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SIMULATION OF NON-LINEAR STOCHASTIC DIFFERENTIAL EQUATION

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This paper describes a numerical technique to solve non-linear stochastic differential equations of Itô and Stratonovich type. We consider Euler, fourth-order Runge-Kutta (R-K) schemes, and other schemes with intermediate accuracy. For the purpose of investigating the convergence of numerical solutions and to apply variable integration step length techniques the special Wiener process generator was developed. The main result of the paper is the FORTRAN program combining Euler and R-K methods both with constant and variable integration step lengths. In an example the accuracy of these methods is compared.

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1. INTRODUCTION

This report discusses the problem of digital simulation of the vector nonlinear stochastic differential equation

$$\frac{dx}{dt} = a(x(t), t) + b(x(t), t)\ \xi(t), \qquad t \in [t_0 = t_{min},\ t_{max}] \tag{1}$$

where $x = (x_1, x_2, \ldots, x_n)$ is a $n \times 1$ state variables vector, $a(x,t) = \| a_i \|$ is a $n \times 1$ vector functions of x and t, $b(x,t) = \| b_{ij} \|$ is a $n \times m$ matrix function. $a(x,t)$ and $b(x,t)$ are satisfied to the Lipschiz conditions. Random forcing $\xi(t) = \frac{d}{dt} w(t)$ is a $m \times 1$ vector of white noises, $w(t)$ is a $m \times 1$ standard Wiener process

$$w_i(t_0) = 0, \qquad i = \overline{1,m},$$

$$E\{w_i(t)\} = 0, \tag{2}$$

$$E\{w_i(t_2) w_j(t_1)\} = \delta_{ij} |t_2 - t_1|.$$

The initial condition of state variables equals $x(t_0)$.

There are few papers on numerical integration of the stochastic differential equation (1), see [1-9,14,15]. The theoretical estimations of accuracy are discussed only in [2] and [9]. The articles [3,5,8] present the experiment experience. The papers [5,6,14,15] are devoted to numerical schemes providing only the statistical convergence, i.e. the convergence of the sample moments. The sample path convergence is discussed in the Ph.D. thesis [2], the book [4, p.186], and the paper [9].

This report deals with:

(a) the application of the Picard iterations [1, p. 279] to obtain several integration schemes (not only Euler and Runge-Kutta schemes) which provide a sample path convergence;

(b) the development of a special Wiener process generator applicable to sample path convergence investigations and to numerical schemes with variable integration step lengths;

(c) the development of routines to integrate the n-dimensional stochastic differential equation (1) in Itô and Stratonovich senses both with constant and variable integration step lengths. The routines are written for FORTRAN-4 compiler at computer PDP-15 (versus FOR at the Department of Automatic Control).

2. DIFFERENCE SCHEMES TO OBTAIN ITÔ AND STRATONOVICH SOLUTIONS

2.1 Preliminary

In the theoretical approach it is more convenient to work with the integral equation equivalent to the differential equation (1):

$$x(t) = x(t_0) + \int_{t_0}^{t} a(x(s),s)\ ds + \int_{t_0}^{t} b(x(s),s)\ dw(s). \qquad (3)$$

To simplify the calculations we consider equation (3) to be the one-dimensional one. The extension to the n-dimensional equation is rather clear, some necessary remarks are pointed out in Section 2.4.

We use the Picard iterations [1, p. 279] to obtain difference schemes for solving equation (3):

$$x_t^{[r+1]} = x_{t_0} + \int_{t_0}^{t} a(x_s^{[r]}, s)\ ds + \int_{t_0}^{t} b(x_s^{[r]}, s)\ dw_s \qquad (4)$$

where $x_t = x(t)$, $w_t = w(t)$, index in square brackets indicates the number of iteration.

Let us choose the zero approximation as $x_t^{[0]} = x_{t_0}$. Then, according to (4), the first approximation is

$$x_t^{[1]} = x_{t_0} + \int_{t_0}^{t} a(x_{t_0},s)\ ds + \int_{t_0}^{t} b(x_{t_0},s)\ dw_s \approx \qquad (5)$$

$$\approx x_{t_0} + a(x_{t_0},t_0)\ \Delta t + b(x_{t_0},t_0)\ \Delta w_t$$

where $\Delta t = t - t_0$, $\Delta w_t = w(t) - w(t_0)$.

Putting (5) into (4) we get the second approximation:

$$x_t^{[2]} = x_{t_0} + \int_{t_0}^{t} a\Big(x_{t_0} + a(x_{t_0}, t_0)\cdot(s-t_0) + b(x_{t_0}, t_0)\cdot$$

$$\cdot(w(s) - w(t_0)), s\Big)\, ds + \int_{t_0}^{t} b\Big(x_{t_0} + a(x_{t_0}, t_0)\cdot(s-t_0) +$$

$$+ b(x_{t_0}, t_0)\cdot(w(s)-w(t_0)),\ s\Big)\, dw_s \approx \qquad (6)$$

$$\approx x_{t_0} + a\cdot\Delta t + (a'_t + a\cdot a'_x)\,\frac{(\Delta t)^2}{2} + a'_x\, b \int_{t_0}^{t} \Delta w_s\, ds + \ldots +$$

$$+ b\cdot\Delta w_t + (b'_t + a\cdot b'_x) \int_{t_0}^{t} \Delta s\, dw_s + b\cdot b'_x \int_{t_0}^{t} \Delta w_s\, dw_s + \ldots$$

In this expression we used the notations

$$a = a(x_{t_0}, t_0), \quad b = b(x_{t_0}, t_0),$$

$$a'_t = \frac{\partial}{\partial t}\, a(x,t)\Big|_{x_{t_0}, t_0} \qquad \text{etc.}$$

Expression (6) contains stochastic integrals. Therefore the results of calculations depend on the definition of the stochastic integral. Consider below both the Itô and the Stratonovich solutions of the stochastic differential equation (1).

2.2 Stratonovich Solution

It can be shown [1] that the first approximation (5) converges to the Itô solution of (1). The simplest difference scheme to find the Stratonovich solution runs from the

second approximation (6) retaining only the terms order Δt and $(\Delta w_t)^2$:

$$x_t = x_{t_0} + a \cdot \Delta t + \frac{1}{2} b\, b'_x (\Delta w_t)^2 \qquad (7)$$

where we use the well-known Stratonovich integral [10]

$$\int_{t_0}^{t} \Delta w_s \, dw_s = \frac{1}{2} (\Delta w_t)^2. \qquad (8)$$

A more accurate difference scheme follows from the full expression for the second approximation (6). This expression contains two stochastic integrals

$$\int_{t_0}^{t} \Delta w_s \, ds \quad \text{and} \quad \int_{t_0}^{t} \Delta s \, dw_s$$

which are the same both in the Itô and Stratonovich senses. Therefore we can use the Itô lemma [6]

$$\Delta w_t \cdot \Delta t = \int_{t_0}^{t} \Delta s \, dw_s + \int_{t_0}^{t} \Delta w_s \, ds \qquad (9)$$

and calculate only one of them, for example

$$\eta = \int_{t_0}^{t} \Delta w_s \, ds. \qquad (10)$$

The integral η (10) can't be expressed in the closed form. So, we find the optimal estimate in the mean square sense

$$\hat{\eta} = E\{\eta \mid w(t), w(t_0)\} = \frac{\Delta t \cdot \Delta w_t}{2}. \qquad (11)$$

The estimate $\hat{\eta}$ converges to η as Δt is refined for every realization of $w(t)$.

Hence we have got a difference scheme more accurate than (7):

$$x_t = x_{t_0} + a\cdot\Delta t + b\cdot\Delta w_t + \frac{1}{2}\, b\cdot b'_x\cdot(\Delta w_t)^2 + \\ + (a'_t + a\cdot a'_x)\frac{(\Delta t)^2}{2} + (a'_x\cdot b + b'_t + a\cdot b'_x)\,\frac{\Delta t\cdot\Delta w_t}{2} \qquad (12)$$

providing Stratonovich solution of equation (1).

We can't estimate other stochastic integrals of higher order such as $\int_{t_0}^{t} (\Delta w_s)^2 ds$ and so on. But we can partly improve the accuracy retaining the higher derivatives of function $a(x,t)$ at the second approximation (6). Thus we get the scheme analogous to the fourth-order Runge-Kutta (R-K) scheme for deterministic equations [2,11]

$$x_t = x_{t_0} + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) \qquad (13)$$

where

$$K_1 = a(x_{t_0}, t_0)\Delta t + b(x_{t_0}, t_0)\Delta w_t,$$

$$K_2 = a(x_{t_0} + \frac{K_1}{2}, t_0 + \frac{\Delta t}{2})\Delta t + b(x_{t_0} + \frac{K_1}{2}, t_0 + \frac{\Delta t}{2})\ \Delta w_t,$$

$$K_3 = a(x_{t_0} + \frac{K_2}{2}, t_0 + \frac{\Delta t}{2})\Delta t + b(x_{t_0} + \frac{K_2}{2}, t_0 + \frac{\Delta t}{2})\ \Delta w_t,$$

$$K_4 = a(x_{t_0} + K_3, t_0 + \Delta t)\Delta t + b(x_{t_0} + K_3, t_0 + \Delta t)\ \Delta w_t,$$

the Wiener process increment $\Delta w_t = w(t) - w(t_0)$.

When the coefficient $b(x,t)$ in equation (1) is rather small, scheme (13) has advantages compared to scheme (12). But for large $b(x,t)$ both schemes have approximately the same accuracy. A similar phenomenon was at first mentioned in [2] and then in [4].

The Runge-Kutta method (13) is the most convenient method to provide the Stratonovich solution of equation (2). Compared with schemes (7) and (12) this method has the better accuracy and demands to calculate only the coefficients $a(x,t)$ and $b(x,t)$, not their higher derivatives.

Therefore just method (13) was used in the FORTRAN program (see Section 5) to obtain the Stratonovich solution.

2.3 Itô Solution of One-dimensional Equation

As it was mentioned above, the first approximation (5)

$$x_t = x_{t_0} + a\cdot\Delta t + b\cdot\Delta w_t \qquad (5')$$

converges to the Itô solution of equation (1). We call this (5') as the Euler scheme has expanded to the stochastic differential equations.

Putting the Itô integral [10]

$$\int_{t_0}^{t} \Delta w_s dw_s = \frac{1}{2}\,[(\Delta w_t)^2 - \Delta t]$$

into the expression for the second approximation (6) and retaining only the terms order Δt and $(\Delta w_t)^2$ we get a more accurate scheme than (5'),

$$x_t = x_{t_0} + a\cdot\Delta t + \frac{1}{2}\, b\, b'_x[(\Delta w_t)^2 - \Delta t] \qquad (14)$$

Note that if we put the approximate equality $(\Delta w_t)^2 \approx \Delta t$ into the right-hand side of (14) we'll get the previous scheme (5').

The difference scheme similar to (12) is

$$x_t = x_{t_0} + a\cdot\Delta t + b\cdot\Delta w_t + \frac{1}{2}\, b\cdot b'_x[(\Delta w_t)^2 - \Delta t] + \\ + (a'_t + a\cdot a'_x)\,\frac{(\Delta t)^2}{2} + (a'_x\cdot b + b'_t + a\cdot b'_x)\,\frac{\Delta t\cdot\Delta w_t}{2}\,. \qquad (15)$$

Scheme (15), providing the Itô solution, differs from the Stratonovich solution scheme (12) only by the term $\frac{1}{2}b\cdot b'_x\cdot\Delta t$. Hence the R-K scheme to obtain the Itô solution of equation (1) differs from expression (13) by the same term:

$$x_t = x_{t_0} + \frac{1}{6}\,(K_1 + 2K_2 + 2K_3 + K_4) - \frac{1}{2}\, b\cdot b'_x\cdot\Delta t \qquad (16)$$

where

$$K_1 = a(x_{t_0}, t_0)\Delta t + b(x_{t_0}, t_0)\ \Delta w_t,$$

$$K_2 = a(x_{t_0} + \frac{K_1}{2},\ t_0 + \frac{\Delta t}{2})\Delta t + b(x_{t_0} + \frac{K_1}{2},\ t_0 + \frac{\Delta t}{2})\Delta w_t,$$

$$K_3 = a(x_{t_0} + \frac{K_2}{2},\ t_0 + \frac{\Delta t}{2})\Delta t + b(x_{t_0} + \frac{K_2}{2},\ t_0 + \frac{\Delta t}{2})\Delta w_t,$$

$$K_4 = a(x_{t_0} + K_3,\ t_0 + \Delta t)\Delta t + b(x_{t_0} + K_3,\ t_0 + \Delta t)\Delta w_t.$$

Another way to get an R-K scheme to obtain the Itô solution of equation (1) is to apply the standard R-K scheme (13) to the next Stratonovich equation equivalent to the Itô one (1):

$$dx = [a(x,t) - \frac{1}{2}\, b(x,t)b'_x(x,t)]dt + b(x,t)dw(t). \qquad (17)$$

Hence we get an R-K scheme like (13), applied to the equivalent Stratonovich equation (17):

$$x_t = x_{t_0} + \frac{1}{6}\ (K_1 + 2K_2 + 2K_3 + K_4) \qquad (18)$$

where

$$K_1 = [a(x_{t_0}, t_0) - \frac{1}{2}\ b\ b'_x(x_{t_0}, t_0)]\Delta t + b(x_{t_0}, t_0)\Delta w_t,$$

$$K_2 = [a(x_{t_0} + \frac{K_1}{2},\ t_0 + \frac{\Delta t}{2}) - \frac{1}{2}\ b\ b'_x(x_{t_0} + \frac{K_1}{2},\ t_0 + \frac{\Delta t}{2})]\Delta t + b(x_{t_0} + \frac{K_1}{2},\ t_0 + \frac{\Delta t}{2})\Delta w_t,$$

$$K_3 = [a(x_{t_0} + \frac{K_2}{2},\ t_0 + \frac{\Delta t}{2}) - \frac{1}{2}\ b\ b'_x(x_{t_0} + \frac{K_2}{2},\ t_0 + \frac{\Delta t}{2})]\Delta t + b(x_{t_0} + \frac{K_2}{2},\ t_0 + \frac{\Delta t}{2})\Delta w_t,$$

$$K_4 = [a(x_{t_0} + K_3,\ t_0 + \Delta t) - \frac{1}{2}\ b\ b'_x(x_{t_0} + K_3,\ t_0 + \Delta t)]\Delta t + b(x_{t_0} + K_3,\ t_0 + \Delta t)\Delta w_t.$$

Scheme (16) has some advantages compared to the standard R-K scheme (18) as it demands to calculate the term $\frac{1}{2}\, b\, b'_x \Delta t$ only

once per each integration step. At the same time it has drawbacks from the point of programming.

To solve the Itô equation (1) we use the Euler method (5) in the FORTRAN program (see Section 5). In order to apply the standard R-K method (18) it is preliminarily necessary to calculate the term $\frac{1}{2}b(x,t)b'_x(x,t)$ of the equivalent Stratonovich equation (17) analytically (see also next section, 2.4).

2.4 Generalization to Many-dimensional Equations

The generalization of the R-K scheme (13) and the Euler scheme (5') for equation (1) to be n-dimensional is rather clear. It is necessary to replace scalar values by vectors and matrices [11]. That is only a formal thing to do. As for the R-K scheme (18) for the equivalent Stratonovich equation (17), it is necessary to calculate the term $\frac{1}{2} b\, b'_x$. Consider three cases:

2.4.1. In the general case the scalar form of the Itô vector equation (1) is

$$\begin{cases} \dfrac{dx_1}{dt} = a_1(\vec{x},t) + \sum\limits_{j=1}^{m} b_{ij}(\vec{x},t)\ \xi_j(t) \\ \vdots \\ \dfrac{dx_n}{dt} = a_n(\vec{x},t) + \sum\limits_{j=1}^{m} b_{ij}(\vec{x},t)\ \xi_j(t). \end{cases} \tag{19}$$

Using the Stratonovich rule [10] we find that the scalar term $\frac{1}{2} b\, b'_x$ in equation (17) corresponds to the term

$$\frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{m} \frac{\partial b_{ik}(\vec{x},t)}{\partial x_j}\, b_{jn}(\vec{x},t), \quad i = \overline{1,n}. \tag{20}$$

2.4.2. In the second case there is only one white noise excitation, $\xi(t)$:

$$\begin{cases} \dfrac{dx_1}{dt} = a_1(\vec{x},t) + g_1(\vec{x},t)\ \xi(t), \\ \vdots \\ \dfrac{dx_n}{dt} = a_n(\vec{x},t) + g_n(\vec{x},t)\ \xi(t) \end{cases} \tag{21}$$

so the term $\frac{1}{2} b\, b'_x$ corresponds to

$$\frac{1}{2} \sum_{j=1}^{n} \frac{\partial g_i(\vec{x},t)}{\partial x_j}\, g_j(\vec{x},t), \quad i = \overline{1,n}. \tag{22}$$

2.4.3. In the third case every scalar equation contains only one white noise:

$$\begin{cases} \dfrac{dx_1}{dt} = a_1(\vec{x},t) + g_1(\vec{x},t)\ \xi_1(t), \\ \vdots \\ \dfrac{dx_n}{dt} = a_n(\vec{x},t) + g_n(\vec{x},t)\ \xi_n(t) \end{cases} \tag{23}$$

so the term $\frac{1}{2} b\, b'_x$ corresponds to

$$\frac{1}{2} \frac{\partial g_i(\vec{x},t)}{\partial x_i}\, g_i(\vec{x},t), \quad i = \overline{1,n} \tag{24}$$

Remark. A difference scheme to solve the n-dimensional Itô equation (19), similar to (15), is shown in Appendix.

3. WIENER PROCESS GENERATOR

3.1 Algorithm

While digital simulating the stochastic differential equations it is necessary to check the accuracy and to be sure that the approximate solution converges to the exact one (we consider the convergence in the sample path sense). So, we are to repeat simulations with different integration step lengths taking the same realization of the vector Wiener process $\vec{w}(t)$. It means that the samples of an approximate Wiener process at any point t_r equal the values of finer approximation at that time. The same problem takes place at the variable integration step method (see Section 4).

To do this, let us choose the smallest integration step length

$$h_{min} = \frac{t_{max} - t_{min}}{2^{K_{max}}}. \tag{25}$$

The current step length equals

$$h = \frac{t_{max} - t_{min}}{2^{K}} \tag{26}$$

where $K = K_{min}, K_{min}+1, \ldots, K_{max}$. The value of K is appointed by the user in the constant integration step length method or is automatically chosen in the variable integration step length method.

So, if we have a sample of one-dimensional standard Wiener process $w(t_r)$ at any point t_r and are to generate the next sample $w(t_{r+1})$ at the point $t_{r+1} = t_r + h_r$, where step h_r is equal to (26), we use the algorithm (see Fig. 1):

$$w(t_{r+1}) = w(t_r) + \sqrt{h_{min}} \sum_{\ell=1}^{h/h_{min}} \zeta_{J_r+\ell} \tag{27}$$

where ζ_i is Gaussian random numbers with zero mean and unite

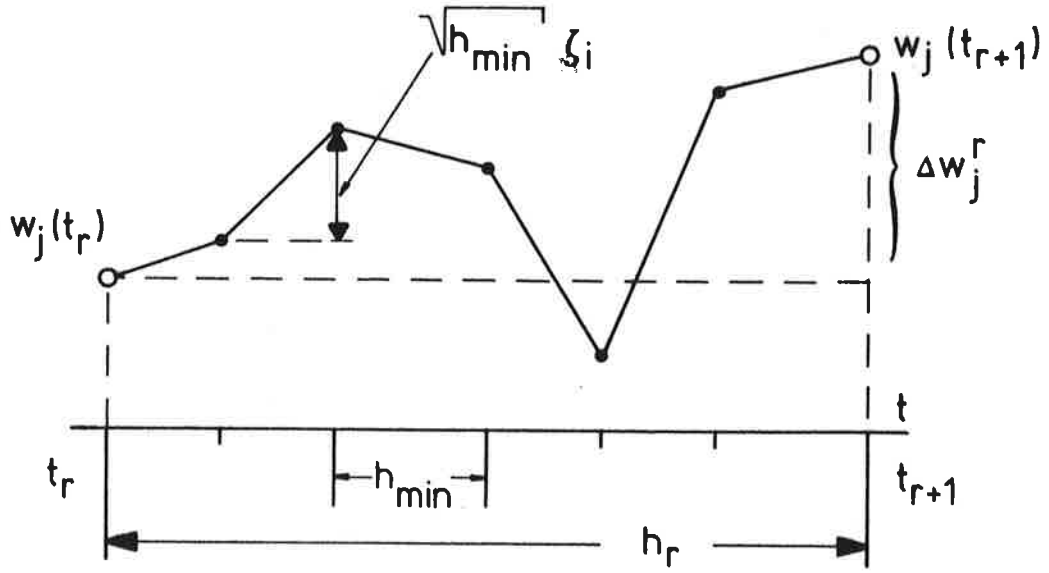


Fig. 1 - Wiener process generator.

variance, J_r is the number of the latest sample of the random number generator. The increment of one-dimensional Wiener process is equal to

$$\Delta w_r = w(t_{r+1}) - w(t_r) = \sqrt{h_{min}} \sum_{\ell=1}^{h/h_{min}} \zeta_{J_r+\ell}. \qquad (28)$$

Generating the m-dimensional Wiener process $\vec{w}(t) = (w_1(t), \ldots, w_m(t))$ by one random number generator we use the algorithm

$$w_j(t_{r+1}) = w_j(t_r) + \sqrt{h_{min}} \sum_{\ell=1}^{h/h_{min}} \zeta_{J_r+(\ell-1)m+j} \qquad (29)$$

where $j = 1,2,\ldots,m$.

Algorithm (29) was written in FORTRAN as

```
 9 DO 11 J=1,M
11 R(J)=0.0
   DO 12 I=1,KR
   DO 12 J=1,M
   CALL MCNODI(NU,GS)                                    (30)
12 R(J)=R(J)+GS
   DO 13 J=1,M
   R(J)=R(J)*SHM
13 W(J)=W(J)+R(J)
```

where M is the number of scalar Wiener processes, KR = h/h_{min}, = $2^{K_{max}-K}$, SHM = $\sqrt{h_{min}}$, MCNODI(NU,GS) is the Gaussian routine name, NU is an odd number, GS is N(0,1) Gaussian random numbers, R(J) is equal to the Wiener process increment $w_j(t_{r+1}) - w_j(t_r)$.

3.2 Comparison of two Gaussian Random Number Generators

At first we tested algorithm (29) with a PDP-15 Gaussian random number routine MCNODI like (30). It was obtained that this routine provides Wiener processes with periodical components, see Fig. 2a. We divided the time interval $t \in [0,1]$ on $2^{15} = 32\ 768$ points and tested the MCNODI routine with several initial numbers NU (Fig. 2a corresponds to NU = 9) but at every realization a period of approximately 5460 numbers was obtained.

To increase the period we used nonlinear transformation of two rectangular random variables [12, p.176]

$$y_1 = (-2 \ln \zeta_1)^{1/2} \cos 2\pi\zeta_2$$
$$y_2 = (-2 \ln \zeta_1)^{1/2} \sin 2\pi\zeta_2. \qquad (31)$$

Algorithm (31) transforms two independent random variables, ζ_1 and ζ_2, rectangular over [0,1] to two independent variables, y_1 and y_2, from a Gaussian distribution with zero mean and unit variance.

To simplify the program we use only one random number, y_1. Rectangular variables provides the routine MCREDI which gives one zero sample per period. The FORTRAN program to generate the Wiener process (29) is

(a)

(b)

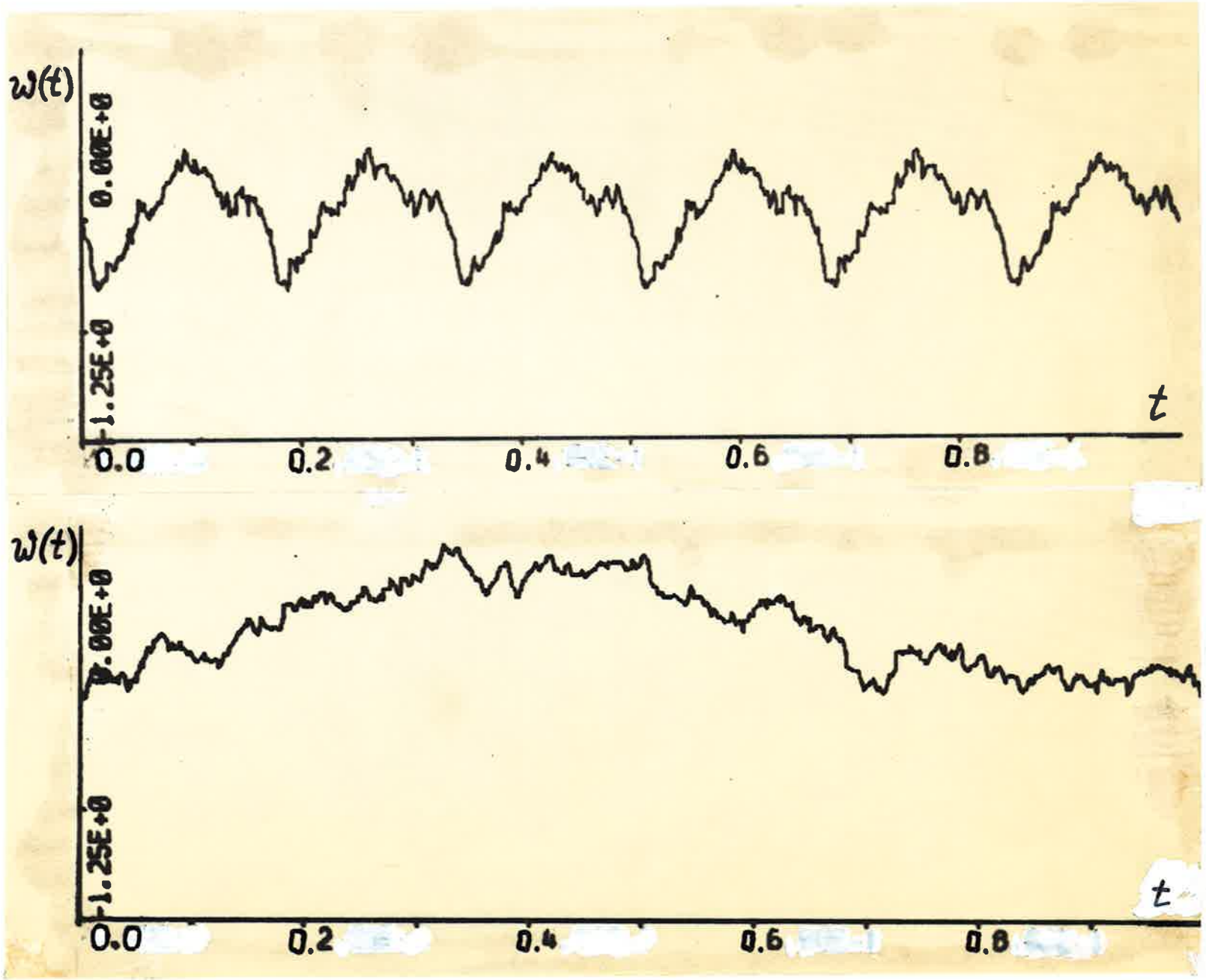


Fig.2. Wiener process: (a) MCNODI generator, (b) nonlinear transformation of rectangular random variables.

```
 9 DO 11 J=1,M
11 R(J)=0.0
   DO 12 I=1,KR
   DO 12 J=1,M
99 CONTINUE
   CALL MCREDI(NU,REC1)
   CALL MCREDI(NU,REC2)                                    (32)
   IF(REC1.EQ.0.0)GOTO 99
   GS=SQRT(-2.0*ALOG(REC1))*COS(PI2*REC2)
12 R(J)=R(J)+GS
   DO 13 J=1,M
   R(J)=R(J)*SHM
13 W(J)=W(J)+R(J)
```

The period of such a generator on PDP-15 is equal to $2^{17}-1 = 131071$. A realization of a Wiener process with initial number NU=9 is shown in Fig. 2b.

In conclusion, we point out the dependence between the mean value m and the standard deviation σ of the Gaussian generator (32) on N numbers of sampling:

N	m	σ
2	1.0080227554	1.3937898577
4	0.7538282275	1.0280693471
8	0.1870607361	1.2018311917
16	0.0475175977	1.2200787663
32	0.1173253767	1.0714786648
64	-0.0137966501	0.9706972837
128	-0.0495720468	1.0398858189
256	0.0026516137	1.0168921351
512	0.0438656453	0.9735938460
1024	0.0572415916	0.9886081666
2048	-0.0095007990	1.0021583736
4096	-0.0236054291	1.0073654949
8192	-0.0117221705	0.9965249449
16384	-0.0032938370	0.9994821548
32768	0.0002269202	1.0002311468

4. VARIABLE INTEGRATION STEP SIZE TECHNIQUE (STEP DOUBLING AND HALVING)

The estimation of local truncation error and integration step size control developed for ordinary differential equation [11] doesn't prove for stochastic ones. However, we have no alternative, as it is desirable to reduce the computation time applying the variable integration step length method. Therefore let's try to do this.

Each basic step of size h_r is done twice, once as two steps of size $h_r/2$ and once as one step of size h_r (see Fig. 3).

If the result of one step of size h_r is $x_{r+1}^{(1)}$, while the result of two steps of size $h_r/2$ is $x_{r+1}^{(2)}$, we have a local truncation error

$$\delta_{r+1} = \left| x_{r+1}^{(1)} - x_{r+1}^{(2)} \right| . \tag{33}$$

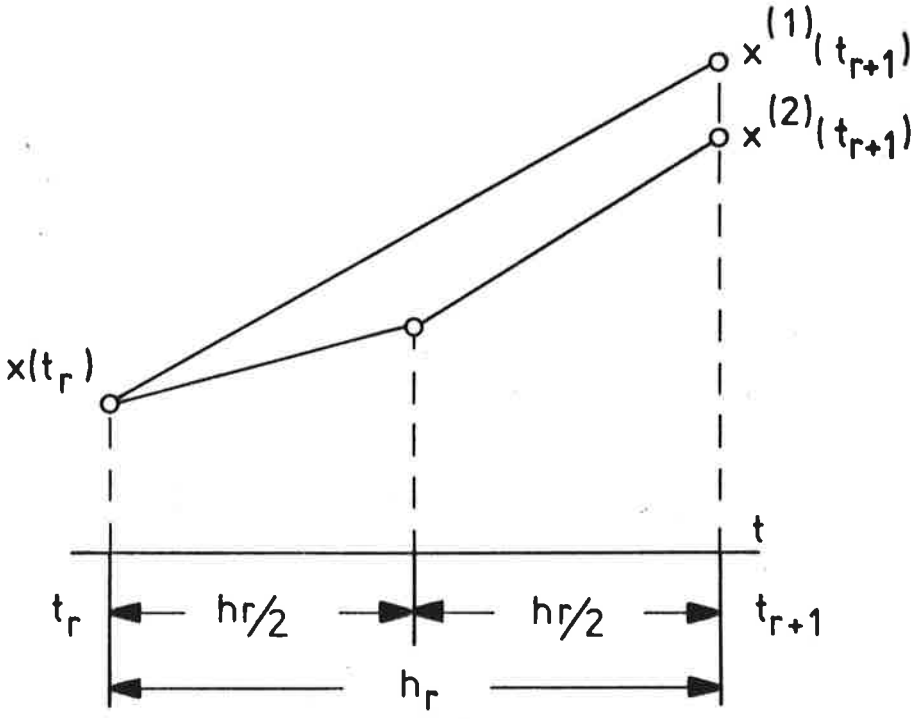


Fig. 3 - Variable step size.

In the n-dimensional case $\vec{x} = (x_1, x_2, \ldots, x_n)$ is a vector, so

$$\delta_{r+1} = \max_i \left| x_i^{(1)}(t_{r+1}) - x_i^{(2)}(t_{r+1}) \right|. \qquad (34)$$

Then compare δ_{r+1} with some limit error ε.

If $\delta_{r+1} > \varepsilon$, restart from point t_r halving the integration step length $h_r = h_r/2$. To get the same samples of Wiener process we have to store the state of the random number generator (NU in program (32)) and the Wiener process value $w(t_r)$ at each initial point t_r.

If $\varepsilon/10 < \delta_{r+1} < \varepsilon$, calculate the next step with the same step length $h_{r+1} = h_r$.

If $\delta_{r+1} < \varepsilon/10$, calculate the next step with double step length $h_{r+1} = 2h_r$.

The integration method can be both Runge-Kutta and Euler. But some examples show (see Section 6) that the Euler method with variable integration step length has no advantages compared to the Euler method with constant integration step length. In fact the R-K local truncation error decreases as $h \to 0$ faster than the Euler error. Therefore, if we can consider solution $x_{r+1}^{(2)}$ in expression (30) to be rather close to the exact one, it is not true for the Euler method. Hence we recommend to use the variable integration step method only for R-K schemes.

5. DESCRIPTION OF THE FORTRAN PROGRAM

The program package for simulating the n-dimensional stochastic differential equation (1) contains three standard subroutines STOCH, STH, STEP, the main program, and at least two user subroutines SIDE and OUTP which are called on by standard subroutines.

5.1 Main Program

The main program calls on the standard subroutine

```
CALL STOCH(XI,TMIN,TMAX,NX,NR,EPS,K,KMAX,KMIN,ISTEP,
METHOD,TP1,DP,NU,IS,W)
```

where the input arguments are:

XI - vector of initial conditions of state variables

TMIN - initial time of integration

TMAX - upper limit of integration time

NX - number of first-order equation n in (1)

NR - number of input white noises m in (1)

EPS - local truncation error in the variable integration step length method, ε

K - defines the value of the integration step h (26) if ISTEP=0, otherwise K indicates the initial value of h

KMAX - indicates the minimum value of the integration step length h_{min} (25)

KMIN - indicates the maximum value h_{max}

ISTEP - if ISTEP=0, the integration interval $[t_{min}, t_{max}]$ is divided into 2^K equal segments of length h (26), otherwise the variable integration step method is used

METHOD - if METHOD=0, the subroutine obtains Itô solution by the Euler method, otherwise Stratonovich solution by the fourth-order Runge-Kutta method

TP1 - first point of time to output the solution by means of the user subroutine OUTP

DP - increment of time to output the solution

NU - initial state of the random number generator (odd number).

The argument IS is the output one which equals the number of integration steps. Vector W is the allocation vector.

In the main program the user has to define the input arguments and point out the real dimensions of vectors XI(NX) and W(7·NX+4·NR).

5.2 Standard Subroutines

Subroutine STOCH (Table 1) is the auxiliary subroutine for dynamic allocation of vectors in FORTRAN-4 language (see report [13]). This subroutine calls on the basic subroutine STH (Table 2) which organizes the numerical integration with a constant, if ISTEP=0, or with variable step length otherwise. The subroutine STH also contains the Wiener process generator (32) which calls on the library subroutine MCREDI.

Subroutine STH calls on the standard subroutine STEP (Table 3) to perform integration on one step by the Euler method (5'), if METHOD=0, or by R-K method (13) otherwise.

5.3 User Subroutines

5.3.1 Subroutine SIDE. Subroutine STEP calls on the user subroutine

```
SIDE(T+H,H,XH,R,DX)
```

where the input arguments are

T - current time, t_r
H - current integration step length, h_r, when using constant step technique, $h_r/2$ otherwise
X - initial vector $\vec{x}(t_r)$ on elementary integration interval (t_r, t_{r+1})
R - vector of Wiener process increments $\Delta\vec{w}_r$.

The user has to write the program to calculate the output vector argument DX of dimension NX in such a way.

For the Euler method (5') DX is the vector state variables increments, for the R-K method (13) it is the vector coefficient $\vec{K}_\ell (\ell=1,2,3,4)$.

Vector DX contains coefficients of the stochastic differential equation (1). Let the n×1 vector $A(I) = \alpha_i(\vec{x}(t_r), t_r)$. Note that in many practical cases matrix $\| b_{ij} \|$ in equation (1) contains a lot of zero elements. Therefore the user has to write down the n×m matrix $\| b_{ij} \|$ as a vector $B(J) = b_{ij}(\vec{x}_r, t_r)$ for non-zero elements b_{ij}. The dimension of B is equal to data of non-zero elements in matrix $\| b_{ij} \|$.

In the FORTRAN notation the i:th component of vector DX is equal to

$$DX(I) = A(I)*H+B(...)*R(1)+...+B(...)*R(NR), \qquad I = 1,2,...,NX. \tag{35}$$

Thus, in subroutine SIDE, the user has to define the vector coefficients A(I) and B(J) and write down expression (35) where H and R(J) are input arguments defined into basic subroutine STEP.

Due to allocation vector techniques the dimension of the vectors X, R, and DX can be described like this:

```
DIMENSION X(1),DX(1),R(1)
```

For vectors A and B the user has to point out their real dimensions.

5.3.2 Subroutine OUTP. Basic subroutine STH calls on the subroutine

```
OUTP(T,X,WIENER)
```

to output the current time T, the vector state variables X and the Wiener process WIENER.

All the arguments of subroutine OUTP are the input arguments defined into the basic subroutine STH.

Dimensions of the vector X and WIENER are described in such a manner:

```
DIMENSION X(1),WIENER(1).
```

TABLE 1

```
001      C      NAME: STOCH
002      C      ----
003      C      SUBTITLE: SOLUTION OF THE NONLINEAR ITO OR STRATONOVICH STOCHASTIC
004      C      DIFFERENTIAL EQUATIONS
005      C
006      C      KEYWORDS:
007      C      --------
008      C      SOLUTION, SYSTEM OF NONLINEAR STOCHASTIC DIFFERENTIAL EQUATIONS,
009      C      ITO EQUATION, STRATONOVICH EQUATION
010      C
011      C      IMPLEMENTOR: VSEVOLOD D.RAZEVIG      DATE: 1977-05-19
012      C      -----------                          -----
013      C
014      C      INSTITUTE:
015      C      ---------
016      C      DEPARTMENT OF AUTOMATIC CONTROL
017      C      LUND INSTITUTE OF TECHNOLOGY, SWEDEN
018      C
019      C      ACCEPTED:                                  VERSION: 1
020      C      --------                                   -------
021      C
022      C=============================================================
023      C
024      C      PURPOSE
025      C      =======
026      C
027      C      ITO OR STRATONOVICH SOLUTION OF THE SYSTEM OF N FIRST-ORDER NONLINEAR
028      C      STOCHASTIC DIFFERENTIAL EQUATIONS X'=A(X,T) + B(X,T)*W' WITH
029      C      CONSTANT OR VARIABLE LENGTH STEPS
030      C
031      C      USAGE
032      C      =====
033      C
034      C      PROGRAM TYPE: SUBROUTINE
035      C      -------------
036      C
037      C      ARGUMENTS:
038      C      ---------
039      C      STOCH(XI,TMIN,TMAX,NX,NR,EPS,K,KMAX,KMIN,ISTEP,METHOD,
040           1TP1,DP,NU,IS,W)
041      C
042      C      XI     - VECTOR OF INITIAL CONDITIONS OF STATE VARIABLES
043      C      TMIN   - INITIAL TIME OF INTEGRATION
044      C      TMAX   - UPPER LIMIT OF INTEGRATION TIME
045      C      NX     - NUMBER OF EQUATIONS
046      C      NR     - NUMBER OF INPUT WHITE NOISES
047      C      EPS    - LOCAL TRUNCATION ERROR
048      C      K      - INDICATE THE INITIAL VALUE OF THE INTEGRATION STEP
049      C               H=(TMAX-TMIN)/2**K
050      C      KMAX   - INDICATE THE MINIMUM VALUE OF THE INTEGRATION STEP
051      C               HMIN=(TMAX-TMIN)/2**KMAX
052      C      KMIN   - INDICATE THE MAXIMUM VALUE OF THE INTEGRATION STEP
053      C               HMAX=(TMAX-TMIN)/2**KMIN
054      C      ISTEP  - IF ISTEP=0, INTEGRATION INTERVAL DIVIDED INTO 2**K EQUAL
055      C               SEGMENTS OF LENGTH H, OTHERWISE ARE USED THE VARIABLE
056      C               INTEGRATION STEP
057      C      METHOD- IF METHOD=0, SUBROUTINE OBTAINS ITO SOLUTION BY EULER METHOD,
058      C               OTHERWISE STRATONOVICH SOLUTION BY THE FOURTH-ORDER
059      C               RUNGE-KUTTA METHOD
060      C      TP1    - FIRST POINT OF TIME TO OUTPUT THE SOLUTION BY MEANS USER
061      C               SUBROUTINE OUTP
062      C      DP     - INCREMENT OF TIME TO OUTPUT THE SOLUTION
063      C      NU     - INITIAL STATE OF RANDOM NUMBER GENERATOR
```

TABLE 1 (continued)

```
C      IS     - NUMBER OF THE INTEGRATION STEPS
C      W      - ALLOCATION VECTOR HAVING DIMENSION 7*NX+4*NR
C
C      NOTES:
C      -----
C      1) USER COMPOSES THE MAIN PROGRAM WHICH CALLS ON SUBROUTINE STOCH
C      2) USER COMPOSES SUBROUTINE SIDE(T,H,X,R,DX)
C
C         INPUT ARGUMENTS:
C         ----------------
C         T    - CURRENT TIME
C         H    - CURRENT INTEGRATION STEP LENGTH
C         X    - CURRENT STATE VARIABLES VECTOR
C         R    - WIENER PROCESS INCREMENT VECTOR
C
C         OUTPUT ARGUMENT
C         ---------------
C         DX   - UNIT INCREMENT VECTOR EQUALS TO DX(I)=A(X,T)*H+B(X,T)*R(J)
C                (I=1,2,...,NX,  J=1,2,...,NR). NONLINEAR VECTOR COEFFICIENTS
C                A(X,T) AND B(X,T) ARE DESCRIBED BY USER. DIMENSION
C                OF A(X,T) EQUALG TO NX, MAXIMUM DIMENSION OF B(X,T)
C                EQUALS TO NX*NR.
C      3) USER COMPOSES SUBROUTINE OUTP(T,X,WIENER) TO OUTPUT
C         THE SOLUTION X(I) AND WIENER PROCESS WIENER(J).
C      4) SUBROUTINE STOCH CALLS ON INNER SUBROUTINES STH AND STEP
C      5) SUBROUTINE STH CALLS ON LIBRARY SUBROUTINE MCREDI (RECTANGULAR
C         RANDOM NUMBER GENERATOR)
C
C      METHOD
C      ======
C
C      IN STH IS USED EULER OR FORTH-ORDER RUNGE-KUTTA METHOD BOTH WITH
C      CONSTANT AND VARIABLE INTEGRATION STEP LENGTH . THE CURRENT STEP
C      IS HALFING AND DOUBING IN SUCH A MANNER TO GET THE SAME SAMPLES
C      OF NOISE WHILE SOLVING THE EQUATION WITH DIFFERENT VALUES OF
C      EPS OR K.
C
C      REFERENCES:
C      ----------
C      1. D.J.WRIGHT, IEEE TRANS. ON AUTOMATIC CONTROL, V.AC-19, N 1, 1974
C      2. N.NIKITIN, S.PERVACHEV, V.RAZEVIG, AUTOMATION AND
C         REMOTE CONTROL, N 4, 1975.
C      3. C.W.GEAR, NUMERICAL INITIAL VALUE PROBLEMS IN ORDINARY
C         DIFFERENTIAL EQUATIONS, 1971.
C
C      CHARACTERISTICS
C      ===============
C
C      REVISIONS:
C      ---------
C
C-----------------------------------------------------------------------
C
       SUBROUTINE STOCH(XI,TMIN,TMAX,NX,NR,EPS,K,KMAX,KMIN,ISTEP,METHOD,
      1TP1,DP,NU,IS,W)
C
       DIMENSION XI(1),W(1)
C
C
       KW1=1
       KW2=KW1+NX
       KW3=KW2+NX
       KW4=KW3+NX
       KW5=KW4+NR
```

TABLE 1 (continued)

```
      KW6=KW5+NR
      KW7=KW6+NR
      KW8=KW7+NR
      KW9=KW8+NX
      KW10=KW9+NX
      KW11=KW10+NX
C
C     CALL ON THE BASIC SUBROUTINE STH
C
      CALL STH(X,TMIN,TMAX,NX,NR,EPS,K,KMAX,KMIN,ISTEP,METHOD,
     1TP1,DP,NU,IS,W(KW1),W(KW2),W(KW3),W(KW4),W(KW5),W(KW6),W(KW7),
     2W(KW8),W(KW9),W(KW10),W(KW11))
      RETURN
      END
```

TABLE 2

```
      C
      C     THIS IS BASIC SUBROUTINE TO SOLVE STOCHASTIC DIFFERENTIAL EQUATION
      C
            SUBROUTINE STH(X,TMIN,TMAX,NX,NR,EPS,K,KMAX,KMIN,ISTEP,METHOD,
           1TP1,DP,NU,IS,X1,X2,XH2,WIENER,WIENE1,R,R1,S1,S2,S3,S4)
      C
            DIMENSION X(1),X1(1),X2(1),XH2(1),WIENER(1),WIENE1(1),R(1),R1(1),
           1S1(1),S2(1),S3(1),S4(1)
      C
            DATA PI2/6.283185307/
      C
      C     DETERMINE ZERO INITIAL CONDITIONS FOR WIENER PROCESSES
      C
            DO 1 J=1,NR
          1 WIENER(J)=0.0
      C
      C      ... FOR TIME T, SWITCH IS AND VARIABLE TPR RUNNIG
      C      THE OUTPUTING SUBROUTINE OUTP
      C
            T=TMIN
            IS=0
            TPR=TP1
            EPS1=EPS/10.0
      C
      C     DETERMINE THE MINIMUM AND MAXIMUME VALUES  OF INTEGRATION
      C     STEP LENGTH
      C
            HMIN=(TMAX-TMIN)/2**KMAX
            HMAX=(TMAX-TMIN)/2**KMIN
            SHM=SQRT(HMIN)
      C
      C     CALL ON USER SUBROUTINE OUTP TO OUTPUT THE INITIAL CONDITIONS
      C
            CALL OUTP(T,X,WIENER)
      C
      C     THE BEGINING OF THE INTEGRATION,
      C     DETERMINE THE CURRENT INTEGRATION LENGTH STEP
      C
          2 H=(TMAX-TMIN)/2**K
            H2=H+H
      C
      C     STORAGE THE PRECEDING VALUE OF STATE NU OF RANDOM NUMBER GENERATOR
      C     AND WIENER PROCESS FOR A RESTART
      C
            N1=NU
            DO 3 J=1,NR
            WIENE1(J)=WIENER(J)
          3 R(J)=0.0
      C
      C     CALCULATE THE NUMBER OF CONSECUTIVE SAMPLINGS KR
      C
            KR=INT(H/HMIN+0.1)
      C
      C     GENERATE GAUSSIAN  INCREMENTS R(J) OF WIENER PROCESS AND
      C     THE WIENER PROCESS  AT THE POINT T+H
      C
            DO 4 I=1,KR
            DO 4 J=1,NR
         31 CALL MCREDI(NU,REC1)
            CALL MCREDI(NU,REC2)
            IF(REC1.EQ.0.0)GOTO 31
            GS=SQRT(-2.0*ALOG(REC1))*COS(PI2*REC2)
          4 R(J)=R(J)+GS
```

TABLE 2 (continued)

```
164           DO 5 J=1,NR
165           R(J)=R(J)*SHM
166         5 WIENER(J)=WIENER(J)+R(J)
167       C
168       C     CALL ON SUBROUTINE STEP TO CALCULATE THE STATE VARIABLES
169       C     X1(I) ON THE FIRST STEP LENGTH H
170       C
171           CALL STEP(T,H,X,R,X1,NX,NR,METHOD,S1,S2,S3,S4)
172       C
173       C     IF ISTEP=0, PUT THE VALUES X1(I) TO THE CURRENT STATE VARIABLES
174       C     VECTOR X(I) AND GO TO THE BOTTOM OF SUBROUTINE
175       C
176           IF(ISTEP.NE.0)GOTO 7
177           T=T+H
178           DO 6 I=1,NX
179         6 X(I)=X1(I)
180           IS=IS+1
181           GOTO 16
182       C
183       C     IF ISTEP  NOT EQUALS TO ZERO, COMPUTE THE NEXT STEP
184       C     OF THE LENGTH H AND THE LARGER STEP OF THE LENGTH 2*H
185       C
186       C     GENERATE GAUSSIAN INCREMENTS OF WIENER PROCESS ON THE NEXT STEP R1(J)
187       C     AND CALCULATE WIENER PROCESS AT THE POINT T+2*H
188       C
189         7 DO 8 J=1,NR
190         8 R1(J)=0.0
191           DO 10 I=1,KR
192           DO 10 J=1,NR
193         9 CALL MCREDI(NU,REC1)
194           CALL MCREDI(NU,REC2)
195           IF(REC1.EQ.0.0)GOTO 9
196           GS=SQRT(-2.0*ALOG(REC1))*COS(PI2*REC2)
197        10 R1(J)=R1(J)+GS
198           DO 11 J=1,NR
199           R1(J)=R1(J)*SHM
200           WIENER(J)=WIENER(J)+R1(J)
201        11 R(J)=R(J)+R1(J)
202       C
203       C     CALL ON SUBROUTINE STEP TO CALCULATE XH2(I) ON THE LARGE STEP 2*H
204       C
205           CALL STEP(T,H2,X,R,XH2,NX,NR,METHOD,S1,S2,S3,S4)
206       C
207       C     CALL ON SUBROUTINE STEP TO CALCULATE X2(I) ON THE SECOND STEP H
208       C
209           CALL STEP(T+H,H,X1,R1,X2,NX,NR,METHOD,S1,S2,S3,S4)
210       C
211       C     CALCULATE THE LARGEST ERROR DM
212       C
213           DM=0.0
214           DO 12 I=1,NX
215           DE=ABS(XH2(I)-X2(I))
216           XA=ABS(X2(I))
217           IF(XA.GT.1.0)DE=DE/XA
218           IF(DE.GT.DM)DM=DE
219        12 CONTINUE
220       C
221       C     IS DM LARGER EPS?
222       C
223           IF(DM-EPS)13,13,20
224       C
225       C     IF DM<EPS, DETERMINE CURRENT TIME T=T+H AND CURRENT STATE VARIABLES
226       C     X(I)=X2(I), IF IN ADDITION DM<EPS/10  DOUBL THE LENGTH STEP
227       C
```

TABLE 2 (continued)

```
      13 IF(DM.LT.EPS1)K=K-1
         IF(K.LT.KMIN)K=KMIN
      14 T=T+H2
         IS=IS+2
         DO 15 I=1,NX
      15 X(I)=X2(I)
C
C     CHECK THE NECESSERITY TO CALL ON SUBROUTINE OUTP
C
      16 IF(T-TPR)2,19,19
      19 TPR=TPR+DP
         CALL OUTP(T,X,WIENER)
C
C     IS T LARGER THAN TMAX?
C
         IF(T+HMIN/2.0.LT.TMAX)GOTO 2
         GOTO 25
C
C     IF DM>EPS,  REPEAT THE CALCULATIONS HALFING THE LENGTH STEP
C
      20 K=K+1
C
C     IS K LARGER THAN KMAX?
C
         IF(KMAX-K)21,22,22
C
C     IF K>KMAX, ELIMINATE K AND GO TO THE NEXT STEP
C
      21 K=KMAX
         GOTO 14
C
C     IF K LESS OR EQUAL KMAX RENEW THE STATE OF THE RANDOM NUMBER
C     GENERATOR AND WIENER PROCESSES AND REPEAT THE CALCULATIONS
C
      22 NU=N1
         DO 23 J=1,NR
      23 WIENER(J)=WIENE1(J)
         GOTO 2
      25 RETURN
         END
```

TABLE 3

```
001       C
002       C     THIS IS INNER SUBROUTINE TO PERFORM INTEGRATION ON ONE STEP,
003       C     IF METHOD=0, SUBROUTINE STEP OBTAINS ITO SOLUTION BY EULER
004       C     METHOD, OTHERWISE STRATONOVICH SOLUTION BY THE FORTH-ORDER
005       C     RUNGE-KUTTA METHOD.
006       C
007             SUBROUTINE STEP(T,H,X,R,XH,NX,NR,METHOD,S1,S2,S3,S4)
008       C
009             DIMENSION X(1),R(1),XH(1),S1(1),S2(1),S3(1),S4(1)
010       C
011       C     CALL ON USER SUBROUTINE SIDE TO DETERMINE THE INCREMENT OF
012       C     STATE VARIABLES ON THE STEP LENGTH H WITH INITIAL
013       C     CONDITIONS X(I) AT THE TIME T.
014       C
015             CALL SIDE(T,H,X,R,S1)
016       C
017       C     IS METHOD=0?
018       C
019             IF(METHOD.NE.0)GOTO 2
020       C
021       C     IF METHOD=0 CALCULATE THE NEXT SAMPLE XH(I) OF EULER SOLUTION
022       C     AT THE POIN T+H AND GO TO THE BOTTOM OF THE SUBROUTINE
023       C
024             DO 1 I=1,NX
025           1 XH(I)=S1(I)+X(I)
026             GOTO 7
027       C
028       C     IF METHOD NOT EQUAL TO ZERO CALCULATE THE NEXT RUNGE-KUTTA
029       C     COEFFICIENTS
030       C
031           2 DO 3 I=1,NX
032           3 XH(I)=X(I)+S1(I)*0.5
033             CALL SIDE(T+H*0.5,H,XH,R,S2)
034             DO 4 I=1,NX
035           4 XH(I)=X(I)+S2(I)*0.5
036             CALL SIDE(T+H*0.5,H,XH,R,S3)
037             DO 5 I=1,NX
038           5 XH(I)=X(I)+S3(I)
039             CALL SIDE(T+H,H,XH,R,S4)
040       C
041       C     DETERMINE THE NEXT SAMPLE XH(I) OF STRATONOVICH SOLUTION
042       C
043             DO 6 I=1,NX
044           6 XH(I)=(S1(I)+2.0*S2(I)+2.0*S3(I)+S4(I))/6.0+X(I)
045           7 RETURN
046             END
```

6. EXAMPLES

Example 1. Consider the Itô solution of equation

$$dx(t) = cx\,dt + gx\,dw(t) \qquad (36)$$

by the Euler method (5') with constant integration step length on time interval $t \in [0,1]$ with initial condition $x(0) = 1$.

Choose the number of samples x(t) to be printed equal to 100 (TP1=DP=1/100), the integration step size $h = 1/2^{10}$ (K=10), $h_{min} = 1/2^{15}$ (KMAX=15).

The number of equations in (36) equals NX=1 and the number of white noises NR=1. Therefore the dimension of the allocation vector W equals 7*NX+1*NR=11. Choose the initial value of random number generator NU = 65317.

The main program is shown in Table 4. Note that variables KMIN and EPS are not used in the constant integration step length method.

The coefficients of equation (36) and solution increment (35) are described at the user subroutine SIDE, Table 5, where coefficients c = -1, g = 1.

The user subroutine OUTP (Table 6) outputs the current time T, the solution X(1) and the Wiener process WIENER(1) on a line-printer.

Besides, at Fig. 4 we compare the rate of convergence of several Itô integration procedures:

1) Euler method (5')

2) modified Euler method (14)

3) second approximation (15)

TABLE 4

```
001     C       PROGRAM MAIN
002             DIMENSION XI(1),W(11)
003             COMMON /OUT/ISTEP,METHOD
004             TMIN=0.0
005             TMAX=1.0
006             DP=(TMAX-TMIN)/100.0
007             TP1=DP
008             KMAX=15
009             K=10
010             ISTEP=0
011             METHOD=0
012             XI(1)=1.0
013             NX=1
014             NR=1
015             NU=65317
016             WRITE(6,1)TMIN,TMAX,EPS,TP1,DP,KMAX,KMIN,K,NX,NR,ISTEP,METHOD
017           1 FORMAT(5F11.8,7I8)
018             CALL STOCH(XI,TMIN,TMAX,NX,NR,EPS,K,KMAX,KMIN,ISTEP,METHOD,
019            1TP1,DP,NU,IS,W)
020             WRITE(6,100)IS
021         100 FORMAT(I10)
022             STOP
023             END
```

TABLE 5

```
001             SUBROUTINE SIDE(T,H,X,R,DX)
002             DIMENSION X(1),DX(1),R(1)
003             DIMENSION A(1),B(1)
004             DATA C,G/-1.0,1.0/
005             A(1)=C*X(1)
006             B(1)=G*X(1)
007             DX(1)=A(1)*H+B(1)*R(1)
008             RETURN
009             END
```

TABLE 6

```
001             SUBROUTINE OUTP(T,X,WIENER)
002             DIMENSION X(1),WIENER(1)
003             WRITE(6,10)T,X(1),WIENER(1)
004          10 FORMAT(3F15.8)
005             RETURN
006             END
```

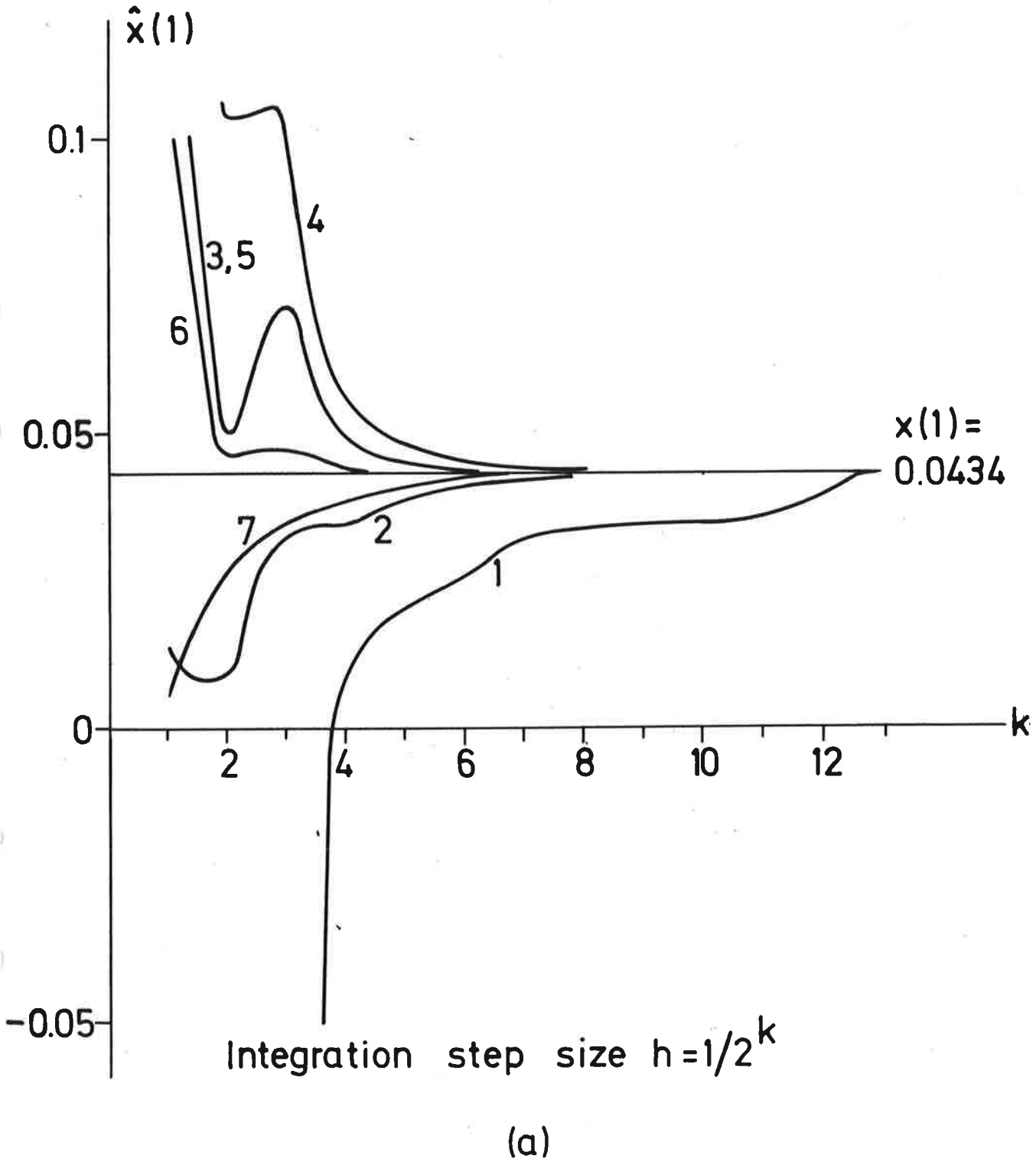


Fig.4a. Equation (36)

1 -Euler method (5')

2 - modified Euler method (14)

3 - second approximation (15)

4 - standard Euler-Cauchy method applied to equation (17)

5 - Euler-Cauchy method analogues to (16)

6 - standard Runge-Kutta method applied to equation (17)

7 - Runge-Kutta method (16)

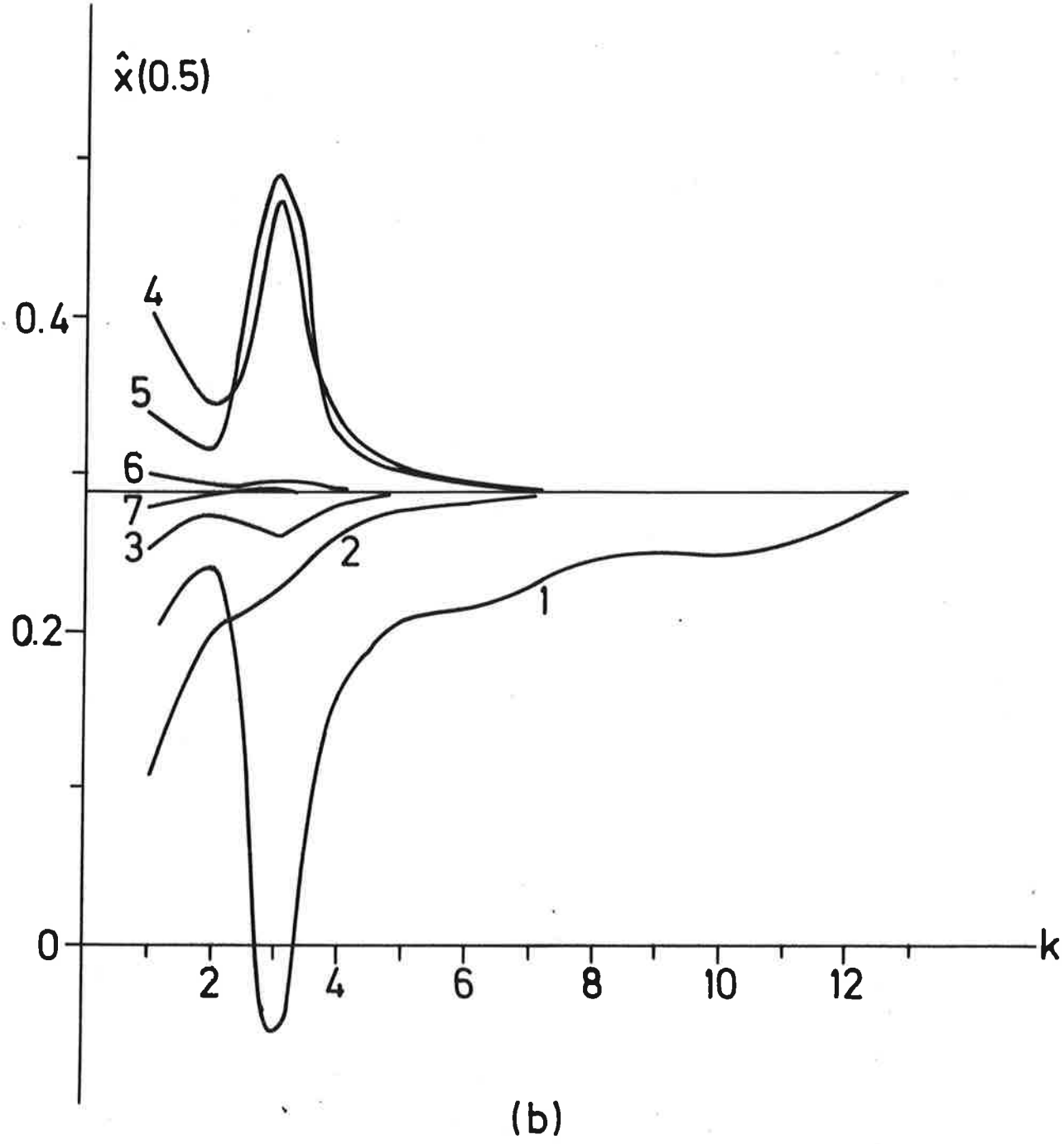


Fig.4b. Equation $dx = -\sin(x) + \sqrt{|x|}\, dw(t)$

4) standard Euler-Cauchy method (second-order R-K method) applied to equivalent Stratonovich equation (17)

5) Euler-Cauchy method analogues to (16)

6) standard Runge-Kutta method (18) applied to equivalent Stratonovich equation (17)

7) Runge-Kutta method (16).

Fig. 4a,b shows that the relative rates of convergence were as follows:

quickest - Runge-Kutta methods (16), (18)
- second approximation (15)

intermediate - Euler-Cauchy methods
- modified Euler method (14)

slowest - Euler method (5').

Example 2. Let's find the Itô solution of equation (36) by the step variable fourth-order Runge-Kutta method (18) applied to equivalent Stratonovich equation (17)

$$dx(t) = (c - \frac{1}{2} g^2)\, x\, dt + gx\, dw(t). \qquad (37)$$

The new main program and the subroutine SIDE are shown in Tables 7 and 8 accordingly. Subroutine OUTP is the same as in Example 1 (see Table 6).

Equation (36) has the analytical Itô solution

$$x(t) = x(0) \exp\{(c - \frac{1}{2} g^2)t + gw(t)\}. \qquad (38)$$

Hence it is possible to find the accuracy of numerical solution

$$\delta(t) = \left| \frac{x(t) - \hat{x}(t)}{x(t)} \right| \qquad (39)$$

where $x(t)$ is the exact solution (38), $\hat{x}(t)$ is the numerical solution.

TABLE 7

```
001     C       PROGRAM MAIN
002             DIMENSION XI(1),W(11)
003             COMMON /OUT/ISTEP,METHOD
004             TMIN=0.0
005             TMAX=1.0
006             DP=(TMAX-TMIN)/100.0
007             TP1=DP
008             KMAX=15
009             K=10
010             KMIN=4
011             EPS=1.0E-06
012             ISTEP=1
013             METHOD=1
014             XI(1)=1.0
015             NX=1
016             NR=1
017             NU=65317
018             WRITE(6,1)TMIN,TMAX,EPS,TP1,DP,KMAX,KMIN,K,NX,NR,ISTEP,METHOD
019           1 FORMAT(5F11.8,7I8)
020             CALL STOCH(XI,TMIN,TMAX,NX,NR,EPS,K,KMAX,KMIN,ISTEP,METHOD,
021            1TP1,DP,NU,IS,W)
022             WRITE(6,100)IS
023         100 FORMAT(I10)
024             STOP
025             END
```

TABLE 8

```
001             SUBROUTINE SIDE(T,H,X,R,DX)
002             DIMENSION X(1),DX(1),R(1)
003             DIMENSION A(1),B(1)
004             DATA C,G/-1.0,1.0/
005             A(1)=C*X(1)
006             B(1)=G*X(1)
007             DX(1)=(A(1)-0.5*G*G*X(1))*H+B(1)*R(1)
008             RETURN
009             END
```

In order to compare the accuracy of Euler and R-K schemes with constant and variable step lengths, Fig. 5 shows the accuracy δ (39) at time $t = 1$ as a function of number of steps. We can conclude that the R-K method with variable integration step length is the most available integration method for stochastic differential equations.

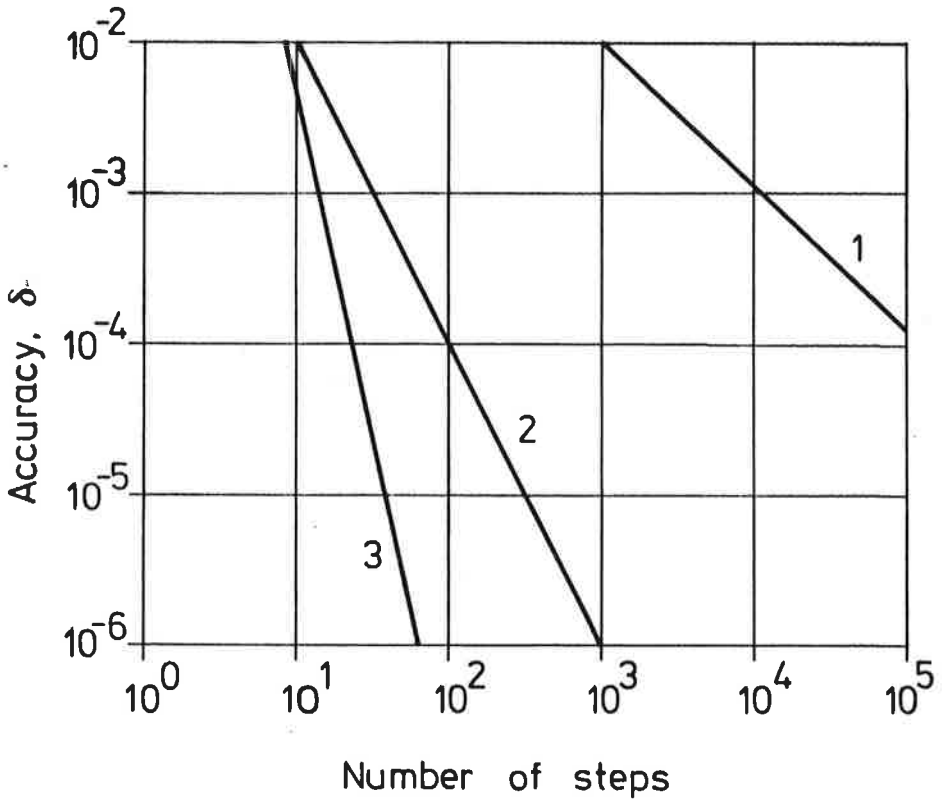


Fig. 5 - Accuracy of methods with constant and variable step size.

1 - Euler method with constant step length.
2 - Runge-Kutta method with constant step length.
3 - Runge-Kutta method with variable step length.

Example 3. Consider the system of two first-order stochastic differential equations

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 + b_1\, \xi_1(t) \\ \dfrac{dx_2}{dt} = v - x_2 - a\, x_1\, e^{-x_1^2/L} + b_2(x_2)\, \xi_2(t) \end{cases} \tag{40}$$

where

$$b_2(x_2) = \frac{g \cdot a}{\tau}\ \frac{1+10\ x_2^2}{5+10\ x_2^2}\ .$$

The user routines to obtain Itô solution of (40) by constant step Euler method and to display results with IDPAC system are shown in Tables 9-11. The dialog on teletype is shown in Table 12. The displayed phase plane of system (40) is shown in Fig. 6.

Example 4. In article [8] second-order phase-lock loop differential equation was studied and it was discovered in practice that "the Euler integration is greatly superior to Runge-Kutta methods from a convergence-rate accuracy criterion". It is difficult to comment on such a statement as no mathematical expressions to the numerical schemes used was pointed out.

We consider that the Runge-Kutta methods (13), (16), (18) (and their multidimensional analogues) have better accuracy that the Euler method (5') or have, in some special examples [4, p.186], at least the same accuracy.

That's why we implement our computer program to simulate the phase-lock loop [8]:

$$\begin{cases} \dfrac{dx_1}{dt} = x_2, \\ \dfrac{dx_2}{dt} = -\sin(x_1) - \cos(x_1)\, \xi_1(t) - \sin(x_1)\, \xi_2(t) \end{cases} \tag{41}$$

TABLE 9

```
001      C      PROGRAM MAIN
002             DIMENSION XI(2),W(22)
003             COMMON /PLOT/IP
004             TMIN=0.0
005             TMAX=1.0
006             NPR=1000
007             IP=NPR+1
008             DP=(TMAX-TMIN)/NPR
009             TP1=DP
010             KMAX=12
011             K=12
012             ISTEP=0
013             METHOD=0
014             XI(1)=-2.0
015             XI(2)=1.25
016             NX=2
017             NR=2
018             NU=65317
019             CALL OUTP(T,XI,W)
020             IP=0
021             WRITE(6,1)TMIN,TMAX,EPS,TP1,DP,KMAX,KMIN,K,NX,NR,ISTEP,METHOD
022           1 FORMAT(5F11.8,7I8)
023             CALL STOCH(XI,TMIN,TMAX,NX,NR,EPS,K,KMAX,KMIN,ISTEP,METHOD,
024            1TP1,DP,NU,IS,W)
025             WRITE(6,100)IS
026         100 FORMAT(I10)
027             IP=-1
028             CALL OUTP(T,XI,W)
029             STOP
030             END
```

TABLE 10

```
001             SUBROUTINE SIDE(T,H,X,R,DX)
002             DIMENSION X(1),DX(1),R(1)
003             DIMENSION A(2),B(2)
004             DATA G,V,AK,TAY,BB/0.3,1.25,5.0,0.5,0.4/
005             B(2)=BB
006             XX=X(1)*X(1)
007             X10=10.0*X(2)*X(2)
008             A(1)=X(2)
009             A(2)=(V-X(2)-AK*X(1)*EXP(-XX))/TAY
010             B(1)=G*AK/TAY*(1.0+X10)/(5.0+X10)
011             DX(1)=A(1)*H+B(2)*R(2)
012             DX(2)=A(2)*H+B(1)*R(1)
013             RETURN
014             END
```

TABLE 11

```
001             SUBROUTINE  OUTP(T,X,WIENER)
002             DIMENSION X(1),WIENER(1),IT(10),FILN(2)
003             COMMON /PLOT/IP
004             DATA FILN(2)/4H BIN/
005             IF(IP)300,200,100
006         100 IT(1)=IP
007             IT(2)=3
008             IT(4)=0
009             IT(9)=0
010             WRITE(9,1000)
011             READ(8,1001)FILN(1)
012             CALL ENTER(1,FILN)
013             WRITE(1)IT
014             RETURN
015        1000 FORMAT(' ENTER FILE NAME')
016        1001 FORMAT(A5)
017         200 WRITE(1)T,X(1),X(2)
018             RETURN
019         300 CALL CLOSE(1)
020             RETURN
021             END
```

TABLE 12

```
$FOR

FPF4X V3A000
>BL←MAIN

END PASS1
FPF4X V3A000
>BL←SIDE


DOS-15 UV3A000
$A RK <NEW> -1

$GLOAD

LOADER V3A000
><←MAIN,SIDE,OUTP,STOCH,STH,STEP

ENTER FILE NAME
DATA

STOP  000000


DOS-15 YV3A999
$A RK <PAC> -4/RK 3,4,5,7,15,16

$BUFFS 5

$E IDPAC

IDPAC V2D
>
PLOT 4 DATA(2)←DATA(3) -2 2
```

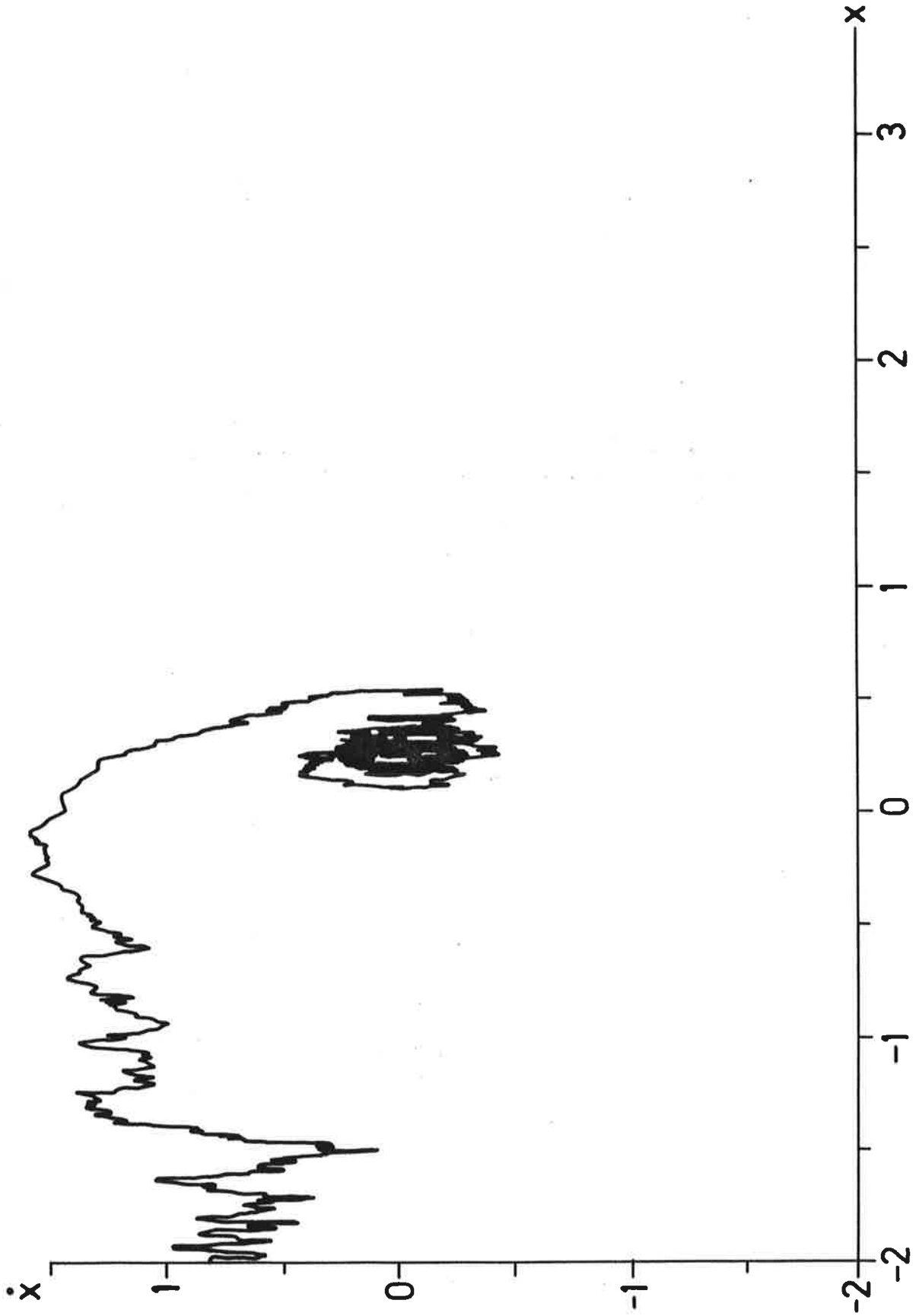


Fig.6. Phase plane

where $\xi_1(t)$ and $\xi_2(t)$ are uncorrelated white random processes with unite spectral density.

In [8] is shown that equation (41) has the same Itô and Stratonovich solutions as the term (20) is equal to zero. So, we solve (41) with Euler (5') and Runge-Kutta (13) methods. Listings of user subroutines are shown in Tables 13-15. Results of simulation during time $t \in [0,1]$ and initial conditions $x_1(0) = \pi/4$, $x_2(0) = \pi/4$ with different integration step size $h = 1/2^K$, $K = 1,2,\ldots,14$, are shown in Tables 16-17. The solutions $x_1(1)$, $x_2(1)$ as functions of parameter K are shown in Fig. 7 which proves that the Runge-Kutta method (13) has a higher rate of convergence than the Euler method (5').

TABLE 13

```
001      C      PROGRAM MAIN
002             DIMENSION XI(2),W(22)
003             TMIN=0.0
004             TMAX=1.0
005             DP=TMAX
006             TP1=DP
007             KMAX=14
008             ISTEP=0
009             METHOD=1
010             NX=2
011             NR=2
012             WRITE(6,33)
013          33 FORMAT(5X,19H RUNGE-KUTTA METHOD)
014             DO 1 K=1,KMAX
015             WRITE(6,22)K
016          22 FORMAT(I6)
017             NU=65317
018             XI(1)=0.785
019             XI(2)=0.785
020             CALL STOCH(XI,TMIN,TMAX,NX,NR,EPS,K,KMAX,KMIN,ISTEP,METHOD,
021            1TP1,DP,NU,IS,W)
022           1 CONTINUE
023             STOP
024             END
```

TABLE 14

```
001             SUBROUTINE SIDE(T,H,X,R,DX)
002             DIMENSION X(1),DX(1),R(1)
003             DIMENSION A(2)
004             S=SIN(X(1))
005             C=COS(X(1))
006             A(1)=X(2)
007             A(2)=-S
008             B=C*R(1)+S*R(2)
009             DX(1)=A(1)*H
010             DX(2)=A(2)*H-B
011             RETURN
012             END
```

TABLE 15

```
001             SUBROUTINE OUTP(T,X,WIENER)
002             DIMENSION X(1),WIENER(1)
003             WRITE(6,1)T,X(1),X(2)
004           1 FORMAT(3F18.10)
005             RETURN
006             END
```

TABLE 16

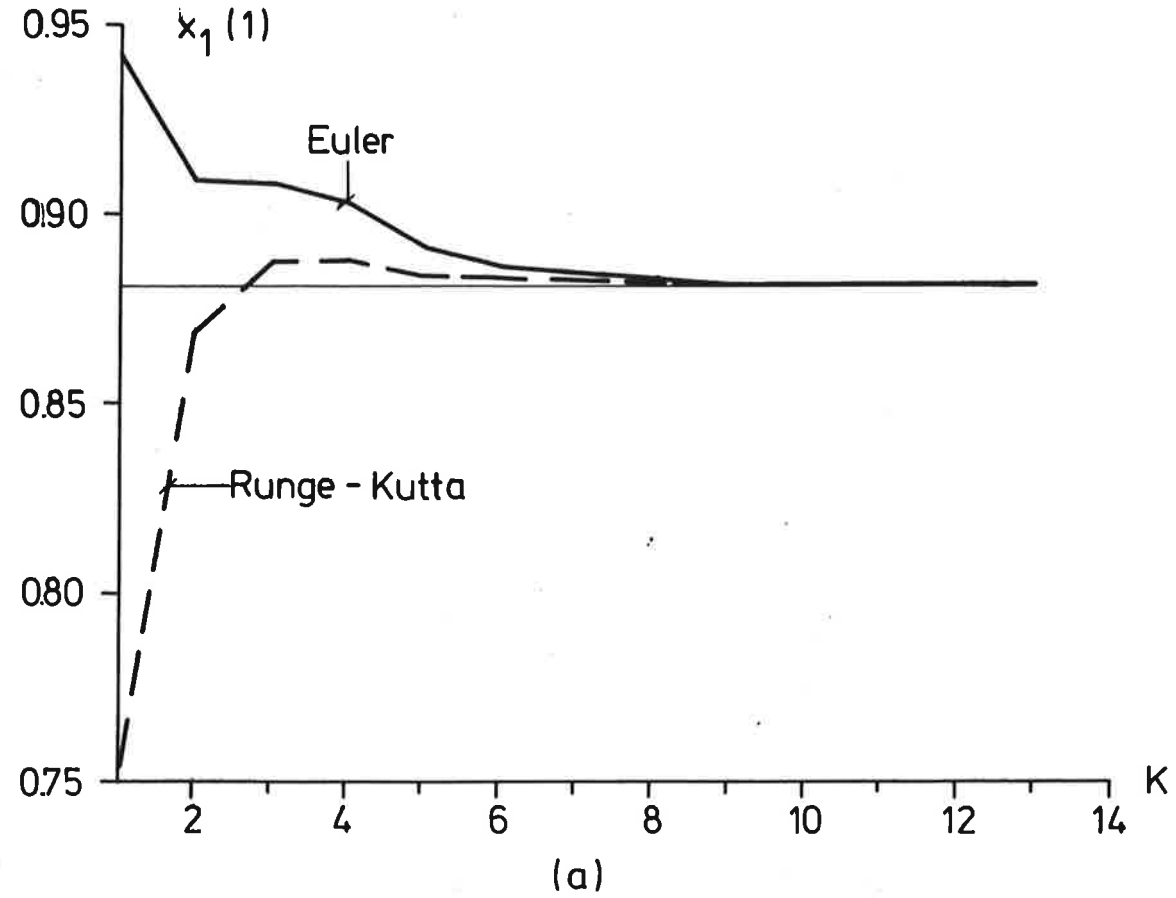
RUNGE-KUTTA METHOD

K=1	t	x_1	x_2
	0.0000000000	0.7849999964	0.7849999964
	1.0000000000	0.7559317946	0.0567011535
2			
	0.0000000000	0.7849999964	0.7849999964
	1.0000000000	0.8679136485	0.0010905191
3			
	0.0000000000	0.7849999964	0.7849999964
	1.0000000000	0.8875938505	0.0400029011
4			
	0.0000000000	0.7849999964	0.7849999964
	1.0000000000	0.8886059821	0.0224689320
5			
	0.0000000000	0.7849999964	0.7849999964
	1.0000000000	0.8837629259	0.0077346563
6			
	0.0000000000	0.7849999964	0.7849999964
	1.0000000000	0.8835852146	0.0092189689
7			
	0.0000000000	0.7849999964	0.7849999964
	1.0000000000	0.8829277456	0.0078043751
8			
	0.0000000000	0.7849999964	0.7849999964
	1.0000000000	0.8827768266	0.0072949543
9			
	0.0000000000	0.7849999964	0.7849999964
	1.0000000000	0.8813270479	0.0067779524
10			
	0.0000000000	0.7849999964	0.7849999964
	1.0000000000	0.8813672960	0.0066111484
11			
	0.0000000000	0.7849999964	0.7849999964
	1.0000000000	0.8814257830	0.0067376704
12			
	0.0000000000	0.7849999964	0.7849999964
	1.0000000000	0.8814252317	0.0067664690
13			
	0.0000000000	0.7849999964	0.7849999964
	1.0000000000	0.8814192563	0.0067955683
14			
	0.0000000000	0.7849999964	0.7849999964
	1.0000000000	0.8814185113	0.0067782488

TABLE 17

EULER METHOD

K=1	t	x_1	x_2
	0.0000000000	0.7849999964	0.7849999964
	1.0000000000	0.9426864088	-0.4416966140
2			
	0.0000000000	0.7849999964	0.7849999964
	1.0000000000	0.9193312973	-0.1712644026
3			
	0.0000000000	0.7849999964	0.7849999964
	1.0000000000	0.9185541421	-0.0245606415
4			
	0.0000000000	0.7849999964	0.7849999964
	1.0000000000	0.9033740759	-0.0134939626
5			
	0.0000000000	0.7849999964	0.7849999964
	1.0000000000	0.8915344328	-0.0119556952
6			
	0.0000000000	0.7849999964	0.7849999964
	1.0000000000	0.8868568391	-0.0003889427
7			
	0.0000000000	0.7849999964	0.7849999964
	1.0000000000	0.8847047687	0.0031411243
8			
	0.0000000000	0.7849999964	0.7849999964
	1.0000000000	0.8837036937	0.0050881854
9			
	0.0000000000	0.7849999964	0.7849999964
	1.0000000000	0.8817466199	0.0055904603
10			
	0.0000000000	0.7849999964	0.7849999964
	1.0000000000	0.8815876842	0.0060214628
11			
	0.0000000000	0.7849999964	0.7849999964
	1.0000000000	0.8815371841	0.0064465657
12			
	0.0000000000	0.7849999964	0.7849999964
	1.0000000000	0.8814813346	0.0066198139
13			
	0.0000000000	0.7849999964	0.7849999964
	1.0000000000	0.8814466149	0.0067221449
14			
	0.0000000000	0.7849999964	0.7849999964
	1.0000000000	0.8814319670	0.0067414588



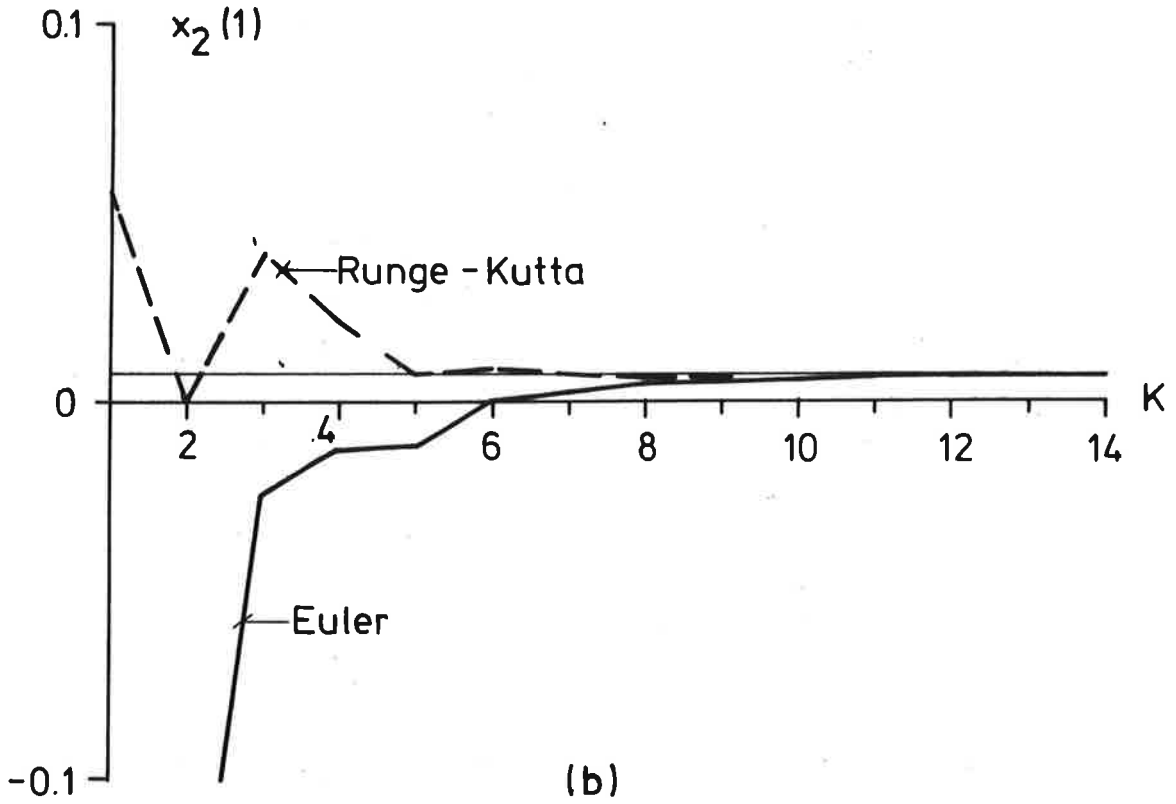


Fig.7. Phase-lock loop (41)

7. ACKNOWLEDGEMENTS

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9. APPENDIX

Iterative solution of the n-dimensional Itô stochastic differential equation (19) provides the next difference scheme, similar to the one-dimensional scheme (15):

$$x_i(t) = x_i(t_0) + a_i\,\Delta t + \sum_{j=1}^{m} b_{ij}\,\Delta w_j(t_0) +$$

$$+ \sum_{j=1}^{m}\sum_{k=1}^{n}\sum_{\ell=1}^{m} \frac{\partial b_{ij}}{\partial x_k}\, b_{k\ell}\,\eta_{\ell j}(t_0) + \frac{\Delta t}{2}\sum_{j=1}^{m}\left(\frac{\partial b_{ij}}{\partial t} + \right. \tag{41}$$

$$\left. + \sum_{k=1}^{n} \frac{\partial a_i}{\partial x_k}\, b_{kj} + \sum_{k=1}^{n} \frac{\partial b_{ij}}{\partial x_k}\, a_k\right)\Delta w_j(t_0) +$$

$$+ \left(\frac{\partial a_i}{\partial t} + \sum_{k=1}^{n} a_k\,\frac{\partial a_i}{\partial x_k}\right)\frac{(\Delta t)^2}{2}$$

where $\Delta w_j(t_0) = w_j(t) - w_j(t_0)$,

$$\eta_{\ell j}(t_0) = \begin{cases} \frac{1}{2}\,[(\Delta w_j(t_0))^2 - \Delta t], & \ell = j, \\ \frac{1}{2}\,\Delta w_\ell(t_0)\,\Delta w_j(t_0), & \ell \neq j. \end{cases} \tag{42}$$

Algorithm (41) is not used in our routine. We have published it, as expression (41) is more general than expressions (15) and (23) in [14]. Our scheme (41) provides not only sample moments convergence as (15) and (23) in [14], but also the sample path convergence. Besides expression (23) in [14] contains some misprints and expression (15) in [14] doesn't consider the correlation between dependent random variables Z_{1n} and Z_{2n}. Such a correlation is taken into account in scheme (3.2) in [6] but this algorithm is implementable only for the one-dimension case (and provides only sample moments convergence).

In order to obtain the Stratonovich solution of equation (19) by means of (41) it is necessary to replace expression (42) by

$$\eta_{\ell j}(t_0) = \frac{1}{2}\,\Delta w_\ell(t_0)\cdot\Delta w_j(t_0)\,. \tag{42}$$