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Event-Based State Estimation Using an Improved Stochastic Send-on-Delta Sampling Scheme

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***Abstract*—Event-based sensing and communication holds the promise of lower resource utilization and/or better performance for remote state estimation applications found in e.g. networked control systems. Recently, stochastic event-triggering rules have been proposed as a means to avoid the complexity of the problem that normally arises in event-based estimator design. By using a scaled Gaussian function in the stochastic triggering scheme, the optimal remote state estimator becomes a linear Kalman filter with a case dependent measurement update. In this paper we propose a modified version of the stochastic send-on-delta triggering rule. The idea is to use a very simple predictor in the sensor, which allows the communication rate to be reduced while preserving estimation performance compared to regular stochastic send-on-delta sampling. We derive the optimal mean-square error estimator for the new scheme and present upper and lower bounds on the error covariance. The proposed scheme is evaluated in numerical examples, where it compares favorably to previous stochastic sampling approaches, and is shown to preserve estimation performance well even at large reductions in communication rate.**

I. Introduction

In recent years, networked control systems (NCSs) have been finding application in a broad range of areas, such as industrial automation, health care, public transport and aerospace [1], [2]. While NCSs offers a lot in terms of flexibility, the communication between devices over band-limited, often wireless, shared networks can pose a constraint in several applications. Examples of NCSs where communication is severely limited include unmanned aerial vehicles (UAVs) with stealth requirements, vehicles with tight power-budgets such as planetary rovers, long endurance sensor networks with limited energy supply and underwater vehicles [2]. Additionally, having a large number of sensors sharing a network with limited bandwidth may risk congestion [3]. Abandoning the traditional approach of periodic sampling and communication in favor of event-based schemes has the potential of substantially reducing communication and computation rate while still preserving good application performance.

A simple and intuitive idea is to let the sensor communicate only when a significant change has occurred in the measured variable. This is known as send-on-delta sampling [4], where the collected measurement is commonly compared to the last transmitted value. A variant is to instead compare with the estimator's prediction, see e.g. [5]. Since the sensor then knows the exact prediction of the estimator, it is possible to achieve very efficient communication. However, this kind of scheme requires either a local copy of the estimator in the sensor, or estimator-to-sensor feedback, which come with the cost of more energy-consuming computations in the sensor or more network usage respectively.

A fundamental challenge with deterministic send-on-delta approaches is that they involve nonlinear measurements, which implies that the design of an optimal state estimator becomes intractable for higher-order systems [6], [7]. While approximative estimation methods such as particle filters exist, these typically involve very heavy on-line computations and do not easily lend themselves to analysis. A survey of various event-based estimation methods is found in [8].

Recently, stochastic event-trigger conditions have been proposed to alleviate the problem of intractable optimal estimator design [9]–[11]. By using a stochastic sampler involving a scaled Gaussian function in the event generator, the optimal remote estimator for a linear system driven by white Gaussian noise turns into a Kalman filter with case dependent measurement update. In [9], two stochastic sampling schemes—one open-loop (OL) and one closed-loop (CL)—are proposed and analyzed. In the OL scheme, the probability of sampling increases with the magnitude of the measured output. In the CL scheme, the probability of sampling increases with the magnitude of the innovation generated by the measurement, requiring sensor-to-estimator feedback. Bounds on the asymptotic error covariance are presented as well as a closed-form formula for the average communication rate under the OL scheme. In [11], stochastic send-on-delta sampling (SSOD) is proposed as a third possible scheme, where the probability of sampling increases with the distance between the measured output and the previously transmitted value. The SSOD scheme is however to our knowledge not analyzed nor evaluated in any examples.

In this paper, we propose a modified SSOD scheme. By introducing a very simple predictor in the sensor, the measured variable is not compared with the last transmitted measurement but rather with a scaled version of it. This allows for a performance closer to that of the CL scheme while still running in open loop (i.e., there is no feedback communication from the estimator to the sensor).

The remainder of this paper is outlined as follows. In Section II the remote estimation problem is formulated. The basic framework of stochastic event-triggering, a brief review of the schemes proposed in [9] and [11] and a derivation of the modified stochastic send-on-delta scheme are presented in Section III. The optimal estimator for the new scheme and bounds on the resulting error covariance are presented

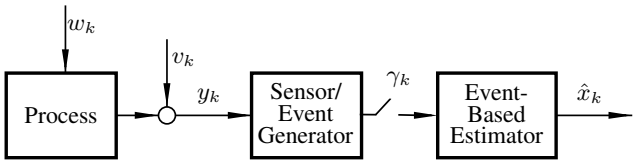

Fig. 1. The remote state estimation problem. The event-based estimator should produce optimal estimates of the process states based on irregularly communicated measurements from the sensor.

in Section IV. The new scheme is evaluated in numerical examples, with comparisons to the earlier stochastic approaches and a simulated case study, in Section V. Finally, the paper is concluded in Section VI. The derivation of the MMSE estimator presented in Section IV is given in the Appendix.

II. The Remote Estimation Problem

A. Problem Formulation

We study a prototypical remote state estimation problem involving a process subject to disturbances, a sensor with an event generator, and a remote event-based state estimator, see Figure 1. Assume that the process is described by a discrete-time linear system

$$\begin{aligned} x_{k+1} &= Ax_k + w_k \\ y_k &= Cx_k + v_k \end{aligned} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $y_k \in \mathbb{R}^m$ is the measurement vector collected by the sensor and $w_k \in \mathbb{R}^n$ and $v_k \in \mathbb{R}^m$ are mutually uncorrelated white Gaussian noises with covariance matrices $Q > 0$ and $R > 0$. The initial state vector x_0 is assumed to be zero-mean and Gaussian with covariance matrix $\Sigma_0 > 0$. The initial state is uncorrelated with w_k and v_k for all $k \geq 0$. The system is assumed to be observable.

The goal of the remote estimator is to produce the optimal estimate of the state vector x_k in the minimum mean-square error (MMSE) sense based on the history of transmitted measurements. In the case when the sensor transmits y_k at every time instant k, the MMSE state estimator for (1) is the standard Kalman filter [12]. In the case when the sensor transmits measurements based on some triggering condition the problem instead becomes event-based, where each transmitted measurement marks an event.

Adopting the notation of [9]–[11], let the variable γ_k signal events, such that $\gamma_k = 1$ means that the measurement was transmitted from the sensor, and $\gamma_k = 0$ that it was not. Assuming that no data is lost in the transmission, the information set I_k available to the remote estimator at time k is

$$I_k \triangleq \{\gamma_0, \ldots, \gamma_k, y_0^r, \ldots, y_k^r\}$$

where y_k^r is the received measurement at time k given as

$$y_k^r = \begin{cases} y_k & \text{if } \gamma_k = 1 \\ \emptyset & \text{if } \gamma_k = 0 \end{cases}$$

Furthermore, we define $I_{-1} \triangleq \emptyset$. Conditioned on the information set, the MMSE predictions of the state and measurement vectors, $\hat{x}_k^-$, $\hat{y}_k^-$, and the MMSE estimate of the state vector, $\hat{x}_k$, are

$$\hat{x}_k^- \triangleq \mathbb{E}\{x_k \mid I_{k-1}\},\ \hat{y}_k^- \triangleq \mathbb{E}\{y_k \mid I_{k-1}\},\ \hat{x}_k \triangleq \mathbb{E}\{x_k \mid I_k\}$$

and the covariance matrices for the corresponding prediction and estimation errors of the state vector are

$$\begin{aligned} P_k^- &\triangleq \mathbb{E}\{(x_k - \hat{x}_k^-)(x_k - \hat{x}_k^-)^T \mid I_{k-1}\} \\ P_k &\triangleq \mathbb{E}\{(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T \mid I_k\} \end{aligned}$$

In the standard Kalman filter the MMSE estimates and covariances above are easily computed through closed-form recursive equations. It is desirable to preserve that property also in the event-based case. Following [9]–[11], we will consider stochastic triggering conditions for the event generator. Below we describe the basic framework of stochastic triggering, make a brief review of the previously proposed schemes and then propose an new stochastic triggering condition based on the SSOD scheme.

III. Stochastic Triggering Conditions

A. Basic Framework

The principle of stochastic triggering is as follows. At each time instant k, the event generator generates an independent and identically distributed random variable ζ_k, which is uniformly distributed over $[0, 1]$. The random variable ζ_k is compared to the value of a decision function $\varphi(y_k - \mu_k) : \mathbb{R}^m \to [0, 1]$ that depends on the difference between the collected measurement y_k and a prediction μ_k of the measurement. The prediction is based on information available to both the sensor and the remote estimator. The event generator then decides if the sensor should transmit or not according to

$$\gamma_k = \begin{cases} 1, & \text{if } \zeta_k > \varphi(y_k - \mu_k) \\ 0, & \text{if } \zeta_k \leq \varphi(y_k - \mu_k) \end{cases}$$

Since ζ_k is uniformly distributed, this rule has the property

$$\begin{aligned} \Pr(\gamma_k = 0) &= \varphi(y_k - \mu_k) \\ \Pr(\gamma_k = 1) &= 1 - \varphi(y_k - \mu_k) \end{aligned}$$

i.e., the probability of not transmitting y_k is equal to the value of the decision function. Intuitively, $\varphi(y_k - \mu_k)$ should be chosen such that it attains a value close to 1 when y_k does not deviate much from the prediction μ_k, and close to 0 when the deviation is large. In [9], the authors propose $\varphi(y_k - \mu_k)$ as a scaled Gaussian function, which leads to closed-form recursive equations for the corresponding MMSE estimator.

B. Previously Suggested Schemes

We now make a brief review of the versions of $\varphi(y_k - \mu_k)$ proposed in [9] and [11]. These are:

Open-Loop (OL):

$$\varphi_{OL}(y_k) \triangleq \exp(-\frac{1}{2} y_k^T Y y_k) \tag{2}$$

Closed-Loop (CL):

$$\varphi_{CL}(y_k - \hat{y}_k^-) \triangleq \exp(-\frac{1}{2}(y_k - \hat{y}_k^-)^T Y (y_k - \hat{y}_k^-)) \tag{3}$$

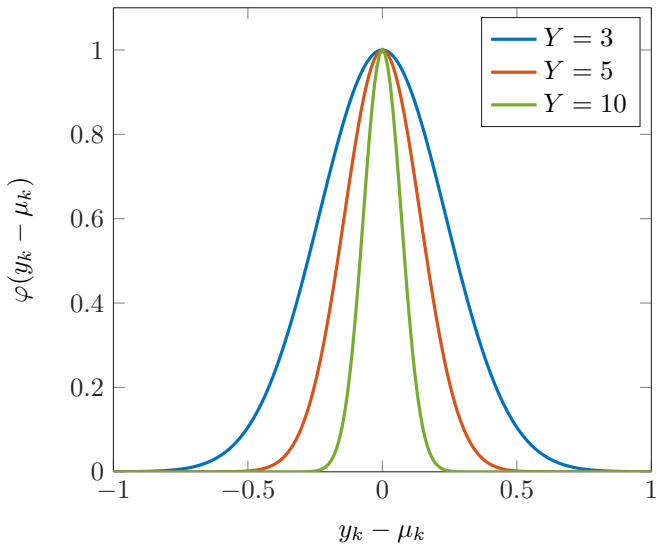

Fig. 2. The decision function $\varphi(y_k - \mu_k)$ in the scalar case for different values of Y.

Stochastic Send-on-Delta (SSOD):

$$\varphi_{SSOD}(y_k - y_{k-l}) \triangleq \exp(-\frac{1}{2}(y_k - y_{k-l})^T Y (y_k - y_{k-l})) \tag{4}$$

In these expressions, the matrix $Y > 0$ is a design parameter that influences the shape of the decision function, and $l \geq 1$ is the number of time steps since the latest transmission made by the sensor. Figure 2 illustrates the shape of the decision function for different values of Y in the scalar case. By varying the value of Y the probability of transmitting and hence the expected communication rate can be varied. From the expressions above, we see that the schemes only differ in the choice of prediction, where OL uses $\mu_k = 0$, CL uses $\mu_k = \hat{y}_k^-$ and SSOD uses $\mu_k = y_{k-l}$.

The OL scheme compares the collected measurement to the prediction zero in each time instant, resulting in high probability of transmission when the magnitude of y_k is great. The prediction is based on the fact that when the system (1) is stable, then y_k will be zero-mean in stationarity. The main strength of the OL scheme is that the prediction requires no computations from the sensor. However, it is only suitable for stable systems, since for unstable systems the scheme will trivially result in $\gamma_k = 1$ almost always after a sufficiently long time.

The CL scheme works in both the stable and the unstable case, since $\hat{y}_k^-$ will follow the dynamics of the system. However, a major drawback of the CL scheme is that it requires transmission of $\hat{y}_k^-$ to the sensor at every time-instant k, counter-acting the objective of reducing communication rate in the network. It is however still interesting as an indicator of the achievable performance with stochastic triggering.

The need for a simple prediction makes the SSOD scheme interesting. With $\mu_k = y_{k-l}$, there are no computations required like in the OL scheme, but intuitively it should offer a reasonable prediction of y_k for small values of l and slow-varying processes. However, y_{k-l} will generally be a poor prediction as l grows due to the dynamics in the system. This motivates us to improve the SSOD scheme by introducing a simple predictor for y_k in the sensor, while still not requiring any estimator-to-sensor communication.

C. Stochastic Send-on-Delta with Simple Prediction

To derive a simple prediction of y_k for the stochastic send-on-delta scheme, we start by computing $\mathbb{E}\{y_k \mid y_{k-l}\}$ using the following theorem:

Theorem 1: (Theorem 3.2 in [13], Ch. 7)
Let x and y be two vectors which are jointly Gaussian. The conditional distribution of x given y is Gaussian with mean

$$\mathbb{E}\{x \mid y\} = \mathbb{E}\{x\} + R_{xy} R_y^{-1} (y - \mathbb{E}\{y\})$$

where

$$R_{xy} = \mathbb{E}\{(x - \mathbb{E}\{x\})(y - \mathbb{E}\{y\})^T\}$$
$$R_y = \mathbb{E}\{(y - \mathbb{E}\{y\})(y - \mathbb{E}\{y\})^T\}$$

The covariance of x conditioned on y is

$$\mathbb{E}\{(x - \mathbb{E}\{x|y\})(x - \mathbb{E}\{x|y\})^T|y\} = R_x - R_{xy} R_y^{-1} R_{xy}^T$$

where

$$R_x = \mathbb{E}\{(x - \mathbb{E}\{x\})(x - \mathbb{E}\{x\})^T\}$$

Furthermore, the stochastic variables y and $x - \mathbb{E}\{x \mid y\}$ are independent.

The proof is stated in [13]. Since y_k and y_{k-l} are jointly Gaussian, we thus have

$$\mathbb{E}\{y_k \mid y_{k-l}\} = \mathbb{E}\{y_k\} + R_{y_k y_{k-l}} R_{y_{k-l}}^{-1} (y_{k-l} - \mathbb{E}\{y_{k-l}\})$$

Assuming (1) is stable and has converged to its stationary distribution, we have $\mathbb{E}\{y_k\} = \mathbb{E}\{y_{k-l}\} = 0$ which leads to

$$\mathbb{E}\{y_k \mid y_{k-l}\} = R_{y_k y_{k-l}} R_{y_{k-l}}^{-1} y_{k-l} \tag{5}$$

Next we compute the matrices $R_{y_k y_{k-l}}$ and $R_{y_{k-l}}$ under the same assumption:

$$\begin{aligned} R_{y_k y_{k-l}} &= \mathbb{E}\{y_k y_{k-l}^T\} = \mathbb{E}\{(Cx_k + v_k)(Cx_{k-l} + v_{k-l})^T\} \\ &= C\mathbb{E}\{x_k x_{k-l}^T\} C^T \\ &= C\mathbb{E}\{(\sum_{j=1}^{l} A^{j-1} w_{k-j} + A^l x_{k-l}) x_{k-l}^T\} C^T \\ &= CA^l \mathbb{E}\{x_{k-l} x_{k-l}^T\} C^T = CA^l \Sigma C^T \end{aligned}$$

Here $\Sigma > 0$ is the steady-state covariance of the state vector, which, when (1) is stable, can be obtained as the solution to the discrete-time Lyapunov equation

$$\Sigma = A\Sigma A^T + Q \tag{6}$$

Using similar calculations, the matrix $R_{y_{k-l}}$ is

$$\begin{aligned} R_{y_{k-l}} &= \mathbb{E}\{y_{k-l} y_{k-l}^T\} = \mathbb{E}\{(Cx_{k-l} + v_{k-l})(Cx_{k-l} + v_{k-l})^T\} \\ &= C\Sigma C^T + R \end{aligned}$$

We thus see that with the assumption on stationarity the matrix $R_{y_k y_{k-l}} R_{y_{k-l}}^{-1}$ only depends on l. Motivated by (5) we propose a new stochastic send-on-delta scheme, which uses a simple prediction dependent on the number of samples since the last transmission:

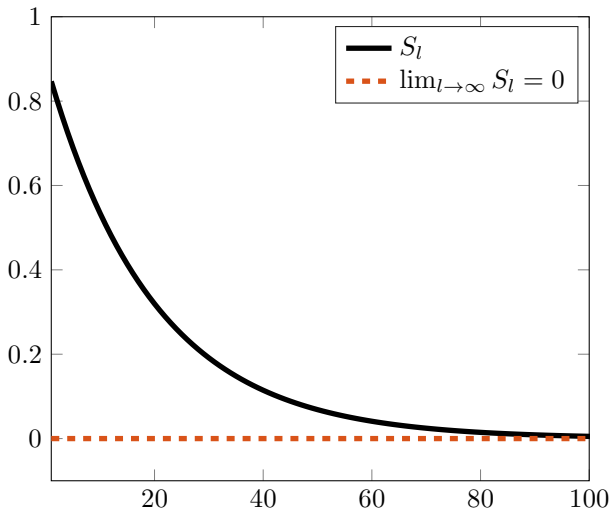

Fig. 3. Convergence of the scaling factor S_l for the system (8).

Stochastic Send-on-Delta with Simple Prediction (SSOD-P):

$$\varphi_{SSOD-P}(z_{k,l}) \triangleq \exp(-\frac{1}{2} z_{k,l}^T Y z_{k,l}) \tag{7}$$

where

$$z_{k,l} \triangleq y_k - S_l y_{k-l}$$
$$S_l \triangleq CA^l \Sigma C^T (C\Sigma C^T + R)^{-1}$$

The proposed scheme is useful for stable systems and takes into account the cross-covariance between the last transmitted and the current measurement. As the matrix S_l is known beforehand, a look-up table can be computed offline and stored in the sensor for improved efficiency. In practice, S_l can for sufficiently large l be approximated with its limit

$$\lim_{l\to\infty} S_l = 0$$

We observe that the limit case of the proposed scheme is identical to the OL scheme, i.e., when y_{k-l} no longer offers any information on predicting the value of y_k. Depending on the A matrix, the convergence speed to the limit differs and hence also the required memory storage. As an example, consider the scalar system with slow dynamics [11]:

$$A = 0.95,\ C = 1,\ Q = 0.8,\ R = 1 \tag{8}$$

In Figure 3 the values of S_l are computed for the system (8) and compared to the limit value. The decreasing value of S_l reflects the loss of correlation between y_k and y_{k-l} as l increases. In this example, S_l has essentially converged to zero after 100 steps, which for single-precision (32 bit) arithmetics would require a look-up table of size of 400 bytes.

IV. The MMSE Estimator

In this section we present the recursive closed-form equations for the MMSE estimator when using the SSOD-P scheme (7). We also present bounds on the covariance P_k^-.

A. Estimator

The MMSE estimator for the SSOD-P is given by the following theorem:

Theorem 2: Consider the system (1) where the measurement vector y_k is transmitted to a remote estimator based on a stochastic decision according to the SSOD-P scheme in (7). Then x_k conditioned on I_{k-1} and I_k respectively are Gaussian distributed, and their mean and covariance satisfy to following recursive equations:

Initial Conditions:

$$\hat{x}_0^- = 0$$
$$P_0^- = \Sigma_0$$

Time Update:

$$\hat{x}_k^- = A\hat{x}_{k-1} \tag{9}$$
$$\hat{y}_k^- = C\hat{x}_k^- \tag{10}$$
$$P_k^- = AP_{k-1}A^T + Q \tag{11}$$

Measurement Update:

$$\hat{x}_k = \hat{x}_k^- + K_k(\gamma_k z_{k,l} - \hat{z}_{k,l}^-) \tag{12}$$
$$\hat{z}_{k,l}^- \triangleq \mathbb{E}\{z_{k,l} \mid I_{k-1}\} = \hat{y}_k^- - S_l y_{k-l} \tag{13}$$
$$P_k = P_k^- - K_k C P_k^- \tag{14}$$
$$K_k = P_k^- C^T [C P_k^- C^T + R + (1-\gamma_k) Y^{-1}]^{-1} \tag{15}$$

Remark 1: The MMSE estimator for a sensor using the regular SSOD scheme in (4) can be shown to be identical to the equations in Theorem 2 with $S_l = 1$.

The proof of Theorem 2 is stated in the Appendix. The resulting MMSE estimator resembles the standard Kalman filter, with recursive equations for time and measurement update. In fact, the measurement update in the case when $\gamma_k = 1$ and the time update are identical to the standard Kalman filter. When $\gamma_k = 0$, the predicted measurement $\hat{y}_k^-$ is compared to the scaled previous measurement $S_l y_{k-l}$ instead of y_k, and the added uncertainty is reflected in the addition of Y^{-1} in the expression for K_k. The case when $\gamma_k = 0$ can be interpreted as a measurement update with additional Gaussian noise with covariance Y^{-1}. This simple representation of the added uncertainty in the MMSE estimator is due to the Gaussian shape of the decision function in the event generator.

B. Covariance Bounds

The bounds on the covariance matrix P_k^- for the OL scheme derived in [11] can also be used to derive the same bounds for the proposed SSOD-P scheme since the MMSE estimators in both cases have the same equations for updating the error covariance. Thus we have the following bounds on the covariance matrix P_k^-:

Theorem 3: (See Theorem 6.3.3 in [11], Ch. 6) Assume the SSOD-P scheme in (7) is applied to the system (1) and the states are estimated with the MMSE estimator in Theorem 2. If the system is stable, then for any $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $k \geq N$ the following limits on P_k^- hold:

$$\underline{X} - \epsilon I \leq P_k^- \leq \bar{X} + \epsilon I$$

Here, $\underline{X}$ and $\bar{X}$ are the unique solutions to the discrete-time algebraic Riccati equations

$$\underline{X} = A\underline{X}A^T + Q - A\underline{X}C^T(C\underline{X}C^T + R)^{-1}C\underline{X}A^T$$

and

$$\bar{X} = A\bar{X}A^T + Q - A\bar{X}C^T(C\bar{X}C^T + R + Y^{-1})^{-1}C\bar{X}A^T$$

The proof is identical to the proof of Theorem 6.3.3 in [11] and is hence omitted. Note that the matrices $\bar{X}$ and $\underline{X}$ correspond to the stationary value of P_k^- in the extreme cases when $\gamma_k = 0$ and $\gamma_k = 1$ for all k respectively.

V. Numerical Examples

To demonstrate the advantage of using the SSOD-P scheme in (7) compared to the previously proposed schemes in (2)–(4), and to showcase the potential of reducing the communication rate, we here present some numerical examples. In Section V-A the asymptotic mean trace of the prediction error covariance, $\lim_{k\to\infty} \mathbb{E}\{\text{tr}(P_k^-)\}$, is compared between the schemes for different communication rates $0 \leq \gamma \leq 1$, here defined as the mean number of transmissions per time unit. The examples in this section show that the proposed scheme give comparable performance to the CL scheme without any estimator-to-sensor communication, and outperforms both the regular SSOD and the OL schemes. In Section V-B we simulate the SSOD-P scheme with different communication rates in a state feedback position control case study. The example shows that large reductions in communication rate can be achieved while maintaining good control and estimation performance.

A. Error Covariance versus Communication Rate

Consider the first order stable case of (1) with $A = 0.95$, $C = 1$, $Q = 0.8$ and $R = 1$ which was evaluated for the OL and CL schemes in [11]. In this example we also compare the performance of the MMSE estimators using the regular SSOD and the proposed SSOD-P schemes. The Y parameter is varied between 0.05 and 20 to obtain different communication rates. The asymptotic mean error variance is computed for a simulation of 50,000 time steps at each Y value, and the results are presented in Figure 4. We observe that both the SSOD and the SSOD-P schemes has a performance comparable to the CL scheme without any estimator-to-sensor feedback, and they clearly outperform the OL scheme. To illustrate the advantage of featuring the scaling S_l in the SSOD-P scheme, we consider another example of a highly oscillatory second order system:

$$A = \begin{bmatrix} -0.85 & -0.35 \\ 0.35 & -0.85 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 10^{-3} & 0 \\ 0 & 1 \end{bmatrix}, \; R = 0.1$$

The trace of the asymptotic mean error covariance matrix is computed for a simulation of 50,000 time steps at each communication rate, and the results are presented in Figure 5. In this example the SSOD-P scheme outperforms the SSOD, which performs worse than also the OL scheme. The reason is that the fast oscillatory dynamics make y_k deviate strongly from y_{k-l} even for small l, which forces the sensor to transmit more often in the SSOD case than in the SSOD-P case. Still in this highly oscillatory example the proposed SSOD-P scheme performs comparably well to the CL scheme.

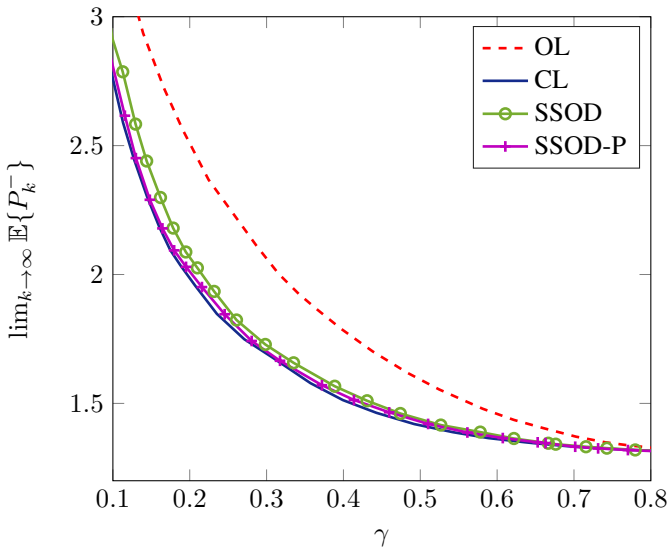

Fig. 4. Asymptotic expected error variance $\lim_{k\to\infty} \mathbb{E}\{P_k^-\}$ versus communication rate γ under the OL, CL, SSOD and the SSOD-P schemes for a first order system.

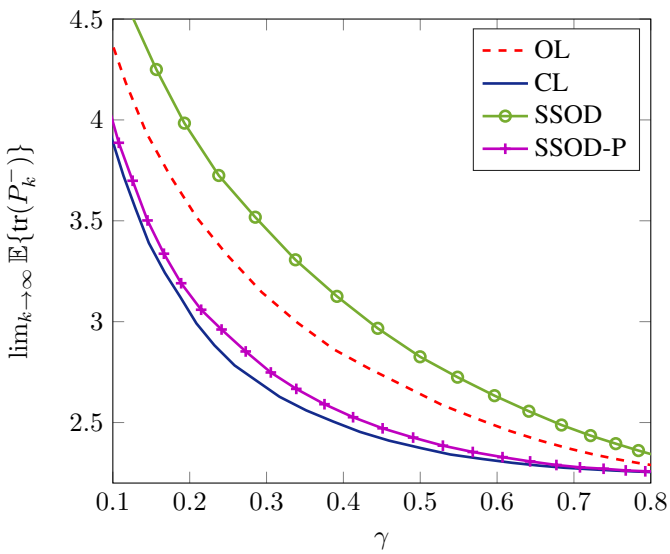

Fig. 5. Trace of the asymptotic error covariance matrix $\lim_{k\to\infty} \mathbb{E}\{\text{tr}(P_k^-)\}$ versus communication rate γ under the OL, CL, SSOD and SSOD-P schemes for a second order oscillatory system.

B. Illustrative Case Study

To showcase the potential of reducing the communication rate using the SSOD-P scheme, we here present a simulated case study of position control of elastic connected masses using a linear servo. The process consists of two masses on carts, connected together with a spring, which both can move in one dimension. A linear servo is used to apply a force to the first cart, while the available sensor measurement from the process is the position of the second cart. Using state feedback, the control objective is to reject disturbances and bring both cart positions and velocities to zero.

The process is described by the differential equations

$$m\ddot{p_1}(t) = -d_1\dot{p_1}(t) - k(p_1(t) - p_2(t)) + k_m u(t)$$
$$m\ddot{p_2}(t) = -d_2\dot{p_2}(t) + k(p_1(t) - p_2(t))$$

where the positions of the carts are p_1 and p_2 respectively, and u is the voltage input to the linear servo. The mass of each cart is $m = 2$ kg, the damping acting on the carts are $d_1 = 3$ and $d_2 = 4$ Nm^{-1}s respectively, the spring constant is $k = 400$ Nm^{-1} and the linear servo gain is $k_m = 3$ NV^{-1}. We use a sampling period of 0.01 s to represent the process in discrete time. Adding Gaussian process- and measurement noise, the resulting state space model is given as

$$x_{k+1} = Ax_k + Bu_k + w_k$$
$$y_{k+1} = Cx_k + v_k$$
$$w_k \sim N(0, Q),\ v_k \sim N(0, R)$$

where

$$A = \begin{bmatrix} 0.99 & 0.01 & 0.01 & 0 \\ -1.97 & 0.98 & 1.97 & 0.01 \\ 0.01 & 0 & 0.99 & 0.01 \\ 1.97 & 0.01 & -1.97 & 0.97 \end{bmatrix}$$
$$B = \begin{bmatrix} 0 \\ 0.01 \\ 0 \\ 0 \end{bmatrix}, \quad C = [0 \quad 0 \quad 1 \quad 0]$$
$$Q = 5 \cdot 10^{-7} \cdot \mathrm{I}_{4\times 4}, \quad R = 10^{-4}$$

The state vector is $x_k = [x_{1,k}\, x_{2,k}\, x_{3,k}\, x_{4,k}]^T$ where the states $x_{1,k}$, $x_{2,k}$ represent the first cart's position and velocity respectively, and $x_{3,k}$, $x_{4,k}$ are the corresponding states for the second cart. To make the system asymptotically stable a state feedback law, $u_k = -Lx_k$, obtained through LQR-design is used, where $L = [108.14,\ 14.39,\ -14.85,\ 5.33]$. Since the true process state vector is not available, the estimates of the SSOD-P, $\hat{x}_k$, are used in the state feeback control law.

The controlled process using the SSOD-P scheme for state estimation was simulated in stationarity for 10^6 time steps for three different values of Y corresponding to the communication rates $\gamma = 0.7$, 0.3 and 0.1. The nominal case of $\gamma = 1$, i.e., estimation using the standard Kalman filter, was also simulated for reference. The standard deviations of the control error and estimation error for each state are presented in Table I and II respectively.

TABLE I. STANDARD DEVIATION OF CONTROL ERROR

Y	γ	x_1 [mm]	x_2 [mms^{-1}]	x_3 [mm]	x_4 [mms^{-1}]
-	1.0	3.71	42.59	3.93	45.66
$8.90 \cdot 10^4$	0.7	3.73	42.77	3.95	45.78
$9.05 \cdot 10^3$	0.3	4.11	44.89	4.27	47.14
$1.86 \cdot 10^3$	0.1	5.46	49.04	5.53	50.04

TABLE II. STANDARD DEVIATION OF ESTIMATION ERROR

Y	γ	x_1 [mm]	x_2 [mms^{-1}]	x_3 [mm]	x_4 [mms^{-1}]
-	1.0	3.18	39.69	3.14	39.56
$8.90 \cdot 10^4$	0.7	3.20	39.90	3.17	39.76
$9.05 \cdot 10^3$	0.3	3.55	42.52	3.55	42.35
$1.86 \cdot 10^3$	0.1	4.88	47.77	4.92	47.60

From Tables I and II we see that the impact on control and estimation performance is small in relation to the reduction in communication rate. When transmitting 70% of the measurements, the performance degradation is minimal. The relative increase in standard deviation of the control error is then in the approximate range 0.3–0.5%. The performance degradation in the estimation is of similar magnitude, with an increase of 0.5–1.0%. Even when transmitting only 30% of the measurements, the increase relative to the nominal case is only 3.2–10.8% for the control error and 7.1–13.1% for the estimation error. However, in the extreme case of transmitting only 10% of the collected measurements the performance starts to be significantly degraded. The relative increase in standard deviation is then 9.6–47.2% for the control error and 20.3–56.7% for the estimation error.

To see the impact of reduced communication rate in presence of a large disturbance, an impulse force on the second cart was simulated, where the cart velocity jumps to 1 ms^{-1}. Using the same values of Y, the resulting trajectories of the cart positions are presented in Figure 6. Also in this example it becomes apparent that the degradation of control performance is small in relation to the reduction in communication rate. In Figure 6, the simulation lasts for 5 seconds, corresponding to 500 discrete time steps, with the impulse occurring after 1 second. The response with $Y = 8.90 \cdot 10^4$, resulting in 400 transmissions, is practically identical to the nominal case where all 500 measurements are transmitted. Even with $Y = 9.05 \cdot 10^3$ and 274 transmissions the degradation is minimal. Only with $Y = 1.86 \cdot 10^3$ and 191 transmissions is there a clearly visible deviations from the nominal case, where the disturbance is somewhat slower rejected.

VI. CONCLUSION

We have proposed a new stochastic send-on-delta sampling scheme with a simple predictor in the event generator, which can reduce the communication rate in remote state estimation applications while still yielding a simple estimator design. The event generator in the sensor must store a look-up table of a scaling factor in order to correctly predict the measurement vector at each time step, before stochastically deciding whether to communicate the measurement or not to the remote state estimator. The solution is hence slightly more complex than the previously proposed open-loop and stochastic send-on-delta schemes, but offers better estimation performance.

The corresponding MMSE remote estimator has been derived for the scheme, and we have also presented bounds on the resulting error covariance matrix. However, there is room for further analysis, including derivations of expected communication rate and asymptotic error covariances, similar to what is done for the open-loop and closed-loop schemes in [9] and [11].

The performance of the proposed modified stochastic send-on-delta scheme has been compared to the other stochastic triggering schemes proposed in [9] and [11] in numerical examples. Considering the case of no estimator-to-sensor feedback, the proposed scheme outperforms the other schemes. The proposed scheme also show a comparable performance to the closed-loop scheme which utilizes estimator-to-sensor feedback. Also, a simulated case study using the proposed

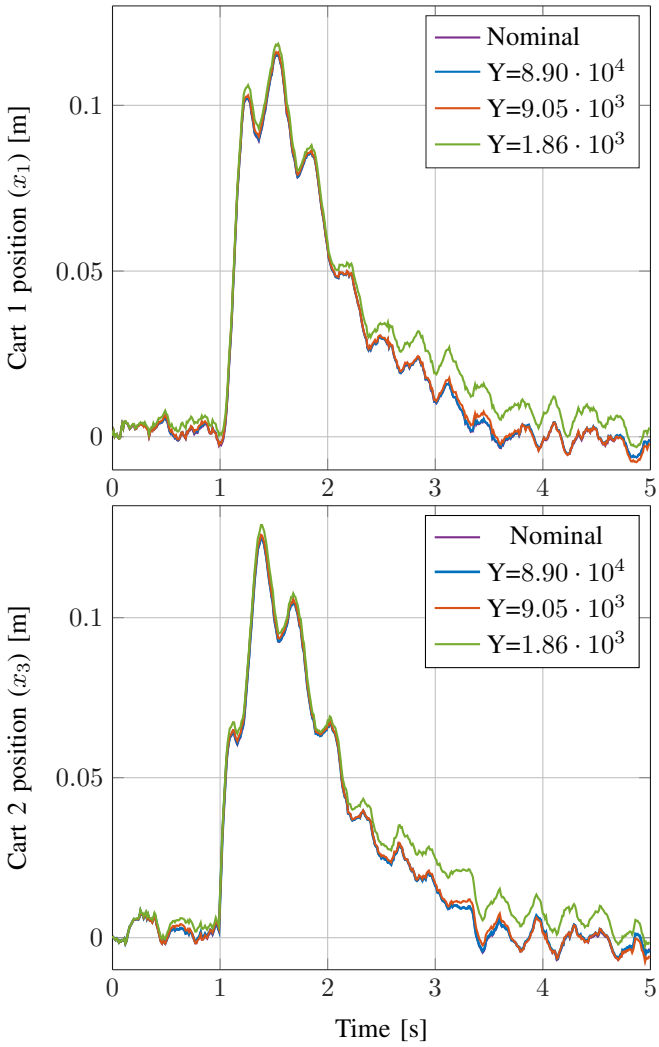

Fig. 6. Comparison of impulse disturbance rejection using state feedback and state estimation with the SSOD-P scheme for different communication rates. The impulse occurs at time 1 s.

scheme showcases the potential of reducing the communication rate significantly in relation to the price paid in estimation performance.

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Appendix

The following lemma will be used in the proof of Theorem 2:

Lemma 1: (Lemma 6.1 in [11], Ch 6)
Let $\Phi > 0$ be partitioned as

$$\Phi = \begin{bmatrix} \Phi_{xx} & \Phi_{xz} \\ \Phi_{xz}^T & \Phi_{zz} \end{bmatrix}$$

where $\Phi_{xx} \in \mathbb{R}^{n\times n}$, $\Phi_{xz} \in \mathbb{R}^{n\times m}$ and $\Phi_{zz} \in \mathbb{R}^{m\times m}$. Then it holds that

$$\Phi^{-1} + \begin{bmatrix} 0 & 0 \\ 0 & Y \end{bmatrix} = \Theta^{-1}$$

where

$$\Theta = \begin{bmatrix} \Theta_{xx} & \Theta_{xz} \\ \Theta_{xz}^T & \Theta_{zz} \end{bmatrix}$$

and

$$\begin{aligned} \Theta_{xx} &= \Phi_{xx} - \Phi_{xz}(\Phi_{zz} + Y^{-1})^{-1}\Phi_{xz}^T \\ \Theta_{xz} &= \Phi_{xz}(I + Y\Phi_{zz})^{-1} \\ \Theta_{zz} &= (\Phi_{zz}^{-1} + Y)^{-1} \end{aligned}$$

The proof of Lemma 1 is stated in [11]. We now continue with the proof of Theorem 2:

Proof of Theorem 2: The theorem is proved by induction in the same manner as how the OL and CL MMSE estimators are derived in [11]. In (1) x_0 was assumed zero-mean Gaussian with covariance Σ_0, and since $I_{-1} = \emptyset$ we have the initial conditions

$$\begin{aligned} \hat{x}_0^- &= \mathbb{E}\{x_0 \mid I_{-1}\} = \mathbb{E}\{x_0\} = 0 \\ P_0^- &= \mathbb{E}\{(x_0 - \hat{x}_0^-)(x_0 - \hat{x}_0^-)^T\} = \Sigma_0 \end{aligned}$$

Assume that x_k conditioned on I_{k-1} is Gaussian with mean $\hat{x}_k^-$ and covariance P_k^-. For the measurement update there are two possibilities. We start with the case when $\gamma_k = 0$:

Measurement Update, $\gamma_k = 0$: Consider the joint conditional probability density function of x_k and $z_{k,l}$,

$$\begin{aligned} f(x_k, z_{k,l} \mid I_k) &= f(x_k, z_{k,l} \mid \gamma_k = 0, I_{k-1}) \\ &= \frac{\Pr(\gamma_k = 0 \mid x_k, z_{k,l}, I_{k-1}) f(x_k, z_{k,l} \mid I_{k-1})}{\Pr(\gamma_k = 0 \mid I_{k-1})} \\ &= \frac{\Pr(\gamma_k = 0 \mid z_{k,l}) f(x_k, z_{k,l} \mid I_{k-1})}{\Pr(\gamma_k = 0 \mid I_{k-1})} \end{aligned} \tag{16}$$

where Bayes' theorem was used in the second equality, and the fact that the outcome of γ_k only depends on $z_{k,l}$ in the third equality. We now continue by evaluating (16). From (7) we have

$$\begin{aligned} \Pr(\gamma_k = 0 \mid z_{k,l}) &= \Pr(\zeta_k \le \varphi_{SSOD-P}(z_{k,l}) \mid z_{k,l}) \\ &= \exp(-\frac{1}{2} z_{k,l}^T Y z_{k,l}) \end{aligned} \tag{17}$$

To evaluate $f(x_k, z_{k,l} \mid I_{k-1})$ we require the covariance matrix Φ_k of the joint probability vector $[x_k^T, z_{k,l}^T]^T$ conditioned on I_{k-1}. Observing the fact that $z_{k,l} - \hat{z}_{k,l}^- = y_k - \hat{y}_k^-$, Φ_k is

$$\Phi_k = \begin{bmatrix} P_k^- & P_k^- C^T \\ C P_k^- & C P_k^- C^T + R \end{bmatrix} \tag{18}$$

From (16)–(18) we get

$$f(x_k, z_{k,l} \mid I_k) = \alpha_k \exp(-\frac{1}{2}\theta_k)$$

where

$$\alpha_k = \frac{1}{\Pr(\gamma_k = 0 \mid I_{k-1})\sqrt{\det(\Phi_k)(2\pi)^{m+n}}}$$

and

$$\theta_k = \begin{bmatrix} x_k - \hat{x}_k^- \\ z_{k,l} - \hat{z}_{k,l}^- \end{bmatrix}^T \Phi_k^{-1} \begin{bmatrix} x_k - \hat{x}_k^- \\ z_{k,l} - \hat{z}_{k,l}^- \end{bmatrix} + z_{k,l}^T Y z_{k,l}$$

By using Lemma 1 and completing the square we get

$$\theta_k = \begin{bmatrix} x_k - \hat{x}_k \\ z_{k,l} - \hat{z}_{k,l} \end{bmatrix}^T \Theta_k^{-1} \begin{bmatrix} x_k - \hat{x}_k \\ z_{k,l} - \hat{z}_{k,l} \end{bmatrix} + c_k$$

where

$$\begin{aligned} \hat{x}_k &= \hat{x}_k^- - P_k^- C^T (C P_k^- C^T + R + Y^{-1})^{-1} \hat{z}_{k,l}^- \\ \hat{z}_{k,l} &= \hat{z}_{k,l}^- - [(C P_k^- C^T + R)^{-1} + Y]^{-1} Y \hat{z}_{k,l}^- \\ c_k &= \hat{z}_{k,l}^{-T} Y [I - ((C P_k^- C^T + R)^{-1} + Y)^{-1} Y] \hat{z}_{k,l}^- \end{aligned}$$

and

$$\Theta_k = \begin{bmatrix} \Theta_{xx,k} & \Theta_{xz,k} \\ \Theta_{xz,k}^T & \Theta_{zz,k} \end{bmatrix}$$

where

$$\begin{aligned} \Theta_{xx,k} &= P_k^- - P_k^- C^T (C P_k^- C^T + R + Y^{-1})^{-1} C P_k^- \\ \Theta_{xz,k} &= P_k^- C^T [I + Y (C P_k^- C^T + R)]^{-1} \\ \Theta_{zz,k} &= [(C P_k^- C^T + R)^{-1} + Y]^{-1} \end{aligned}$$

The joint conditional probability density is thus

$$f(x_k, z_{k,l} \mid I_k) = \beta_k \exp(-\frac{1}{2} \begin{bmatrix} x_k - \hat{x}_k \\ z_{k,l} - \hat{z}_{k,l} \end{bmatrix}^T \Theta_k^{-1} \begin{bmatrix} x_k - \hat{x}_k \\ z_{k,l} - \hat{z}_{k,l} \end{bmatrix}) \tag{19}$$

$$\beta_k = \alpha_k \exp(-\frac{1}{2} c_k)$$

where the fact that (19) integrates to 1 implies that

$$\beta_k = \frac{1}{\sqrt{\det(\Theta_k)(2\pi)^{n+m}}}$$

Thus x_k and $z_{k,l}$ conditioned on I_k are jointly Gaussian, where in particular x_k has the mean $\hat{x}_k$ and covariance $\Theta_{xx,k}$. This proves (12)–(15) when $\gamma_k = 0$.

Measurement Update, $\gamma_k = 1$: When $\gamma_k = 1$ the measurement y_k is transmitted. Thus

$$f(x_k \mid I_k) = f(x_k \mid y_k, I_{k-1}) \tag{20}$$

Computing (20) corresponds to the measurement update of the standard Kalman filter [12]. Thus, x_k conditioned on I_k when $\gamma_k = 1$ is Gaussian with mean

$$\hat{x}_k = \hat{x}_k^- + K_k (y_k - \hat{y}_k^-)$$

and covariance

$$P_k = P_k^- - K_k C P_k^-$$

which proves (12)-(15) when $\gamma_k = 1$.

Time Update: We now consider the time update. Assume x_{k-1} conditioned on I_{k-1} is Gaussian with mean $\hat{x}_{k-1}$ and covariance P_{k-1}. Then from (1) x_k conditioned on I_{k-1} will also be Gaussian since it is formed by a linear combination of Gaussian variables. We have

$$\begin{aligned} \hat{x}_k^- &= \mathbb{E}\{x_k \mid I_{k-1}\} = \mathbb{E}\{A x_{k-1} + w_{k-1} \mid I_{k-1}\} \\ &= A \mathbb{E}\{x_{k-1} \mid I_{k-1}\} = A \hat{x}_{k-1} \end{aligned}$$

and the covariance

$$\begin{aligned} P_k^- &= \mathbb{E}\{(x_k - A\hat{x}_{k-1})(x_k - A\hat{x}_{k-1})^T \mid I_{k-1}\} \\ &= A P_k A^T + Q \end{aligned}$$

Finally, the measurement vector y_k conditioned on I_{k-1} is

$$\begin{aligned} \hat{y}_k^- &= \mathbb{E}\{y_k \mid I_{k-1}\} = \mathbb{E}\{C x_k + v_k \mid I_{k-1}\} \\ &= C \mathbb{E}\{x_k \mid I_{k-1}\} = C \hat{x}_k^- \end{aligned}$$

This proves (9)–(11). Since x_0 conditioned on I_{-1} is Gaussian, we have by induction that x_k conditioned on I_k and I_{k-1} respectively will be Gaussian for all k. Thus the proof of Theorem 2 is concluded. By setting $S_l = 1$, the same proof can be used to derive the MMSE estimator for the regular stochastic send-on-delta scheme in (4). ■

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