y(t) + b_0(t)$$

It was brought into the field of control and estimation in [Kal60]. It has many other applications. Some good survey papers are collected in [BLW91]. A recent text book is [LR95].

First notations will be introduced. Then an application of the Riccati equation will be briefly reviewed. Furthermore different numerical methods will be discussed. Their advantages and disadvantages will be investigated. Finally, different reduction methods used to extend the applicability of different numerical solvers will be surveyed.

2.1 Notations

There are many ways of writing the discrete-time Riccati equation. In its simplest form it reads

$$S(k+1) = A^T S(k) A + Q_1 - A^T S(k) B \left[B^T S(k) B + Q_2\right]^{-1} B^T S(k) A$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $Q_1 \in \mathbb{R}^{n \times n}$, $Q_2 \in \mathbb{R}^{m \times m}$, and $S(k+1) \in \mathbb{R}^{n \times n}$. The equation should be solved for $S(k)$, $k = 1, 2, \ldots$, and different initial conditions $S(0)$ may be used depending on the circumstances. The focus in this work, however, is on the DARE. This is obtained by taking $S(k+1) = S(k)$ in the above equation, i.e.

$$S = A^T S A + Q_1 - A^T S B (B^T S B + Q_2)^{-1} B^T S A$$

In many interesting applications of the Riccati equation, or for numerical stability, it is not possible or sensible to compute the inverses. This can be circumvented by considering

$$\begin{aligned} S &= A^T S A + Q_1 - L^T G L \\ GL &= B^T S A \\ G &= B^T S B + Q_2 \end{aligned}$$

where L and G are new unknown variables. Notice that this L is the optimal L in the LQ problem of Section 2.2. Usually it is desirable to look for solutions $S = S^T \geq 0$ such that $\lambda(A - BL)$ has absolute value less than or equal to one. Often it is assumed that (A, B) is stabilizable. To further generalize the equations the following replacement may be considered

$$GL = B^T S A \leftrightarrow GL = B^T S A + Q_{12}^T$$

This will enable applications to more general and interesting problems, e.g. cross-terms in the LQ problems. With

$$Q = \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix}$$

it holds that the DARE may be written

$$\begin{pmatrix} I & 0 \end{pmatrix}^T S \begin{pmatrix} I & 0 \end{pmatrix} + \begin{pmatrix} L & I \end{pmatrix}^T G \begin{pmatrix} L & I \end{pmatrix} = \begin{pmatrix} A & B \end{pmatrix}^T S \begin{pmatrix} A & B \end{pmatrix} + Q$$

There always exist C, and D such that

$$Q = \begin{pmatrix} C & D \end{pmatrix}^T J \begin{pmatrix} C & D \end{pmatrix}$$

where

$$J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

Now it is possible to further rewrite the DARE as

$$\begin{pmatrix} I & 0 \\ L & I \end{pmatrix}^T \begin{pmatrix} S & 0 \\ 0 & G \end{pmatrix} \begin{pmatrix} I & 0 \\ L & I \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} S & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1}$$

The corresponding spectral factorization reads

$$H^*(\lambda) J H(\lambda) = [I + L\Psi(\lambda)]^* G [I + L\Psi(\lambda)]$$

where $\Psi(\lambda) = (\lambda I - A)^{-1} B$, $H(\lambda) = C\Psi(\lambda) + D$. The zeros of the pencil

$$P(\lambda) = \begin{pmatrix} -\lambda I + A & B \\ C & D \end{pmatrix} \tag{2}$$

will play an important role in the analysis. These are by the spectral factorization related to the closed loop eigenvalues of $A - BL$. In the rest of the paper it will be assumed that $J = I$, i.e. $Q \geq 0$. The reason is that the ill-posedness is much easier to analyze and circumvent for this case, see Section 6 for a further discussion.

2.2 Applications of the Riccati Equation

The DARE can be used to solve many different problems in control theory such as the Linear Quadratic (LQ) control problem, and the Kalman filter problem. For the purpose of this presentation we will only need the relation to the LQ control problem.

Consider the linear time-invariant dynamic system

$$x(k+1) = Ax(k) + Bu(k) + v(k), \quad x(0) = 0$$

where $x(k) \in R^n$ is the state, $u(k) \in R^m$ is the control signal, and where $v(k) \in R^n$ is a sequence of independent random variables with zero mean value and unit covariance. Assume that $u(k)$ is given by a feedback from the states as

$$u(k) = -Lx(k)$$

in such a way that $|\lambda(A - BL)| < 1$. Introduce the performance index

$$V(L) = \lim_{k\to\infty} \mathrm{E}\left\{ \begin{pmatrix} x(k) \\ u(k) \end{pmatrix}^T Q \begin{pmatrix} x(k) \\ u(k) \end{pmatrix} \right\}$$

and the optimization problem

$$\inf_L V(L)$$

If there exists a solution to this problem it is given by the solution to the DARE such that $|\lambda(A - BL)| \leq 1$, and the infimal loss is $\mathrm{E}v^T Sv = \text{trace } S$. The existence of a solution is equivalent to the stabilizability of (A, B). Notice that the performance index also can be written

$$V(L) = \lim_{k\to\infty} \mathrm{E}\left\{ z^T(k) z(k) \right\}$$

where $z(k) = Cx(k) + Du(k)$.

2.3 Numerical Methods for Riccati Equations

Many methods for solving the ARE have been proposed in the literature, [AL84]. It is not within the scope of this presentation to survey all of them. Some of them are only applicable to special cases of the ARE, and many do not have satisfactory numerical behavior. The most promising ones are the Schur methods and Newton's methods. These will be discussed in more detail below.

2.3.1 The Schur Methods

These methods have their roots in the classical eigenvector approach for solving Riccati equations going back to [vE98]. Some early references in control literature are [MF63, Pot66, Vau70, Mar72]. However, the use of eigenvectors to form the solution of the Riccati equation is not attractive from a numerical point of view, since it essentially boils down to computing the Jordan form. A numerically stable algorithm was first obtained after it had been shown that Schur vectors could be used, [Lau79]. The Schur-decomposition goes back to [Sch09].

The idea is the following. Introduce the Symplectic matrix

$$\mathcal{A} = \begin{pmatrix} A + BQ_2^{-1}B^T A^{-T} Q_1 & -BQ_2^{-1}B^T A^{-T} \\ -A^{-T}Q_1 & A^{-T} \end{pmatrix}$$

Then, [MW31], there exists an orthogonal similarity transformation U such that $T = U^T \mathcal{A} U$ is block-upper-triangular with maximum block-size of two; this is called the real Schur form. Moreover T can be partioned such that

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}$$

where the spectrum of T_{11} has absolute value less than one and the spectrum of T_{22} has absolute value larger than one; this is called the ordered real Schur form. If then U is partioned conformably as

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$$

it holds that $S = S^T = U_{21}U_{11}^{-1} \geq 0$ solves the DARE with $\lambda(A - BL) = \lambda(T_{11})$ having absolute value less than one, see [Lau79]. For the so called ordered QR-algorithm it holds that the complexity for obtaining the ordered real Schur form decomposition is about $75n^3$ floating point operations, where n is the number of rows and columns in A.

One drawbacks with the ordered Schur-form method for solving Riccati equations presented above is that is assumes that A and Q_2 are invertible. This drawback can be overcome by considering instead of the standard eigenvalue problem associated with the regular pencil $\lambda I - \mathcal{A}$, the generalized eigenvalue problem associated with the pencil

$$P_E(\lambda) = \begin{pmatrix} 0 & -\lambda I + A & B \\ -I + \lambda A^T & Q_1 & Q_{12} \\ \lambda B^T & Q_{12}^T & Q_2 \end{pmatrix}$$

[vD81a, ENF82]. Notice how easily the Q_{12} term has been incorporated. As is the case with the classical way of solving the DARE, by means of a standard eigenvalue problem, there are different canonical forms also for the generalized eigenvalue problem. Corresponding to the Jordan form is the Kronecker form, [Kro90]. Computing the Kronecker form is, however, just as unstable from a numerical point of view as computing the Jordan form, [GvL89]. Corresponding to the Schur form and the real Schur form are the generalized Schur form, [MS73], and the generalized real Schur form, [Ste72], respectively. The ordered version of the latter one is of interest in this context, [vD81a]. Notice that it is not always possible to just apply the ordered version of the real QZ algorithm as presented in [vD81a] to compute the ordered version of the generalized real Schur form. This is because of the fact that $P_E(\lambda)$ may be irregular, i.e. $\det P_E(\lambda) \equiv 0$. When this happens the solution to the DARE is often discontinuous with respect to parameter variations.

Example 1 *Consider the LQ control problem with*

$$x(k+1) = x(k)/2 + bu(k) + v(k)$$

and

$$V = \lim_{k\to\infty} \mathrm{E}\left\{x(k)^2\right\}$$

Then the solution to the DARE is given by

$$S = \begin{cases} 4/3, & \text{if } b = 0 \\ 1, & \text{otherwise} \end{cases}$$

Example 2 *Consider the LQ control problem with*

$$x(k+1) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} x(k) + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u(k) + v(k)$$

and

$$V = \lim_{k \to \infty} \mathrm{E}\left\{[cx_1(k) + x_2(k)]^2 + u_2(k)^2\right\}$$

Then the solution to the DARE is given by

$$S = \begin{pmatrix} c^2 & c \\ c & s_{22} \end{pmatrix}$$

where

$$s_{22} = \begin{cases} 2+\sqrt{5}, & \text{if } c = 0 \\ 4, & \text{otherwise} \end{cases}$$

In the first example the input is redundant for $b = 0$ and rank $B < m$. In the second example u_1 is dynamically redundant together with x_1 for $c = 0$. There are other perturbations of (A, B, C, D) for which the solutions S are continuous, [SS95].

The way to overcome this problem is to make reductions either on the pencil $P_E(\lambda)$, or directly on the DARE before applying the ordered version of the real QZ algorithm. This will be surveyed in more detail in Section 2.4. Notice that the reductions made in [vD81a], before computing the eigenvalues, only remove the infinite eigenvalues and not the arbitrary ones.

A more sever drawback, irrespective of whether standard eigenvalue or generalized eigenvalue problems are considered for solving the DARE, is that the ordering of the the eigenvalues becomes difficult when there are eigenvalues close to the unit circle. This is usually referred to as the closed loop system $A - BL$ being poorly damped, and it was noted already in [Lau79] in an example. Later on it has been extensively studied, see e.g. [Arn83, Bye83, AL84]. It can be related to the invertibility of the Lyapunov operator associated with the DARE, [GL90]. When this happens the solution of the DARE is often continuous but not differentiable with respect to differentiable parameter variations.

Example 3 *Consider the LQ control problem with*

$$x(k+1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(k) + v(k)$$

and

$$V = \lim_{k \to \infty} \mathrm{E}\left\{c^2 x_1^2(k) + u^2(k)\right\}$$

Then for small values of c the solution to the DARE is given by

$$S = \begin{pmatrix} c^{3/2}\sqrt{2} & c \\ c & c^{1/2}\sqrt{2} \end{pmatrix}$$

Hence the solution is not differentiable for $c = 0$, and the infimal cost for the LQ problem is zero. Moreover

$$L = \begin{pmatrix} c & c^{1/2}\sqrt{2} \end{pmatrix}$$

So $A - BL$ has all its eigenvalues on the unit circle for $c = 0$, and it is only possible to come arbitrarily close to the infimum using stabilizing control.

Ways to overcome this numerical problem have been proposed. The idea is to make transformations on the matrix $\mathcal{A}$ which respect its Symplectic structure, [Bye86, BGM86]. This has been extended to the generalized eigenvalue approach in [Meh88, BGBM92]. However, these structure preserving algorithms have a complexity of order n^4 when the general case is considered, [CG87]; the ordered QZ algorithm only has a complexity of order n^3.

There are also numerical problems with the Schur methods related to bad scaling of the problem. This has been reported in [AL84, PCK87, PCK88]. The reason for these problems is to be found in the formula $S = U_{21}U_{11}^{-1}$. If U_{11} is ill-conditioned with respect to inversion or if the elements of U_{11} or U_{21} are small, S will be inaccurate; U_{11} is for example singular if (A, B) is not stabilizable, [AL84]. Ways to overcome this problem have been proposed in [KLW89, GKL92]. The idea is to multiply the matrices Q_1, Q_{12}, Q_2, and S with a suitable constant.

To summarize it should be stressed that the scaling problem is not an inherent problem to the DARE itself but to the Schur method if not properly implemented. The former two problems, however, are inherent to the DARE itself, since they may cause non-continuity or non-differentiability of the solution.

2.3.2 Newton's Method

This method is an iterative methods yielding a sequence of S_i converging to the solution S of the ARE. In continuous time the method is due to Kleinman, [Kle68] and in discrete time the method is due to Hewer, [Hew71]. The recursion reads

$$\begin{aligned}
S_i &= (A - BL_i)^T S_i (A - BL_i) + \begin{pmatrix} I \\ -L_i \end{pmatrix}^T Q \begin{pmatrix} I \\ -L_i \end{pmatrix} \\
G_i L_{i+1} &= B^T S_i A + Q_{12}^T \\
G_i &= B^T S_i B + Q_2
\end{aligned}$$

At each iteration step a Lyapunov equation and a linear system of equations have to be solved. There are several disadvantages with Newton's methods. The first one is that an initial stabilizing L_0 has to be found. This is usually computationally expensive. The second disadvantage is that the global convergence rate may be very slow. However, if the method is applied to an initial L_0

which is obtained from a Schur method, the convergence is often quadratic. To use an initial L_0 from a Schur method is sometimes sensible, since this might give a good improvement of a solution obtained from a badly scaled Schur solution, [AL84, GL90, KLW90]. Hence with this two-step solution procedure the Schur method does not have to be be implemented with rescaling. Also Newton's methods can successfully be used when solving many DARE:s with slightly perturbed parameters.

Newton's method also has problems corresponding to "arbitrary eigenvalues" of the closed loop system. Then there is no unique L_{i+1}, and a stabilizing one has to bee chosen among all the possible solutions. When there are poorly damped eigenvalues of the closed loop system, Newton's methods also have problems, [AL84, GL90, KLW90]. This is related to the fact that the Lyapunov equation then has no unique solution. Then the method does not converge to the desirable solution when implemented in finite precision.

2.3.3 Summary

The Schur methods and Newton's methods all have problems when there are arbitrary eigenvalues or eigenvalues on the unit circle of the closed loop system. The solution to the DARE is then usually not continuous or not differentiable with respect to parameter variations, respectively. Hence it can be argued that these problems are inherent to the DARE itself and not to the specific method used for solving it, i.e. the DARE is ill-posed, The well-posedness or ill-posedness of an equation is related to the conditioning of the equation. A lot of work has been done on the conditioning on the ARE. A general reference on conditioning in systems and control is [PLvD93]. There the condition number of a solution to an equation is defined, and the equation is said to be well conditioned if the condition number is small and badly conditioned if it is large. Also the equation is said to be well-posed if the condition number is finite, and otherwise it is said to be ill-posed. Depending on what norms are used several different condition numbers can be obtained, and the literature is vast, see e.g. [PCK87, KPC90, KLW90, GL90, Gha95].

2.4 Reduction Schemes

Many different reduction schemes have been proposed for the Riccati equation. Several of them apply also to the time-varying equations, including time varying systems, and not only to the ARE. They typically reduce parts of the equation which can be trivially computed. Early references are [Buc59, Dey64, BJ65, Hen68, MB68, Buc67, KG68, Sil69]. The modern reduction methods fall into two different categories: invariant direction methods and unobservable subspace methods, both of which will be surveyed below.

2.4.1 Invariant Direction Methods

In the early references to reduction methods the reductions of the Riccati equation are obtained by differentiation or differencing of the measurement signal in the Kalman filter problem associated with the Riccati equation until white noise components appear in this signal. More reductions can be done in the discrete time case as compared to the continuous time case. This was shown for the SISO case with $Q_2 = 0$ in [BRS70] using the concept of invariant directions. An invariant direction is a vector x such that $S(k)x$ is constant for $k \geq n$ for some constant n. It can be shown that this part of the Riccati equation can be solved without iterating the equation for $k \geq n$. These results were then generalized in [Rap72, GK73, CA77b, CA77a] to the general MIMO case using concepts such as constant, predictable and degenerate directions. Notice that constant and invariant directions are just different names for the same type of directions. A predictable direction is a vector x such that $S(k)x$ is zero for $k \geq n$ for some constant n. Degenerate directions is a common name for constant and predictable directions. In continuous time the predictable and constant directions coincide and are equal to the number of process derivatives that do not contain white noise. In discrete time the situation is more complicated, [GK73]. The final word about using invariant directions for reducing Riccati equatons can be found in [CA77a]. There also indefinite weighting matrices are considered.

2.4.2 Unobservable Subspace Methods

The idea here is that the solution to the LQ problem associated with the DARE for SISO systems can be obtained by taking the feedback matrix L such that the closed loop system $(C - DL, A - BL)$ is maximally unobservable. This idea goes back to [Kuc72], and in [PS73] it is generalized to minimum phase MIMO systems. There the so called structure algorithm of [SP71] is used to compute a preliminary feedback such that $(C - DL, A - BL)$ is maximally unobservable, then a reduced order DARE for which $S > 0$ is solved, and the results are combined to yield the solution to the full order DARE. Notice that this reduced order DARE has the same property as the reduced order DARE obtained from an invariant direction method, [CA77a], i.e. its solution is positive definite.

2.4.3 Summary

Notice that the reduction methods surveyed above are not numerically stable, since they rely on nonorthogonal transformations, [Cle93]. Furthermore they remove more than is necessary in order to obtain a well-posed reduced order DARE. Also many of the methods do not consider the general case. This motivates the need for the new reduction scheme proposed in this paper.

2.5 Summary

In this section the well-known Schur methods and Newton's method for solving Riccati equations have been reviewed. Also different reduction methods for

Riccati equations have been surveyed. It has been made plausible that there is still need for more research in order to implement good numerical solvers.

3 Preliminaries

In this section some preliminary results on existence and uniqueness of the solution to the DARE will be given. It will be defined when the solution is well-posed, and this will be related to the differentiability of the solution. The aim is to obtain sufficient conditions for well-posedness that motivates the reductions described in the next section.

3.1 Some Definitions

The Discrete-Time Algebraic Riccati Equation (DARE) is given by (1) with $J = I$, i.e.

$$\begin{pmatrix} I & 0 \\ L & I \end{pmatrix}^T \begin{pmatrix} S & 0 \\ 0 & G \end{pmatrix} \begin{pmatrix} I & 0 \\ L & I \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} S & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3}$$

The known variables are $A \in R^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{p\times m}$; the unknown variables are $S \in \mathbb{R}^{n\times n}$, $L \in \mathbb{R}^{m\times n}$, and $G \in \mathbb{R}^{m\times m}$. Notice that G is a trivial function of S, and hence it will often be said that (S, L) are the unknown variables. Associated to the DARE is the linear time-invariant system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ z(k) &= Cx(k) + Du(k) \end{aligned}$$

which is abbreviated (A, B, C, D). The first system equation is also abbreviated (A, B). Of interest is also the pencil $P(\lambda)$ defined in (2).

Definition 1 *The system (A, B) is said to be stabilizable if there exist L such that $|\lambda(A - BL)| < 1$.*

Definition 2 *The system (A, B, C, D) is said to be left invertible if the normal column rank of $P(\lambda)$ is full, i.e.* $\max_{\lambda\in\mathbb{C}} \operatorname{rank} P(\lambda) = m + n$.

Definition 3 *The system (A, B, C, D) is said to have no zeros on the unit circle if* $\operatorname{rank}_{|\lambda|=1} P(\lambda) = \max_{\lambda\in\mathbb{C}} \operatorname{rank} P(\lambda)$.

The following definition of condition number is taken from [PLvD93] and is under the assumption that limes and supremum can be taken in reversed order equivalent to the norm of the Fréchet derivative, [GKL92].

Definition 4 *The condition number of the solution S to the DARE is defined as*

$$\kappa\left[S(\mathcal{X})\right] = \lim_{\delta\to 0} \sup_{d_2(\mathcal{X},\mathcal{X}^*)=\delta} \frac{d_1\left(S(\mathcal{X}), S(\mathcal{X}^*)\right)}{\delta}$$

where $\mathcal{X} = (A, B, C, D)$, $\mathcal{X}^* = (A^*, B^*, C^*, D^*)$, and where d_i are the metric functions induced by e.g. the l_1 vector norm. The DARE is said to be ill-posed if $\kappa\left[S(\mathcal{X})\right]$ is infinite, and it is said to be well-posed if $\kappa\left[S(\mathcal{X})\right]$ is finite. When $\kappa\left[S(\mathcal{X})\right]$ is large the DARE is said to be ill-conditioned, and when $\kappa\left[S(\mathcal{X})\right]$ is small it is said to be well-conditioned.

3.2 Existence and Uniqueness

The existence and uniqueness of the DARE has been described in many different references. One of the more general ones is [HK95], which is based on writing the Popov equation associated to the DARE in its Smith-form. Here another approach will be taken which is based on the Kleinman-iteration.

Theorem 1 *Assume that (A, B) is stabilizable and that (A, B, C, D) is left invertible. Then there is always a real solution (S, L) to the DARE such that $S = S^T \geq 0$ and with $|\lambda(A - BL)| \leq 1$. It also holds that $G > 0$, and that both S and L are unique. Furthermore $|\lambda(A - BL)| < 1$, if and only if (A, B, C, D) has no zeros on the unit circle.*

Proof: See Appendix. □

Theorem 2 *Assume that (A, B) is stabilizable. Then there is always a real solution (S, L) to the DARE such that $S = S^T \geq 0$ and with $|\lambda(A - BL)| \leq 1$. It also holds that $G \geq 0$, and that S is unique. However, L is not unique. Furthermore $|\lambda(A - BL)| < 1$, if and only if (A, B, C, D) has no zeros on the unit circle.*

Proof: See the next section. □

3.3 Well-Posedness

Now sufficient conditions for the DARE to be well-posed will be derived.

Lemma 1 *Assume that (A, B) is stabilizable and that (A, B, C, D) is left invertible and has no zeros on the unit circle. Then the real symmetric positive semidefinite solution S to the DARE such that $|\lambda(A - BL)| < 1$ is differentiable at (A, B, C, D).*

Proof: Formal implicit derivation with respect to (A, B, C, D) gives

$$\begin{aligned}
\dot{S} + \dot{L}^T GL + L^T \dot{G} L + L^T G \dot{L} &= \dot{A}^T SA + A^T \dot{S} A + A^T S \dot{A} \\
&+ \dot{C}^T C + C^T \dot{C} \qquad (4) \\
\dot{G} L + G \dot{L} &= \dot{B}^T SA + B^T \dot{S} A + B^T S \dot{A} \\
&+ \dot{D}^T C + D^T \dot{C} \qquad (5) \\
\dot{G} &= \dot{B}^T SB + B^T \dot{S} B + B^T S \dot{B} \\
&+ \dot{D}^T D + D^T \dot{D} \qquad (6)
\end{aligned}$$

Substitution of (5) and (6) into (4) gives

$$\begin{aligned}
\dot{S} &= (A - BL)^T \dot{S} (A - BL) \\
&+ (\dot{A} - \dot{B}L)^T S (A - BL) + (A - BL)^T S (\dot{A} - \dot{B}L) \\
&+ (\dot{C} - \dot{D}L)^T S (C - DL) + (C - DL)^T S (\dot{C} - \dot{D}L)
\end{aligned}$$

Now, $A - BL$ is stable by assumption and the equation for $\dot{S}$ is a Lyapunov-equation. Hence it follows that $\dot{S}$ is unique. By (6) it follows that $\dot{G}$ is unique. Further, since $G > 0$, it follows by (5) that $\dot{L}$ is unique as well. Hence S is differentiable at (A, B, C, D) by the implicit function theorem. □

Theorem 3 *Assume that (A, B) is stabilizable and that (A, B, C, D) is left invertible and has no zeros on the unit circle. Then the DARE is well-posed.*

Proof: Assume that the DARE is not well-posed. Then $\kappa\left[S(\mathcal{X})\right] = \infty$. This implies that there exist a $\mathcal{Y}$ such that

$$\infty = \lim_{\delta \to 0} \sup_{\eta \mathcal{Y}: d_2(\mathcal{X}, \mathcal{X}+\eta\mathcal{Y})=\delta} \frac{d_1\left(S(\mathcal{X}), S(\mathcal{X}+\eta\mathcal{Y})\right)}{\delta} = \lim_{\eta \to 0} \frac{d_1\left(S(\mathcal{X}), S(\mathcal{X}+\eta\mathcal{Y})\right)}{d_2\left(\mathcal{X}, \mathcal{X}+\eta\mathcal{Y}\right)}$$

which proves that S is not differentiable at (A, B, C, D). This is by by Lemma 1 a contradiction. □

3.4 Summary

In this section the existence and uniqueness of the solution of the DARE has been investigated. Also sufficient conditions for when the DARE is well-posed have been given. It will be seen in the next section that any DARE can be reduced to a well-posed one in a numerically sound way.

4 Reduction Scheme

In this section it will be shown how to reduce the DARE to a well-posed equation. The reductions are performed on the associated linear time invariant system (A, B, C, D), and they have very intuitive signal interpretations connected to the LQ-problem. The transformations performed are very similar to the ones described in [CSS93].

4.1 Statically Redundant Input Signals

Make an SVD transformation on $\begin{pmatrix} B \\ D \end{pmatrix}$ such that

$$U^T \begin{pmatrix} B \\ D \end{pmatrix} V = \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}$$

and let

$$\begin{pmatrix} A & \bar{B} \\ C & \bar{D} \end{pmatrix} = \begin{pmatrix} A & \bar{B}_1 & 0 \\ C & \bar{D}_1 & 0 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & V \end{pmatrix}$$

where $\begin{pmatrix} \bar{B}_1 \\ \bar{D}_1 \end{pmatrix}$ has full column rank. In these new coordinates the DARE reads

$$\begin{pmatrix} I & 0 \\ \bar{L} & I \end{pmatrix}^T \begin{pmatrix} S & 0 \\ 0 & \bar{G} \end{pmatrix} \begin{pmatrix} I & 0 \\ \bar{L} & I \end{pmatrix} = \begin{pmatrix} A & \bar{B} \\ C & \bar{D} \end{pmatrix}^T \begin{pmatrix} S & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & \bar{B} \\ C & \bar{D} \end{pmatrix}$$

where $L = V\bar{L}$ and $\bar{G} = V^T G V$. Some calculations show that with

$$\bar{L} = \begin{pmatrix} \bar{L}_1 \\ \bar{L}_2 \end{pmatrix}; \quad \bar{G} = \begin{pmatrix} \bar{G}_1 & \bar{G}_{12} \\ \bar{G}_{12}^T & \bar{G}_2 \end{pmatrix}$$

this DARE is equivalent to

$$\begin{pmatrix} I & 0 \\ \bar{L}_1 & I \end{pmatrix}^T \begin{pmatrix} S & 0 \\ 0 & \bar{G}_1 \end{pmatrix} \begin{pmatrix} I & 0 \\ \bar{L}_1 & I \end{pmatrix} = \begin{pmatrix} A & \bar{B}_1 \\ C & \bar{D}_1 \end{pmatrix}^T \begin{pmatrix} S & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & \bar{B}_1 \\ C & \bar{D}_1 \end{pmatrix}$$

and $\bar{G}_2 = 0$, $\bar{G}_{12} = 0$, and $\bar{L}_2$ arbitrary.

Now rename all variables so that the above reduced DARE reads

$$\begin{pmatrix} I & 0 \\ L & I \end{pmatrix}^T \begin{pmatrix} S & 0 \\ 0 & G \end{pmatrix} \begin{pmatrix} I & 0 \\ L & I \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} S & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

4.2 Decoupling of Direct Term

Make an SVD transformation on D such that

$$U^T D V = \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}$$

and let

$$\begin{pmatrix} A & \bar{B} \\ \bar{C} & \bar{D} \end{pmatrix} = \begin{pmatrix} A & \bar{B}_1 & \bar{B}_2 \\ \bar{C}_1 & \Sigma & 0 \\ \bar{C}_2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & U^T \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & V \end{pmatrix}$$

The DARE is now equivalent to

$$\begin{pmatrix} I & 0 \\ \bar{L} & I \end{pmatrix}^T \begin{pmatrix} S & 0 \\ 0 & \bar{G} \end{pmatrix} \begin{pmatrix} I & 0 \\ \bar{L} & I \end{pmatrix} = \begin{pmatrix} A & \bar{B} \\ \bar{C} & \bar{D} \end{pmatrix}^T \begin{pmatrix} S & 0 \\ 0 & \bar{I} \end{pmatrix} \begin{pmatrix} A & \bar{B} \\ \bar{C} & \bar{D} \end{pmatrix}$$

where $L = V\bar{L}$ and $\bar{G} = V^T G V$. Now let $\tilde{L}_1$ be a solution to

$$\Sigma \tilde{L}_1 = \bar{C}_1$$

and define

$$\tilde{L} = \begin{pmatrix} \tilde{L}_1 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} = \begin{pmatrix} A & \bar{B} \\ \bar{C} & \bar{D} \end{pmatrix} \begin{pmatrix} I & 0 \\ -\tilde{L} & I \end{pmatrix} = \begin{pmatrix} \tilde{A} & \bar{B}_1 & \bar{B}_2 \\ 0 & \Sigma & 0 \\ \bar{C}_2 & 0 & 0 \end{pmatrix}$$

where $\tilde{A} = A - \bar{B}\tilde{L}$. The DARE is now equivalent to

$$\begin{pmatrix} I & 0 \\ \hat{L} & I \end{pmatrix}^T \begin{pmatrix} S & 0 \\ 0 & \bar{G} \end{pmatrix} \begin{pmatrix} I & 0 \\ \hat{L} & I \end{pmatrix} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}^T \begin{pmatrix} S & 0 \\ 0 & \bar{I} \end{pmatrix} \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}$$

where $\hat{L} = \bar{L} - \tilde{L}$. Rename all variables so that the above DARE reads

$$\begin{pmatrix} I & 0 \\ L & I \end{pmatrix}^T \begin{pmatrix} S & 0 \\ 0 & G \end{pmatrix} \begin{pmatrix} I & 0 \\ L & I \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} S & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B_1 & B_2 \\ 0 & \Sigma & 0 \\ C_2 & 0 & 0 \end{pmatrix}$$

with B_2 full collumn rank, and Σ invertible. Notice that the computation of $\tilde{L}_1$ is numerically stable in the sense that it is possible to choose the condition number of Σ by selection of the tolerance in the SVD computation.

4.3 Reduction to Left Invertability

Apply the numerically stable algorithm of van Dooren in [vD81b] to compute the supremal (A, B_2)-controllability subspace in the kernel of C_2. This transformation can be written

$$\begin{aligned} \begin{pmatrix} \bar{A} & \bar{B}_2 \\ \bar{C}_2 & 0 \end{pmatrix} &= \begin{pmatrix} \bar{A}_1 & 0 & \bar{A}_{13} & \bar{B}_{12} & 0 \\ \bar{A}_{21} & \bar{A}_2 & \bar{A}_{23} & \bar{B}_{22} & \bar{B}_{23} \\ 0 & 0 & \bar{A}_3 & \bar{B}_{32} & 0 \\ 0 & 0 & \bar{C}_{23} & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} T & 0 \\ 0 & I \end{pmatrix}^T \begin{pmatrix} A & B_2 \\ C_2 & 0 \end{pmatrix} \begin{pmatrix} T & 0 \\ -\bar{L}_2 & V_2 \end{pmatrix} \end{aligned}$$

for unitary T and V_2, $\bar{L}_2$ partioned as

$$\bar{L}_2 = \begin{pmatrix} \bar{L}_{21} & \bar{L}_{22} & 0 \\ \bar{L}_{31} & \bar{L}_{32} & 0 \end{pmatrix}$$

and where $\bar{B}_{32}$ has full collumn rank. It holds with

$$\bar{L} = \begin{pmatrix} 0 \\ \bar{L}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \bar{L}_{21} & \bar{L}_{22} & 0 \\ \bar{L}_{31} & \bar{L}_{32} & 0 \end{pmatrix}; \quad V = \begin{pmatrix} I & 0 \\ 0 & V_2 \end{pmatrix}$$

that

$$\begin{aligned} \begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & D \end{pmatrix} &= \begin{pmatrix} \bar{A}_1 & 0 & \bar{A}_{13} & B_{11} & \bar{B}_{12} & 0 \\ \bar{A}_{21} & \bar{A}_2 & \bar{A}_{23} & B_{21} & \bar{B}_{22} & \bar{B}_{23} \\ 0 & 0 & \bar{A}_3 & B_{31} & \bar{B}_{32} & 0 \\ 0 & 0 & 0 & \Sigma & 0 & 0 \\ 0 & 0 & \bar{C}_{23} & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} T & 0 \\ 0 & I \end{pmatrix}^T \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} T & 0 \\ -\bar{L} & V \end{pmatrix} \end{aligned}$$

where $(\bar{A}_2, \bar{B}_{23})$ is the supremal (A, B_2)-controllability subspace in the kernel of C_2. The DARE now reads

$$\begin{pmatrix} I & 0 \\ \hat{L} & I \end{pmatrix}^T \begin{pmatrix} \bar{S} & 0 \\ 0 & \bar{G} \end{pmatrix} \begin{pmatrix} I & 0 \\ \hat{L} & I \end{pmatrix} = \begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{pmatrix}^T \begin{pmatrix} \bar{S} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{pmatrix}$$

where $V\hat{L} = LT - \bar{L}$, $\bar{G} = V^T G V$, and $\bar{S} = T^T S T$. Partition $\hat{L}$ as

$$\hat{L} = \begin{pmatrix} \hat{L}_1 \\ \hat{L}_2 \\ \hat{L}_3 \end{pmatrix} = \begin{pmatrix} \hat{L}_{11} & \hat{L}_{12} & \hat{L}_{13} \\ \hat{L}_{21} & \hat{L}_{22} & \hat{L}_{23} \\ \hat{L}_{31} & \hat{L}_{32} & \hat{L}_{33} \end{pmatrix}$$

Then it holds that

$$\begin{aligned} \bar{S} &= \begin{pmatrix} \bar{S}_1 & 0 & \bar{S}_{13} \\ 0 & 0 & 0 \\ \bar{S}_{13}^T & 0 & \bar{S}_3 \end{pmatrix}; \quad \hat{L} = \begin{pmatrix} \hat{L}_1 \\ \hat{L}_2 \\ \hat{L}_3 \end{pmatrix} = \begin{pmatrix} \hat{L}_{11} & 0 & \hat{L}_{13} \\ \hat{L}_{21} & 0 & \hat{L}_{23} \\ \hat{L}_{31} & \hat{L}_{32} & \hat{L}_{33} \end{pmatrix} \\ \bar{G} &= \begin{pmatrix} \bar{G}_1 & \bar{G}_{12} & 0 \\ \bar{G}_{12}^T & \bar{G}_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

and $\hat{L}_3$ arbitrary but such that $\bar{A}_2 - \bar{B}_{23}\hat{L}_{32}$ is stable, is a solution with

$$\tilde{S} = \begin{pmatrix} \bar{S}_1 & \bar{S}_{13} \\ \bar{S}_{13}^T & \bar{S}_3 \end{pmatrix}; \quad \tilde{L} = \begin{pmatrix} \hat{L}_{11} & \hat{L}_{13} \\ \hat{L}_{21} & \hat{L}_{23} \end{pmatrix}; \quad \tilde{G} = \begin{pmatrix} \bar{G}_1 & \bar{G}_{12} \\ \bar{G}_{12}^T & \bar{G}_2 \end{pmatrix}$$

satisfying the reduced DARE

$$\begin{pmatrix} I & 0 \\ \tilde{L} & I \end{pmatrix}^T \begin{pmatrix} \tilde{S} & 0 \\ 0 & \tilde{G} \end{pmatrix} \begin{pmatrix} I & 0 \\ \tilde{L} & I \end{pmatrix} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}^T \begin{pmatrix} \tilde{S} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}$$

where

$$\begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} = \begin{pmatrix} \bar{A}_1 & \bar{A}_{13} & B_{11} & \bar{B}_{12} \\ 0 & \bar{A}_3 & B_{31} & \bar{B}_{32} \\ 0 & 0 & \Sigma & 0 \\ 0 & \bar{C}_{23} & 0 & 0 \end{pmatrix}$$

Rename all variables so that the above DARE reads

$$\begin{pmatrix} I & 0 \\ L & I \end{pmatrix}^T \begin{pmatrix} S & 0 \\ 0 & G \end{pmatrix} \begin{pmatrix} I & 0 \\ L & I \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} S & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_1 & A_{13} & B_{11} & B_{12} \\ 0 & A_3 & B_{31} & B_{32} \\ 0 & 0 & \Sigma & 0 \\ 0 & C_{23} & 0 & 0 \end{pmatrix}$$

It now remains to show that this system is left invertible, i.e. $P(\lambda)$ has full normal collumn rank. This follows by similar arguments as in [Kai80], pp 543–546, by noting that Σ is invertible. It is now also clear that Theorem 2 has been proven.

4.4 Removal of Finite Zeros

It holds that all the finite zeros of $P(\lambda)$ are given by the eigenvalues of A_1. It turns out that some of them also give zero contribution to S, and that is the topic of this subsection. Let U be a unitary transformation on A_1 such that

$$A_1 U = U \begin{pmatrix} \bar{A}_1 & \bar{A}_{12} \\ 0 & \bar{A}_2 \end{pmatrix}$$

where $\bar{A}_1$ has all its eigenvalues inside or on the unit circle, and where $\bar{A}_2$ has all its eigenvalues strictly outside the unit circle. Such a transformation can be obtained in a numerically stable way from the real ordered QR-algorithm. Now define

$$\begin{aligned} \begin{pmatrix} \bar{\bar{A}} & \bar{\bar{B}} \\ \bar{C} & D \end{pmatrix} &= \begin{pmatrix} \bar{A}_1 & \bar{A}_{12} & \bar{A}_{13} & \bar{B}_{11} & \bar{B}_{12} \\ 0 & \bar{A}_2 & \bar{A}_{23} & \bar{B}_{21} & \bar{B}_{22} \\ 0 & 0 & A_3 & B_{31} & B_{32} \\ 0 & 0 & 0 & \Sigma & 0 \\ 0 & 0 & C_{23} & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \bar{U} & 0 \\ 0 & I \end{pmatrix}^T \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \bar{U} & 0 \\ 0 & I \end{pmatrix} \end{aligned}$$

where

$$\bar{U} = \begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix}$$

The DARE is now equivalent to

$$\begin{pmatrix} I & 0 \\ \bar{L} & I \end{pmatrix}^T \begin{pmatrix} \bar{S} & 0 \\ 0 & G \end{pmatrix} \begin{pmatrix} I & 0 \\ \bar{L} & I \end{pmatrix} = \begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & D \end{pmatrix}^T \begin{pmatrix} S & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & D \end{pmatrix}$$

where $\bar{L} = L\bar{U}$, and $\bar{S} = \bar{U}^T S \bar{U}$. It holds that

$$\bar{S} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \bar{S}_2 & \bar{S}_{23} \\ 0 & \bar{S}_{23}^T & \bar{S}_3 \end{pmatrix}; \quad \bar{L} = \begin{pmatrix} 0 & \bar{L}_2 & \bar{L}_3 \end{pmatrix}$$

is a solution with

$$\tilde{S} = \begin{pmatrix} \bar{S}_2 & \bar{S}_{23} \\ \bar{S}_{23}^T & \bar{S}_3 \end{pmatrix}; \quad \tilde{L} = \begin{pmatrix} \bar{L}_2 & \bar{L}_3 \end{pmatrix}$$

satisfying the reduced DARE

$$\begin{pmatrix} I & 0 \\ \tilde{L} & I \end{pmatrix}^T \begin{pmatrix} \tilde{S} & 0 \\ 0 & G \end{pmatrix} \begin{pmatrix} I & 0 \\ \tilde{L} & I \end{pmatrix} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & D \end{pmatrix}^T \begin{pmatrix} \tilde{S} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & D \end{pmatrix}$$

where

$$\begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & D \end{pmatrix} = \begin{pmatrix} \bar{A}_2 & \bar{A}_{23} & \bar{B}_{21} & \bar{B}_{22} \\ 0 & A_3 & B_{31} & B_{32} \\ 0 & 0 & \Sigma & 0 \\ 0 & C_3 & 0 & 0 \end{pmatrix}$$

Rename all variables so that the above DARE reads

$$\begin{pmatrix} I & 0 \\ L & I \end{pmatrix}^T \begin{pmatrix} S & 0 \\ 0 & G \end{pmatrix} \begin{pmatrix} I & 0 \\ L & I \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} S & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_1 & A_{12} & B_{11} & B_{12} \\ 0 & A_2 & B_{21} & B_{22} \\ 0 & 0 & \Sigma & 0 \\ 0 & C_{23} & 0 & 0 \end{pmatrix}$$

where all the finite zeros of $P(\lambda)$ are given by the eigenvalues of A_1, and where the eigenvalues of A_1 are strictly outside the unit circle. Hence by Theorem 3 it holds that this reduced DARE is well-posed in the sense of Definition 4.

The algorithm described so far provides a more stable way of making the reductions described in [PS73, CA77a]. It also generalizes [PS73] to the cases when $\begin{pmatrix} B \\ D \end{pmatrix}$ does not have full column rank and when $P(\lambda)$ is non-minimum-phase. Notice that it is unnecessary to remove the strictly stable modes of $\bar{A}_1$ in order to get a well-posed DARE. However, it is easier from a numerical point of view.

4.5 Numerical Stability

What numerical properties does the reduction scheme have? Unfortunately it is not possible to claim the the transformations performed on

$$P(0) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

are backward stable. This is due to the computation of the feedback gains in sections 4.2 and 4.3. However, these computations can be given desirable condition number by choosing the tolerance in the computation of the SVD decompositions yielding the left hand sides of the linear systems of equations to be solved to obtain the feedback gains. Moreover the feedback gain in Section 4.3 does not have to be computed in case there are no zeros on the unit circle. This is because of the fact that only the input signals, not the states have to be removed in order to obtain left invertability, [vD81b].

What solution to the DARE is obtained by performing the above reductions? Losely speaking they yield the largest solution in the following sense. With a certain accuracy the above reduction algorithm tries to remove as many input signals as possible. This implies for the associated LQ control problem that there is less freedom in computing the infimal value of trace$S$. Hence trace$S$ is maximized. However, it is not possible to say that this maximization is done over some reasonable set close to P(0), such as e.g.

$$\{\bar{P}(0) : ||P(0) - \bar{P}(0)|| \leq \epsilon\}$$

since most likely slight perturbations within this set would make trace$S$ slightly larger. However, due to the differentiability of the reduced DARE this solution would not be significantly larger. Hence it is fair to say that the reduction scheme approximately finds the solution that maximizes trace$S$ for a problem close to (A, B, C, D). This is a very tractable choice from the point of view of the application to LQ control, since the largest solution corresponds to lower feedback gain, i.e. use of less energy in the control signal.

4.6 Summary

In this section it has been shown how to reduce the DARE to a well-posed one. These reductions involved removing statically redundant input signals, a supremal controllability subspace, and finite zeros inside or on the unit circle. Also Theorem 2 has been proven. It has been argued that the reduction scheme is numerically sound.

5 Evaluation

In this section the reduction scheme of the previous section will be evaluated. It has been implemented in Matlab, [Mat92]. To compute the supremal (A, B) controllability subspace in the kernel of C Algorithms 1–3 of [vD81b] have been implemented, and to compute the invariant subspaces to remove the stable finite zeros the fortran LAPACK routine DGEESX, see [ABB+92], has been interfaced to Matlab using cmex, [Mat93]. To solve the reduced DARE the Schur method using the generalized eigenvalue approach described in Section 2 has been implemented in Matlab using the routine qzorder by Cleve Moler to solve and order the generalized eigenvalue problem. Unfortunately this routine only computes and orders the complex Schur-form and not the real one. However, it has not been possible to find any implementation of the real one, and it is not within the scope of this work to implement such an algorithm. Benchmark problems for the DARE have been proposed in [BL96]. However, these problems do not address ill-posedness, and cannot be used for evaluating the reduction scheme proposed in this paper.

In Section 5.1 Examples 1–2 will be revisited, and in Section 5.2 Example 3 will be revisited. In Section 5.3 an application examples of LQ control will be investigated, and in Section 5.4 some concluding remarks will be given.

5.1 Non Left Invertability

For Example 1 both the standard routine, i.e. the Schur form method of Section 2 without any reductions, and the reduction routine give the same correct answer, i.e. $S = 1.3333$. For Example 2 the two routines give different answers. The standard routine results in $S = \infty$ whereas the reduction routine gives the answer

$$S = \begin{pmatrix} 0 & 0 \\ 0 & 4.23606797749979 \end{pmatrix}$$

which is the correct answer with at least 14 decimals. The reduced system is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

from which it is seen that one control signal and one state has been removed.

5.2 Zeros on the Unit Circle

For Example 3 the two routines also give different answers. The standard routine results in

$$S = \begin{pmatrix} 0.00000000099804 & 0.00000831479363 \\ 0.00000831482721 & 0.13180676188135 \end{pmatrix} \times 10^{-3}$$

whereas the reduction routine gives the correct answer $S = 0$.

5.3 Linear Quadratic Control

Here an application example concerning level control of a tank with two pumps as actuators will be investigated. It is assumed that the pumps have dynamics. The continuous time transfer functions relating the voltages u_1 and u_2 applied to the pumps to the flows q_1 and q_2 going into the tank are given by

$$\begin{aligned} q_1(s) &= \frac{1}{sT_1+1}u_1(s) \\ q_2(s) &= \frac{1}{sT_2+1}u_2(s) \end{aligned}$$

where $T_1 = 1$ and $T_2 = 0.1$ are the time constants of the pumps. The level z in the tank is then given by

$$z(s) = \frac{1}{s}\left[q_1(s) + q_2(s)\right]$$

Sampling these equations with zero order hold and sample interval 0.02 and introducing a state space description results in

$$\begin{aligned} x(k+1) &= \begin{pmatrix} 0.9802 & 0 & 0 \\ 0 & 0.8187 & 0 \\ 0.0198 & 0.0181 & 1.0000 \end{pmatrix} x(k) + \begin{pmatrix} 0.0198 & 0 \\ 0 & 0.1813 \\ 0.0002 & 0.0019 \end{pmatrix} u(k) \\ z(k) &= \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} x(k) \end{aligned}$$

The solution to the DARE is for the standard routine

$$\begin{aligned} S &= \begin{pmatrix} 0.00000000000000 & 0.00000000000000 & 0.00000000000000 \\ 0.00000000000000 & 0.00000000000000 & 0.00000000000000 \\ 0.00000000000000 & 0.00000000000000 & 1.00000000000014 \end{pmatrix} \\ L &= \begin{pmatrix} 0.12053532522949 & 0.06889785060581 & 7.12712277493106 \\ -0.00221337867926 & 0.00236977171475 & -0.22207812340223 \end{pmatrix} \times 10^3 \end{aligned}$$

with closed loop eigenvalues

$$\begin{pmatrix} -1.01741070514381 \\ 0.00001404447402 \\ -0.00000000000000 \end{pmatrix}$$

The solution to the DARE is for the reduction routine

$$\begin{aligned} S &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ L &= \begin{pmatrix} 0.65311429051678 & 0.15334235086238 & 0.55997718084119 \\ 0.03644108214024 & 0.08051155999052 & 5.27941799275496 \end{pmatrix} \times 10^2 \end{aligned}$$

with closed loop eigenvalues

$$\begin{pmatrix} -0.95375050187009 \\ -0.00000000178825 \\ 0.00000000178825 \end{pmatrix}$$

So not only does the standard routine give the wrong S as solution but it also provides an L which is not stabilizing.

5.4 Summary

In this section the new reduction routine has been evaluated an compared to a standard solver. It has been shown that the reduction routine is always giving as good results as the standard routine, and in most cases it performs better.

6 Conclusions

In this paper has been discussed how to solve ill-posed semidefinite DARE:s in a numerically sound way. Different methods proposed in the literature have been surveyed. It was found that none of the existing methods are satisfactory. Either they do not address the problem in its full generality, or they are too inefficient.

A method that transforms the DARE into a trivial part and a reduced order well-posed DARE has been developed. It has been argued that the reduction scheme is numerically sound. It has been implemented in Matlab, and it has in several examples been shown that the new scheme is very promising.

Future Research

So far only the semidefinite case has been addressed. Many applications of the DARE are indefinite, such as e.g. H_∞-control. Hence it would be of great interest to extend the results to that case. Unfortunately the DARE is not invariant under output-injection, the dual of state-feedback, so it is not possible to make dual reductions to get right invertability, which by the spectral factorization would have guaranteed uniqueness of the solution, a key part of the derivations in this paper. Thus it is not clear if it is possible to make reductions only on (A, B, C, D) in order to obtain a well-posed reduce order DARE, or if J also has to be taken into consideration for the indefinite case.

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7 Appendix

Here Theorem 1 will be proven. To this end the following lemmas are needed.

Lemma 2 *Assume $S \geq 0$ satisfies $S = A^T SA + Q, Q \geq 0$. Then $Q \neq 0$, only if A has an eigenvalue $|\lambda| < 1$.*

Proof: Assume first that A has a complete set of eigenvectors. There is then for $Q \neq 0$ an eigenvector x with $x^*Qx > 0$, and $x^*Sx = |\lambda|^2 x^*Sx + x^*Qx$ then requires $|\lambda| < 1$. For defective A the proof is extended using generalized eigenvectors. □

Lemma 3 *For any real symmetric solution $S \geq 0$ and any corresponding $A - BL$ introduce $T = \begin{pmatrix} T_- & T_0 & T_+ \end{pmatrix}$ with $(A-BL)T = TJ$, $J = \mathrm{diag}(J_-, J_0, J_+)$ and $|\lambda(J_-)| > 1$, $|\lambda(J_0)| = 1$, $|\lambda(J_+)| < 1$. Then*

$$(C - DL)\begin{pmatrix} T_- & T_0 \end{pmatrix} = 0, \quad S\begin{pmatrix} T_- & T_0 \end{pmatrix} = 0$$

Proof: The proof follows from Lemma 2 using stabilizability and

$$T^T ST = J^T T^T STJ + T^T (C - DL)^T (C - DL)T$$

□

Lemma 4 *The system (A, B, C, D) is left invertible if and only if the normal column rank of $H(\lambda) = C(\lambda I - A)^{-1}B + D$ is full, i.e. $\max_{\lambda \in \mathbb{C}} \mathrm{rank} H(\lambda) = m$.*

Proof: The proof follows from the following identity:

$$P(\lambda) = \begin{pmatrix} -\lambda I + A & 0 \\ C & H(\lambda) \end{pmatrix} \begin{pmatrix} I & -\Psi(\lambda) \\ 0 & I \end{pmatrix}$$

where $\Psi(\lambda) = (\lambda I - A)^{-1}B$, and the fact that there exist λ such that $\mathrm{rank}(\lambda I - A) = n$. □

7.1 Proof of Theorem 1

A straight-forward proof, cf. [Kuc91], can be made using the Kleinman recursion

$$\begin{aligned} A_i &= A - BL_i, \quad C_i = C - DL_i && (7) \\ S_i &= A_i^T S_i A_i + C_i^T C_i && (8) \\ G_i &= D^T D + B^T S_i B && (9) \\ G_i L_{i+1} &= D^T C + B^T S_i A && (10) \end{aligned}$$

for $i = 0, 1, \ldots$ with initial value L_0 such that A_0 is stable. It will first be shown that the sequence of L_i is well defined, and then the question about convergence will be investigated. Assume that A_i is stable. Then there exists a unique $S_i \geq 0$

that solves (8), since it is a Lyapunov-equation, and there exists an L_{i+1} that solves (10), since

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} S_i & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \geq 0$$

If it can be concluded that A_{i+1} is stable, it thus follows by induction that A_i is stable for all $i \geq 0$. Assume that A_{i+1} is not stable. Then there exist λ and x such that $|\lambda| \geq 1$ and $A_{i+1}x = x\lambda$. Now use $\Delta_i = (L_i - L_{i+1})^T G_i (L_i - L_{i+1})$ in (7–10) to obtain

$$S_i = A_{i+1}^T S_i A_{i+1} + C_{i+1}^T C_{i+1} + \Delta_i \tag{11}$$

and

$$(1 - |\lambda|^2) x^* S_i x = x^* C_{i+1}^T C_{i+1} x + x^* \Delta_i x$$

From $|\lambda| \geq 1$ and $S_i \geq 0$ follows that $x^* \Delta_i x = 0$. Thus $L_i x = L_{i+1} x$ provided $G_i > 0$, and hence the contradiction that λ is also an eigenvalue of A_i. To show that $G_i > 0$, rewrite (7–10) and (11) as

$$\begin{aligned} &\begin{pmatrix} I & 0 \\ L_{i+1} & I \end{pmatrix}^T \begin{pmatrix} S_i - \Delta_i & 0 \\ 0 & G_i \end{pmatrix} \begin{pmatrix} I & 0 \\ L_{i+1} & I \end{pmatrix} \\ = &\begin{pmatrix} A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} S_i & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{aligned} \tag{12}$$

Let $\Psi(\lambda) = (\lambda I - A)^{-1} B$, and let $H(\lambda) = C\Psi(\lambda) + D$. Notice that $A\Psi(\lambda) + B = -\lambda\Psi(\lambda)$. Thus by multiplying (12) by $\begin{pmatrix} \Psi(\lambda) \\ I \end{pmatrix}$ from the right and its adjoint from the left the following equality is obtained

$$H^*(\lambda)H(\lambda) + \Psi^*(\lambda)\Delta_i\Psi(\lambda) = [I + L_{i+1}\Psi(\lambda)]^* G_i [I + L_{i+1}\Psi(\lambda)] \tag{13}$$

Now by Lemma 4 the left invertability of (A, B, C, D) implies that rank $H(\lambda) = m$ for some λ, which by (13) and $\Delta_i \geq 0$ implies that $G_i > 0$. Thus it is proven that the sequence of L_i is well defined and A_i is stable for all $i \geq 0$.

It will now be shown that the sequence S_i converges to some limit S. Further manipulations show that the following Lyapunov-equation holds

$$S_i - S_{i+1} = A_{i+1}^T (S_i - S_{i+1}) A_{i+1} + \Delta_i \tag{14}$$

Since A_{i+1} is stable and since $\Delta_i \geq 0$ it follows that $S_i - S_{i+1} \geq 0$. Thus it holds that $0 \leq S_{i+1} \leq S_i$, which implies that $S_i \to S \geq 0$. The equation (9) implies that $G_i \to G = D^T D + B^T S B$, and there exists L such that $GL = D^T C + B^T S A$, since

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} S & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \geq 0$$

Further $A_{i+1}^T (S_i - S_{i+1}) A_{i+1} \to 0$ and $\Delta_i \to 0$, since both matrices are positive semidefinite. Thus it is proven that the limit S solves the DARE. Similarly to (13) it also holds that

$$H^*(\lambda)H(\lambda) = [I + L\Psi(\lambda)]^* G [I + L\Psi(\lambda)] \tag{15}$$

Now by Lemma 4 the left invertability of (A, B, C, D) implies that $G > 0$, and hence L is a unique solution. The sequence L_i therefore converges to L, and since the eigenvalues of A_i are inside the unit circle, it follows that in the limit the eigenvalues of $A_c = A - BL$ are inside or on the unit circle.

Now the right-hand side of (15) looses rank on the unit circle when λ is a closed loop eigenvalue on the unit circle, since $G > 0$ and

$$\begin{pmatrix} -\lambda I + A & 0 \\ L & I + L\Psi(\lambda) \end{pmatrix} = \begin{pmatrix} -\lambda I + A_c & B \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ L & I \end{pmatrix} \begin{pmatrix} I & \Psi(\lambda) \\ 0 & I \end{pmatrix}$$

Similarly the left-hand side looses rank on the unit circle when $P(\lambda)$ does, since

$$P(\lambda) = \begin{pmatrix} -\lambda I + A & 0 \\ C & H(\lambda) \end{pmatrix} \begin{pmatrix} I & -\Psi(\lambda) \\ 0 & I \end{pmatrix}$$

It can here be assumed that A is stable, since an initial stabilizing feedback L_0 just corresponds to multiplying the DARE by $\begin{pmatrix} I & 0 \\ -L_0 & I \end{pmatrix}$ from the right and its transpose from the left. It is thus proven that L is stabilizing if and only if $\mathrm{rank}_{|\lambda|=1} P(\lambda) = n + m$. To show the uniqueness of S consider two solutions S_1 and S_2 with corresponding closed loop matrices $A_1 = A - BL_1$ and $A_2 = A - BL_2$. Then it holds that

$$A_2^T (S_1 - S_2) A_1 = S_1 - S_2$$

Let $T_1 = [T_{10}, T_{1+}]$ and $T_1 = [T_{20}, T_{2+}]$ be spectrum-splitting transformations with

$$\begin{aligned} A_1 T_1 &= T_1 \,\mathrm{diag}(J_{10}, J_{1+}) \\ A_2 T_2 &= T_2 \,\mathrm{diag}(J_{20}, J_{2+}) \end{aligned}$$

where J_{1+} and J_{2+} are the blocks with eigenvalues inside the unit circle. It then holds that for $k \to \infty$

$$\begin{aligned} (S_1 - S_2) T_{1+} &= (A_2^T)^k (S_1 - S_2) T_{1+} J_{1+}^k \to 0 \\ T_{2+}^T (S_1 - S_2) &= (J_{2+}^T)^k T_{2+}^T (S_1 - S_2) A_1^k \to 0 \end{aligned}$$

Furthermore $S_1 T_{10} = 0$ and $S_2 T_{20} = 0$ by Lemma 3, so $S_1 = S_2$.